# An alternative to Witt vectors

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(Communicated by Urs Hartl)

Dedicated to our friend and colleague Peter Schneider

**Abstract.** The ring of Witt vectors associated to a ring R is a classical tool in algebra. We introduce a ring C(R) which is more easily constructed and which is isomorphic to the Witt ring W(R) for a perfect  $\mathbb{F}_p$ -algebra R. It is obtained as the completion of the monoid ring  $\mathbb{Z}R$ , for the multiplicative monoid R, with respect to the powers of the kernel of the natural map  $\mathbb{Z}R \to R$ .

#### 1. Introduction

Since the work of Witt on discretely valued fields with given perfect residue field in [9] the "vectors" that carry his name have become important in many branches of mathematics. In [6], [7] Lazard gave a new approach to Witt vectors generalizing the theory to the case of a perfect  $\mathbb{F}_p$ -residue algebra. This approach is the one used in Serre [8] for example. Addition and multiplication of Witt vectors are defined by certain universal polynomials. This description is cumbersome. While thinking about periodic cyclic cohomology for  $\mathbb{F}_p$ -algebras we found an alternative C(R) to the p-typical Witt ring W(R) of a perfect  $\mathbb{F}_p$ -algebra R. The rings C(R) and W(R) are canonically isomorphic but the construction of C(R) as a completion (hence the name) of a monoid algebra  $\mathbb{Z}R$  is much simpler. We have therefore made an effort to develop the properties of C(R) independently of the theory of W(R).

Using the approach in [3], [4] we can define periodic cyclic homology for a ring R using completed extensions by free (noncommutative)  $\mathbb{Z}$ -algebras. If one applies this procedure to an  $\mathbb{F}_p$ -algebra R, the completion C(R) of the free  $\mathbb{Z}$ -module  $\mathbb{Z}R$  appears as a natural intermediate step.

If the  $\mathbb{F}_p$ -algebra R is not perfect, C(R) is still defined and different from W(R). However a somewhat more involved construction in the same spirit does

give W(R) in general. We will address this in a subsequent paper together with applications.

In this note all rings are commutative with 1 and all ring homomorphisms map 1 to 1. The background reference is [8, II §4–§6].

## 2. Construction and properties of C(R)

A p-ring A is a commutative ring with unit which is Hausdorff and complete for the topology defined by a sequence of ideals  $\mathfrak{a}_1 \supset \mathfrak{a}_2 \supset \ldots$  with the following properties:

- 1)  $\mathfrak{a}_i\mathfrak{a}_j\subset\mathfrak{a}_{i+j}$  for  $i,j\geq 1$ ,
- 2)  $A/\mathfrak{a}_1 = R$  is a perfect  $\mathbb{F}_p$ -algebra, i.e. the Frobenius homomorphism  $x \mapsto x^p$  is an isomorphism of R.

For a p-ring we have  $p \in \mathfrak{a}_1$  and hence  $\mathfrak{a}_i \supset p^i A$ . A p-ring is called strict if  $\mathfrak{a}_i = p^i A$  and if p is not a zero divisor in A. It is known that for every perfect  $\mathbb{F}_p$ -algebra R there is a strict p-ring A = W(R) with A/pA = R. The pair  $(W(R), W(R) \to R)$  is unique up to a unique isomorphism.

View R as a monoid under multiplication and let  $\mathbb{Z}R$  be the monoid algebra of  $(R,\cdot)$ . Its elements are formal sums of the form  $\sum_{r\in R} n_r[r]$  with almost all  $n_r=0$ . Addition and multiplication are the obvious ones. Note that [1]=1 but  $[0]\neq 0$ . Multiplicative maps  $R\to B$  into commutative rings mapping 1 to 1 correspond to ring homomorphisms  $\mathbb{Z}R\to B$ . The identity map R=R induces the surjective ring homomorphism  $\pi:\mathbb{Z}R\to R$  which sends  $\sum n_r[r]$  to  $\sum n_r r$ . Let I be its kernel, so that we have an exact sequence

$$0 \longrightarrow I \longrightarrow \mathbb{Z}R \xrightarrow{\pi} R \longrightarrow 0.$$

It is not difficult to see that as a  $\mathbb{Z}$ -module I is generated by elements of the form [r] + [s] - [r+s] for  $r, s \in R$ . We will not use this fact in the sequel. The multiplicative isomorphism  $r \mapsto r^p$  of R induces a ring isomorphism  $F : \mathbb{Z}R \to \mathbb{Z}R$  mapping  $\sum n_r[r]$  to  $\sum n_r[r^p]$ . It satisfies F(I) = I.

Let  $C(R) = \varprojlim_{\nu} \mathbb{Z}R/I^{\nu}$  be the *I*-adic completion of  $\mathbb{Z}R$ . By construction C(R) is Hausdorff and complete for the topology defined by the ideals  $\mathfrak{a}_i = \widehat{I}^i$  where

$$\widehat{I}^i = \varprojlim_{\nu} I^i / I^{\nu} \subset C(R).$$

Note that at this stage we do not know that  $\hat{I}^i = \hat{I}^i$  since  $\mathbb{Z}R$  is not noetherian in general. Condition 1) above is satisfied and 2) as well since

$$C(R)/\hat{I} = \mathbb{Z}R/I = R.$$

Hence C(R) is a p-ring. The construction of C(R) is functorial in R.

**Theorem 1.** Let R be a perfect  $\mathbb{F}_p$ -algebra. Then C(R) is a strict p-ring with C(R)/pC(R) = R.

The result is an immediate consequence of the universal properties shared by C(R) and any strict p-ring with residue algebra R, once one knows that

such a strict p-ring exists; see remark 6 below. In the following we give a self-contained proof of Theorem 1 which does not use this information.

Consider the "arithmetic derivation"  $\delta: \mathbb{Z}R \to \mathbb{Z}R$  defined by the formula

$$\delta(x) = \frac{1}{p}(F(x) - x^p).$$

It is well defined since  $F(x) \equiv x^p \mod p\mathbb{Z}R$  and since  $\mathbb{Z}R$  being a free  $\mathbb{Z}$ -module has no  $\mathbb{Z}$ -torsion. The following formulas for  $x, y \in \mathbb{Z}R$  are immediate

(1) 
$$\delta(x+y) = \delta(x) + \delta(y) - \sum_{\nu=1}^{p-1} \frac{1}{p} \binom{p}{\nu} x^{\nu} y^{p-\nu}$$

and

(2) 
$$\delta(xy) = \delta(x)F(y) + x^p \delta(y).$$

Applying (2) inductively gives the relation

(3) 
$$\delta(x_1 \cdots x_n) = \sum_{\nu=1}^n x_1^p \cdots x_{\nu-1}^p \delta(x_{\nu}) F(x_{\nu+1}) \cdots F(x_n) \text{ for } x_i \in \mathbb{Z}R.$$

Equation (1) shows that we have

(4) 
$$\delta(x+y) \equiv \delta(x) + \delta(y) \mod I^n \text{ if } x \text{ or } y \text{ is in } I^n.$$

Together with (3) it follows that

(5) 
$$\delta(I^n) \subset I^{n-1} \text{ for } n \ge 1.$$

**Lemma 2.** Let R be a perfect  $\mathbb{F}_{n}$ -algebra and  $n \geq 1$  an integer.

- a) If  $pa \in I^n$  for some  $a \in \mathbb{Z}R$  then  $a \in I^{n-1}$ .
- b)  $I^n = I^{\nu} + p^n \mathbb{Z}R$  for any  $\nu \geq n$ .

*Proof.* a) According to formula (5) we have  $\delta(pa) \in I^{n-1}$ . On the other hand, by definition:

$$\delta(pa) = F(a) - p^{p-1}a^p,$$

and therefore since  $pa \in I^n$ 

$$\delta(pa) \equiv F(a) \bmod I^n$$
.

It follows that  $F(a) \in I^{n-1}$  and hence  $a \in I^{n-1}$  since F is an automorphism with F(I) = I.

b) We prove the inclusion  $I^n \subset I^{\nu} + p^n \mathbb{Z}R$  for  $\nu \geq n$  by induction with respect to  $n \geq 1$ . The other inclusion is clear. For  $y \in \mathbb{Z}R$  and  $\nu \geq 1$  we have

$$F^{\nu}(y) \equiv y^{p^{\nu}} \mod p\mathbb{Z}R.$$

Applying this to  $y = F^{-\nu}(x)$ , we get for all  $x \in \mathbb{Z}R$ 

$$x \equiv F^{-\nu}(x)^{p^{\nu}} \mod p\mathbb{Z}R.$$

For  $x \in I$  this shows that  $x \in I^{\nu} + p\mathbb{Z}R$  settling the case n = 1 of the assertion. Now assume that  $I^n \subset I^{\nu} + p^n\mathbb{Z}R$  has been shown for a given  $n \geq 1$  and all  $\nu \geq n$ . Fix some  $\nu \geq n + 1$  and consider an element  $x \in I^{n+1}$ . By the induction assumption  $x = y + p^n z$  with  $y \in I^{\nu}$  and  $z \in \mathbb{Z}R$ . Hence  $p^nz=x-y\in I^{n+1}$ . Using assertion a) of the lemma repeatedly shows that  $z\in I$ . Hence  $z\in I^{\nu}+p\mathbb{Z}R$  by the case n=1. Writing z=a+pb with  $a\in I^{\nu}$  and  $b\in \mathbb{Z}R$  we find

$$x = (y + p^n a) + p^{n+1} b \in I^{\nu} + p^{n+1} \mathbb{Z}R.$$

Thus we have shown the induction step  $I^{n+1} \subset I^{\nu} + p^{n+1}\mathbb{Z}R$ .

After these preparations the proof of Theorem 1 follows easily: We have to show that  $p^nC(R) = \widehat{I^n}$  for all  $n \geq 1$  and that p is not a zero divisor in C(R). Let  $p^{-n}(I^{\nu})$  be the inverse image of  $I^{\nu}$  under  $p^n$ -multiplication on  $\mathbb{Z}R$ . Then for any  $\nu \geq n \geq 1$  we have an exact sequence where the surjectivity on the right is due to part b) of Lemma 2:

$$0 \longrightarrow p^{-n}(I^{\nu})/I^{\nu} \longrightarrow \mathbb{Z}R/I^{\nu} \xrightarrow{p^n} I^n/I^{\nu} \longrightarrow 0.$$

From this we get an exact sequence of projective systems whose transition maps for  $\nu \geq n$  are the reduction maps. Set  $N_{\nu} = p^{-n}(I^{\nu})/I^{\nu}$ . Then we have an exact sequence

$$0 \longrightarrow \varprojlim_{\nu} N_{\nu} \longrightarrow C(R) \xrightarrow{p^n} \widehat{I^n} \longrightarrow \varprojlim_{\nu} {}^{(1)}N_{\nu}.$$

The transition map  $N_{\nu+n} \to N_{\nu}$  is the zero map since  $a \in p^{-n}(I^{\nu+n})$  implies  $p^n a \in I^{\nu+n}$  and hence  $a \in I^{\nu}$  by part a) of Lemma 2. In particular  $(N_{\nu})$  is Mittag-Leffler, so that  $\varprojlim_{\nu}^{(1)}(N_{\nu}) = 0$ . It is also clear now that  $\varprojlim_{\nu} N_{\nu} = 0$ . It follows that  $p^n$ -multiplication on C(R) is injective with image  $\widehat{I^n}$ . Hence p is not a zero divisor and  $\widehat{I^n} = p^n C(R)$ .

## Remarks.

1) If R is a perfect  $\mathbb{F}_p$ -algebra there is an isomorphism

$$R \xrightarrow{\sim} I^n/I^{n+1}$$
 given by  $r \longmapsto p^n[r]$ .

This follows because:

$$I^n/I^{n+1} = \widehat{I^n}/\widehat{I^{n+1}} = p^nC(R)/p^{n+1}C(R) \stackrel{p^{-n}}{=} C(R)/pC(R) = R.$$

2) The automorphism F of  $\mathbb{Z}R$  satisfies F(I) = I. Hence it induces an automorphism F of C(R) which lifts the Frobenius automorphism of the perfect  $\mathbb{F}_p$ -algebra R. The Verschiebung  $V:C(R)\to C(R)$  is the additive homomorphism defined by  $V(x)=pF^{-1}(x)$ . By definition  $\operatorname{Im} V^i=p^iC(R)$  and  $V\circ F=F\circ V=p$ . The projection  $\pi:C(R)\to R$  has a multiplicative splitting defined as the composition  $\omega:R\hookrightarrow \mathbb{Z}R\to C(R)$ . Frobenius F, Verschiebung V and Teichmüller lift  $\omega$  are well known extra structures on rings of Witt vectors.

**Proposition 3.** If R = K is a perfect field of characteristic p then C(K) is a discrete valuation ring of mixed characteristic with residue field K.

Proof. This is true for any strict p-ring W with residue field K. The well known argument is as follows. By assumption pW is a maximal ideal of W. For  $x \in W \setminus pW$  choose  $y \in W$  with  $xy \equiv 1 \mod p$ . Then  $(xy)^{p^{\nu}} \equiv 1 \mod p^{\nu}$  by [8, II §4 Lemma 1]. Hence  $x \mod p^{\nu}$  is a unit in  $W/p^{\nu}W$  and therefore  $x = (x \mod p^{\nu})_{\nu \geq 0}$  is a unit in W. Hence the ring W is local with unique maximal ideal pW. Since W is separated i.e.  $\bigcap_{\nu=1}^{\infty} p^{\nu}W = 0$  it follows that for every  $0 \neq a \in W$  there is a unique integer  $v(a) \geq 0$  with  $a = p^{v(a)}x$  and  $x \in W \setminus pW$  i.e.  $x \in W^*$ . Since multiplication with p is injective on W, it follows that W is an integral domain. The map  $v: W \setminus \{0\} \to \mathbb{Z}$  satisfies v(ab) = v(a) + v(b) by definition and  $v(a+b) \geq \min(v(a), v(b))$  because as seen above, an element of W is a unit if and only if its reduction mod p is nonzero. The valuation v extends uniquely to a discrete valuation on the quotient field Q of W with valuation ring W.

**Remark.** In particular C(K) is noetherian while in general  $\mathbb{Z}K$  is very far from being noetherian.

As a topological additive group, C(R) has another description which is sometimes useful. Let  $\mathfrak{b}$  be a basis of the  $\mathbb{F}_p$ -algebra R and let  $\mathbb{Z}\mathfrak{b}$  be the free  $\mathbb{Z}$ -module with basis  $\mathfrak{b}$ . The inclusion  $\mathfrak{b} \subset R$  induces an additive homomorphism

$$\mathbb{Z}\mathfrak{b} \hookrightarrow \mathbb{Z}R \longrightarrow C(R)$$

and hence a map

$$\widehat{\mathbb{Z}\mathfrak{b}}=\varprojlim_n\mathbb{Z}\mathfrak{b}/p^n\mathbb{Z}\mathfrak{b}\longrightarrow C(R).$$

**Proposition 4.** If R is a perfect  $\mathbb{F}_p$ -algebra, the map  $\widehat{\mathbb{Zb}} \to C(R)$  is a topological isomorphism of additive groups. In particular any inclusion  $R_1 \hookrightarrow R_2$  resp. surjection  $R_1 \to R_2$  of perfect  $\mathbb{F}_p$ -algebras induces an inclusion  $C(R_1) \hookrightarrow C(R_2)$  resp. surjection  $C(R_1) \to C(R_2)$  with a continuous additive splitting.

*Proof.* By Theorem 1 we have to show that for each  $n \geq 1$  the additive map

$$\alpha_n: \mathbb{Z}\mathfrak{b}/p^n\mathbb{Z}\mathfrak{b} \longrightarrow C(R)/p^nC(R)$$

is an isomorphism. For n=1 this is true because  $\mathbb{F}_p\mathfrak{b}=R$  since  $\mathfrak{b}$  is an  $\mathbb{F}_p$ -basis for R. Now assume that  $\alpha_n$  is an isomorphism and consider the commutative diagram

$$0 \longrightarrow \mathbb{Z}\mathfrak{b}/p\mathbb{Z}\mathfrak{b} \xrightarrow{p^n} \mathbb{Z}\mathfrak{b}/p^{n+1}\mathbb{Z}\mathfrak{b} \longrightarrow \mathbb{Z}\mathfrak{b}/p^n\mathbb{Z}\mathfrak{b} \longrightarrow 0$$

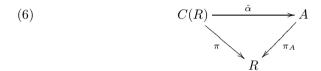
$$\downarrow^{\alpha_1} \qquad \downarrow^{\alpha_{n+1}} \qquad \downarrow^{\alpha_n}$$

$$0 \longrightarrow C(R)/pC(R) \xrightarrow{p^n} C(R)/p^{n+1}C(R) \longrightarrow C(R)/p^nC(R) \longrightarrow 0$$

The upper sequence is exact and because of Theorem 1 the lower sequence is exact as well. Hence  $\alpha_{n+1}$  is an isomorphism. The remaining assertions follow immediately.

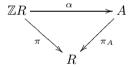
**Remark.** If the basis  $\mathfrak{b}$  happens to be closed under multiplication then  $\mathbb{Z}\mathfrak{b}$  is a ring and  $\widehat{\mathbb{Z}\mathfrak{b}} \to C(R)$  an isomorphism of rings. This is the case in the following example. The perfect  $\mathbb{F}_p$ -algebra  $R = \mathbb{F}_p[t_1^{p^{-\infty}}, \dots, t_d^{p^{-\infty}}]$  has a basis  $\mathfrak{b}$  consisting of monomials. This basis is multiplicatively closed and hence C(R) is the p-adic completion of the monoid algebra  $\mathbb{Z}\mathfrak{b}$  i.e. of the algebra  $\mathbb{Z}[t_1^{p^{-\infty}}, \dots, t_d^{p^{-\infty}}]$ .

**Proposition 5.** Let A be a p-ring with perfect residue algebra R as above. Then there is a unique homomorphism of rings  $\hat{\alpha}: C(R) \to A$  such that the following diagram commutes:



**Remark.** This is true for any strict p-ring instead of C(R), cp. [8, II §5 Prop. 10]. However in our case the argument is particularly simple and we do not even have to know that C(R) is strict.

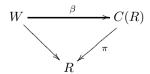
*Proof.* Since A is a p-ring, there is a unique multiplicative section  $\alpha_0: R \to A$  of  $\pi_A$ , cp. [8, II §4 Prop. 8]. Hence there is a unique ring homomorphism  $\alpha: \mathbb{Z}R \to A$  such that the diagram



commutes. Since  $\alpha(I) \subset \mathfrak{a}_1$  we have  $\alpha(I^{\nu}) \subset \mathfrak{a}_1^{\nu} \subset \mathfrak{a}_{\nu}$  and therefore  $\alpha$  extends to a unique and automatically continuous homomorphism  $\hat{\alpha}: C(R) \to A$  such that (6) commutes.

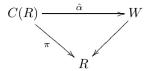
**Remark 6.** As we saw above it is immediate that C(R) is a p-ring with residue algebra R. Showing directly that C(R) is a strict p-ring required some thought. If one already knows that there is a strict p-ring W with residue algebra R, then it is easy to see that C(R) is isomorphic to W and hence strict. Here is the argument:

The universal property of strict p-rings [8, II §5 Prop. 10] gives us a unique homomorphism  $\beta: W \to C(R)$  such that the diagram



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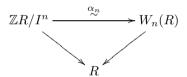
commutes. On the other hand by Proposition 5 there is a unique homomorphism  $\hat{\alpha}: C(R) \to W$  such that



commutes. The map  $\alpha \circ \beta$  is the identity on W because of the universal property for the strict p-ring W. The map  $\beta \circ \alpha$  is the identity on C(R) by Proposition 5 because C(R) is a p-ring. It follows that  $C(R) \cong W$  is a strict p-ring. An equally simple proof may be given by using the characterization of the triple  $(W(R), R \hookrightarrow W(R), \pi : W(R) \to R)$  in [2, Prop. 3.1] which is based on [5, Thm. 1.2.1].

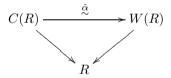
From the preceding remark we get the following corollary:

**Corollary 7.** Let  $W_n(R)$  be the truncated (p-typical) Witt ring of the perfect  $\mathbb{F}_p$ -algebra R. There is a unique homomorphism of rings  $\alpha_n : \mathbb{Z}R/I^n \to W_n(R)$  inducing the standard multiplicative embedding  $R \hookrightarrow W_n(R)$  and making the following diagram commute



Moreover,  $\alpha_n$  is an isomorphism.

*Proof.* Let W(R) be the p-typical Witt ring of R. According to Remark 6 there is a commutative diagram



Reducing mod  $p^n$  and noting that  $W(R)/p^nW(R)=W_n(R)$  and

$$C(R)/p^nC(R) = C(R)/\widehat{I^n} = \mathbb{Z}R/I^n$$

we get an isomorphism  $\alpha_n$  as desired. There is a unique ring homomorphism  $\alpha: \mathbb{Z}R \to W_n(R)$  prolonging the multiplicative embedding  $R \hookrightarrow W_n(R)$ . Hence  $\alpha_n$  is uniquely determined.

As a set  $W_n(R)$  is  $R^n$ . Addition and multiplication are given by certain universal polynomials in 2n variables over  $\mathbb{Z}$ . We now describe the isomorphism  $\alpha_2$ . Note that (4) and (5) imply that  $\delta$  induces a (nonadditive) map

$$\overline{\delta}: \mathbb{Z}R/I^2 \longrightarrow \mathbb{Z}R/I = R.$$

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We also have the ring homomorphism of reduction  $\pi: \mathbb{Z}R/I^2 \to \mathbb{Z}R/I = R$ .

**Proposition 8.** The isomorphism

$$\alpha_2: \mathbb{Z}R/I^2 \xrightarrow{\sim} W_2(R) = R^2$$

is given by the map  $(\pi, \overline{\delta})$ .

*Proof.* The composition  $R \to \mathbb{Z}R/I^2 \to W_2(R) = R^2$  is the standard multiplicative embedding. One checks that  $\alpha_2$  is a ring homomorphism using the formulas for addition and multiplication on  $W_2(R) = R^2$ :

$$(x,y) + (x',y') = \left(x + x', y + y' - \frac{1}{p} \sum_{\nu=1}^{p-1} \binom{p}{\nu} x^{\nu} x'^{p-\nu}\right)$$

and

$$(x,y) \cdot (x',y') = (xx', x'^p y + y'x^p + pyy').$$

Using Corollary 7 the assertion follows.

**Remark.** With respect to the ordinary R-module structure on  $R^2$  the map  $\alpha_2$  is nonlinear. Hence the simple addition and multiplication on  $\mathbb{Z}R/I^2$  become something nonobvious on  $R^2$ . We have not tried to describe  $\alpha_n$  for  $n \geq 3$  by explicit formulas.

It is interesting to compare the I-adic completion C(R) of  $\mathbb{Z}R$  with its p-adic completion i.e. the completion with respect to powers of the ideal  $p\mathbb{Z}R$ . Lemma 2b) shows that the projective system  $(I^n/p^n\mathbb{Z}R)_n$  satisfies the Mittag-Leffler condition. Therefore we obtain the following exact sequence

(7) 
$$0 \to \varprojlim I^n/p^n \mathbb{Z}R \to \varprojlim \mathbb{Z}R/p^n \mathbb{Z}R \to \varprojlim \mathbb{Z}R/I^n \to 0$$

which describes the kernel of the natural map from the p-adic completion to C(R).

Now, if R is a finite perfect  $\mathbb{F}_p$ -algebra, then the p-adic completion of  $\mathbb{Z}R$  is  $\mathbb{Z}_pR$  the monoid algebra of R over  $\mathbb{Z}_p$  and C(R) has an instructive description as a complete subring of  $\mathbb{Z}_pR$ .

**Proposition 9.** Assume that R is a finite perfect  $\mathbb{F}_p$ -algebra. Then there is an idempotent e in  $\mathbb{Z}_p R$  such that  $e(\mathbb{Z}_p R)$  is the kernel of the natural map  $\mathbb{Z}_p R \to C(R)$  and such that  $(1-e)\mathbb{Z}_p R$  is topologically isomorphic to C(R).

*Proof.* For each n, the quotient  $\mathbb{Z}R/p^n\mathbb{Z}R$  is finite and in particular an Artin ring (descending chains of ideals become stationary). Using Lemma 2b), we see that the image  $A_n$  of  $I^n$  in  $\mathbb{Z}R/p^n\mathbb{Z}R$  is an ideal such that  $A_n^2 = A_n$ .

According to the structure theorem for Artin rings (see e.g. [1, Thm. 8.7]),  $\mathbb{Z}R/p^n\mathbb{Z}R$  is (uniquely) a finite direct product  $\prod_i B_i$  of local Artin rings  $B_i$ . Any idempotent ideal in a local Artin ring B is either 0 or equal to B since the maximal ideal in B is nilpotent, [1, 8.2 and 8.4]. Therefore the projection of  $A_n$  to any of the components  $B_i$  is either 0 or  $B_i$ . If we let  $e_n$  denote the sum of the identity elements of the  $B_i$  in which the component of  $A_n$  is nonzero

we get an idempotent  $e_n$  in  $\mathbb{Z}R/p^n\mathbb{Z}R$  such that  $e_n(\mathbb{Z}R/p^n\mathbb{Z}R) = A_n$  ( $e_n$  is a unit element for  $A_n$  and therefore uniquely determined).

Since, by Lemma 2b), the image of  $I^{n+1}$  in  $\mathbb{Z}R/p^{n+1}\mathbb{Z}R$  maps surjectively to the image of  $I^n$  in  $\mathbb{Z}R/p^n\mathbb{Z}R$  under the natural map, the sequence  $(e_n)$  defines an element e in  $\mathbb{Z}_pR = \varprojlim \mathbb{Z}R/p^n\mathbb{Z}R$ . By construction it is an idempotent in  $A = \varprojlim A_n = \varprojlim I^n/p^n\mathbb{Z}R$  such that ex = x for each x in A. It follows that  $A = e(\mathbb{Z}_pR)$ .

The exact sequence (7) then shows that the map  $(1 - e)\mathbb{Z}_p R \to C(R)$  is a continuous bijective homomorphism between compact rings and therefore a topological isomorphism.

**Remark.** The proof shows that  $A = e(\mathbb{Z}_p R)$  in the preceding proposition is a unital ring which is the projective limit of a system  $(A_n)$  of unital rings with unital transition maps. In the case  $R = \mathbb{F}_p$  looking at the canonical decomposition of  $\mathbb{Z}_p R$  under the action of  $\mathbb{F}_p^{\times}$  we see that e has the following explicit description

$$1 - e = (p - 1)^{-1} \sum_{r \in \mathbb{F}_p^{\times}} \omega(r)^{-1} [r] \text{ in } \mathbb{Z}_p R.$$

Here  $\omega$  is the Teichmüller character  $\omega: \mathbb{F}_p^{\times} \to \mathbb{Z}_p^{\times}$ .

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