The L^2-Torsion Polytope of Groups and the Integral Polytope Group

DISSERTATION

zur

Erlangung des Doktorgrades (Dr. rer. nat.)
der
Mathematisch-Naturwissenschaftlichen Fakultät
der
Rheinischen Friedrich-Wilhelms-Universität Bonn

vorgelegt von

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aus

Beckum

Bonn 2017

Angefertigt mit Genehmigung der Mathematisch-Naturwissenschaftlichen Fakultät der Rheinischen Friedrich-Wilhelms-Universität Bonn

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Tag der Promotion: 22.01.2018
Erscheinungsjahr: 2018

Table of Contents

1 Introduction

In the mid-80s, Thurston [Thu86] defined a seminorm x_M on the first cohomology $H^1(M;\mathbb{R})$ of a compact connected orientable 3-manifold M by measuring the complexity of surfaces dual to a given cohomology class. This *Thurston norm* is closely related to the question if and how the 3-manifold fibers over the circle and can concisely be described by its unit ball B_{x_M}. Thurston showed that this is not just some convex subset of $H^1(M;\mathbb{R})$, but in fact a polyhedron whose vertices are contained in the lattice $H^1(M;\mathbb{Z})$. So if x_M is even a norm, which is always the case if M is hyperbolic, then B_{x_M} is an *integral polytope*, by which we mean the convex hull inside $H^1(M;\mathbb{R})$ of a finite set of points in $H^1(M;\mathbb{Z})$.

Since the Thurston norm is defined by minimizing the complexity of dual surfaces, it has become costumary to search for lower bounds or even completely detect it, see for example [McM00, Tur02, Coc04, Har05, FK06, Fri07, FK08, FV15, FSW16]. Most approaches involve generalizations of the Alexander polynomial such as higher-order or twisted versions and share a common paradigm: One considers the Reidemeister torsion of the chain complex of M twisted with coefficients in suitable skew-fields. Every cohomology class in $H^1(M;\mathbb{Z})$ determines a *degree* of this Reidemeister torsion which is shown to be a lower bound for the Thurston norm. This degree can often be described in terms of the homology of M with coefficients in twisted Laurent polynomial rings.

This strategy has most recently been extended with a view towards L^2-invariants and has been implemented there in two different ways. One possibility is to replace the Reidemeister torsion with the L^2-torsion of twisted chain complexes associated to M, bringing about *twisted L^2-torsion functions*. It was first defined by Li-Zhang [LZ06a, LZ06b, LZ08] for knots and by Dubois-Friedl-Lück [DFL16, DFL15a] in general. The connection of a suitable notion of *degree* of these functions to the Thurston norm was detected by Liu [Liu17] and Friedl-Lück [FL15].

Another way to apply L^2-invariants in the context of the Thurston norm is to keep the aforementioned common paradigm and find new skew-field coefficients to which it applies. This has been carried out by Friedl-Lück [FL16a] with the aid of the *Atiyah Conjecture*. While the conjecture originally predicts the possible values of L^2-Betti numbers of G-CW-complexes, its study brought about a skew-field commonly called $\mathcal{D}(G)$ containing the integral group ring $\mathbb{Z}G$. It can be used as a replacement of the Ore localization of the integral group ring for non-amenable groups. This approach produces *twisted L^2-Euler characteristics* which satisfy similar inequalities with the Thurston norm as the degree of L^2-torsion functions. As the name suggests, these invariants can also be described in terms of twisted L^2-Betti numbers.

Under basic L^2-acyclicity assumptions the classical L^2-torsion, twisted L^2-torsion functions and twisted L^2-Euler characteristics enjoy a common set of basic properties, including simple homotopy invariance as well as sum, product, induction, and restriction formulas. This led Friedl-Lück [FL16b] to formalize the concept of an *additive L^2-torsion invariant* to be an assignment that, very roughly speaking, associates to every (finite based free) L^2-acyclic $\mathbb{Z}G$-chain complex an element in some fixed abelian group such that short exact sequences translate into sum relations. They also construct a *universal* L^2-torsion invariant $\rho_u^{(2)}$ which encapsulates all other L^2-torsion invariants. Motivated by the definition of classical torsion invariants, it takes values in a weak version of the reduced K_1-group $\widetilde{K}_1^w(\mathbb{Z}G)$ which is adjusted to the L^2-setting. Namely, instead of automorphisms of finitely generated projective $\mathbb{Z}G$-modules one takes as generators $\mathbb{Z}G$-endomorphisms $\mathbb{Z}G^n \to \mathbb{Z}G^n$ that become weak isomorphisms after passing to $L^2(G)$. Similarly, instead of a chain contraction of a contractible chain complex one considers a *weak* chain contraction of an L^2-acyclic chain complex C_* in order to construct $\rho_u^{(2)}(C_*)$ as an element in $\widetilde{K}_1^w(\mathbb{Z}G)$. If one wants to apply this to G-CW-complexes, then it is necessary to pass to the *weak Whitehead group* $\mathrm{Wh}^w(G)$ of G, i.e., the quotient of $\widetilde{K}_1^w(\mathbb{Z}G)$ by the subgroup containing the right multiplications with elements of the form $\pm g$ for $g \in G$.

Since the universal L^2-torsion invariant encodes twisted L^2-Euler characteristics, and these detect in many situations the Thurston norm, the universal L^2-torsion invariant also detects the Thurston norm. This slogan can be strengthened by going back to Thurston's polytopes. Namely, in between the weak Whitehead group $\mathrm{Wh}^w(G)$ and norms on the first cohomology $H^1(G;\mathbb{R})$, which we view as continuous maps $H^1(G;\mathbb{R}) \to \mathbb{R}$, one can squeeze in a geometric gadget called *the integral polytope group*: If H is a finitely generated free-abelian group, then pointwise addition, sometimes called *Minkowski sum*, turns the set of polytopes in $H \otimes_{\mathbb{Z}} \mathbb{R}$ with vertices in H into a commutative monoid, denoted by $\mathfrak{P}(H)$. The integral polytope group $\mathcal{P}(H)$ is the Grothendieck group of this commutative monoid. Identifying polytopes which are translates of each other produces a quotient called $\mathcal{P}_T(H)$ which fits into a sequence

$$\mathrm{Wh}^w(G) \xrightarrow{\mathbb{P}} \mathcal{P}_T(H_1(G)_f) \xrightarrow{\mathfrak{N}} \mathrm{Map}(H^1(G;\mathbb{R}),\mathbb{R}),$$

where $H_1(G)_f$ denotes the free part of the first integral homology $H_1(G)$ of G. The right-hand map $\mathfrak{N}$ called *norm homomorphism* is classical, namely any integral polytope $P \subseteq H \otimes_{\mathbb{Z}} \mathbb{R}$ determines a norm on $\mathrm{Hom}(H,\mathbb{R})$ by measuring the thickness of P in the direction of a given homomorphism $H \to \mathbb{R}$. (Here we identify $\mathrm{Hom}(H_1(G)_f,\mathbb{R}) = H^1(G;\mathbb{R})$.) In sharp contrast to this, the left-hand map $\mathbb{P}$ called *polytope homomorphism* has only recently been defined [FL16a, FL16b] and heavily relies on the structure of $\mathcal{D}(G)$. Forerunner versions have at least implicitly been examined in the context of higher-order Alexander polynomials by Cochran, Harvey, and Friedl [Coc04, Har05, FH07, Fri07].

It is one of the main results of Friedl-Lück's theory [FL16b, Theorem 3.27] that if M is a sufficiently nice (or in their words *admissible*) 3-manifold unequal to the solid torus and whose fundamental group satisfies the Atiyah Conjecture, then the image of the negative of the universal L^2-torsion invariant $-\rho_u^{(2)}(\widetilde{M})$ under the composition $\mathfrak{N} \circ \mathbb{P}$ is the Thurston norm. Even stronger, the image of $-\rho_u^{(2)}(\widetilde{M})$ under the map $\mathbb{P}$ is dual to the unit ball of the Thurston norm $B_{x_M} \subseteq H^1(M;\mathbb{R})$, see [FL16b, Theorem 3.35]. So *for 3-manifolds we have come full circle:* from the polytopes of the Thurston norm to Reidemeister torsion as lower bounds for the Thurston norm, to L^2-torsion invariants and the universal L^2-torsion, and back to polytopes by virtue of the polytope homomorphism.

Even though all the research described so far is motivated by and mostly carried out for 3-manifolds, the theory applies in much greater generality. By Friedl-Lück's work, we now have a universal L^2-torsion invariant $\rho_u^{(2)}(X;\mathcal{N}(G))$ associated to any finite free L^2-acyclic G-CW-complex X. If G additionally satisfies the Atiyah Conjecture, then we get on top the *L^2-torsion polytope* $P(X;G)$ which is defined as the image of $-\rho_u^{(2)}(X;\mathcal{N}(G))$ under the polytope homomorphism $\mathbb{P}$. This can in particular be applied to groups themselves: If G is a group with finite L^2-acyclic classifying space and whose Whitehead group vanishes, then

$$\rho_u^{(2)}(G) := \rho_u^{(2)}(EG;\mathcal{N}(G)) \in \mathrm{Wh}^w(G)$$

only depends on G. If G additionally satisfies the Atiyah Conjecture, then the L^2-torsion polytope

$$P(G) := P(EG; G) \in \mathcal{P}(H_1(G)_f)$$

is also available. These two objects as well as the integral polytope group itself are the main objects of study in this dissertation.

Results

The universal L^2-torsion is a rather abstract invariant, the L^2-torsion polytope on the other hand is quite concrete (which does not at all mean that it is easy to compute). As the place in which this new invariant lives, the integral polytope group deserves attention, and yet almost nothing is known about its structure. We take this lack of information as motivation for a thorough investigation of the integral polytope group. There is a canonical involution $*\colon \mathcal{P}(H) \to \mathcal{P}(H)$ induced by reflection about the origin which also passes to the quotient $*\colon \mathcal{P}_T(H) \to \mathcal{P}_T(H)$. Using the interplay between geometry and algebra, we will establish a set of techniques for computations in these groups which will eventually allow us to prove the following list of results.

Theorem 4.1 (Structure of the integral polytope group). *Let H be a finitely generated free-abelian group.*

(1) (Symmetric elements) We have

$$\ker\big(\mathrm{id} - *\colon \mathcal{P}(H) \to \mathcal{P}(H)\big) = \mathrm{im}\big(\mathrm{id} + *\colon \mathcal{P}(H) \to \mathcal{P}(H)\big).$$

(2) (Antisymmetric elements) We have

$$\ker\big(\mathrm{id} + *\colon \mathcal{P}(H) \to \mathcal{P}(H)\big) = \mathrm{im}\big(\mathrm{id} - *\colon \mathcal{P}(H) \to \mathcal{P}(H)\big)$$

and

$$\ker\big(\mathrm{id} + *\colon \mathcal{P}_T(H) \to \mathcal{P}_T(H)\big) = \mathrm{im}\big(\mathrm{id} - *\colon \mathcal{P}_T(H) \to \mathcal{P}_T(H)\big).$$

(3) (Basis) There are sets $\mathcal{B}_1 \subseteq \mathcal{B}_2 \subseteq \ldots \subseteq \mathcal{B}_n \subseteq \mathcal{P}_T(H)$ such that $\mathcal{B}_m \setminus \mathcal{B}_{m-1}$ contains only polytopes of dimension m and $\mathcal{B}_m \cap \mathcal{P}_T(G)$ is a basis for $\mathcal{P}_T^m(G)$ for every pure subgroup $G \subseteq H$ and $1 \leq m \leq n$. In particular, $\mathcal{B}_n$ is a basis for $\mathcal{P}_T(H)$.

Moreover, if $\mathcal{A} \subseteq H$ denotes a basis of H and $\mathcal{B}'_n \subseteq \mathcal{P}(H)$ is a set of representatives for $\mathcal{B}_n \subseteq \mathcal{P}_T(H)$, then $\mathcal{A} \cup \mathcal{B}'_n$ is a basis for $\mathcal{P}(H)$.

(4) (Involution as face Euler characteristic) For any polytope $P \in \mathfrak{P}(H)$ we have in $\mathcal{P}(H)$

$$*P = -\sum_{F \in \mathcal{F}(P)} (-1)^{\dim(F)} \cdot F,$$

where $\mathcal{F}(P)$ denotes the set of faces of P (including P itself).

We then turn over to an investigation of the L^2-torsion polytope of groups with an emphasis on two quite different classes of L^2-acyclic groups. The first one is the class of infinite amenable groups. In the context of L^2-invariants and related fields, amenable groups stand out as a class of groups satisfying strong vanishing results. Among others, all infinite amenable groups have vanishing L^2-Betti numbers, vanishing L^2-torsion (provided that the group admits a finite classifying space), vanishing rank gradients and homology gradients (provided that the group is finitely generated), and fixed price 1 (see Chapter 5 for references). Wegner [Weg00, Weg09] showed that any group with finite classifying space which is of so-called det ≥ 1-class and contains an infinite elementary amenable normal subgroup has vanishing L^2-torsion.

Motivated by this latter result, we will introduce the notion of groups *of* $P \geq 0$*-class* and even stronger *of polytope class* by virtue of the polytope homomorphism. We then show that torsion-free infinite amenable groups satisfying the Atiyah Conjecture possess these properties. As a byproduct we obtain the homotopy invariance of the L^2-torsion polytope over infinite amenable groups. In a second step we then adjust Wegner's strategy towards a program to prove the following vanishing result for the L^2-torsion polytope. It partially confirms a conjecture of Friedl-Lück-Tillmann [FLT16, Conjecture 6.4].

Theorem 5.15 (The L^2-torsion polytope and elementary amenability). *Let G be a group of type F (i.e., G admits a finite classifying space) which is of $P \geq 0$-class. Suppose that G contains a non-abelian elementary amenable normal subgroup. Then G is L^2-acyclic and we have*

$$P(G) = 0.$$

In particular, the L^2-torsion polytope of an elementary amenable group of type F vanishes.

Beyond elementary amenable groups we apply our study of the integral polytope group to obtain some evidence for the vanishing of the L^2-torsion polytope.

Proposition 5.19 (The L^2-torsion polytope and amenability). *Let $G \neq \mathbb{Z}$ be an amenable group of type F satisfying the Atiyah Conjecture. Then $P(G)$ lies in the kernel of the norm homomorphism $\mathfrak{N}\colon \mathcal{P}_T(H_1(G)_f) \to \mathrm{Map}(H^1(G;\mathbb{R}),\mathbb{R})$ and there is an integral polytope $P \in \mathfrak{P}(H_1(G)_f)$ such that in $\mathcal{P}_T(H_1(G)_f)$ we have*

$$P(G) = P - *P.$$

The second class of groups whose L^2-torsion polytope is studied in this thesis lies on the other side of the universe of groups. This is the class of ascending HNN extensions of finitely generated free groups F_n, which are determined by monomorphisms $F_n \to F_n$. Here the L^2-torsion polytope has the potential to play a significant role in the study of the outer automorphism group $\mathrm{Out}(F_n)$ of free groups. This group is a prominent player in geometric group theory and notoriously hard to handle since powerful invariants of free group automorphisms are rare.

We will first show that the L^2-torsion polytope induces a norm on the first cohomology of ascending HNN extensions of free groups. Then we concentrate on the class of *unipotent polynomially growing*, short *UPG*, automorphisms for which we can fully compute the universal L^2-torsion and alongside all other L^2-torsion invariants.

Theorem 6.15 (Universal L^2-torsion of UPG automorphisms). *Let $\pi_\alpha = F_n \rtimes_\alpha \mathbb{Z}$ with $n \geq 1$ and $\alpha\colon F_n \to F_n$ a UPG automorphism. Then there are elements $g_1, ..., g_{n-1} \in \pi_\alpha \setminus F_n$ such that for any admissible homomorphism $\mu\colon \pi_\alpha \to G$ to a torsion-free group G, we have $\mu(g_i) \neq 0$ and*

$$\rho_u^{(2)}(\alpha;\mu) = -\sum_{i=1}^{n-1} [r_{1-\mu(g_i)}\colon \mathbb{Z}G \to \mathbb{Z}G].$$

As an important corollary we show that the L^2-torsion polytope determines another invariant on the first cohomology called the *Sigma invariant* or *Bieri-Neumann-Strebel (BNS) invariant* $\Sigma(G)$. This is defined by measuring finiteness properties of the kernel of homomorphisms $G \to \mathbb{R}$, and it is in general quite hard to compute.

Corollary 6.20 (L^2-torsion polytope determines BNS invariant for UPG automorphisms). *Let $\pi_\alpha = F_n \rtimes_\alpha \mathbb{Z}$ with $n \geq 2$ and $\alpha\colon F_n \to F_n$ a UPG automorphism. Then $P(\pi_\alpha)$ is represented by a symmetric polytope. Moreover, for any $\varphi \in H^1(\pi_\alpha;\mathbb{R}) = \operatorname{Hom}(\pi_\alpha,\mathbb{R}) = \operatorname{Hom}(H_1(\pi_\alpha)_f,\mathbb{R})$ we have $\varphi \in \Sigma(\pi_\alpha)$ if and only if there is a unique vertex of $P(\pi_\alpha)$ maximizing φ.*

This result is inspired by a similar theorem of Friedl-Tillmann [FT15] for the case where the group is defined in terms of a presentation with two generators, one relation, and has first Betti number equal to 2.

Organization of the thesis

We collect previous work on which this thesis is built in Chapters 2 and 3. More precisely, Chapter 2 arranges the invariants on the first cohomology we will be dealing with, i.e., the Thurston norm, the Alexander norms, and the BNS-invariant. These are the more classical invariants occurring in this thesis.

Chapter 3 then presents a concise collection of L^2-torsion invariants. Beginning with the classical L^2-torsion, we will work our way along twisted L^2-torsion functions and twisted L^2-Euler characteristics towards the recent construction of the universal L^2-torsion. Since all these L^2-torsion invariants are constructed and examined in numerous original papers, it seemed worthwhile collecting them in a survey-type chapter for the first time. We restrict our attention to a rather dense presentation only highlighting the main results along the way.

Chapter 4 is a self-contained study of the integral polytope group. While it is formally independent of the previous chapters, it should be read as an attempt to get a feeling for the L^2-torsion polytope. We introduce techniques which play off geometry against algebra. These enable us to prove the four points occurring in the aforementioned Theorem 4.1 one by one. The construction of a geometrically tangible basis of the integral polytope group lies in some sense at the heart of this chapter.

Chapter 5 and Chapter 6 then present the investigations of the L^2-torsion polytope of amenable groups on the one hand and free group HNN extensions on the other hand. Small parts of the polytope language introduced in Chapter 4 will be used again in Chapter 5, but other than that it is independent. Chapter 6 is completely independent of the previous two chapters so that the final three chapters of this thesis can be read in arbitrary order.

Conventions

Throughout this thesis we will use the following conventions without further notice.

(1) Given a finitely generated abelian group H, we denote by $\operatorname{tors}(H) \subseteq H$ the torsion subgroup and by $H_f = H/\operatorname{tors}(H)$ the free part of H.

(2) Given a space X we use the identifications

$$H^1(X;\mathbb{Z}) = \operatorname{Hom}(\pi_1(X),\mathbb{Z}) = \operatorname{Hom}(H_1(X),\mathbb{Z}) = \operatorname{Hom}(H_1(X)_f,\mathbb{Z}).$$

(3) If R is a ring, then we denote the set of $m \times n$-matrices over R by $M_{m,n}(R)$. An element $A \in M_{m,n}(R)$ will be viewed as an R-homomorphism of left R-modules $R^m \to R^n$ by right multiplication, often also denoted by $r_A\colon R^m \to R^n$.

(4) Most invariants we encounter in this thesis are defined for CW-complexes and are simple homotopy invariants. Since every compact topological manifold carries up to simple homotopy equivalence a preferred CW-structure [KS69, Theorem IV], we can apply these invariants without harm to those manifolds.

(5) If V is a finite-dimensional real vector space, then a halfspace in V is a subset of the form $\{v \in V \mid \varphi(v) \leq c\}$ for some $\varphi \in V^*$ and $c \in \mathbb{R}$. A *polyhedron* in V is the intersection of finitely many halfspaces. A *polytope* in V is a compact polyhedron, or equivalently, the convex hull of finitely many points.

Acknowledgments

I thank my advisor Wolfgang Lück for his ongoing encouragement spanning three theses and six years. His support guided me from my first wobbly steps in L^2-invariants to the present dissertation. I am particularly grateful for his taking me on board as a Ph.D. student after I had already opted out of the academic world and for his understanding when mathematics was only of secondary importance.

I thank Stefan Friedl for his interest in my early work, for an invitation to and a tour around Regensburg, and for the “true Bavarian experience”.

I thank Fabian Henneke and Dawid Kielak for discussions inspiring parts of this work.

I gratefully acknowledge the financial support I have received successively by the GRK 1150 “Homotopy and Cohomology” funded by the Deutsche Forschungsgemeinschaft, the Max Planck Institute for Mathematics in Bonn, and the Deutsche Telekom Stiftung.

I am indebted to my wife Janine for your caring support in everything I do, for unconditional initiative in everything you do, and most of all for your courage to start a family in the midst of these incalculable times that Ph.D. studies inevitably are.

2 Invariants on the First Cohomology

In this chapter we collect the classical invariants which not only show up later, but in fact motivate large portions of this thesis.

2.1 The Thurston norm

The Thurston norm was introduced by Thurston [Thu86] in relation to the question if and how a 3-manifold fibers over the circle. We briefly recall its definition.

Given a compact surface S, we put

$$\chi_-(S) = \sum_{C \in \pi_0(S)} \max\{0, -\chi(C)\}.$$

Let M be a compact connected orientable 3-manifold. Given a cohomology class $\varphi \in H^1(M;\mathbb{Z})$ we define its *Thurston norm* to be

$$x_M(\varphi) = \min\{\chi_-(S) \mid S \subseteq M \text{ properly embedded surface dual to } \varphi\}.$$

We call a cohomology class $\varphi \in H^1(M;\mathbb{Q})$ *fibered* if there is a fibration $F \to M \xrightarrow{p} S^1$ and a positive integer k such that $H_1(p)\colon H_1(M) \to H_1(S^1) = \mathbb{Z}$ coincides with $k \cdot \varphi$.

Theorem 2.1 (Properties of the Thurston norm)**.** *Let M be a compact connected orientable 3-manifold. Then:*

(1) x_M is a seminorm on $H^1(M;\mathbb{Z})$ which can be extended to a seminorm on $H^1(M;\mathbb{R})$ (denoted by the same symbol).

(2) If M is hyperbolic, then x_M is a norm on $H^1(M;\mathbb{R})$.

(3) If $F \to M \xrightarrow{p} S^1$ is a fiber bundle with compact surface F as fiber, then we get for $\varphi = H_1(p)\colon H_1(M) \to H_1(S^1) = \mathbb{Z}$

$$x_M(\varphi) = \begin{cases} -\chi(F) & \textit{if } \chi(F) \leq 0; \\ 0 & \textit{if } \chi(F) \geq 0. \end{cases}$$

(4) The unit norm ball B_{x_M} is a polyhedron. If x_M is a norm, then B_{x_M} is a polytope.

(5) There are open codimension 1 faces (see Definition 4.6) of B_{x_M} such that $\varphi \in H^1(M;\mathbb{Z})$ is fibered if and only if φ lies in the cone over these faces.

(6) If $p\colon N \to M$ is a k-sheeted covering and $\varphi \in H^1(M;\mathbb{R})$, then

$$x_N(p^*\varphi) = k \cdot x_M(\varphi).$$

Proof. (1) to (5) are Thurston's work [Thu86] while (6) is due to Gabai [Gab83, Corollary 6.13]. Here (2) follows directly from (1) since hyperbolic compact orientable 3-manifolds are atoroidal, see [BP92, Proposition D.3.2.8]. □

2.2 The Alexander norm

The Alexander norm originates in the Alexander polynomial first defined by Alexander [Ale28] as a knot invariant. It was generalized to arbitrary groups by Fox [Fox53, Fox54] as one of the early applications of what is now a common tool called *Fox calculus* (see Remark 6.5). Milnor [Mil62] was the first to describe the Alexander polynomial of 3-manifolds and 2-complexes in terms of Reidemeister torsion, see Theorem 2.9 (1). We will later also explain its relation to the universal L^2-torsion, see Remark 6.9.

Let G be a finitely presented group. Let X be a finite CW-complex with fixed basepoint x and fixed isomorphism $\pi_1(X,x) \cong G$. Let $\mu\colon G \to H$ be an epimorphism onto a free-abelian group. Let $p\colon \overline{X} \to X$ be the H-covering associated to μ. The *Alexander module of G with respect to μ* is defined as the $\mathbb{Z}H$-module $A_{G,\mu} = H_1(\overline{X}, p^{-1}(x))$, which only depends on G and μ. It is a finitely presented $\mathbb{Z}H$-module and we pick a finite $\mathbb{Z}H$-presentation

$$\mathbb{Z}H^r \xrightarrow{M} \mathbb{Z}H^s \to A_{G,\mu} \to 0,$$

for some matrix $M \in M_{r,s}(\mathbb{Z}H)$. The *Alexander ideal* $I_{G,\mu}$ is the ideal generated by all $(s-1)\times(s-1)$-minors of the matrix M. It does not depend on the choice of a finite presentation for $A_{G,\mu}$. The *Alexander polynomial*

$$\Delta_{G,\mu} \in \mathbb{Z}H/\{\pm h \mid h \in H\}$$

is defined as the greatest common divisor of all elements in $I_{G,\mu}$. If $\mu = \mathrm{pr}\colon G \to H_1(G)_f$ is the projection onto the free part of the first homology, we use the shorter notation $\Delta_G = \Delta_{G,\mu}$.

Now write $\Delta_G = \sum_{h\in H_1(G)_f} x_h \cdot h$ for elements $x_h \in \mathbb{Z}$ almost all of which vanish. McMullen [McM02] defines the *Alexander norm*

$$\|\cdot\|_A\colon H^1(G;\mathbb{Z}) = \mathrm{Hom}(H_1(G)_f, \mathbb{Z}) \to \mathbb{Z}$$

by

$$\|\varphi\|_A = \max\{\varphi(g) - \varphi(h) \mid x_g, x_h \neq 0\}.$$

It is easy to see that this defines indeed a seminorm on $H^1(G;\mathbb{Z})$. In fact, this passage from multivariable polynomials to seminorms is the simplest case of the polytope homomorphism of Section 3.7.2.

Recall that the *deficiency* of a finitely presented group G is the maximum of all values $g - r$ such that there exists a presentation of G with g generators and r relations.

Theorem 2.2 (Properties of the Alexander polynomial and norm)**.** *Let G be a finitely presented group and $H = H_1(G)_f$.*

(1) If $\operatorname{def}(G) \geq 2$, then $\Delta_G = 0$.

(2) Let $\operatorname{def}(G) = 1$ and $b_1(G) \geq 2$. For a fixed isomorphism $\mathbb{Z}H \cong \mathbb{Z}[t_1, ..., t_b]$, we have

for any $\varphi \in H^1(G;\mathbb{Z}) = \mathrm{Hom}(H,\mathbb{Z})$

$$\Delta_{G,\varphi}(t) = (t-1)\cdot \Delta_G(t^{\varphi(t_1)},...,t^{\varphi(t_b)})$$

(3) Let M *be a closed orientable* 3*-manifold with* $b_1(G) \geq 2$*. Then for any* $\varphi \in H^1(M;\mathbb{Z})$ *we have*

$$\Delta_{G,\varphi}(t) = (t-1)^2\cdot \Delta_G(t^{\varphi(t_1)},...,t^{\varphi(t_b)})$$

(4) If $\varphi\colon G \to \mathbb{Z}$ *is surjective, then* $\Delta_{G,\varphi} \in \mathbb{Z}\mathbb{Z} = \mathbb{Z}[t^{\pm}]$ *has degree* $b_1(\ker\varphi)$*. In particular,* $\Delta_{G,\varphi} = 0$ *if and only if* $b_1(\ker\varphi)$ *is infinite.*

(5) If $\varphi\colon G\to\mathbb{Z}$ *is surjective and* $\ker\varphi$ *is finitely presented, then* $\Delta_{G,\varphi}$ *is monic.*

(6) Let M *be a compact connected orientable* 3*-manifold with empty or toroidal boundary. If* $b_1(M)\geq 2$*, then for any* $\varphi\in H^1(M;\mathbb{Z})$ *we have*

$$\|\varphi\|_A \leq x_M(\varphi).$$

If $b_1(M) = 1$ *and* φ *is a generator of* $H^1(M;\mathbb{Z})$*, then*

$$\|\varphi\|_A \leq x_M(\varphi) + 1 + b_3(M).$$

Proof. (1) is obvious.

(2) and (3) are implicitly proved by McMullen [McM02, Theorem 5.1], see also Button [But07, Theorem 3.1 and Theorem 3.6].

(4) is well-known, see for example [McM02, Equation (4.1)].

(5) is proved in [But07, Proposition 2.1].

(6) is proved in [McM02, Theorem 1.1]. □

The last inequality is a generalization of the well-known inequality

$$\deg \Delta_K(t) \leq 2\cdot g(K)$$

for a knot K, where $g(K)$ denotes the knot genus of K, i.e., the minimal genus of a Seifert surface for K. This is because in this case the Thurston norm of a generator φ of $H^1(M_K;\mathbb{Z})$ satisfies $x_{M_K}(\varphi) = 2\cdot g(K)-1$, where $M_K = S^3\smallsetminus \nu K$. We collect a more conceptual way to define the Alexander polynomial and more properties of the Alexander norm in Theorem 2.9 after introducing its higher-order versions.

2.3 Higher-order Alexander norms

The definition of higher-order Alexander norms originates in work of Cochran [Coc04] for knots, Harvey [Har05] for finite CW-complexes and certain solvable quotients, and Friedl [Fri07] in general. The construction uses algebraic concepts which require a few preliminary remarks.

2.3.1 Twisted Laurent polynomial rings, crossed products, and Ore localization. Let R be ring and $t\colon R\to R$ a ring automorphism. We define the t*-twisted Laurent* *polynomial ring* $R_t[u^{\pm}]$ as the usual Laurent polynomial ring $R[u^{\pm}]$, but with multiplication determined by

$$(r \cdot u^m) \cdot (s \cdot u^n) = rt^m(s) \cdot u^{m+n}$$

for $r, s \in R$ and $m, n \in \mathbb{Z}$. As for the untwisted version, the ring $R_t[u^{\pm}]$ carries a natural degree function $\deg$. Twisted Laurent polynomial rings occur naturally as shown by the next example.

Example 2.3. Let $0 \to K \to G \xrightarrow{p} \mathbb{Z} \to 0$ be a group extension, and let R be a ring. Pick a preimage $g \in G$ of $1 \in \mathbb{Z}$ under p. Then there is an isomorphism $RG \cong RK_t[u^{\pm}]$ with $t\colon RK \to RK,\ k \mapsto g^{-1}kg$.

Twisted Laurent polynomial rings on the one hand and group rings on the other hand have a common generalization called *crossed product*. Since we do not need the technical details in what follows, we only describe this concept very roughly and refer to [Lüc02, Section 10.3.2] for more information. Let R be a ring and G be a group, and take maps (of sets) $c\colon G \to \mathrm{Aut}(R)$ and $\tau\colon G \times G \to R^{\times}$. The crossed product $R *_{c,\tau} G$ has as underlying abelian group the free R-module with basis G and as multiplication

$$(r \cdot g) \cdot (s \cdot h) = rc(g)(s)\tau(g,h) \cdot gh$$

for $r, s \in R$ and $g, h \in G$. The multiplication is associative under certain conditions on c and τ. We can now generalize the example above.

Example 2.4. Let $0 \to K \to G \xrightarrow{p} Q \to 0$ be a group extension, and let R be a ring. Pick a set-theoretic section $s\colon Q \to G$ of p. We can identify $RG \cong (RK) *_{c,\tau} Q$, where the structure maps c and τ are defined by

$$c(q)\left(\sum_{k \in K} a_k \cdot k\right) = \sum_{k \in K} a_k \cdot s(q)ks(q)^{-1}$$

and

$$\tau(q, q') = s(q)s(q')s(qq')^{-1} \in K.$$

The isomorphism $(RK) * Q \to RG$ is given by

$$\sum_{q \in Q} \lambda_q \cdot q \mapsto \sum_{q \in Q} \lambda_q \cdot s(q).$$

Let R be a ring without zero-divisors and $S \subseteq R$ a multiplicatively closed subset. Then R satisfies the *(left) Ore condition* with respect to S if for any $r \in R, s \in S$ there are $r' \in R, s' \in S$ such that $s'r = r's$. In this case, one can define the *(left) Ore localization $S^{-1}R$ of R at S* completely analogous to the concept of localization in commutative rings. There is a ring homomorphism $R \to S^{-1}R$ and $S^{-1}R$ is a flat R-module. Note that if R satisfies the Ore condition with respect to $S = R \setminus \{0\}$, then $S^{-1}R$ is a skew-field. The notions *right Ore condition* and *right Ore localization* are defined similarly. If R is a ring with an involution which respects S, then the left and right Ore condition are equivalent. We refer to [Ste75, Chapter II] for more information on non-commutative localization.

Ore localizations will occur in this thesis almost exclusively in the following situation.

Lemma 2.5. *Let G be a torsion-free elementary amenable group and k a skew-field. Then any crossed product $k * G$ satisfies the Ore condition with respect to $S = k * G \setminus \{0\}$.*

Proof. This follows from [Lin06, Theorem 2.3], see also [KLM88, Theorem 1.2]. □

2.3.2 Definition of higher-order Alexander norms.

Definition 2.6 (Large homomorphism)**.** A homomorphism $f\colon \pi \to G$ of groups is *large* if the canonical projection $\mathrm{pr}\colon \pi \to H_1(\pi)_f$ factors over f, i.e., there exists $g\colon G \to H_1(\pi)_f$ such that $\mathrm{pr} = g \circ f$.

Let X be a finite CW-complex and let $\mu\colon \pi_1(X) \to G$ be a large epimorphism. Let $p\colon \overline{X} \to X$ be the G-covering associated to μ. Assume that G is a torsion-free elementary amenable group. Then by Lemma 2.5, $\mathbb{Z}G$ satisfies the Ore condition with respect to $S = \mathbb{Z}G \smallsetminus \{0\}$ and we denote the corresponding Ore localization by $Q(G) = S^{-1}\mathbb{Z}G$.

Let $\varphi\colon G \to \mathbb{Z}$ be an epimorphism and denote by $K = \ker(\varphi\colon G \to \mathbb{Z})$. Then $\mathbb{Z}K$ satisfies the Ore condition with respect to $T = \mathbb{Z}K \smallsetminus \{0\}$. The twisting $t\colon \mathbb{Z}K \to \mathbb{Z}K$ described in Example 2.3 extends to $t\colon Q(K) \to Q(K)$, and we have a chain of ring embeddings

$$\mathbb{Z}G \cong \mathbb{Z}K_t[u^{\pm}] \subseteq Q(K)_t[u^{\pm}] \subseteq Q(G),$$

It is easy to see that the last embedding localizes to an isomorphism

$$U^{-1}Q(K)_t[u^{\pm}] \cong Q(G) \tag{2.1}$$

for U the set of non-trivial elements in $Q(K)_t[u^{\pm}]$.

Definition 2.7 (Higher-order Alexander norms)**.** Suppose that the homology $H_*(X; Q(G))$ of the chain complex $Q(G) \otimes_{\mathbb{Z}G} C_*(\overline{X})$ vanishes, where $C_*(\overline{X})$ denotes the cellular $\mathbb{Z}G$-chain complex of $\overline{X}$. Then the homology of the chain complex $Q(K)_t[u^{\pm}] \otimes_{\mathbb{Z}G} C_*(\overline{X})$ is finite-dimensional over $Q(K)$ (compare also Lemma 3.23 and Theorem 3.24). The *higher-order Alexander norm* associated to μ is defined as

$$\delta(X;\mu)(\varphi) = \dim_{Q(K)} H_1(Q(K)_t[u^{\pm}] \otimes_{\mathbb{Z}G} C_*(\overline{X})).$$

The above definition is due to Harvey for certain quotients coming from the rational derived series of $\pi_1(X)$. There is an alternative and more conceptual way to define these norms that will also foreshadow the relationship between universal L^2-torsion and twisted L^2-Euler characteristics (see Theorem 3.52). This approach is, to the best of our knowledge, due to Friedl [Fri07].

Denote by

$$\tau(X;\mu) \in K_1(Q(G))/\{(\pm g) \mid g \in G\}$$

the Reidemeister torsion of the finite based free acyclic $Q(G)$-chain complex $Q(G)\otimes_{\mathbb{Z}G} C_*(\overline{X})$ equipped with some cellular basis (see Section 3.1.1 below for explanations). The *Dieudonné determinant* [Die43] induces an isomorphism

$$\det_{Q(G)}\colon K_1(Q(G)) \xrightarrow{\cong} Q(G)^{\times}_{\mathrm{ab}},$$

where $Q(G)^{\times} = Q(G) \smallsetminus \{0\}$ denotes the units of $Q(G)$, and $Q(G)^{\times}_{\mathrm{ab}}$ the abelianization thereof (see [Ros94, Corollary 2.2.6] or [Sil81, Corollary 4.3]). Using the localization isomorphism of (2.1) we can extend the degree function $\deg_\varphi$ on $Q(K)_t[u^{\pm}]$ to $Q(G)^{\times}$ by

$$\deg_\varphi(b^{-1}a) = \deg_\varphi(a) - \deg_\varphi(b)$$

for $a, b \in Q(K)_t[u^{\pm}]$. This assignment descends to the quotient $Q(G)^{\times}_{\mathrm{ab}}/\{[\pm g] \mid g \in G\}$. Then we have the following.

Theorem 2.8 (Alexander norms as torsion degrees)**.** *If in the above situation X is a compact connected orientable 3-manifold with empty or toroidal boundary or X is a finite* *connected 2-complex with* $\chi(X) = 0$*, then we have*

$$\delta(X;\mu)(\varphi) = \deg_\varphi \det_{Q(G)}(\tau(X;\mu)).$$

provided that $G \neq \mathbb{Z}$*. If* $G = \mathbb{Z}$ *and* φ *is an isomorphism, then*

$$\delta(X;\mu)(\varphi) = \begin{cases} \deg_\varphi \det(\tau(X;\mu)) + 2 & \textit{if } X \textit{ is a closed } 3\textit{-manifold;} \\ \deg_\varphi \det(\tau(X;\mu)) + 1 & \textit{otherwise.} \end{cases}$$

Proof. This follows from [Fri07, Corollary 3.6, Lemma 4.3, and Lemma 4.4]. □

Theorem 2.9 (Properties of Alexander norms)**.** *Let* X *be a compact connected orientable* 3*-manifold with empty or toroidal boundary or a finite connected* 2*-complex with* $\chi(X) = 0$*. Let* $\mu\colon \pi_1(X) \to G$ *be a large epimorphism onto a torsion-free elementary amenable group. Fix some* $\varphi \in H^1(X;\mathbb{Z})$*. Then:*

(1) (Alexander polynomial as Reidemeister torsion) Consider the case $\mu = \mathrm{pr}\colon \pi_1(X) \to H_1(X)_f$ *and* $H_*(X; Q(H_1(X)_f) = 0$*. If* $b_1(X) \geq 2$*, then*

$$\Delta_{\pi_1(X)} = \det \tau(X;\mathrm{pr})$$

under the inclusion $\mathbb{Z}H_1(X)_f/\{\pm h\} \to Q(H_1(X)_f)/\{\pm h\}$*. If* $b_1(X) = 1$ *and* t *denotes a generator of* $H_1(X)_f$*, then we have*

$$\Delta_{\pi_1(X)} = \begin{cases} \det \tau(X;\mathrm{pr}) \cdot (t-1)^2 & \textit{if } X \textit{ is a closed } 3\textit{-manifold;} \\ \det \tau(X;\mathrm{pr}) \cdot (t-1) & \textit{otherwise.} \end{cases}$$

(2) Consider the case $\mu = \mathrm{pr}\colon \pi_1(X) \to H_1(X)_f$ *and* $H_*(X; Q(H_1(X)_f) = 0$*. Then*

$$\delta(X;\mathrm{pr})(\varphi) = \|\varphi\|_A.$$

(3) $\delta(X;\mu)$ *is a seminorm on* $H^1(X;\mathbb{Z}) = H^1(G;\mathbb{Z})$*.*

(4) Let $\mu'\colon \pi_1(X) \to G'$ *be an epimorphism onto a torsion-free elementary amenable group such that* μ *factorizes as* $\pi_1(X) \xrightarrow{\mu'} G' \xrightarrow{\beta} G$ *for some group epimorphism* β*. If* $H_*(X;Q(G)) = 0$*, then* $H_*(X;Q(G')) = 0$*, and in this case we have*

$$\delta(X;\mu)(\varphi) \leq \delta(X;\mu')(\varphi).$$

(5) Suppose that $H_*(X;Q(G)) = 0$*. If* $b_1(X) \geq 2$ *or* $\mu \neq \mathrm{pr}$*, then we have in the* 3*-manifold case*

$$\delta(X;\mu)(\varphi) \leq x_X(\varphi).$$

If φ *is fibered, then the inequality is an equality.*

Proof. (1) is due to Milnor [Mil62] for link exteriors, but his argument generalizes to any case where X is not a closed 3-manifold. This latter case is handled by Turaev [Tur75].

(2) was proved by Harvey [Har05, Proposition 5.12]. It also follows from part (1) and Theorem 2.8.

(3) was proved by Friedl-Harvey [FH07, Theorem 1.1] for 3-manifolds. The proof also applies to 2-complexes.

(4) and (5) were proved by Harvey in [Har06, Corollary 2.10] and [Har05, Theorem 10.1] for the quotients of the rational derived series and by Friedl [Fri07, Theorem 1.2 and Theorem 1.3] in general (where also twistings by representations of $\pi_1(X)$ were allowed). □

Higher-order Alexander norms will later be shown to be a special case of twisted L^2-Euler characteristics, see Corollary 3.29.

2.4 The Bieri-Neumann-Strebel-invariant

In this section we recall the definition of the BNS-invariant due to Bieri-Neumann-Strebel [BNS87]. A forerunner version of it was defined by Bieri-Strebel [BS81] for the case of abelian groups. Close connections of the BNS-invariant to the previously presented Thurston and Alexander norms have been established early on. For example, the BNS-invariant was used by Dunfield [Dun01] to show that the Thurston and Alexander norm of a *fibered* 3-manifold do not always coincide on all cohomology classes. We use the monograph of Strebel [Str12] as our main reference.

Let G be a finitely generated group. Put $S(G) = \operatorname{Hom}(G, \mathbb{R})/\mathbb{R}^{>0}$, where the positive reals act on $\operatorname{Hom}(G, \mathbb{R})$ by multiplication. Pick a finite generating set S of G and denote by $\operatorname{Cay}(G, S)$ the Cayley graph of G with respect to S. Given $\varphi \in \operatorname{Hom}(G, \mathbb{R})$, denote by $\operatorname{Cay}(G, S)_\varphi$ the subgraph of $\operatorname{Cay}(G, S)$ induced by the vertex subset $\{g \in G \mid \varphi(g) \geq 0\}$. Then the (first) *Sigma-invariant* or *BNS-invariant* is defined as

$$\Sigma(G) = \{[\varphi] \in S(G) \mid \operatorname{Cay}(G, S)_\varphi \text{ is connected}\}.$$

This definition is independent of the choice of generating set, see [Str12, Theorem A2.3]. The following examples are taken from [Str12, Section A2.1a].

Example 2.10. (1) For a finitely generated free-abelian group H we have $\Sigma(H) = S(H)$. Namely, if we take the standard generating set for $\mathbb{Z}^n$, then the set $\{h \in \mathbb{Z}^n \mid \varphi(h) \geq 0\}$ is the intersection of a halfspace in $\mathbb{R}^n$ with the lattice $\mathbb{Z}^n$. This is easily seen to be connected.

(2) Let $G = A * B$ be a free product of finitely generated groups A and B. We claim that $\Sigma(G)$ is empty, which applies in particular to free groups. Let S and T be generating sets of A and B respectively. Let $\varphi\colon G \to \mathbb{R}$ be non-trivial. Without loss of generality there is an element $a \in A$ such that $\varphi(a) > 0$. Take a non-trivial $b \in B$ with $\varphi(b) \geq 0$. Then $a^{-1}ba$ lies in $\operatorname{Cay}(G, S \cup T)_\varphi$, but we claim that it cannot be connected to 1. A path from 1 to $a^{-1}ba$ in $\operatorname{Cay}(G, S \cup T)$ corresponds to a sequence $w_1, w_2, ..., w_k$ such that w_i is a word in either S or T. We may assume that every w_i is non-trivial in G since the path contains a loop otherwise. But then the normal form theorem for free products implies that w_1 represents a^{-1}. Since $\varphi(a^{-1}) < 0$, the path does not lie inside $\operatorname{Cay}(G, S \cup T)_\varphi$.

We give a collection of properties of $\Sigma(G)$.

Theorem 2.11 (Properties of the BNS-invariant)**.** *Let G be a finitely generated group and $\varphi \in \operatorname{Hom}(G, \mathbb{Z})$.*

(1) $\Sigma(G)$ is an open subset of $S(G)$.

(2) We have $[\varphi] \in \Sigma(G) \cap -\Sigma(G)$ if and only if $\ker(\varphi)$ is finitely generated.

(3) *If φ is surjective, then $\varphi \in \Sigma(G)$ if and only if there is a finitely generated subgroup $A \subseteq G$, a monomorphism $\alpha\colon A \to A$ and a commutative diagram*

$$\begin{tikzcd} G \arrow[r, "\cong"] \arrow[dr, "\varphi"'] & \langle\, A, t \mid t^{-1}at = \alpha(a) \,\rangle \arrow[d, "\psi"] \\ & \mathbb{Z} \end{tikzcd}$$

such that the upper map is an isomorphism, $\psi(t) = 1$, and $\psi(A) = 0$.

(4) *If $G = \pi_1(M)$ for a compact orientable irreducible 3-manifold M, then*

$$\Sigma(G) = -\Sigma(G)$$

and $\Sigma(G)$ is the projection of the fibered faces described in Theorem 2.1 (5) to $S(G)$.

Proof. These results are due to Bieri-Neumann-Strebel, see [BNS87, Theorem A, Theorem B1, Proposition 4.3 and Theorem E]. □

3 L^2-Torsion Invariants

This chapter is a concise (and necessarily incomplete) survey of L^2-torsion invariants. Beginning with the classical notion of L^2-torsion, we will review twisted L^2-torsion, from which twisted L^2-torsion functions can comfortably be defined. We then turn over to the twisted L^2-Euler characteristics constructed by Friedl-Lück.

All these invariants share similar features such as (simple) homotopy invariance as well as sum, product, induction and restriction formulas. These features were originally proved case by case and this tedious work serves as the main motivation for the universal L^2-torsion constructed by Friedl-Lück [FL16b]. As the name suggests, the previous L^2-torsion invariants (and their basic properties) can be derived from the universal L^2-torsion. Finally, we present the L^2-torsion polytope, a geometric invariant which is the central object of study for the rest of this thesis.

3.1 Preliminaries on L^2-invariants

In this preliminary section we collect some terminology, notation and basics concerning L^2-invariants, following the standard reference [Lüc02].

3.1.1 From $\mathbb{Z}G$-modules to Hilbert $\mathcal{N}(G)$-modules. Let R be a ring. A *based free* R-module $(M, [B])$ is a free R-module M equipped with an equivalence class of R-basis $[B]$, where two R-bases B and B' are *equivalent* if there is a bijection $\sigma\colon B \to B'$ such that $\sigma(b) = \pm b$. Let R-FBMOD be the category whose objects are finitely generated based free R-modules and whose morphisms are R-linear maps. Let R-FBCC be the category of chain complexes over R-FBMOD. Moreover, we call an R-chain complex *finite* if each chain module is finitely generated and all but finitely many chain modules vanish.

Let G be a (discrete) group. We denote by $L^2(G)$ the (complex) Hilbert space with Hilbert basis G. It carries a canonical left G-action induced by the multiplication in G. The *group von Neumann algebra* $\mathcal{N}(G)$ of G is the algebra of bounded G-equivariant operators on $L^2(G)$. Denote by $\mathcal{N}(G)$-FGHIL the category of finitely generated Hilbert $\mathcal{N}(G)$-modules (see [Lüc02, Definition 1.5]) and by $\mathcal{N}(G)$-FGHCC the category of chain complexes over $\mathcal{N}(G)$-FGHIL. We define a functor

$$\Lambda\colon \mathbb{Z}G\text{- FBMOD} \to \mathcal{N}(G)\text{- FGHIL}$$

that sends an object $(M, [B])$ to $L^2(G) \otimes_{\mathbb{C}G} M$ equipped with the Hilbert space structure for which the map

$$\bigoplus_{b\in B} L^2(G) \to L^2(G) \otimes_{\mathbb{Z}G} M, \quad (x_b)_{b\in B} \mapsto \sum_{b\in B} x_b \otimes b$$

becomes an isometry. A morphism $f\colon (M, [B]) \to (N, [C])$ is sent to the bounded G-equivariant operator $\mathrm{id} \otimes_{\mathbb{Z}G} f\colon L^2(G) \otimes_{\mathbb{Z}G} M \to L^2(G) \otimes_{\mathbb{Z}G} N$. This functor can be extended to a functor

$$\Lambda\colon \mathbb{Z}G\text{- FBCC} \to \mathcal{N}(G)\text{- FGHCC}$$

by applying Λ to each chain module and differential.

Let M be a free $\mathbb{Z}G$-module. Then two $\mathbb{Z}G$-bases B and B' of M are *cellularly equivalent* if there is a bijection $\sigma\colon B \to B'$ and elements $\epsilon(b) \in \{\pm 1\}$, $g(b) \in G$ such that $\sigma(b) = \epsilon(b) \cdot g(b) \cdot b$. If X is a finite free G-CW-complex, then the chain modules $C_n(X)$ carry a canonical cellular equivalence class of basis induced from the cellular structure. We call an equivalence class of basis of $C_n(X)$ a *cellular basis* if its cellular equivalence class agrees with this canonical cellular equivalence class of basis.

3.1.2 L^2-Betti numbers. The *von Neumann dimension* of a Hilbert $\mathcal{N}(G)$-module M (see [Lüc02, Definition 1.10]) is denoted by $\dim_{\mathcal{N}(G)}(M)$. The same notation will be used for the *(extended) von Neumann dimension* of $\mathcal{N}(G)$-modules (see [Lüc02, Definition 6.20]), where $\mathcal{N}(G)$ is just considered as a ring. This overload of notation is justified in view of [Lüc02, Theorem 6.24].

The *L^2-Betti numbers* of a $\mathbb{Z}G$-chain complex C_* are defined as

$$b_n^{(2)}(C_*;\mathcal{N}(G)) = \dim_{\mathcal{N}(G)} H_n(\mathcal{N}(G) \otimes_{\mathbb{Z}G} C_*).$$

Applying this to the singular $\mathbb{Z}G$-chain complex of a G-space X produces the *L^2-Betti numbers* $b_n^{(2)}(X;\mathcal{N}(G))$ of X. In practice, however, we will exclusively work with G-CW-complexes, where we can take the cellular $\mathbb{Z}G$-complex instead, see [Lüc02, Lemma 6.52]. We call a $\mathbb{Z}G$-chain complex or a G-space *L^2-acyclic* if all its L^2-Betti numbers vanish. We refer to [Lüc02, Chapters 1 and 6] for a detailed account of L^2-Betti numbers.

3.1.3 Fuglede-Kadison determinant. Let $f\colon V \to W$ be a morphism of finite-dimensional Hilbert $\mathcal{N}(G)$-modules. Then there is an associated *spectral density function* $F(f)\colon [0,\infty) \to [0,\infty]$ (see [Lüc02, Definition 2.1]), which is used in the construction of the *Fuglede-Kadison determinant* $\det_{\mathcal{N}(G)}(f)$ as follows: The morphism f is *of determinant class* if $\int_{0+}^{\infty} \log(\lambda)\, dF > -\infty$, and in this case we define

$$\det_{\mathcal{N}(G)}(f) = \exp\left(\int_{0+}^{\infty} \log(\lambda)\, dF\right).$$

Otherwise we put $\det_{\mathcal{N}(G)}(f) = 0$. The basic properties of this notion of determinant are collected in [Lüc02, Theorem 3.14]. We also point out the following useful fact.

Remark 3.1. If $g \in G$ is an element of infinite order and z is a complex number, then $\Lambda(r_{1-z\cdot g}\colon \mathbb{Z}G \to \mathbb{Z}G)$ is a weak isomorphism and we have by [Lüc02, Theorem 3.14 (6) and Equation (3.24)]

$$\det_{\mathcal{N}(G)}(\Lambda(r_{1-z\cdot g}\colon \mathbb{Z}G \to \mathbb{Z}G)) = \max\{1, |z|\}.$$

A Hilbert $\mathcal{N}(G)$-chain complex C_* with finite-dimensional chain modules is of determinant class if all of its differentials are of determinant class. We call C_* *det-L^2-acyclic* if it is both of determinant class and L^2-acyclic. A free G-CW-complex of finite type, i.e., with finitely many cells in each dimension, is of determinant class (respectively, det-L^2-acyclic) if its cellular chain complex is of determinant class (respectively, det-L^2-acyclic) after applying Λ.

If $A \in M_{m,n}(\mathbb{Z}G)$ is a matrix, then the morphism $r_A\colon L^2(G)^m \to L^2(G)^n$ given by right multiplication with A is conjectured to be of determinant class [Lüc02, Conjecture 3.94 (3)]. This would imply that every free G-CW-complex of finite type is of determinant class. The (stronger) *Determinant Conjecture* predicts that we even have $\det_{\mathcal{N}(G)}(r_A) \geq 1$. An affirmative answer to the Determinant Conjecture is known by the work of Elek-Szabó [ES05, Theorem 5] if G belongs to the class of sofic groups, which contains among others all residually amenable and in particular all residually finite groups. Previous special instances of this statement were proved by Lück, Clair, and Schick [Lüc94a, Cla99, Sch01]. We refer to [Lüc02, Section 3.2] for a thorough investigation of the Fuglede-Kadison determinant and to [Lüc02, Chapter 13] for an account of the Determinant Conjecture.

3.2 Classical L^2-torsion

We use [Lüc02, Chapter 3] as main reference for the classical L^2-torsion.

Definition 3.2 (L^2-torsion)**.** Let C_* be a finite based free $\mathbb{Z}G$-chain complex of determinant class. Then its L^2*-torsion* is defined as

$$\rho^{(2)}(C_*;\mathcal{N}(G)) = -\sum_{n\in\mathbb{Z}}(-1)^n \cdot \log\det{}_{\mathcal{N}(G)}(\Lambda(c_n)) \in \mathbb{R}.$$

If X is a finite free G-CW-complex, then its L^2-torsion is defined as

$$\rho^{(2)}(X;\mathcal{N}(G)) = \rho^{(2)}(C_*(X),[B]);\mathcal{N}(G)),$$

where $C_*(X)$ is the cellular $\mathbb{Z}G$-chain complex of X equipped with some cellular basis $[B]$ (in the sense of Section 3.1.1).

While the definition makes sense as soon as C_* is of determinant class, it is in practice often necessary to restrict the attention to L^2-acyclic chain complexes. Otherwise, even rudimentary properties fail. The following collection of basic properties is taken from [Lüc02, Theorem 3.93].

Theorem 3.3 (Basic properties of L^2-torsion)**.**

(1) (Homotopy invariance) Let $f\colon X \to Y$ be a G-homotopy equivalence of finite free G-CW-complexes. If X or Y is $\det$*-L^2-acyclic, then both are* $\det$*-L^2-acyclic and we have*

$$\rho^{(2)}(Y;\mathcal{N}(G)) - \rho^{(2)}(X;\mathcal{N}(G)) = \Phi(\tau(f)),$$

where $\tau(f) \in \mathrm{Wh}(G)$ denotes the Whitehead torsion and $\Phi\colon \mathrm{Wh}(G) \to \mathbb{R}$ is the homomorphism induced by taking the Fuglede-Kadison determinant.

(2) (Sum formula) Let

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & X_1 \arrow[d] \\ X_2 \arrow[r] & X \end{tikzcd}$$

be a G-pushout of finite free G-CW-complexes such that the upper horizontal map is cellular, the left-hand map is an inclusion of G-CW-complexes and X carries the G-CW-structure induced from the X_i. If X_i for $i = 0,1,2$ is $\det$*-L^2-acyclic, then X is* $\det$*-L^2-acyclic and we have*

$$\rho^{(2)}(X;\mathcal{N}(G)) = \rho^{(2)}(X_1;\mathcal{N}(G)) + \rho^{(2)}(X_2;\mathcal{N}(G)) - \rho^{(2)}(X_0;\mathcal{N}(G)).$$

(3) *(Product formula) Let X_i be a finite free G_i-CW-complex for $i = 1, 2$. If X_1 is* $\det$-L^2*-acyclic, then the $(G_1 \times G_2)$-CW-complex $X_1 \times X_2$ is* $\det$-L^2*-acyclic and we have*
$$\rho^{(2)}(X_1 \times X_2; \mathcal{N}(G_1 \times G_2)) = \chi(X_2/G_2) \cdot \rho^{(2)}(X_1; \mathcal{N}(G_1)).$$

(4) *(Induction) Let $i\colon H \to G$ be an inclusion of groups. Let X be a finite free H-CW-complex. Then the finite free G-CW-complex $i_*X = G \times_H X$ is* $\det$-L^2*-acyclic if and only if X is* $\det$-L^2*-acyclic, and in this case we have*
$$\rho^{(2)}(i_*X; \mathcal{N}(G)) = \rho^{(2)}(X; \mathcal{N}(H)).$$

(5) *(Restriction) Let $i\colon H \to G$ be an inclusion of groups with finite index. Let X be a finite free G-CW-complex. Let i^*X be the finite free H-CW-complex obtained from X by restriction. Then i^*X is* $\det$-L^2*-acyclic if and only if X is* $\det$-L^2*-acyclic, and in this case we have*
$$\rho^{(2)}(i^*X; \mathcal{N}(H)) = [G : H] \cdot \rho^{(2)}(X; \mathcal{N}(G)).$$

(6) *(Fibrations) Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration such that F and B are finite CW-complexes. Let $\overline{E} \to E$ be a G-covering and $\overline{F} \to F$ the G-covering obtained from it by pullback along i. Assume that* $\mathrm{Wh}(G)$ *vanishes. Assume that $\overline{F}$ is* $\det$-L^2*-acyclic. Then $\overline{E}$ is up to G-homotopy equivalence a finite free* $\det$-L^2*-acyclic G-CW-complex and we have*
$$\rho^{(2)}(\overline{E}; \mathcal{N}(G)) = \chi(B) \cdot \rho^{(2)}(\overline{F}; \mathcal{N}(G)).$$

(7) *(Poincaré Duality) Let M be a free proper cocompact G-manifold of even dimension without boundary. Assume that M is orientable. If M is* $\det$-L^2*-acyclic, then*
$$\rho^{(2)}(M; \mathcal{N}(G)) = 0.$$

The L^2-torsion has been computed in special cases and shown to relate to other, more geometric invariants. For example, if M is a closed hyperbolic 3-manifold, then Lück-Schick [LS99] show that
$$\rho^{(2)}(\widetilde{M}; \mathcal{N}(\pi_1(M))) = -\frac{1}{6\pi} \cdot \mathrm{vol}(M).$$
This result has generalizations for all odd dimensions by Hess-Schick [HS98]

If X is an aspherical finite CW-complex such that $\pi_1(X)$ is of $\det \geq 1$-class and contains an infinite elementary amenable normal subgroup, then $\widetilde{X}$ is det-L^2-acyclic and $\rho^{(2)}(\widetilde{X}; \mathcal{N}(\pi_1(X))) = 0$ by Wegner [Weg09]. This result motivates the computation of the L^2-torsion polytope of amenable groups of Chapter 5.

Finally, if F_n is a finitely generated free group and $\alpha\colon F_n \to F_n$ is an automorphism, then Clay [Cla17] showed for the semidirect product $\pi_\alpha = F_n \rtimes_\alpha \mathbb{Z}$ that $-\rho^{(2)}(E\pi_\alpha; \mathcal{N}(\pi_\alpha))$ is a lower bound for the growth rates of α. In particular, the L^2-torsion vanishes for polynomially growing automorphisms. We will reprove this result in Corollary 6.21.

3.3 Twisted L^2-torsion

In this section we recall L^2-torsion twisted with finite-dimensional representations. This notion can be seen as a convenient basis for the construction of twisted L^2-torsion functions in Section 3.4. Chronologically, however, twisted L^2-torsion functions were defined first, but important ingredients for a good behavior of L^2-torsion, i.e., L^2-acyclicity and determinant class, had to be checked case by case. A systematic study of how these notions are affected by twisting the chain complex was eventually contributed by Lück [Lüc15], which also serves as the main reference of this section.

Let V be a finite-dimensional (complex left) G-representation and $[B_V]$ be an equivalence class of $\mathbb{C}$-basis in the sense of Section 3.1.1. We define a twisting functor

$$\eta_V : \mathbb{C}G\text{-}\operatorname{FBMOD} \to \mathbb{C}G\text{-}\operatorname{FBMOD}$$

that sends a finitely generated based free $\mathbb{C}G$-module M equipped with the equivalence class of basis $[B_M]$ to the $\mathbb{C}G$-module $M \otimes_{\mathbb{C}} V$ equipped with the equivalence class of basis $[\{b \otimes v \mid b \in B_M,\ v \in B_V\}]$. On morphisms η_V sends a $\mathbb{C}G$-linear map $f\colon M \to N$ to $f \otimes_{\mathbb{C}} V\colon M \otimes_{\mathbb{C}} V \to N \otimes_{\mathbb{C}} V$. For simplicity, we suppress the equivalence class $[B_V]$ in the notation although the functor η_V depends on it. This functor has an obvious extension to

$$\eta_V : \mathbb{C}G\text{-}\operatorname{FBCC} \to \mathbb{C}G\text{-}\operatorname{FBCC}.$$

Central questions about this functor are how it manipulates L^2-Betti numbers, determinant class and L^2-torsion.

Definition 3.4 (Twisted L^2-torsion)**.** Let V be a based finite-dimensional G-representation. Let C_* be a finite based free $\mathbb{C}G$-chain complex. Then C_* is *of V-twisted determinant class* (respectively, *V-twisted L^2-acyclic*, or *V-twisted* det-*L^2-acyclic*) if $\Lambda(\eta_V(C_*))$ is of determinant class (respectively, L^2-acyclic or det-L^2-acyclic).

If C_* is of V-twisted determinant class, then we define the *V-twisted L^2-torsion of C_** to be

$$\rho^{(2)}(C_*; V) = \rho^{(2)}(\Lambda(\eta_V(C_*))).$$

These notions carry over to a finite free G-CW-complex X by applying them to the cellular $\mathbb{C}G$-chain complex of X endowed with some cellular basis. It follows from [Lüc02, Theorem 3.35 (5)] and [Lüc15, Lemma 2.2 (1)] that this does not depend on the choice of cellular basis for $C_*(X)$.

The basic properties of twisted L^2-torsion are collected in [Lüc15, Theorem 5.7], including homotopy invariance as well as sum, product, restriction and induction formulas in the same spirit as Theorem 3.3. We omit these here and rather concentrate on the question when twisted L^2-torsion applies. The definition requires the chain complex C_* to be at least of twisted determinant class, but just as for the classical L^2-torsion, good behavior of this invariant only arises under the additional assumption of twisted L^2-acyclicity. It will in practice be therefore convenient to know when these properties can be reduced to those of the untwisted chain complex. This is answered in an important special case by the following technical result [Lüc15, Theorem 6.7].

Theorem 3.5 (L^2-acyclicity and determinant class after twisting)**.** *Let G be a countable residually finite group and $\nu\colon G \to \mathbb{Z}^d$ be an epimorphism. Let V be a based finite-dimensional $\mathbb{Z}^d$-representation. Denote by ν^*V its pullback to G (equipped with the same equivalence class of basis). Let C_* be a finite based free $\mathbb{Z}G$-chain complex. Then:*

(1) If $\Lambda(C_*)$ is of determinant class, then so is $\Lambda(\eta_{\nu^*V}(C_*))$.

(2) For every $n \in \mathbb{Z}$, we have

$$b_n^{(2)}(\Lambda(\eta_{\nu^*V}(C_*))) = \dim_{\mathbb{C}}(V) \cdot b_n^{(2)}(\Lambda(C_*)).$$

In particular, if $\Lambda(C_*)$ is L^2-acyclic, then so is $\Lambda(\eta_{\nu^* V}(C_*))$.

3.4 Twisted L^2-torsion functions

Twisted L^2-torsion functions were first constructed for knots by Li-Zhang [LZ06a, LZ06b, LZ08] and further examined by Dubois-Wegner [DW10, DW15] and Ben Aribi [BA16] (appearing there under the alternative names L^2-Alexander-Conway invariant or L^2-Alexander invariant). Dubois-Friedl-Lück [DFL16, DFL15a, DFL15b] generalize this notion to finite CW-complexes and investigate it for 3-manifolds (there called L^2-Alexander torsion). The relation of the asymptotic behavior of twisted L^2-torsion functions to the Thurston norm is especially noteworthy. This relation was further strengthened by Liu [Liu17], who also answered the question of continuity, and Friedl-Lück [FL15], see Theorem 3.14. The aforementioned work by Lück [Lüc15], which we presented in Section 3.3, clarified questions of when twisted L^2-torsion functions are available, see Theorem 3.11.

3.4.1 Definition of twisted L^2-torsion functions. With twisted L^2-torsion in our toolbox, it is now easy to give the definition of twisted L^2-torsion functions.

Definition 3.6 (Twisted L^2-torsion function)**.** Let $\varphi\colon G \to \mathbb{R}$ be a group homomorphism. Fix $t \in \mathbb{R}^{>0}$. Let $\mathbb{C}_t$ be the based 1-dimensional $\mathbb{R}$-representation $\mathbb{C}$, where $r \in \mathbb{R}$ acts by multiplication with t^r, equipped with the equivalence class of the standard basis. Denote by $\varphi^*\mathbb{C}_t$ the G-representation obtained from $\mathbb{C}_t$ by restriction along φ.

Let C_* be a finite based free $\mathbb{Z}G$-chain complex. Then C_* is of *φ-twisted determinant class* (respectively, *φ-twisted L^2-acyclic*, or *φ-twisted det-L^2-acyclic*) if it is for all $t \in \mathbb{R}^{>0}$ of $\varphi^*\mathbb{C}_t$-twisted determinant class (respectively, $\varphi^*\mathbb{C}_t$-twisted L^2-acyclic, or $\varphi^*\mathbb{C}_t$-twisted det-L^2-acyclic) in the sense of Definition 3.4.

If C_* is of φ-twisted determinant class, then we define the *φ-twisted L^2-torsion function* as

$$\rho^{(2)}(C_*;\varphi)\colon \mathbb{R}^{>0} \to \mathbb{R},\ \ t \mapsto \rho^{(2)}(C_*;\varphi^*\mathbb{C}_t),$$

where the right-most term is defined in Definition 3.4.

If we want to apply this set of definitions to the cellular chain complex of a finite free G-CW-complex X, then the choice of cellular basis possibly affects the twisted L^2-torsion. In order to get rid of this ambiguity, we introduce the following notion.

Definition 3.7. Two functions $f, g\colon \mathbb{R}^{>0} \to \mathbb{R}$ are *equivalent* if there is a real number r such that $f(t) - g(t) = r \cdot \log(t)$. In this case we use the notation $f \doteq g$.

Definition 3.8 (Twisted L^2-torsion function for G-CW-complexes)**.** Let X be a finite free G-CW-complex and let $\varphi\colon G \to \mathbb{R}$ be a group homomorphism. Then X is *of φ-twisted determinant class*, *φ-twisted L^2-acyclic*, or *φ-twisted* det*-L^2-acyclic* if the cellular chain complex $C_*(X)$ equipped with some cellular basis has this property. If X is of φ-twisted determinant class, then we define the *φ-twisted L^2-torsion function $\rho^{(2)}(X;\varphi)$ of X* to be the equivalence class of the function

$$\rho^{(2)}(C_*(X);\varphi)\colon \mathbb{R}^{>0} \to \mathbb{R}.$$

We introduce one more notation. Let X be a finite CW-complex and take group homomorphisms $\pi_1(X) \xrightarrow{\mu} G \xrightarrow{\varphi} \mathbb{R}$. Denote by $\overline{X} \to X$ the G-covering associated to

μ. If $\overline{X}$ is of φ-twisted determinant class, then we write

$$\rho^{(2)}(X;\mu,\varphi) = \rho^{(2)}(\overline{X};\varphi).$$

This definition does not depend on the choice of cellular basis [Lüc15, Theorem 7.3 (3)]. Next we illustrate that L^2-torsion functions are computable in special cases.

Example 3.9. (1) Let $f\colon X \to X$ be a cellular self-homotopy equivalence of a connected finite CW-complex X. Denote its mapping torus by T_f. Let $\pi_1(T_f) \xrightarrow{\mu} G \xrightarrow{\varphi} \mathbb{Z}$ be a factorization of the canonical epimorphism such that G is a residually finite group. It is proved in [Lüc15, Theorem 7.10] that the G-covering associated to μ is φ-twisted det-L^2-acyclic and there is a constant $T > 0$ such that

$$\rho^{(2)}(T_f;\mu,\varphi) \doteq \left(t \mapsto \begin{cases} 0 & \text{if } t \leq 1/T; \\ \chi(F)\cdot\log(t) & \text{if } t \geq T. \end{cases} \right)$$

(2) Let $K \subseteq S^3$ be a knot and consider the 3-manifold $X_K = S^3 \setminus \nu K$, where νK is an open tubular neighbourhood. Let $\varphi \in H^1(X_K;\mathbb{Z})$ be a generator. Then it is proved by Ben-Aribi [BA16] that K is trivial if and only if

$$\rho^{(2)}(\widetilde{X_K};\varphi) \doteq \left(t \mapsto \begin{cases} 0 & \text{if } t \leq 1; \\ \log(t) & \text{if } t \geq 1. \end{cases} \right)$$

(3) Let M be a Seifert fiber space unequal to $S^1 \times S^2$ and $S^1 \times D^2$. Take homomorphisms $\pi_1(M) \xrightarrow{\mu} G \xrightarrow{\varphi} \mathbb{Z}$ such that the image of a regular fiber under μ has infinite order. Then Herrmann [Her16] calculates

$$\rho^{(2)}(M;\mu,\varphi) \doteq \left(t \mapsto \begin{cases} 0 & \text{if } t \leq 1; \\ -x_M(\varphi)\cdot\log(t) & \text{if } t \geq 1. \end{cases} \right)$$

The following set of properties is the analogue of Theorem 3.3 for twisted L^2-torsion functions and appears in [Lüc15, Theorem 7.5].

Theorem 3.10 (Basic properties of twisted L^2-torsion functions)**.** *Let $\varphi\colon G \to \mathbb{R}$ be a group homomorphism.*

(1) (Homotopy invariance) Let $f\colon X \to Y$ be a G-homotopy equivalence of finite free G-CW-complexes. If X or Y is φ-twisted det-L^2-acyclic, then both are φ-twisted det-L^2-acyclic. If, additionally, f is simple (or $\mathbb{Z}G$ satisfies the K-theoretic Farrell-Jones Conjecture), then

$$\rho^{(2)}(X;\varphi) \doteq \rho^{(2)}(Y;\varphi).$$

(2) (Sum formula) Let

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & X_1 \arrow[d] \\ X_2 \arrow[r] & X \end{tikzcd}$$

be a G-pushout of finite free G-CW-complexes such that the upper horizontal map is cellular, the left-hand map is an inclusion of G-CW-complexes and X carries the G-CW-structure induced from the X_i. If X_i for $i = 0, 1, 2$ is φ-twisted det-L^2-acyclic, *then X is φ-twisted det-L^2-acyclic and we have*

$$\rho^{(2)}(X;\varphi) \doteq \rho^{(2)}(X_1;\varphi) + \rho^{(2)}(X_2;\varphi) - \rho^{(2)}(X_0;\varphi).$$

(3) *(Product formula) Let X_i be a finite free G_i-CW-complex for $i = 1, 2$. Let $\varphi\colon G_1 \times G_2 \to \mathbb{R}$ be a homomorphism and denote by $\varphi_1\colon G_1 \to \mathbb{R}$ the obvious restriction to G_1. If X_1 is φ_1-twisted det-L^2-acyclic, then the $G_1 \times G_2$-CW-complex $X_1 \times X_2$ is φ-twisted det-L^2-acyclic and we have*

$$\rho^{(2)}(X_1 \times X_2;\varphi) \doteq \chi(X_2/G_2) \cdot \rho^{(2)}(X_1;\varphi_1).$$

(4) *(Induction) Let $i\colon H \to G$ be an inclusion of groups. If X is a finite free $(\varphi\circ i)$-twisted det-L^2-acyclic H-CW-complex, then the finite free G-CW-complex $i_*X = G \times_H X$ is φ-twisted det-L^2-acyclic and we have*

$$\rho^{(2)}(i_*X;\varphi) \doteq \rho^{(2)}(X;\varphi\circ i).$$

(5) *(Restriction) Let $i\colon H \to G$ be an inclusion of groups with finite index. Let X be a finite free G-CW-complex and i^*X the finite free H-CW-complex obtained from X by restriction. Then i^*X is $(\varphi\circ i)$-twisted det-L^2-acyclic if and only if X is φ-twisted det-L^2-acyclic. In this case we have*

$$\rho^{(2)}(i^*X;\varphi\circ i) \doteq [G:H] \cdot \rho^{(2)}(X;\varphi).$$

(6) *(Fibrations) Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration of connected finite CW-complexes. Let $\overline{E} \to E$ be a G-covering and $\overline{F} \to F$ the G-covering obtained from it by pullback along i. Assume that $\mathbb{Z}G$ satisfies the K-theoretic Farrell-Jones Conjecture. Assume that $\overline{F}$ is φ-twisted-*det*-L^2-acyclic. Then $\overline{E}$ is up to G-homotopy equivalence a finite free G-CW-complex which is φ-twisted* det*-L^2-acyclic, and we have*

$$\rho^{(2)}(\overline{E};\varphi) \doteq \chi(B) \cdot \rho^{(2)}(\overline{F};\varphi).$$

(7) *(Poincaré Duality) Let M be a free proper cocompact smooth G-manifold without boundary. Assume that M is orientable and the G-action is orientation-preserving. If M is φ-twisted det-L^2-acyclic, then*

$$\rho^{(2)}(M;\varphi)(t) \doteq (-1)^{\dim(M)+1} \cdot \rho^{(2)}(M;\varphi)(t^{-1}).$$

(8) *(Scaling) Given $r \in \mathbb{R}$, a finite free G-CW-complex X is $(r\cdot\varphi)$-twisted-*det*-L^2-acyclic if and only if it is φ-twisted-*det*-L^2-acyclic. In this case we have*

$$\rho^{(2)}(X;r\cdot\varphi)(t) \doteq \rho^{(2)}(X;\varphi)(t^r).$$

In many applications, especially for 3-manifolds, the following theorem renders arguments for why L^2-torsion functions are well-defined and well-behaved obsolete.

Theorem 3.11. *(Pinching estimate and continuity) Let X be a finite free G-CW-complex and let $\varphi\colon G \to \mathbb{R}$ be a homomorphism. Assume that G is finitely generated and residually finite.*

(1) *Then X is of φ-twisted determinant class. More precisely, there are for any represen-*

tative of $\rho^{(2)}(X;\varphi)$ constants $C, D > 0$ such that we get for $0 < t < 1$

$$C \cdot \log(t) - D \leq \rho^{(2)}(X;\varphi)(t) \leq -C \cdot \log(t) + D$$

and for $t \geq 1$

$$-C \cdot \log(t) - D \leq \rho^{(2)}(X;\varphi)(t) \leq C \cdot \log(t) + D.$$

(2) If X is L^2-acyclic, then X is φ-twisted $\det$-L^2-acyclic. In this case, the φ-twisted L^2-torsion function $\rho^{(2)}(X;\varphi)$ is continuous.

Proof. The first statement is proved by Lück [Lüc15, Theorem 7.3 (1)]. Essentially it follows from the technical Theorem 3.5 (1) and the fact that residually finite groups satisfy the Determinant Conjecture, see Section 3.1.3. If X is L^2-acyclic, then X is also φ-twisted L^2-acyclic by Theorem 3.5 (2). The continuity statement follows from Liu's work [Liu17, Theorem 5.1]. □

3.4.2 The degree of twisted L^2-torsion functions. Among the many aspects of twisted L^2-torsion functions, their asymptotic behavior stands out as a particularly promising field of study. More precisely, we are interested in the following gadget.

Definition 3.12 (Degree of functions)**.** Let $[f]$ be an equivalence class of functions in the sense of Definition 3.7. Suppose that for some (and hence every) representative the values $\liminf_{t\to 0} \frac{f(t)}{\log(t)}$ and $\limsup_{t\to\infty} \frac{f(t)}{\log(t)}$ are real numbers. Then we define the *degree* of $[f]$ to be

$$\deg([f]) = \limsup_{t\to\infty} \frac{f(t)}{\log(t)} - \liminf_{t\to 0} \frac{f(t)}{\log(t)}.$$

In view of Theorem 3.11, the degree is always available for the twisted L^2-torsion function of finite free G-CW-complexes *provided that* G is finitely generated and residually finite. Note that this applies in particular to the universal covering of an admissible 3-manifold [AFW15, (C.25)]. The reader is invited to work out the degree in the situations of Example 3.9.

Definition 3.13 (Admissible 3-manifold)**.** Following [FL16b], we say that a 3-manifold is *admissible* if it is connected, orientable, irreducible with empty or toroidal boundary and infinite fundamental group. Note that an admissible 3-manifold is in particular *aspherical*, meaning that its higher homotopy groups $\pi_i(M)$ for $i \geq 2$ vanish, see [AFW15, (C.1)].

The question why one might consider the degree of L^2-torsion functions is answered by the next theorem. We emphasize the striking similarity with Theorem 3.30 below about twisted L^2-Euler characteristics.

Theorem 3.14 (Degree of L^2-torsion functions and the Thurston norm)**.** *Let $M \neq S^1 \times D^2$ be an admissible 3-manifold. Then:*

(1) (Inequality of degree and Thurston norm) Let $\mu\colon \pi_1(M) \to G$ be a homomorphism to a finitely generated and residually finite group such that the G-covering associated to μ is L^2-acyclic. Let $\varphi\colon G \to \mathbb{R}$ be a homomorphism. Then we have

$$-\deg \rho^{(2)}(M;\mu,\varphi) \leq x_M(\varphi).$$

(2) (Equality for quasi-fibered classes) Let $\mu\colon \pi_1(M) \to G$ be a large homomorphism onto a finitely generated residually finite group such that the G-covering associated to μ is

L^2-acyclic. Let $\varphi\colon G \to \mathbb{R}$ be a homomorphism. If $\varphi \circ \mu\colon \pi_1(M) \to \mathbb{R}$ is quasi-fibered, i.e., a limit in $H^1(M;\mathbb{R})$ of fibered classes in $H^1(M;\mathbb{Q})$, then we have

$$-\deg \rho^{(2)}(M;\mu,\varphi) = x_M(\varphi).$$

(3) (Equality of degree and Thurston norm) Assume that M is not a closed graph manifold. Then the canonical projection factors into epimorphisms $\pi_1(M) \xrightarrow{\alpha} \Gamma \xrightarrow{\beta} H_1(M)_f$ where Γ is a virtually finitely generated free-abelian group such that:

If $\pi_1(M) \xrightarrow{\mu} G \xrightarrow{\nu} \Gamma$ is a factorization of α such that G is finitely generated and residually finite, then for any homomorphism $\varphi\colon \Gamma \to \mathbb{Q}$ we have

$$-\deg \rho^{(2)}(M;\mu,\varphi\circ\nu) = x_M(\varphi).$$

In particular, we have for the universal covering and any $\varphi\colon \pi_1(M) \to \mathbb{Q}$

$$-\deg \rho^{(2)}(\widetilde{M};\varphi) = x_M(\varphi).$$

Proof. (1) was proved by Liu [Liu17, Theorem 1.4].

(2) is due to Friedl-Lück [FL15, Theorem 4.15].

(3) appears in [FL15, Theorem 5.1]. Liu simultaneously proved the equality for the universal covering in [Liu17, Theorem 1.2]. ☐

3.5 Twisted L^2-Euler characteristics

Twisted L^2-Euler characteristics are a variation of the classical L^2-Euler characteristic for G-CW-complexes in the presence of a homomorphism $G \to \mathbb{Z}$. This section builds heavily on [FL16a].

3.5.1 Definition of twisted L^2-Euler characteristics. Twisted L^2-Euler characteristics are the only torsion invariant considered here for which the chain complexes in question are not assumed to be finite. The reason for this is twofold. On the one hand, we will now twist chain complexes with infinite-dimensional representations, so even if the original chain complex was finite, the twisted one is not. This will, on the other hand, be remedied by the fact that we then take the von Neumann dimension of their homology with coefficients in the group von Neumann algebra $\mathcal{N}(G)$, which can be done irrespective of any finiteness condition (in sharp contrast to the classical L^2-torsion).

Definition 3.15 (Twisted L^2-Euler characteristic)**.** Let $\varphi\colon G \to \mathbb{Z}$ be a group homomorphism. Let $\varphi^*\mathbb{Z}\mathbb{Z}$ be the $\mathbb{Z}G$-module given by restriction of the $\mathbb{Z}\mathbb{Z}$-module $\mathbb{Z}\mathbb{Z}$ along φ. If C_* is a $\mathbb{Z}G$-chain complex, then we view $C_* \otimes_{\mathbb{Z}} \varphi^*\mathbb{Z}\mathbb{Z}$ as a $\mathbb{Z}G$-chain complex via the diagonal G-action. We put

$$\begin{aligned} b_n^{(2)}(C_*;\mathcal{N}(G),\varphi) &= \dim_{\mathcal{N}(G)} H_n(\mathcal{N}(G) \otimes_{\mathbb{Z}G} (C_* \otimes_{\mathbb{Z}} \varphi^*\mathbb{Z}\mathbb{Z})), \\ h(C_*;\mathcal{N}(G),\varphi) &= \sum_{n\geq 0} b_n^{(2)}(C_*;\mathcal{N}(G),\varphi). \end{aligned}$$

The chain complex C_* is called *φ-L^2-finite* if $h(C_*;\mathcal{N}(G),\varphi)$ is finite. In this case, the

φ-twisted L^2-Euler characteristic is defined as

$$\chi^{(2)}(C_*;\mathcal{N}(G),\varphi) = \sum_{n\geq 0} (-1)^n \cdot b_n^{(2)}(C_*;\mathcal{N}(G),\varphi).$$

These notions carry over to a G-CW-complex X by applying them to the cellular $\mathbb{Z}G$-chain complex $C_*(X)$. We then write $\chi^{(2)}(X;\mathcal{N}(G),\varphi)$ for $\chi^{(2)}(C_*(X);\mathcal{N}(G),\varphi)$.

We introduce one more notation. Let X be a CW-complex and take group homomorphisms $\pi_1(X) \xrightarrow{\mu} G \xrightarrow{\varphi} \mathbb{Z}$. Denote by $\overline{X} \to X$ the G-covering associated to μ. We say that X is (μ,φ)-L^2-finite if $\overline{X}$ is φ-L^2-finite, and in this case we write

$$\chi^{(2)}(X;\mu,\varphi) = \chi^{(2)}(\overline{X};\mathcal{N}(G),\varphi).$$

In complete analogy with Theorem 3.3 and Theorem 3.10 we first collect a set of basic properties of twisted L^2-Euler characteristics, compare [FL16a, Theorem 2.5]. Even though this is routine by now, it provides striking evidence for the idea that L^2-torsion, twisted L^2-torsion, twisted L^2-torsion functions, and twisted L^2-Euler characteristics all come from one common invariant. This will be the universal L^2-torsion of Section 3.6.

Theorem 3.16 (Basic properties of twisted L^2-Euler characteristics)**.** *Let $\varphi\colon G \to \mathbb{Z}$ be a homomorphism.*

(1) (Homotopy invariance) Let $f\colon X \to Y$ be a G-homotopy equivalence of G-CW-complexes. If X or Y is φ-L^2-finite, then both are φ-L^2-finite and we have

$$\chi^{(2)}(X;\mathcal{N}(G),\varphi) = \chi^{(2)}(Y;\mathcal{N}(G),\varphi).$$

(2) (Sum formula) Let

$$\begin{array}{ccc} X_0 & \longrightarrow & X_1 \\ \downarrow & & \downarrow \\ X_2 & \longrightarrow & X \end{array}$$

be a G-pushout of G-CW-complexes such that the upper horizontal map is cellular, the left-hand map is an inclusion of G-CW-complexes and X carries the G-CW-structure induced from the X_i. If X_i for $i = 0,1,2$ is φ-L^2-finite, then X is φ-L^2-finite and we have

$$\chi^{(2)}(X;\mathcal{N}(G),\varphi) = \chi^{(2)}(X_1;\mathcal{N}(G),\varphi) + \chi^{(2)}(X_2;\mathcal{N}(G),\varphi) - \chi^{(2)}(X_0;\mathcal{N}(G),\varphi).$$

(3) (Induction) Let $i\colon H \to G$ be an inclusion of groups. Let X be a H-CW-complex. Then the G-CW-complex $i_*X = G \times_H X$ is φ-L^2-finite if and only if X is $(\varphi \circ i)$-L^2-finite, and in this case we have

$$\chi^{(2)}(i_*X;\mathcal{N}(G),\varphi) = \chi^{(2)}(X;\mathcal{N}(H),\varphi\circ i).$$

(4) (Restriction) Let $i\colon H \to G$ be an inclusion of groups with finite index. Let X be a G-CW-complex and i^*X the finite free H-CW-complex obtained from X by restriction. Then i^*X is $(\varphi\circ i)$-L^2-finite if and only if X is φ-L^2-finite and in this case we have

$$\chi^{(2)}(i^*X;\mathcal{N}(H),\varphi\circ i) = [G:H]\cdot\chi^{(2)}(X;\mathcal{N}(G),\varphi).$$

(5) (Fibrations) Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration of connected CW-complexes such that

B is a finite CW-complex. Let $\overline{E} \to E$ be a G-covering and $\overline{F} \to F$ the G-covering obtained from it by pullback along i. If $\overline{F}$ is φ-L^2-finite, then $\overline{E}$ is φ-L^2-finite, and in this case we have

$$\chi^{(2)}(\overline{E}; \mathcal{N}(G), \varphi) = \chi(B) \cdot \chi^{(2)}(\overline{F}; \mathcal{N}(G), \varphi).$$

(6) (Scaling) Given an integer $k \geq 1$, a G-CW-complex X is φ-L^2-finite if and only if it is $(k \cdot \varphi)$-L^2-finite, and in this case we have

$$\chi^{(2)}(X; \mathcal{N}(G), k \cdot \varphi) = k \cdot \chi^{(2)}(X; \mathcal{N}(G), \varphi).$$

(7) (Epimorphisms) Let X be a connected CW-complex and $\mu\colon \pi_1(X) \to G$ be a homomorphism. Let G' be the image of μ, $\mu'\colon \pi_1(X) \to G'$ be the epimorphism induced by μ, and φ' be the restriction of φ to G'. Then X is (μ, φ)-L^2-finite if and only if it is (μ', φ')-L^2-finite, and in this case we have

$$\chi^{(2)}(X; \mu, \varphi) = \chi^{(2)}(X; \mu', \varphi').$$

By the last two parts of the above theorem, we can in practice always assume that μ and φ are surjective. In this situation we will make frequent use of the following lemma.

Lemma 3.17 (Twisted L^2-Euler characteristic as L^2-Euler characteristic)**.** *Let C_* be a $\mathbb{Z}G$-chain complex and let $\varphi\colon G \to \mathbb{Z}$ be an epimorphism. Let $i\colon K \to G$ be the inclusion of $K = \ker \varphi$. Denote by i^*C_* the $\mathbb{Z}K$-chain complex obtained from C_* by restriction. Then*

$$b_n^{(2)}(C_*; \mathcal{N}(G), \varphi) = b_n^{(2)}(i^*C_*; \mathcal{N}(K)).$$

In particular, C_* is φ-L^2-finite if and only if i^*C is L^2-finite, and in this case we have

$$\chi^{(2)}(C_*; \mathcal{N}(G), \varphi) = \chi^{(2)}(i^*C_*; \mathcal{N}(K)).$$

Proof. This is based on the observation that the twisted chain complex $C_* \otimes_{\mathbb{Z}} \varphi^*\mathbb{Z}\mathbb{Z}$ is $\mathbb{Z}G$-isomorphic to $\mathbb{Z}G \otimes_{\mathbb{Z}K} i^*C_*$. See [FL16a, Lemma 2.6] for the complete argument. ☐

3.5.2 Enter the Atiyah Conjecture. The use of twisted Laurent polynomial rings over skew-fields in the construction of invariants for knots and 3-manifolds has reached adulthood in the evolution of papers [COT03, Coc04, Har05, FH07, Fri07]. We have shed some light on this principle in the definition of higher-order Alexander norms in Section 2.3. However, in all cases treated in the above papers, the skew-fields in question arise as the Ore localization of amenable groups. This means that the theory restricts to G-CW-complexes with amenable G, or to covering spaces with amenable deck transformation group.

In this section, we remedy the lack of Ore localizations for non-amenable groups by introducing the Atiyah Conjecture. Originally, this conjecture made predictions about the possible values of L^2-Betti numbers. For us it will be relevant that groups satisfying the Atiyah Conjecture admit an embedding of their integral group ring into a skew-field $\mathcal{D}(G)$, and that this skew-field can be used in the computation of L^2-Betti numbers instead of the group von Neumann algebra $\mathcal{N}(G)$. We can then invoke almost without change the machinery presented in Section 2.3 in order to define a new set of norms on the first cohomology of 3-manifolds and 2-complexes.

This section follows [FL16a, Chapter 3] to a great extent.

Conjecture 3.18 (Atiyah Conjecture)**.** *A torsion-free group G satisfies the* Atiyah Conjecture *(with rational coefficients) if for any matrix* $A \in M_{m,n}(\mathbb{Q}G)$ *we have*

$$\dim_{\mathcal{N}(G)}\big(\ker(r_A\colon \mathcal{N}(G)^m \to \mathcal{N}(G)^n)\big) \in \mathbb{Z}.$$

Here is a short summary of what is known about the Atiyah Conjecture.

Theorem 3.19 (Results on the Atiyah Conjecture). *(1) If G is a torsion-free group satisfying the Atiyah Conjecture, then each subgroup of G satisfies the Atiyah Conjecture.*

(2) Let $\mathcal{C}$ be the smallest class of groups containing all free groups and which is closed under directed unions as well as extensions with elementary amenable groups. If G is a torsion-free group lying in $\mathcal{C}$, then G satisfies the Atiyah Conjecture.

(3) Let M be an admissible 3-manifold which is not a closed graph manifold or which admits a Riemannian metric of non-positive sectional curvature. Then $\pi_1(M)$ is torsion-free, belongs to the class $\mathcal{C}$, and hence satisfies the Atiyah Conjecture.

Proof. (1) This is [Lüc02, Theorem 6.29 (2)].

(2) This is due to Linnell [Lin93].

(3) This is explained in [FL16a, Theorem 3.2 (3)]. □

We cannot directly work with the Atiyah Conjecture in the form stated above. We now work out a reformulation suitable in our context, which relies on the following objects.

Definition 3.20 ($\mathcal{U}(G)$, $\mathcal{D}(G)$ and $\mathcal{R}(G)$). Let $\mathcal{U}(G)$ denote the algebra of operators affiliated to $\mathcal{N}(G)$, see [Lüc02, Chapter 8]. Algebraically, this is the Ore localization of $\mathcal{N}(G)$ with respect to the set of weak isomorphisms, see [Lüc02, Theorem 8.22 (1)].

Let $\mathcal{D}(G)$ be the smallest subring of $\mathcal{U}(G)$ which contains $\mathbb{Q}G$ and is *division closed*, meaning that every element of $\mathcal{D}(G)$ which is a unit in $\mathcal{U}(G)$ is already a unit in $\mathcal{D}(G)$.

Let $\mathcal{R}(G)$ be the smallest subring of $\mathcal{U}(G)$ which contains $\mathbb{Q}G$ and is *rationally closed*, meaning that any square matrix over $\mathcal{R}(G)$ which becomes invertible over $\mathcal{U}(G)$ is already invertible over $\mathcal{R}(G)$.

Thus we obtain a rectangle of inclusions

$$\begin{tikzcd}
\mathbb{Q}G \arrow[rr] \arrow[d] & & \mathcal{N}(G) \arrow[d] \\
\mathcal{D}(G) \arrow[r] & \mathcal{R}(G) \arrow[r] & \mathcal{U}(G).
\end{tikzcd}$$

Example 3.21. In the case $G = \mathbb{Z}^n$ the above rectangle specializes to

$$\begin{tikzcd}
\mathbb{Q}[u_1^{\pm}, ..., u_n^{\pm}] \arrow[r] \arrow[d] & L^\infty(T^n) \arrow[d] \\
\mathbb{Q}(u_1^{\pm}, ..., u_n^{\pm}) \arrow[r] & L(T^n),
\end{tikzcd}$$

where $\mathbb{Q}[\mathbb{Z}^n] = \mathbb{Q}[u_1^{\pm}, ..., u_n^{\pm}]$ denotes the Laurent polynomial ring in n variables, $\mathcal{D}(\mathbb{Z}^n) = \mathcal{R}(\mathbb{Z}^n) \cong \mathbb{Q}(u_1^{\pm}, ..., u_n^{\pm})$ denotes the field of fractions thereof, $\mathcal{N}(\mathbb{Z}^n) \cong L^\infty(T^n)$ denotes the algebra of (equivalence classes of) essentially bounded measurable functions $T^n \to \mathbb{C} \cup \{\infty\}$ on the n-torus, and $\mathcal{U}(\mathbb{Z}^n) \cong L(T^n)$ denotes the algebra of (equivalence classes of) measurable functions $T^n \to \mathbb{C}$. This follows from Lemma 3.23 (2) below and [Lüc02, Example 1.4 and Example 8.11].

Using these rings we have the following result.

Proposition 3.22. *A torsion-free group G satisfies the Atiyah Conjecture if and only if $\mathcal{D}(G)$ is a skew-field. In this case, we have $\mathcal{D}(G) = \mathcal{R}(G)$.*

Proof. See [Lüc02, Lemma 10.39]. $\square$

It turns out that for the class of amenable groups we have been working with this skew field before (see also Lemma 2.5).

Lemma 3.23 ($\mathcal{D}(G)$ of amenable groups)**.** *(1) Let G be a torsion-free elementary amenable group. Then G satisfies the Atiyah Conjecture.*

(2) Let G be a torsion-free amenable group satisfying the Atiyah Conjecture. Then $\mathbb{Q}G$ satisfies the Ore condition with respect to $T = \mathbb{Q}G \smallsetminus \{0\}$ and there is an isomorphism of skew-fields

$$T^{-1}\mathbb{Q}G \cong \mathcal{D}(G).$$

In particular, $\mathcal{D}(G)$ is flat over $\mathbb{Q}G$.

Proof. (1) This is a special case of Theorem 3.19 (2).

(2) $\mathbb{Q}G$ satisfies the Ore condition by [Lüc02, Example 8.16]. Recalling the notion of division closure, it is then easy to see that the inclusion $\mathbb{Q}G \to \mathcal{D}(G)$ localizes to an isomorphism $T^{-1}\mathbb{Q}G \xrightarrow{\cong} \mathcal{D}(G)$. $\square$

The following theorem explains how $\mathcal{D}(G)$ and twisted Laurent polynomial rings over skew-fields and crossed products (see Section 2.3.1) enter the context of twisted L^2-Euler characteristics. It will play a crucial role in the rest of this thesis.

Theorem 3.24 (Structure of $\mathcal{D}(G)$ and L^2-Betti numbers)**.** *Let G be a torsion-free group satisfying the Atiyah Conjecture.*

(1) Let $\mu\colon G \to H$ be an epimorphism onto a free-abelian group and denote its kernel by K. Then the structure maps of the crossed product $\mathbb{Z}K * H \cong \mathbb{Z}G$ of Example 2.4 extend to $\mathcal{D}(K)$, the resulting crossed product $\mathcal{D}(K) * H$ satisfies the Ore condition with respect to $T = (\mathcal{D}(K) * H) \smallsetminus \{0\}$, and there is a $\mathcal{D}(K)$-isomorphism

$$T^{-1}(\mathcal{D}(K) * H) \cong \mathcal{D}(G).$$

If H is infinite cyclic, then $\mathcal{D}(K) * H$ is isomorphic to the ring $\mathcal{D}(K)_t[u^{\pm}]$ of twisted Laurent polynomials.

(2) Let C_* be a projective $\mathbb{Z}G$-chain complex. Then

$$\dim_{\mathcal{N}(G)} H_n(\mathcal{N}(G) \otimes_{\mathbb{Z}G} C_*) = \dim_{\mathcal{D}(G)} H_n(\mathcal{D}(G) \otimes_{\mathbb{Z}G} C_*)$$

(3) Let $\varphi\colon G \to \mathbb{Z}$ be an epimorphism with kernel K. Let C_* be a finitely generated projective $\mathbb{Z}G$-chain complex such that

$$\dim_{\mathcal{N}(G)} H_n(\mathcal{N}(G) \otimes_{\mathbb{Z}G} C_*) = \dim_{\mathcal{D}(G)} H_n(\mathcal{D}(G) \otimes_{\mathbb{Z}G} C_*) = 0.$$

Denote by i^*C_* the $\mathbb{Z}K$-chain complex obtained from C_* by restriction along the inclusion $i\colon K \to G$. Then

$$\begin{aligned}\dim_{\mathcal{N}(K)} H_n(\mathcal{N}(K) \otimes_{\mathbb{Z}K} i^*C_*) &= \dim_{\mathcal{D}(K)} H_n(\mathcal{D}(K) \otimes_{\mathbb{Z}K} i^*C_*) \\ &= \dim_{\mathcal{D}(K)} H_n(\mathcal{D}(K)_t[u^{\pm}] \otimes_{\mathbb{Z}G} C_*),\end{aligned}$$

and this common value is finite.

Proof. This is [FL16a, Theorem 3.6]. In order to explain the appearance of twisted Laurent polynomial rings, we repeat the proof of part (3).

The first equality stated there is an instance of part (2). The second equality

$$\dim_{\mathcal{D}(K)} H_n(\mathcal{D}(K) \otimes_{\mathbb{Z}K} i^*C_*) = \dim_{\mathcal{D}(K)} H_n(\mathcal{D}(K)_t[u^{\pm}] \otimes_{\mathbb{Z}G} C_*)$$

follows immediately from the $\mathcal{D}(K)$-chain isomorphism

$$\mathcal{D}(K) \otimes_{\mathbb{Z}K} i^*C_* \cong \mathcal{D}(K)_t[u^{\pm}] \otimes_{\mathbb{Z}G} C_*.$$

It remains to prove that this common value is finite. The ring $\mathcal{D}(K)_t[u^{\pm}]$ is a non-commutative PID (i.e., without zero-divisors and every left/right ideal is a principal left/right ideal) by [Coc04, Proposition 4.5], so an analogue of the structure theorem for modules over PIDs [Coc04, Theorem 5.1] provides an isomorphism

$$H_n(\mathcal{D}(K)_t[u^{\pm}] \otimes_{\mathbb{Z}G} C_*) \cong \mathcal{D}(K)_t[u^{\pm}]^r \oplus \bigoplus_{i=1}^{s} \mathcal{D}(K)_t[u^{\pm}]/(p_i)$$

for some natural numbers r, s and $p_1, ..., p_s \in \mathcal{D}(K)_t[u^{\pm}]$. By part (1), $\mathcal{D}(G)$ is flat over $\mathcal{D}(K)_t[u^{\pm}]$. Moreover, $\mathcal{D}(G) \otimes_{\mathcal{D}(K)_t[u^{\pm}]} \mathcal{D}(K)_t[u^{\pm}]/(p_i) = 0$. The acyclicity assumption implies

$$r = \dim_{\mathcal{D}(G)} H_n(\mathcal{D}(G) \otimes_{\mathbb{Z}G} C_*) = 0.$$

Since each $\mathcal{D}(K)_t[u^{\pm}]/(p_i)$ is a finite-dimensional $\mathcal{D}(K)$-module, $H_n(\mathcal{D}(K)_t[u^{\pm}] \otimes_{\mathbb{Z}G} C_*)$ is a finite-dimensional $\mathcal{D}(K)$-module. □

Definition 3.25 (L^2-acyclic Atiyah pair)**.** Let X be a finite connected CW-complex. Let $\pi_1(X) \xrightarrow{\mu} G \xrightarrow{\varphi} \mathbb{Z}$ be group homomorphisms. The pair (μ, φ) is an *L^2-acyclic Atiyah pair* if the G-covering associated to μ is L^2-acyclic, and G is a torsion-free group satisfying the Atiyah Conjecture.

This notion is convenient in the context of twisted L^2-Euler characteristics for the following reason.

Lemma 3.26. *Let X be a finite connected CW-complex. If (μ, φ) is an L^2-acyclic Atiyah pair, then X is (μ, φ)-L^2-finite and $\chi^{(2)}(X; \mu, \varphi)$ is an integer.*

Proof. This follows by combining Lemma 3.17 and Theorem 3.24 (3). □

3.5.3 Relation to higher-order Alexander norms.

We show that the theory of twisted L^2-Euler characteristics covers the Alexander norms $\delta(X; \mu)$ of Section 2.3.

Lemma 3.27 (Twisted L^2-Euler characteristics of 3-manifolds and 2-complexes)**.** *Let X be an admissible 3-manifold or a finite connected 2-complex with $\chi(X) = 0$. Let (μ, φ) be an L^2-acyclic Atiyah pair such that φ is surjective and $\varphi \circ \mu$ is neither trivial nor injective. Let $\overline{X} \to X$ be the G-covering associated to μ. Let $K \subseteq G$ be the kernel of φ and denote by $i \colon K \to G$ the inclusion.*

Then we have for any $n \neq 1$

$$b_n^{(2)}(i^*\overline{X}; \mathcal{N}(K)) = \dim_{\mathcal{D}(K)} H_n(\mathcal{D}(K)_t[u^{\pm}] \otimes_{\mathbb{Z}G} C_*(\overline{X})) = 0$$

and hence

$$\chi^{(2)}(X;\mu,\varphi) = -b_1^{(2)}(i^*\overline{X};\mathcal{N}(K)) = -\dim_{\mathcal{D}(K)} H_1(\mathcal{D}(K)_t[u^{\pm}] \otimes_{\mathbb{Z}G} C_*(\overline{X})).$$

Proof. For 3-manifolds, this is [FL16a, Theorem 5.5]. The case of 2-complexes is completely analogous. ☐

Remark 3.28. Instead of demanding $b_n(\overline{X};\mathcal{N}(G)) = 0$ for all $n \geq 0$ it would suffice to require $b_1(\overline{X};\mathcal{N}(G)) = 0$.

Corollary 3.29 (Alexander norms and twisted L^2-Euler characteristics)**.** *Let X be an admissible 3-manifold or a finite connected 2-complex with $\chi(X) = 0$. Let $\mu\colon \pi_1(X) \to G$ be a large epimorphism onto a torsion-free elementary amenable group and $\varphi\colon G \to \mathbb{Z}$ be an epimorphism.*

Then (μ,φ) is an L^2-acyclic Atiyah pair if and only if $H_n(X;Q(G))$ vanishes for all $n \geq 0$, and in this case we have

$$\delta(X;\mu)(\varphi) = -\chi^{(2)}(X;\mu,\varphi).$$

Proof. The group G satisfies the Atiyah Conjecture by Lemma 2.5. In the notation of Section 2.3 we have isomorphisms $Q(G) \cong \mathcal{D}(G)$ and $Q(K) \cong \mathcal{D}(K)$ by Lemma 3.23. By Theorem 3.24 (2) we also have

$$b_n^{(2)}(\overline{X};\mathcal{N}(G)) = \dim_{\mathcal{D}(G)} H_n(\mathcal{D}(G) \otimes_{\mathbb{Z}G} C_*(\overline{X})) = \dim_{Q(G)} H_n(X;Q(G)).$$

Hence (μ,φ) is an L^2-acyclic Atiyah pair if and only if $H_n(X;Q(G))$ vanishes for all $n \geq 0$.

The equality then follows by comparing Definition 2.7 with Lemma 3.27. ☐

3.5.4 Relation to the Thurston norm.

The relation of twisted L^2-Euler characteristics to the Thurston norm is examined in [FL16a]. We collect the results proved there in one single theorem. This also highlights the striking similarity with Theorem 3.14 about the degree of twisted L^2-torsion functions and the Thurston norm.

Theorem 3.30 (Twisted L^2-Euler characteristics and the Thurston norm)**.** *Let $M \neq S^1 \times D^2$ be an admissible 3-manifold. Then:*

1. *(Inequality of $\chi^{(2)}$ and Thurston norm) Let (μ,φ) be an L^2-acyclic Atiyah pair. Then M is (μ,φ)-L^2-finite and we have*

$$-\chi^{(2)}(M;\mu,\varphi) \leq x_M(\varphi \circ \mu).$$

2. *(Equality for quasi-fibered classes) Let $\pi_1(M) \xrightarrow{\mu} G \xrightarrow{\nu} H_1(M)_f$ be a factorization of the canonical projection such that G is a torsion-free group satisfying the Atiyah Conjecture. If $\varphi \in \operatorname{Hom}(H_1(M)_f,\mathbb{Z})$ is quasi-fibered, then $(\mu,\varphi\circ\nu)$ is an L^2-acyclic Atiyah pair and we have*

$$-\chi^{(2)}(M;\mu,\varphi\circ\nu) = x_M(\varphi).$$

3. *(Equality of $\chi^{(2)}$ and Thurston norm) Assume that M is not a closed graph manifold. Then the canonical projection factors into epimorphisms $\pi_1(M) \xrightarrow{\alpha} \Gamma \xrightarrow{\beta} H_1(M)_f$ with Γ a virtually finitely generated free-abelian group such that:*

   If $\pi_1(M) \xrightarrow{\mu} G \xrightarrow{\nu} \Gamma$ is a factorization of α such that G is a torsion-free group satisfying the Atiyah Conjecture, then for any $\varphi \in \operatorname{Hom}(H_1(M)_f,\mathbb{Z})$ the pair $(\mu,\varphi\circ$

$\beta \circ \nu$) is an L^2-acyclic Atiyah pair and we have

$$-\chi^{(2)}(M; \mu, \varphi \circ \beta \circ \nu) = x_M(\varphi).$$

Proof. (1) is [FL16b, Theorem 4.1].

(2) is [FL16b, Theorem 6.19].

(3) is [FL16b, Theorem 0.4]. It is noteworthy that the proof uses the work of Agol, Liu, Przytycki-Wise, and Wise on the Virtual Fibering Conjecture [Ago08, Ago13, Liu13, PW12, PW14, Wis12a, Wis12b]. Roughly speaking, their work implies that *virtually* every cohomology class of M is quasi-fibered. Then one can take advantage of part (2). □

3.5.5 Relation to twisted L^2-torsion functions. In view of Theorem 3.14 and Theorem 3.30 it seems natural to compare the degree of twisted L^2-torsion functions and twisted L^2-Euler characteristics. The following theorem establishes this at least for certain coverings.

Theorem 3.31 (Twisted L^2-torsion functions and twisted L^2-Euler characteristics)**.** *Let X be an admissible 3-manifold or a finite connected 2-complex with $\chi(X) = 0$. Let $\mu\colon \pi_1(X) \to G$ be a homomorphism to a torsion-free, elementary amenable, countable, residually finite group such that the G-covering $\overline{X} \to X$ associated to μ is L^2-acyclic. Then for any group homomorphism $\varphi\colon G \to \mathbb{Z}$ the space X is (μ, φ)-L^2-finite, $\overline{X}$ is φ-twisted $\det$-L^2-acyclic, and we have*

$$\chi^{(2)}(X; \mu, \varphi) \leq \deg \rho^{(2)}(X; \mu, \varphi).$$

Proof. This is stated in [FL16a, Theorem 9.1] for admissible 3-manifolds, but not proved. In order to justify that the statement extends to finite connected 2-complexes with vanishing Euler characteristic, we include here a rough outline of the argument.

Given an endomorphism $f\colon \mathbb{Z}G^n \to \mathbb{Z}G^n$ we denote by $\mathrm{el}(f)$ the chain complex

$$... \to 0 \to \mathbb{Z}G^n \xrightarrow{f} \mathbb{Z}G^n \to 0 \to ....$$

where the non-trivial chain modules are concentrated in degree 0 and 1.

Using Remark 3.1 one computes that for an element of infinite order $g \in G$ we have

$$\deg \rho^{(2)}(\mathrm{el}(r_{g-1}); \varphi) = |\varphi(g)| = \chi^{(2)}(\mathrm{el}(r_{g-1}); \mathcal{N}(G), \varphi). \tag{3.1}$$

(Compare also the proof of Corollary 6.21.) Moreover, for any non-trivial element $x \in \mathbb{Z}G$ one has the inequality

$$\deg \rho^{(2)}(\mathrm{el}(r_x); \varphi) \leq \chi^{(2)}(\mathrm{el}(r_x); \mathcal{N}(G), \varphi) \tag{3.2}$$

by [Lüc15, Theorem 6.7 (2)]. The presence of the degree function on $\mathcal{D}(\ker \varphi)_t[u^{\pm}]$ allows us to show that the Dieudonné determinant of a matrix $A \in M_{n,n}(\mathbb{Z}G)$ which becomes invertible over $\mathcal{D}(G)$ can be represented by an element of the form $x \cdot y^{-1}$ for some $x \in \mathbb{Z}G$ and $y \in \mathbb{Z} \ker \varphi$. (This will also be explained and used in the proof of Theorem 5.7.) From this and (3.2) we deduce

$$\deg \rho^{(2)}(\mathrm{el}(r_A); \varphi) \leq \chi^{(2)}(\mathrm{el}(r_A); \mathcal{N}(G), \varphi). \tag{3.3}$$

The chain complex of $\overline{X}$ is homotopic to one of the form

$$0 \to \mathbb{Z}G \xrightarrow{\prod \mu(g_i')-1} \mathbb{Z}G^n \xrightarrow{A} \mathbb{Z}G^n \xrightarrow{\bigoplus \mu(g_i)-1} \mathbb{Z}G \to 0$$

if X is a closed 3-manifold, or homotopic to one of the form

$$0 \to \mathbb{Z}G^{n-1} \xrightarrow{A} \mathbb{Z}G^n \xrightarrow{\bigoplus \mu(g_i)-1} \mathbb{Z}G \to 0$$

in the other cases (also compare Lemma 6.8). In the latter case one has a short exact sequence of L^2-acyclic $\mathbb{Z}G$-chain complexes

$$0 \to \mathrm{el}(\mu(g_i) - 1) \to C_*(\overline{X}) \to \Sigma\, \mathrm{el}(A_i) \to 0,$$

were A_i is obtained from A by deleting the i-th column. Thus we may apply the additivity of both L^2-torsion functions and twisted L^2-Euler characteristics, (3.1) and (3.3) to deduce

$$\begin{aligned}
\chi^{(2)}(X;\mu,\varphi) &= \chi^{(2)}(C_*(\overline{X}); \mathcal{N}(G), \varphi) \\
&= \chi^{(2)}(\mathrm{el}(r_{\mu(g_i)-1}); \mathcal{N}(G), \varphi) + \chi^{(2)}(\Sigma\, \mathrm{el}(r_\mu(A_i)); \mathcal{N}(G), \varphi) \\
&\leq \deg \rho^{(2)}(\mathrm{el}(r_{\mu(g_i)-1}); \varphi) + \deg \rho^{(2)}(\Sigma\, \mathrm{el}(r_\mu(A_i)); \varphi) \\
&= \deg \rho^{(2)}(X; \mu, \varphi).
\end{aligned}$$

The argument for the case of a closed 3-manifold is similar. □

3.6 Universal L^2-torsion

This section presents the main aspects of Friedl-Lück's definition and investigation of their *universal L^2-torsion* [FL16b]. We are led to the notion of universal L^2-torsion and some of its features by considering the following three hints.

- All previous L^2-torsion invariants share identical sets of basic properties which were proved case by case. But *if* these invariants (and all other L^2-torsion invariants) came from one *universal* L^2-torsion invariant, then these properties are in fact encoded in the properties of the universal L^2-torsion invariant.
- Torsion invariants such as Whitehead or Reidemeister torsion are defined for chain complexes satisfying an appropriate contractibility condition, and they take values in certain K_1-groups whose generators reflect that contractibility condition. Since the L^2-torsion invariants so far obtained their best behaviour only for L^2-acyclic chain complexes, this seems to be the right notion of contractibility in the L^2-setting. As generators of the modified K_1-group, it is thus natural to take morphisms $\mathbb{Z}G^n \to \mathbb{Z}G^n$ which after passing to $L^2(G)$ have kernel and cokernel of vanishing von Neumann dimension. Since this is equivalent to being a weak isomorphism, the suitable K_1-group will be called *weak K_1-group* and denoted by $K_1^w(\mathbb{Z}G)$.
- We have seen in Corollary 3.29 that twisted L^2-Euler characteristics of admissible spaces can be viewed as a generalization of higher-order Alexander norms. On the other hand, we have seen in Section 2.3 that these norms can uniformly be described in terms of polynomial degrees of the Reidemeister torsion of the chain complex $\mathcal{D}(G)\otimes_{\mathbb{Z}G} C_*(X)$. The same description applies to twisted L^2-Euler characteristics. The direct passage from the universal L^2-torsion of $C_*(X)$ to its twisted L^2-Euler characteristics

should therefore factor over the Reidemeister torsion of $\mathcal{D}(G) \otimes_{\mathbb{Z}G} C_*(X)$. In other words, there should be a homomorphism $K_1^w(\mathbb{Z}G) \to K_1(\mathcal{D}(G))$ mapping one to the other.

This program will now be made explicit.

3.6.1 A universal L^2-torsion invariant. A short exact sequence of based free $\mathbb{Z}G$-modules

$$0 \to (M', [B']) \xrightarrow{i} (M, [B]) \xrightarrow{p} (M'', [B'']) \to 0$$

is *based exact* if $i(B') \subseteq B$ and p maps $B \setminus i(B')$ bijectively to B'' (up to sign).

Definition 3.32 (L^2-torsion invariant)**.** An *(additive) L^2-torsion invariant* is a pair (A, a) consisting of an abelian group A and an assignment that associates to any finite based free L^2-acyclic $\mathbb{Z}G$-chain complex C_* an element $a(C_*) \in A$ subject to the conditions that

$$a(... \to 0 \to \mathbb{Z}G \xrightarrow{\pm \mathrm{id}} \mathbb{Z}G \to 0 \to ...) = 0$$

and for any based exact sequence $0 \to C_* \to D_* \to E_* \to 0$ of finite based free L^2-acyclic $\mathbb{Z}G$-chain complexes we have

$$a(D_*) = a(C_*) + a(E_*).$$

An additive L^2-torsion invariant (U, u) is *universal* if for every additive L^2-torsion invariant (A, a) there is exactly one homomorphism $f\colon U \to A$ such that we have $f(u(C_*)) = a(C_*)$ for any C_* in question.

Since we have used the term L^2-torsion invariant lavishly so far, we now explicitly argue that all previous invariants indeed fit into this framework.

Example 3.33. (1) The L^2-torsion of Section 3.2 is given by the group $\mathbb{R}$ and the assignment

$$C_* \mapsto \rho^{(2)}(\Lambda(C_*); \mathcal{N}(G)).$$

(2) The twisted L^2-torsion of Section 3.3 is given by the group $\operatorname{Hom}(\operatorname{Rep}_{\mathbb{C}}(H_1(G)_f), \mathbb{R})$ and the assignment

$$C_* \mapsto \Big([V] \mapsto \rho^{(2)}(\Lambda(C_*); \mathcal{N}(G), V)\Big).$$

Here $\operatorname{Rep}_{\mathbb{C}}(G)$ denotes the representation ring of finite-dimensional complex representations of a group G whose group structure comes from the direct sum.

(3) The twisted L^2-torsion function of Section 3.4 is given by $\operatorname{Map}(H^1(G;\mathbb{R}), \operatorname{Fun}(\mathbb{R}^{>0}, \mathbb{R})/\sim)$ and the assignment

$$C_* \mapsto \Bigg(\varphi \mapsto \Big(t \mapsto \rho^{(2)}\big(\Lambda(\eta_{\varphi^*\mathbb{C}_t}(C_*))\big)\Big)\Bigg).$$

Here $\operatorname{Fun}(\mathbb{R}^{>0}, \mathbb{R})/\sim$ denotes the set of functions $\mathbb{R}^{>0} \to \mathbb{R}$ up to the equivalence relation given in Definition 3.7, equipped with pointwise addition.

(4) The twisted L^2-Euler characteristic of Section 3.5 is given by the group $\operatorname{Map}(H^1(G;\mathbb{Z}), \mathbb{Z})$ and the assignment

$$C_* \mapsto \Big(\varphi \mapsto \chi^{(2)}(C_*; \mathcal{N}(G), \varphi)\Big).$$

Definition 3.34 (The group $\widetilde{K}_1^{w,\mathrm{ch}}(\mathbb{Z}G)$)**.** Given an endomorphism $f\colon \mathbb{Z}G^n \to \mathbb{Z}G^n$ we denote by $\mathrm{el}(f)$ the *elementary chain complex associated to* f

$$... \to 0 \to \mathbb{Z}G^n \xrightarrow{f} \mathbb{Z}G^n \to 0 \to ....$$

where the non-trivial chain modules are concentrated in degree 0 and 1.

Let $\widetilde{K}_1^{w,\mathrm{ch}}(\mathbb{Z}G)$ be the abelian group given in terms of generators and relations as follows. Generators are finite based free L^2-acyclic $\mathbb{Z}G$-chain complexes C_* subject to the relations that $[\mathrm{el}(\pm\mathrm{id})] = 0$, and whenever $0 \to C_* \to D_* \to E_* \to 0$ is a based short exact sequence of such complexes, then

$$[D_*] = [C_*] + [E_*].$$

The following is obvious.

Lemma 3.35 (Universal L^2-torsion invariant)**.** *The group $\widetilde{K}_1^{w,\mathrm{ch}}(\mathbb{Z}G)$ together with the assignment $C_* \mapsto [C_*]$ is a universal L^2-torsion invariant.*

Next we work towards a more tractable model of the universal L^2-torsion invariant which mimics Whitehead and Reidemeister torsion.

3.6.2 A better universal L^2-torsion invariant. The hints given at the beginning of this chapter motivate the following definition.

Definition 3.36 (Weak K_1-groups)**.** Let G be a group. Define the *weak K_1-group* $K_1^w(\mathbb{Z}G)$ as the abelian group whose generators $[f]$ are $\mathbb{Z}G$-maps $f\colon \mathbb{Z}G^n \to \mathbb{Z}G^n$ such that $\Lambda(f)$ is a weak isomorphism and the following relations: If $f, g\colon \mathbb{Z}G^n \to \mathbb{Z}G^n$ are two $\mathbb{Z}G$-maps such that $\Lambda(f)$ and $\Lambda(g)$ are weak isomorphisms, then $\Lambda(g \circ f)$ is a weak isomorphism [Lüc02, Lemma 3.37 (1)] and we require

$$[g \circ f] = [f] + [g].$$

If $f\colon \mathbb{Z}G^m \to \mathbb{Z}G^m$, $g\colon \mathbb{Z}G^n \to \mathbb{Z}G^n$, $h\colon \mathbb{Z}G^m \to \mathbb{Z}G^n$ are $\mathbb{Z}G$-maps such that $\Lambda(f)$ and $\Lambda(g)$ are weak isomorphisms, then we require the relation

$$\left[\begin{pmatrix} f & h \\ 0 & g \end{pmatrix}\right] = [f] + [g].$$

This makes sense since the matrix on the left-hand side induces a weak isomorphism by [Lüc02, Lemma 3.37 (2)].

Define the *reduced weak K_1-group* $\widetilde{K}_1^w(\mathbb{Z}G)$ as the quotient of $K_1^w(\mathbb{Z}G)$ by the subgroup $\{[\pm\mathrm{id}\colon \mathbb{Z}G \to \mathbb{Z}G]\}$ and the *weak Whitehead group* $\mathrm{Wh}^w(G)$ as the quotient of $K_1^w(\mathbb{Z}G)$ by the subgroup $\{[r_{\pm g}\colon \mathbb{Z}G \to \mathbb{Z}G] \mid g \in G\}$.

By passing from functional analysis to algebra, we can rephrase the generators of $K_1^w(\mathbb{Z}G)$ in a more algebraic way as follows.

Lemma 3.37. *Given a $\mathbb{Z}G$-map $f\colon \mathbb{Z}G^n \to \mathbb{Z}G^n$, the following statements are equivalent:*

(1) f induces a weak isomorphism $L^2(G)^n \to L^2(G)^n$;

(2) f induces a weak isomorphism $\mathcal{N}(G)^n \to \mathcal{N}(G)^n$ in the sense of [Lüc02, Definition 6.1]*;*

(3) f induces an isomorphism $\mathcal{U}(G)^n \to \mathcal{U}(G)^n$;

(4) f induces an isomorphism $\mathcal{R}(G)^n \to \mathcal{R}(G)^n$.

Proof. (1) $\Leftrightarrow$ (2) follows from [Lüc02, Theorem 6.24].

(2) $\Leftrightarrow$ (3) is [Lüc02, Theorem 8.22 (5)].

(3) $\Leftrightarrow$ (4) follows directly from the definition of rational closure. □

Now there are obvious maps

$$K_1(\mathbb{Z}G) \to K_1^w(\mathbb{Z}G) \to K_1(\mathcal{R}(G)),$$
$$\widetilde{K}_1(\mathbb{Z}G) \to \widetilde{K}_1^w(\mathbb{Z}G) \to \widetilde{K}_1(\mathcal{R}(G)),$$
$$\mathrm{Wh}(G) \to \mathrm{Wh}^w(G) \to K_1(\mathcal{R}(G))/\{[\pm g] \mid g \in G\}.$$

The reduced weak K_1-group is our candidate for a new model of the universal L^2-torsion. We now adjust the definition of Whitehead torsion to the L^2-setting in order to define the invariant itself.

Definition 3.38 (Weak chain contraction)**.** Given a $\mathbb{Z}G$-chain complex, a *weak chain contraction* (γ_*, u_*) consists of a $\mathbb{Z}G$-chain map $u_*\colon C_* \to C_*$ and a $\mathbb{Z}G$-chain homotopy $\gamma_*\colon u_* \simeq 0_*$ such that for all $n \in \mathbb{Z}$ $\Lambda(u_n)$ is a weak isomorphism and $\gamma_n \circ u_n = u_{n+1} \circ \gamma_n$.

The next lemma justifies that this is the right contractibility notion when working with L^2-acyclic chain complexes. It is a partial extension of Lemma 3.37.

Lemma 3.39. *Given a finite based free $\mathbb{Z}G$-chain complex C_*, the following statements are equivalent:*

(1) C_* is L^2-acyclic;

(2) C_* admits a weak chain contraction;

(3) The $\mathcal{U}(G)$-chain complex $\mathcal{U}(G) \otimes_{\mathbb{Z}G} C_*$ is contractible;

(4) The $\mathcal{R}(G)$-chain complex $\mathcal{R}(G) \otimes_{\mathbb{Z}G} C_*$ is contractible;

Proof. This is proved in [FL16b, Lemma 1.5 and Lemma 1.21] using the combinatorial Laplace operators of C_*. □

Let (C_*, c_*) be a finite based free L^2-acyclic $\mathbb{Z}G$-chain complex. Let

$$C_{\mathrm{ev}} = \bigoplus_{n\in\mathbb{Z}} C_{2n} \quad \text{and} \quad C_{\mathrm{odd}} = \bigoplus_{n\in\mathbb{Z}} C_{2n+1}.$$

Pick a weak chain contraction (γ_*, u_*) for C_* which is garantueed to exist by the previous lemma. Let $u_{\mathrm{odd}}\colon C_{\mathrm{odd}} \to C_{\mathrm{odd}}$ denote the obvious map induced by u_* and by $(uc+\gamma)_{\mathrm{odd}}\colon C_{\mathrm{odd}} \to C_{\mathrm{ev}}$ the map sending $x \in C_{2n+1}$ to $u_{2n}c_{2n+1}(x) + \gamma_{2n+1}(x) \in C_{2n} \oplus C_{2n+2}$.

If $f\colon (M,[B]) \to (N,[C])$ is a homomorphism of finitely generated based free $\mathbb{Z}G$-modules such that $\Lambda(f)$ is a weak isomorphism, then we have $|B| = |C|$ by [Lüc02, Lemma 1.13]. Choosing a bijection $b : C \to B$ induces a $\mathbb{Z}G$-isomorphism $b : N \to M$. We then define the class of f in $\widetilde{K}_1^w(\mathbb{Z}G)$ to be

$$[f] = [b \circ f].$$

Definition 3.40 (Universal L^2-torsion)**.** If C_* is a finite based free L^2-acyclic $\mathbb{Z}G$-chain complex, then its *universal L^2-torsion* $\rho_u^{(2)}(C_*;\mathcal{N}(G)) \in \widetilde{K}_1^w(\mathbb{Z}G)$ is defined as

$$\rho_u^{(2)}(C_*;\mathcal{N}(G)) = [(uc+\gamma)_{\mathrm{odd}}] - [u_{\mathrm{odd}}]$$

for some weak chain contraction (γ_*, u_*).

It is proved in [FL16b] that $(uc+\gamma)_{\mathrm{odd}}\colon C_{\mathrm{odd}} \to C_{\mathrm{ev}}$ induces indeed a weak isomorphism and that the above definition is independent of the choice of weak chain contraction. The universal L^2-torsion deserves its name in the following sense.

Theorem 3.41 (Universality of $(\widetilde{K}_1^w(\mathbb{Z}G), \rho_u^{(2)})$)**.** *The homomorphisms*

$$\begin{aligned} \rho_u^{(2)}\colon \widetilde{K}_1^{w,\mathrm{ch}}(\mathbb{Z}G) \to \widetilde{K}_1^w(\mathbb{Z}G),\ [C_*] \mapsto \rho_u^{(2)}(C_*;\mathcal{N}(G)),\\ \mathrm{el}\colon \widetilde{K}_1^w(\mathbb{Z}G) \to \widetilde{K}_1^{w,\mathrm{ch}}(\mathbb{Z}G),\ [f] \mapsto [\mathrm{el}(f)] \end{aligned}$$

are well-defined and inverse to each other. In particular, $(\widetilde{K}_1^w(\mathbb{Z}G), \rho_u^{(2)})$ is a universal L^2-torsion invariant.

Proof. This is [FL16b, Theorem 1.12]. □

In fact slightly more is true: There are chain versions $\widetilde{K}_1^{\mathrm{ch}}(\mathbb{Z}G)$ and $\widetilde{K}_1^{\mathrm{ch}}(\mathcal{R}(G))$ of the usual reduced K_1-group defined like $\widetilde{K}_1^{w,\mathrm{ch}}(\mathbb{Z}G)$ (see Definition 3.34), but replacing *L^2-acyclic* with *contractible* as $\mathbb{Z}G$-chain complex or $\mathcal{R}(G)$-chain complex, respectively. Then taking the usual (Reidemeister) torsion induces maps τ fitting into the commutative diagram

$$\begin{tikzcd} \widetilde{K}_1^{\mathrm{ch}}(\mathbb{Z}G) \arrow[r] \arrow[d, shift right, "\tau"'] & \widetilde{K}_1^{w,\mathrm{ch}}(\mathbb{Z}G) \arrow[r] \arrow[d, shift right, "\rho_u^{(2)}"'] & \widetilde{K}_1^{\mathrm{ch}}(\mathcal{R}(G)) \arrow[d, shift right, "\tau"'] \\ \widetilde{K}_1(\mathbb{Z}G) \arrow[r] \arrow[u, shift right, "\mathrm{el}"'] & \widetilde{K}_1^w(\mathbb{Z}G) \arrow[r] \arrow[u, shift right, "\mathrm{el}"'] & \widetilde{K}_1(\mathcal{R}(G)) \arrow[u, shift right, "\mathrm{el}"'] \end{tikzcd}$$

The horizontal maps are the obvious morphisms. On the right-hand side we refer to Lemma 3.39 for their existence. The vertical maps upwards are induced by taking the elementary chain complex, see Definition 3.34. The proof that the two vertical maps on the left and the two vertical maps on the right are inverse to each other is an adaption of the proof of Theorem 3.41. The commutativity of the left-hand square is obvious since any chain contraction γ of a contractible $\mathbb{Z}G$-chain complex gives the weak chain contraction (γ, id). The commutativity on the right-hand side is the content of the following lemma.

Lemma 3.42. *Let C_* be a finite based free L^2-acyclic $\mathbb{Z}G$-chain complex. Then $C_* \otimes_{\mathbb{Z}G} \mathcal{R}(G)$ is a contractible $\mathcal{R}(G)$-chain complex and the canonical homomorphism $i\colon \widetilde{K}_1^w(\mathbb{Z}G) \to \widetilde{K}_1(\mathcal{R}(G))$ satisfies*

$$i\big(\rho_u^{(2)}(C_*;\mathcal{N}(G))\big) = \tau(C_* \otimes_{\mathbb{Z}G} \mathcal{R}(G)). \tag{3.4}$$

Proof. The chain complex $C_* \otimes_{\mathbb{Z}G} \mathcal{R}(G)$ is contractible by Lemma 3.39.

Let R be any associative ring with 1 and E_* a finite based free contractible R-chain complex. If $u_*\colon E_* \to E_*$ is a chain isomorphism and $\gamma_*\colon u_* \simeq 0_*$ is a chain homotopy such that $\gamma_n \circ u_n = u_{n+1} \circ \gamma_n$, then

$$\tau(E_*) = [(uc+\gamma)_{\mathrm{odd}})] - [u_{\mathrm{odd}}] \in \widetilde{K}_1(R). \tag{3.5}$$

This follows in exactly the same way as the argument leading to [FL16b, Equation (1.8)].

Now the desired equation (3.4) follows from this by comparing (3.5) with the definition of universal L^2-torsion. $\square$

Finally we mention the following result on the horizontal maps above.

Theorem 3.43 ($K_1^w(\mathbb{Z}G)$ vs. $K_1(\mathcal{R}(G))$). *Let $\mathcal{C}$ be the smallest class of groups which contains all free groups and is closed under directed unions and extensions with elementary amenable quotients. Then any torsion-free group G in $\mathcal{C}$ satisfies the Atiyah Conjecture, $\mathcal{R}(G) = \mathcal{D}(G)$ is a skew-field, and the obvious map*

$$K_1^w(\mathbb{Z}G) \to K_1(\mathcal{D}(G))$$

is an isomorphism.

Proof. The first two statements are handled in Theorem 3.19 and Proposition 3.22. The third is a recent result due to Linnell-Lück [LL16, Theorem 0.1]. $\square$

3.6.3 Universal L^2-torsion for G-CW-complexes.

Definition 3.44. Let X be a finite free L^2-acyclic G-CW-complex. Its *universal L^2-torsion* $\rho_u^{(2)}(X;\mathcal{N}(G)) \in \mathrm{Wh}^w(G)$ is defined as the image of $\rho_u^{(2)}(C_*(X);\mathcal{N}(G))$ under the projection $\widetilde{K}_1^w(\mathbb{Z}G) \to \mathrm{Wh}^w(G)$, where $C_*(X)$ denotes the cellular $\mathbb{Z}G$-chain complex of X equipped with a cellular basis.

The following list of basic properties taken from [FL16b, Theorem 2.5] implies Theorem 3.3, Theorem 3.10, and Theorem 3.16 by virtue of Theorem 3.41.

Theorem 3.45 (Basic properties of universal L^2-torsion)**.**

1. *(Homotopy invariance) Let $f\colon X \to Y$ be a G-homotopy equivalence of finite free G-CW-complexes. If X or Y is L^2-acyclic, then both are L^2-acyclic and we have*

$$\rho_u^{(2)}(Y;\mathcal{N}(G)) - \rho_u^{(2)}(X;\mathcal{N}(G)) = \zeta(\tau(f)),$$

where $\tau(f) \in \mathrm{Wh}(G)$ denotes the Whitehead torsion and $\zeta\colon \mathrm{Wh}(G) \to \mathrm{Wh}^w(G)$ is the obvious homomorphism.

2. *(Sum formula) Let*

$$\begin{array}{ccc} X_0 & \longrightarrow & X_1 \\ \downarrow & & \downarrow \\ X_2 & \longrightarrow & X \end{array}$$

be a G-pushout of finite free G-CW-complexes such that the upper horizontal map is cellular, the left-hand map is an inclusion of G-CW-complexes and X carries the G-CW-structure induced from the X_i. If X_i for $i = 0, 1, 2$ is L^2-acyclic, then X is L^2-acyclic and we have

$$\rho_u^{(2)}(X;\mathcal{N}(G)) = \rho_u^{(2)}(X_1;\mathcal{N}(G)) + \rho_u^{(2)}(X_2;\mathcal{N}(G)) - \rho_u^{(2)}(X_0;\mathcal{N}(G)).$$

3. *(Product formula) Let G_1 and G_2 denote groups and $i_*\colon \mathrm{Wh}^w(G_1) \to \mathrm{Wh}^w(G_1 \times G_2)$ the homomorphism induced from the obvious inclusion $i\colon G_1 \to G_1 \times G_2$. Let X_i be finite free G_i-CW-complexes such that X_1 is L^2-acyclic. Then $X_1 \times X_2$ is L^2-acyclic and we have*

$$\rho_u^{(2)}(X_1 \times X_2;\mathcal{N}(G_1 \times G_2)) = \chi(X_2/G_2) \cdot i_*(\rho_u^{(2)}(X_1;\mathcal{N}(G_1))).$$

(4) *(Induction) Let $i\colon H \to G$ be an inclusion of groups and $i_*\colon \mathrm{Wh}^w(H) \to \mathrm{Wh}^w(G)$ the induced homomorphism. If X is a finite free L^2-acyclic H-CW-complex, then $i_*X = G \times_H X$ is a finite free L^2-acyclic G-CW-complex and we have*

$$\rho_u^{(2)}(i_*X; \mathcal{N}(G)) = i_*(\rho_u^{(2)}(X; \mathcal{N}(H))).$$

(5) *(Restriction) Let $i\colon H \to G$ be an inclusion of groups with finite index and $i^*\colon \mathrm{Wh}^w(G) \to \mathrm{Wh}^w(H)$ the restriction homomorphism. Let X be a finite free G-CW-complex and i^*X the finite free H-CW-complex obtained from X by restriction. Then i^*X is L^2-acyclic if and only if X is L^2-acyclic and in this case we have*

$$i^*(\rho_u^{(2)}(X; \mathcal{N}(G))) = \rho_u^{(2)}(i^*X; \mathcal{N}(H)).$$

(6) *(Fibrations) Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration such that F and B are finite CW-complexes. Let $\overline{E} \to E$ be a G covering and $\overline{F} \to F$ the G-covering obtained from it by pullback along i. Assume that $\mathrm{Wh}(G)$ vanishes. Assume that $\overline{F}$ is L^2-acyclic. Then $\overline{E}$ is up to G-homotopy equivalence a finite free L^2-acyclic G-CW-complex and we have*

$$\rho_u^{(2)}(\overline{E}; \mathcal{N}(G)) = \chi(B) \cdot \rho_u^{(2)}(\overline{F}; \mathcal{N}(G)).$$

(7) *(Poincaré Duality) Let M be an orientable n-dimensional manifold with free and proper G-action. Let $w\colon G \to \{\pm 1\}$ denote the orientation homomorphism. Denote by $*\colon \mathrm{Wh}^w(G) \to \mathrm{Wh}^w(G)$ the involution induced from the involution on $\mathbb{Z}G$ determined by $*(x \cdot g) = x \cdot w(g) \cdot g^{-1}$. If M is L^2-acyclic, then*

$$\rho_u^{(2)}(M, \partial M; \mathcal{N}(G)) = (-1)^{n+1} \cdot \rho_u^{(2)}(M; \mathcal{N}(G)).$$

Example 3.46. It is shown in [FL16b, Example 2.7] that for the n-dimensional torus T^n and any non-trivial homomorphism $\mu\colon \pi_1(T^n) \to G$ to a torsion-free group we have

$$\rho_u^{(2)}(\overline{T^n}; \mathcal{N}(G)) = 0$$

for the G-covering $\overline{T^n} \to T^n$ associated to μ.

3.7 The L^2-torsion polytope

Among the various L^2-torsion invariants presented in this chapter, the L^2-torsion polytope constructed in this final section stands out as having a somewhat geometric flavour. It takes values in the Grothendieck group of integral polytopes in certain vector spaces. Unlike L^2-torsion, twisted L^2-torsion functions and twisted L^2-Euler characteristics, the L^2-torsion polytope was constructed *after* the universal L^2-torsion, or to be more precise, in one go by Friedl- Lück [FL16b]. A forerunner version was examined by Friedl-Tillmann [FT15].

3.7.1 Polytope groups. Let V be a finite-dimensional real vector space. By a *polytope in V* we mean a non-empty subset $P \subseteq V$ that is the convex hull of finitely many points. The *Minkowski sum* of two polytopes P and Q in V is defined by pointwise addition, i.e.,

$$P + Q = \{p + q \in V \mid p \in P, q \in Q\}.$$

We denote by $\mathfrak{P}(V)$ the commutative monoid of all polytopes in V with the Minkowski sum as addition. It is cancellative, see e.g. [Sch93, Lemma 3.1.8]. Define the *polytope group* $\mathcal{P}(V)$ to be the Grothendieck group associated to this commutative monoid. Thus elements are given by formal differences $P - Q$ of polytopes $P, Q \in \mathfrak{P}(V)$, and two such differences $P - Q$, $P' - Q'$ are equal if and only if $P + Q' = P' + Q$ as subsets in V. There is an injection of real vector spaces

$$V \to \mathcal{P}(V), \quad v \mapsto \{v\} \tag{3.6}$$

and we let $\mathcal{P}_T(V)$ be the cokernel of this map. The subscript T stands for *translation* since two polytopes become identified in $\mathcal{P}_T(V)$ if and only if there is a translation on V mapping one bijectively to the other. Finally, $\mathcal{P}(V)$ carries a canonical involution determined by reflection about the origin, i.e.,

$$*\colon \mathcal{P}(V) \to \mathcal{P}(V), \quad P \mapsto *P = \{-p \mid p \in P\}. \tag{3.7}$$

This involution descends to $\mathcal{P}_T(V)$.

Next we build an integral version of the polytope group. For this, let H be a finitely generated free-abelian group. A polytope in $H \otimes_{\mathbb{Z}} \mathbb{R}$ is *integral* if it is the convex hull of finitely many points in H, considered as a lattice in $H \otimes_{\mathbb{Z}} \mathbb{R}$. Denote by $\mathfrak{P}(H) \subseteq \mathfrak{P}(H \otimes_{\mathbb{Z}} \mathbb{R})$ the submonoid whose elements are integral polytopes. Then the *integral polytope group* $\mathcal{P}(H)$ is defined as the Grothendieck group of $\mathfrak{P}(H)$. The map (3.6) restricts to an injection $H \to \mathcal{P}(H)$ whose cokernel will be denoted by $\mathcal{P}_T(H)$. We let $\mathfrak{P}_T(H)$ be the image of the composition $\mathfrak{P}(H) \to \mathcal{P}(H) \to \mathcal{P}_T(H)$, thus $\mathfrak{P}_T(H)$ contains precisely those elements of $\mathcal{P}_T(H)$ which can represented by a polytope. The involution (3.7) induces involutions on $\mathcal{P}(H)$ and $\mathcal{P}_T(H)$ which we continue to denote by $*$.

A homomorphism $f\colon H \to H'$ of finitely generated free-abelian groups induces homomorphisms

$$\begin{aligned} \mathcal{P}(f)\colon \mathcal{P}(H) &\to \mathcal{P}(H'); \\ \mathcal{P}_T(f)\colon \mathcal{P}_T(H) &\to \mathcal{P}_T(H') \end{aligned}$$

by sending the class of a polytope P to the class of the polytope $f(P)$. If f is injective, then both $\mathcal{P}(f)$ and $\mathcal{P}_T(f)$ are easily seen to be injective as well. Thus if $G \subseteq H$ is a subgroup, then we will always view $\mathcal{P}(G)$ (respectively $\mathcal{P}_T(G)$) as a subgroup of $\mathcal{P}(H)$ (respectively $\mathcal{P}_T(H)$).

Example 3.47. Integral polytopes in $\mathbb{Z}$ are just intervals $[m, n] \subseteq \mathbb{R}$ starting and ending at integral points. Thus we have $\mathcal{P}(\mathbb{Z}) \cong \mathbb{Z}^2$, where an explicit isomorphism is given by sending the class $[m, n]$ to $(m, n - m)$. Under this isomorphism, the involution corresponds to $*(k, l) = (-l - k, l)$. Similarly, $\mathcal{P}_T(\mathbb{Z}) \cong \mathbb{Z}$, where an explicit isomorphism is given by sending the element $[m, n]$ to $n - m$. The involution $*$ on $\mathcal{P}_T(\mathbb{Z})$ is the identity.

We investigate the structure of the various polytope groups defined above in detail in Chapter 4. At this point, we only use it as the group where another L^2-torsion invariant takes values in.

3.7.2 The polytope homomorphism. Let G be a torsion-free group satisfying the Atiyah Conjecture, let H be a finitely generated free-abelian group, and let $\nu\colon G \to H$ be an epimorphism. In this section we follow Friedl-Lück [FL16b, Section 6.2] to construct a group homomorphism

$$\mathbb{P}_\nu\colon K_1^w(\mathbb{Z}G) \to \mathcal{P}(H)$$

referred to as the *polytope homomorphism*. Earlier versions of it had at least implicitly been considered for torsion-free elementary amenable groups [FH07]. The construction proceeds in multiple steps.

First of all, there is the obvious map

$$K_1^w(\mathbb{Z}G) \to K_1(\mathcal{D}(G)). \tag{3.8}$$

Next we use the non-commutative determinant for skew-fields due to Dieudonné [Die43] which induces an isomorphism (see [Ros94, Corollary 2.2.6] or [Sil81, Corollary 4.3])

$$\det_{\mathcal{D}(G)} \colon K_1(\mathcal{D}(G)) \xrightarrow{\cong} \mathcal{D}(G)^{\times}_{\mathrm{ab}} = \mathcal{D}(G)^{\times}/[\mathcal{D}(G)^{\times}, \mathcal{D}(G)^{\times}]. \tag{3.9}$$

Let $K = \ker(\nu)$. Recall from Theorem 3.24 (1) that the crossed product $\mathcal{D}(K) * H$ embeds into $\mathcal{D}(G)$ and localizing at $T = (\mathcal{D}(K) * H) \smallsetminus \{0\}$ induces an isomorphism $\mathcal{D}(G) \xrightarrow{\cong} T^{-1}(\mathcal{D}(K) * H)$. This induces an isomorphism

$$\mathcal{D}(G)^{\times}_{\mathrm{ab}} \xrightarrow{\cong} (T^{-1}(\mathcal{D}(K) * H))^{\times}_{\mathrm{ab}}, \tag{3.10}$$

For an element $x = \sum_{h \in H} x_h \cdot h \in \mathcal{D}(K) * H$ we define its *support* to be $\mathrm{supp}(x) = \{h \in H \mid x_h \neq 0\}$. For a subset $S \subseteq H$ we denote by $\mathrm{hull}(S) \in \mathfrak{P}(H)$ the convex hull of S inside $H \otimes_{\mathbb{Z}} \mathbb{R}$. By [FL16b, Lemma 6.4], there is a map of monoids

$$(\mathcal{D}(K) * H) \smallsetminus \{0\} \to \mathfrak{P}(H), \;\; x \mapsto P(x) = \mathrm{hull}(\mathrm{supp}(x)).$$

and so we can localize and abelianize to get a map

$$T^{-1}(\mathcal{D}(K) * H)^{\times}_{\mathrm{ab}} \to \mathcal{P}(H), \;\; b^{-1}a \mapsto P(a) - P(b). \tag{3.11}$$

We let

$$\mathbb{P}_{\nu} \colon K_1^w(\mathbb{Z}G) \to \mathcal{P}(H) \quad \text{and} \quad P_{\nu} \colon \mathcal{D}(G)^{\times}_{\mathrm{ab}} \to \mathcal{P}(H) \tag{3.12}$$

be the composition of the maps (3.8), (3.9), (3.10), and (3.11), respectively the composition of the maps (3.10) and (3.11). They induce maps

$$\mathbb{P}_{\nu} \colon \mathrm{Wh}^w(G) \to \mathcal{P}_T(H) \quad \text{and} \quad P_{\nu} \colon \mathcal{D}(G)^{\times}_{\mathrm{ab}}/\{\pm g \mid g \in G\} \to \mathcal{P}_T(H). \tag{3.13}$$

If G satisfies $b_1(G) < \infty$, then this construction can be applied to the canonical projection $\nu = \mathrm{pr} \colon G \to H_1(G)_f$. In this case we omit the subscript ν in the above notation.

3.7.3 The L^2-torsion polytope.

Definition 3.48 (L^2-torsion polytope)**.** Let X be a finite free L^2-acyclic G-CW-complex such that G is a torsion-free group satisfying the Atiyah Conjecture. Let $\nu \colon G \to H$ be an epimorphism onto a finitely generated free-abelian group. Then we define the *L^2-torsion polytope of X with respect to ν* as the image of the negative of its universal L^2-torsion under the polytope homomorphism (3.13), i.e.,

$$P(X; G, \nu) = \mathbb{P}_{\nu}(-\rho_u^{(2)}(X; \mathcal{N}(G))) \in \mathcal{P}_T(H).$$

If G satisfies $b_1(G) < \infty$ and $\nu = \mathrm{pr} \colon G \to H_1(G)_f$ is the canonical projection, then we simply write $P(X; G)$ for $P(X; G, \mathrm{pr})$.

The list of basic properties of the L^2-torsion polytope follows directly from the list of basic properties of the universal L^2-torsion, see Theorem 3.45. We may also turn the universal L^2-torsion and the L^2-torsion polytope into invariants of groups. Recall that a group is *of type F* if it admits a finite classifying space.

Definition 3.49 (Universal L^2-torsion and L^2-torsion polytope of groups)**.** Let G be an L^2-acyclic group of type F such that $\mathrm{Wh}(G) = 0$. Then we define the *universal L^2-torsion* of G to be

$$\rho_u^{(2)}(G) = \rho_u^{(2)}(EG; \mathcal{N}(G)) \in \mathrm{Wh}^w(G).$$

If, additionally, G satisfies the Atiyah Conjecture, then we define the *L^2-torsion polytope* of G as

$$P(G) = P(EG; G) \in \mathcal{P}_T(H_1(G)_f).$$

Remark 3.50 (Assumptions appearing in Definition 3.49)**.** The assumption $\mathrm{Wh}(G) = 0$ appearing above ensures that the universal L^2-torsion of groups is well-defined, see Theorem 3.45 (1). Conjecturally, however, this assumption is obsolete: Any group of type F is torsion-free, and it is conjectured that the Whitehead group of any torsion-free group vanishes, see [LR05, Conjecture 3]. There is also no counterexample to the Atiyah Conjecture known. Thus the L^2-torsion polytope is potentially an invariant for all L^2-acyclic groups of type F.

A forerunner version of the L^2-torsion polytope of groups was examined by Friedl-Tillmann [FT15] in the special case where G is a torsion-free group determined by a presentation with two generators, one relation, and $b_1(G) = 2$.

3.7.4 Relation to twisted L^2-Euler characteristics. Given a finitely generated free-abelian group H, we denote by $\mathrm{Map}(\mathrm{Hom}(H, \mathbb{R}), \mathbb{R})$ the group of continuous maps $\mathrm{Hom}(H, \mathbb{R}) \to \mathbb{R}$ equipped with pointwise addition. A polytope $P \subseteq H \otimes_{\mathbb{Z}} \mathbb{R}$ induces a seminorm on $\mathrm{Hom}(H, \mathbb{R})$ by

$$\|\varphi\|_P = \max\{\varphi(p) - \varphi(q) \mid p, q \in P\}.$$

This seminorm behaves well with respect to Minkowski sums in the sense that

$$\|\varphi\|_{P+Q} = \|\varphi\|_P + \|\varphi\|_Q.$$

Definition 3.51 (Seminorm homomorphism)**.** The homomorphism

$$\mathfrak{N} \colon \mathcal{P}(H) \to \mathrm{Map}(\mathrm{Hom}(H, \mathbb{R}), \mathbb{R}), \quad P - Q \mapsto \|\cdot\|_P - \|\cdot\|_Q$$

is called *seminorm homomorphism.* It passes to the quotient $\mathcal{P}_T(H)$ and the induced map

$$\mathfrak{N} \colon \mathcal{P}_T(H) \to \mathrm{Map}(\mathrm{Hom}(H, \mathbb{R}), \mathbb{R})$$

is denoted by the same symbol.

Twisted L^2-Euler characteristics can be obtained from the universal L^2-torsion. The following theorem shows that twisted L^2-Euler characteristics can still be obtained from the coarser L^2-torsion polytope. It is an extension of Theorem 2.8 (compare also Corollary 3.29).

Theorem 3.52 (L^2-torsion polytope and twisted L^2-Euler characteristics)**.** *Let C_* be a finite based free L^2-acyclic $\mathbb{Z}G$-chain complex such that G is torsion-free and satisfies the Atiyah Conjecture. Let $\nu \colon G \to H$ be an epimorphism onto a finitely generated free-abelian* *group. If $\varphi\colon H \to \mathbb{Z}$ is an epimorphism, then C_* is $(\varphi \circ \nu)$-L^2-finite and we have*

$$\mathfrak{N}\Big(\mathbb{P}_\nu\big(\rho_u^{(2)}(C_*;\mathcal{N}(G))\big)\Big)(\varphi) = \chi^{(2)}(C_*;\mathcal{N}(G), \varphi \circ \nu).$$

In particular, the left-hand side depends on ν and φ only through the composition $\varphi \circ \nu$, and for a finite free G-CW-complex X we have

$$\mathfrak{N}(P(X;G,\nu))(\varphi) = -\chi^{(2)}(X;\mathcal{N}(G), \varphi \circ \nu).$$

Proof. This is essentially [FL16b, Equality (3.26)]. The argument is illuminating and spreads also to the paper [FL16a], so it seems worthwhile roughly outlining it.

Let $K = \ker(\varphi \circ \nu)$, let $i\colon K \to G$ be the inclusion, and let $\mathcal{D}(K)_t[u^{\pm}] \subseteq \mathcal{D}(G)$ be the twisted Laurent polynomial ring associated to $\varphi \circ \nu$ as in Theorem 3.24 (1).

Step 1: For a matrix $A \in M_{n,n}(\mathbb{Z}G)$ that becomes invertible over $\mathcal{D}(G)$, or more generally for a matrix $A \in M_{n,n}(\mathcal{D}(K)_t[u^{\pm}])$ that becomes invertible over $\mathcal{D}(G)$, one first proves by virtue of the Euclidean function on $\mathcal{D}(K)_t[u^{\pm}]$ given by the degree that the Dieudonné determinant $\det_{\mathcal{D}(G)}(A)$ can be represented by an element $x \in \mathcal{D}(K)_t[u^{\pm}]$, and that then

$$\begin{aligned} &\dim_{\mathcal{D}(K)}\big(\operatorname{coker}\big(r_A\colon \mathcal{D}(K)_t[u^{\pm}]^n \to \mathcal{D}(K)_t[u^{\pm}]^n\big)\big) \\ &= \dim_{\mathcal{D}(K)}\big(\operatorname{coker}\big(r_x\colon \mathcal{D}(K)_t[u^{\pm}] \to \mathcal{D}(K)_t[u^{\pm}]\big)\big). \end{aligned}$$

Step 2: For an element $x \in \mathcal{D}(K)_t[u^{\pm}]$, it is a classical fact (and reproved in [FL16a, Lemma 4.3]) that

$$\dim_{\mathcal{D}(K)}\big(\operatorname{coker}\big(r_x\colon \mathcal{D}(K)_t[u^{\pm}] \to \mathcal{D}(K)_t[u^{\pm}]\big)\big) = \deg(x).$$

Step 3: For an element $x \in \mathcal{D}(K)_t[u^{\pm}] \subseteq \mathcal{D}(G)$, it follows right from the definitions that

$$\mathfrak{N}(P_{\varphi\circ\nu}(x))(\mathrm{id}_{\mathbb{Z}}) = \deg(x).$$

Step 4: A special case of [FL16a, Lemma 6.12] states that

$$\mathfrak{N}(P_\nu(x))(\varphi) = \mathfrak{N}(P_{\varphi\circ\nu}(x))(\mathrm{id}_{\mathbb{Z}}).$$

Thus the left-hand side depends on ν and φ only through the composition $\varphi \circ \nu$.

Step 5: Combining these facts with Theorem 3.24 (2) and Lemma 3.17, we calculate

$$\begin{aligned}
&\mathfrak{N}\Big(\mathbb{P}_\nu\big(\rho_u^{(2)}(\mathrm{el}(r_A);\mathcal{N}(G))\big)\Big)(\varphi)\\
&= \mathfrak{N}\big(\mathbb{P}_\nu([r_A])\big)(\varphi)\\
&= \mathfrak{N}(P_\nu(x))(\varphi)\\
&= \mathfrak{N}(P_{\varphi\circ\nu}(x))(\mathrm{id}_{\mathbb{Z}})\\
&= \deg(x)\\
&= \dim_{\mathcal{D}(K)}\big(\operatorname{coker}\big(r_x\colon \mathcal{D}(K)_t[u^{\pm}] \to \mathcal{D}(K)_t[u^{\pm}]\big)\big)\\
&= \dim_{\mathcal{D}(K)}\big(\operatorname{coker}\big(r_A\colon \mathcal{D}(K)_t[u^{\pm}]^n \to \mathcal{D}(K)_t[u^{\pm}]^n\big)\big)\\
&= \dim_{\mathcal{D}(K)}\big(H_0(\mathcal{D}(K)_t[u^{\pm}] \otimes_{\mathbb{Z}G} \mathrm{el}(r_A))\big)\\
&= \dim_{\mathcal{D}(K)}\big(H_0(\mathcal{D}(K) \otimes_{\mathbb{Z}K} i^*\mathrm{el}(r_A))\big)\\
&= \dim_{\mathcal{N}(K)}\big(H_0(\mathcal{N}(K) \otimes_{\mathbb{Z}K} i^*\mathrm{el}(r_A))\big)\\
&= \chi^{(2)}(i^*\mathrm{el}(r_A);\mathcal{N}(K))\\
&= \chi^{(2)}(\mathrm{el}(r_A);\mathcal{N}(G),\varphi\circ\nu).
\end{aligned}$$

Step 6: If C_* is a finite based free L^2-acyclic $\mathbb{Z}G$-chain complex, then the equality

$$\mathfrak{N}\Big(\mathbb{P}_\nu\big(\rho_u^{(2)}(C_*;\mathcal{N}(G))\big)\Big)(\varphi) = \chi^{(2)}(C_*;\mathcal{N}(G),\varphi\circ\nu)$$

now follows from Step 5 and the inverse pair of isomorphisms (see Theorem 3.41)

$$\widetilde{K}_1^{w,\mathrm{ch}}(\mathbb{Z}G) \underset{\mathrm{el}}{\overset{\rho_u^{(2)}}{\rightleftarrows}} \widetilde{K}_1^{w}(\mathbb{Z}G)$$

which identifies the two models of the universal L^2-torsion invariant. □

3.7.5 Relation to the Thurston norm.

Theorem 3.53 (L^2-torsion polytope and the Thurston norm I)**.** *Suppose that $M \neq S^1 \times D^2$ is an admissible 3-manifold that is not a closed graph manifold. Then the canonical projection factors into epimorphisms $\pi_1(M) \xrightarrow{\alpha} \Gamma \xrightarrow{\beta} H_1(M)_f$ with Γ a virtually finitely generated free-abelian group such that:*

If $\pi_1(M) \xrightarrow{\mu} G \xrightarrow{\nu} \Gamma$ is a factorization of α such that G is a torsion-free group satisfying the Atiyah Conjecture and $b_1(G) < \infty$, then the G-covering $\overline{M} \to M$ associated to μ is L^2-acyclic and the composition

$$\mathrm{Wh}^w(G) \xrightarrow{\mathbb{P}} \mathcal{P}_T(H_1(G)_f) \xrightarrow{\mathcal{P}_T(\beta\circ\nu)} \mathcal{P}_T(H_1(M)_f) \xrightarrow{\mathfrak{N}} \mathrm{Map}(H^1(M;\mathbb{R}),\mathbb{R})$$

maps $-\rho_u^{(2)}(\overline{M};\mathcal{N}(G))$ to the Thurston norm x_M.

Proof. This is [FL16a, Theorem 3.24] and follows for surjective integral classes directly from Theorem 3.30 (3) and Theorem 3.52. The homogeneity and continuity of seminorms then imply the general case. □

Without explaining the notion of dualizing polytopes, we mention the following main result of Friedl-Lück's theory.

Theorem 3.54 (L^2-torsion polytope and the Thurston norm II)**.** *Let $M \neq S^1 \times D^2$ be an admissible 3-manifold such that $\pi_1(M)$ satisfies the Atiyah Conjecture. Let $T(M) \subseteq$*

$H^1(M;\mathbb{R})$ be the unit norm ball of the Thurston norm and let $T(M)^* \subseteq H^1(M;\mathbb{R})^* = H_1(M;\mathbb{R})$ be its dual. Then $T(M)^*$ is an integral polytope and we have in $\mathcal{P}_T(H_1(M)_f)$ the equality

$$T(M)^* = P(\widetilde{M};\pi_1(M)).$$

Proof. This is [FL16a, Theorem 3.35] and we include again a short summary.

One has almost by definition that the seminorm map

$$\mathfrak{N}\colon \mathcal{P}_T(H_1(M)_f) \to \mathrm{Map}(H^1(M;\mathbb{R}),\mathbb{R})$$

sends $T(M)^*$ to x_M and that $T(M)^* = *T(M)^*$. The same is true for $P(\widetilde{M};\pi_1(M))$ by the deep Theorem 3.53 and Poincaré duality of the universal L^2-torsion (see Theorem 3.45 (7)). As we shall see in Lemma 4.16 we have in $\mathcal{P}_T(H_1(M)_f)$ the equality of subgroups

$$\ker(\mathfrak{N}) = \ker(\mathrm{id} + *).$$

This implies that $\mathfrak{N}$ restricted to the subgroup $\ker(\mathrm{id} - *)$ is injective, and hence

$$T(M)^* = P(\widetilde{M};\pi_1(M)).$$

□

4 The Integral Polytope Group

The results of this chapter are summarized in the following theorem.

Theorem 4.1 (Structure of the integral polytope group)**.** *Let H be a finitely generated free-abelian group and let V be a real finite-dimensional vector space. Then:*

(1) (Symmetric elements) We have

$$\ker\big(\mathrm{id} - *\colon \mathcal{P}(H) \to \mathcal{P}(H)\big) = \mathrm{im}\big(\mathrm{id} + *\colon \mathcal{P}(H) \to \mathcal{P}(H)\big).$$

(2) (Antisymmetric elements) We have

$$\ker\big(\mathrm{id} + *\colon \mathcal{P}(H) \to \mathcal{P}(H)\big) = \mathrm{im}\big(\mathrm{id} - *\colon \mathcal{P}(H) \to \mathcal{P}(H)\big)$$

and

$$\ker\big(\mathrm{id} + *\colon \mathcal{P}_T(H) \to \mathcal{P}_T(H)\big) = \mathrm{im}\big(\mathrm{id} - *\colon \mathcal{P}_T(H) \to \mathcal{P}_T(H)\big).$$

(3) (Basis) There are sets $\mathcal{B}_1 \subseteq \mathcal{B}_2 \subseteq \ldots \subseteq \mathcal{B}_n \subseteq \mathcal{P}_T(H)$ such that $\mathcal{B}_m \setminus \mathcal{B}_{m-1}$ contains only polytopes of dimension m and $\mathcal{B}_m \cap \mathcal{P}_T(G)$ is a basis for $\mathcal{P}_T^m(G)$ for every pure subgroup $G \subseteq H$ and $1 \leq m \leq n$. In particular, $\mathcal{B}_n$ is a basis for $\mathcal{P}_T(H)$.

Moreover, if $\mathcal{A} \subseteq H$ denotes a basis of H and $\mathcal{B}'_n \subseteq \mathcal{P}(H)$ is a set of representatives for $\mathcal{B}_n \subseteq \mathcal{P}_T(H)$, then $\mathcal{A} \cup \mathcal{B}'_n$ is a basis for $\mathcal{P}(H)$.

(4) (Involution as face Euler characteristic) For any polytope $P \subseteq V$ we have in $\mathcal{P}(V)$

$$*P = -\sum_{F \in \mathcal{F}(P)} (-1)^{\dim(F)} \cdot F,$$

where $\mathcal{F}(P)$ denotes the set of faces of P (including P itself).

A few explanations are in order. The integral and real polytope groups $\mathcal{P}(H)$ and $\mathcal{P}(V)$ as well as their quotients $\mathcal{P}_T(H)$ and $\mathcal{P}_T(V)$ have been introduced in Section 3.7.1. Recall that we denote by $*\colon \mathcal{P}(H) \to \mathcal{P}(H)$ and $*\colon \mathcal{P}(V) \to \mathcal{P}(V)$ the involution induced by reflection about the origin, i.e., $*P = \{-p \mid p \in P\}$. Given a natural number m we denote by $\mathcal{P}_T^m(H)$ the subgroup of $\mathcal{P}_T(H)$ generated by the polytopes of dimension at most m. A subgroup $G \subseteq H$ is *pure* if there is a linear subspace $U \subseteq H \otimes_{\mathbb{Z}} \mathbb{R}$ such that $G = H \cap U$. Equivalently, G is not properly contained in a subgroup $G' \subseteq H$ of the same rank. By considering the simple example $H = \mathbb{Z}$, it is easy to see that the statement of part (3) needs to be restricted to pure subgroups.

The significant role of the integral polytope group in the context of universal L^2-torsion combined with an almost complete lack of information about its structure was the starting point for proving Theorem 4.1. Part (2) directly contributes towards the proof of Proposition 5.19 about the L^2-torsion polytope of amenable groups. Before going into the proof of Theorem 4.1 we first point out precise motivations as well as conclusions of the various parts of the theorem.

Remark 4.2. The inclusions $\supseteq$ are easily seen in both (1) and (2). The analogues of (1) and (2) for the real polytope group $\mathcal{P}(V)$ are trivially true.

Part (1) is established in [CFF17] as a negative result in an approach to define a knot concordance invariant. We emphasize that for the translation quotient we have a proper inclusion

$$\operatorname{im}\big(\mathrm{id} - *\colon \mathcal{P}_T(H) \to \mathcal{P}_T(H)\big) \subsetneq \ker\big(\mathrm{id} + *\colon \mathcal{P}_T(H) \to \mathcal{P}_T(H)\big)$$

as can easily seen for $H = \mathbb{Z}$.

Part (2) is motivated by the question how different integral polytopes P and Q can look if they induce the same seminorm on $\operatorname{Hom}(H, \mathbb{R})$, see Lemma 4.16.

Part (3) of Theorem 4.1 is motivated by the following abstract argument that $\mathcal{P}(H)$ is a free-abelian group. As will be pointed out in (4.11) below, $\mathcal{P}(H)$ embeds into a countable product of infinite cyclic groups. On the other hand, a theorem of Specker [Spe50] states that any such countable subgroup is free-abelian. However, this argument does not yield any *geometric* insight into the structure of the polytope group. Our basis on the other hand is explicit and geometrically tangible. The proof of part (3) will apply almost verbatim to produce a basis of the real vector space $\mathcal{P}(V)$. The only wording that needs to be changed in the formulation is to replace *pure subgroup* by *linear subspace*.

Part (4) of Theorem 4.1 restricts also to the integral polytope group since the faces of an integral polytope are integral. The following corollary of part (4) can be seen as a combinatorial reminiscence of the fact that the Euler characteristic of a closed odd-dimensional manifold vanishes and the Euler characteristic of a closed even-dimensional manifold which bounds a compact manifold is even.

Corollary 4.3. *Let $P \subseteq V$ be a symmetric polytope. Then we have in $\mathcal{P}(V)$*

$$\sum_{\substack{F \in \mathcal{F}(P) \\ F \neq P}} (-1)^{\dim(F)} \cdot F = \begin{cases} 0, & \text{if } \dim(P) \text{ is odd;} \\ -2 \cdot P, & \text{if } \dim(P) \text{ is even.} \end{cases}$$

If we define the *face Euler characteristic* of a polytope $P \subseteq V$ as

$$\chi_{\mathcal{F}}(P) = \sum_{F \in \mathcal{F}(P)} (-1)^{\dim(F)} \cdot F \in \mathcal{P}(V),$$

then we obtain the following second consequence of Theorem 4.1 (4).

Corollary 4.4. *For polytopes $P, Q \subseteq V$ we have*

$$\chi_{\mathcal{F}}(P + Q) = \chi_{\mathcal{F}}(P) + \chi_{\mathcal{F}}(Q).$$

The last three parts of Theorem 4.1 appear as the main results in [Fun16].

Convention 4.5. Throughout this chapter, $\boldsymbol{z} \in \mathbb{R}^n$ will denote the point $(0, ..., 0, 1)$, $\mathcal{Z} \subseteq \mathbb{R}^n$ will denote the 1-dimensional polytope with vertices 0 and $\boldsymbol{z}$, and $\boldsymbol{z}^\perp$ will denote the orthogonal complement of $\boldsymbol{z}$ with respect to the standard inner product.

Given an element $x \in \mathbb{R}^n$, we will consistently refer to its k-th coordinate by x_k.

Given a subset $S \subseteq \mathbb{R}^n$, the *convex hull* of S will be denoted by $\operatorname{hull}(S)$.

4.1 Algebra vs. geometry I: The partition relation

In this section we will use the geometry of polytopes as our main (and only) tool to conveniently manipulate Minkowski sums.

Definition 4.6 (Faces and face maps)**.** Let $\varphi \in \operatorname{Hom}(V, \mathbb{R})$ and let $P \subseteq V$ be a polytope. Then we call

$$F_\varphi(P) = \{p \in P \mid \varphi(p) = \max\{\varphi(q) \mid q \in P\}\}$$

the *face of* P *in* φ*-direction*. A subset $F \subseteq P$ is called a *face* if $F_\varphi(P) = F$ for some $\varphi \in \operatorname{Hom}(V, \mathbb{R})$. The *codimension* of a face $F \subseteq P$ is

$$\operatorname{codim}(F \subseteq P) = \dim(P) - \dim(F).$$

A face is a polytope in its own right, and it is straightforward to check that $F_\varphi(P+Q) = F_\varphi(P) + F_\varphi(Q)$ for any two polytopes P and Q. These two observations imply that we obtain a homomorphism

$$F_\varphi \colon \mathcal{P}(V) \to \mathcal{P}(V), \quad P \mapsto F_\varphi(P) \tag{4.1}$$

that we call *face map (in* φ*-direction)*.

It is allowed to take $\varphi = 0$ in the above definition, where we get $F_\varphi(P) = P$ as the only codimension 0 face. The boundary ∂P is the union of all faces $F \subseteq P$ of codimension at least 1.

Remark 4.7. If H is a finitely generated free-abelian group and $P \subseteq V_H = H \otimes_{\mathbb{Z}} \mathbb{R}$ is an *integral* polytope, then it suffices to consider integral covectors to describe all faces of P. More precisely, for every face F of P there exists φ in the subgroup $\operatorname{Hom}(H, \mathbb{Z}) \subseteq \operatorname{Hom}(H, \mathbb{R}) = \operatorname{Hom}_{\mathbb{R}}(V_H, \mathbb{R})$ such that $F = F_\varphi(P)$.

Definition 4.8 (Hyperplanes and halves)**.** A *hyperplane* $H \subseteq V$ is a subset of the form $H = \{x \in V \mid \varphi(x) = c\}$ for some $\varphi \in \operatorname{Hom}(V, \mathbb{R})$ and $c \in \mathbb{R}$. A hyperplane in $\mathbb{R}^n$ is *flat* if it is a translate of $\boldsymbol{z}^\perp$, and a polytope in $\mathbb{R}^n$ is *flat* if it lies in a flat hyperplane.

Consider a hyperplane $H = \{x \in V \mid \varphi(x) = c\}$ and a subset $S \subseteq V$. Then the two *halves* of S with respect to H are defined as

$$\begin{aligned} S_+ &= \{s \in S \mid \varphi(s) \geq c\} \\ S_- &= \{s \in S \mid \varphi(s) \leq c\}. \end{aligned}$$

Of course, φ is unique only up to a scalar and so the subscripts in the notation are arbitrary. Note that a half of a polytope is either empty, a face of P or a subpolytope of codimension 0. The geometric process of cutting P along H into two halves yields the following algebraic equation.

Lemma 4.9 (Cutting relation)**.** *Let $P \subseteq V$ be a polytope and let P_+ and P_- denote its halves with respect to a hyperplane $H \subseteq V$. Then we have*

$$P_+ + P_- = P + (P \cap H).$$

Proof. We begin with the inclusion $\subseteq$, so let $p \in P_+$ and $q \in P_-$. Then there exists a $0 \leq t \leq 1$ such that $h = t \cdot p + (1-t) \cdot q$ lies in H. Let $r = (1-t) \cdot p + t \cdot q$. Since P is convex, we have $h, r \in P$ and we can write $p + q = r + h \in P + (P \cap H)$.

For the reverse inclusion, let $p \in P$ and $h \in P \cap H$. Without loss of generality suppose that $p \in P_+$. We have $h \in P \cap H \subseteq P_-$ and hence $p + h \in P_+ + P_-$. $\square$

While constructing a basis for the integral and real polytope group, we will be required to decompose a polytope also into more complicated subpolytopes. The following notion, adapted from [Kho97, Paragraph 1], fits nicely into this context.

Definition 4.10 (Partition)**.** A *partition* of a polytope $P \subseteq V$ is a finite set $\mathcal{P}$ of polytopes in V such that

(1) $\bigcup_{Q \in \mathcal{P}} Q = P$;

(2) If $Q \in \mathcal{P}$ and $F \subseteq Q$ is a face, then $Q \in \mathcal{P}$;

(3) If $Q_1, Q_2 \in \mathcal{P}$ and $Q_1 \cap Q_2 \neq \emptyset$, then $Q_1 \cap Q_2$ is a face in both Q_1 and Q_2.

The elements of $\mathcal{P}$ that have the same dimension as P are called the *pieces* of $\mathcal{P}$. For notational convenience that will become clear in Proposition 4.12, let

$$\mathcal{P}^\partial = \{Q \in \mathcal{P} \mid Q \not\subseteq \partial P\}.$$

Example 4.11. (1) Given a polytope P, let $\mathcal{F}(P)$ denote the set of all faces of P (including the codimension 0 face P). Then $\mathcal{F}(P)$ is a partition of P.

(2) Let $P \subseteq V$ be a polytope and let $H_1, ..., H_m \subseteq V$ be a collection of hyperplanes. Let $\mathcal{P}$ be the set that contains the closure of every connected component of $P \setminus \bigcup_{j=1}^m H_j$, together with all its faces. It is easy to see that $\mathcal{P}$ is indeed a partition of P, which we call the *partition of P with respect to $H_1, ..., H_m$*. If $P \cap \bigcup_{j=1}^m H_j \subseteq \partial P$, then we obtain the trivial partition of part (1) as a special case.

The next lemma is a direct analogue of [Kho97, Proposition 3] for the polytope group although the proof is of entirely different nature.

Proposition 4.12 (Partition relation)**.** *Let $P \subseteq V$ be a polytope and $\mathcal{P}$ be a partition of P. Then we have in $\mathcal{P}(V)$ the equation*

$$P = \sum_{Q \in \mathcal{P}^\partial} (-1)^{\operatorname{codim}(Q \subseteq P)} \cdot Q.$$

Proof. We assume without loss of generality that P is full-dimensional, otherwise consider the smallest subspace of V containing P.

We first deal with the special case that $\mathcal{P}$ is the partition of P with respect to a collection of hyperplanes $H_1, ..., H_m \subseteq V$ as in Example 4.11 (2). We proceed by induction on m, where the base case $m = 1$ is taken care of by Lemma 4.9.

For the induction step from $m-1$ to m, we denote the two halves of P with respect to H_m by $P_\pm$, and let $P_H = P \cap H_m$. We may assume that $P_\pm$ are codimension 0 subpolytopes of P since we could otherwise discard H_m in the collection of hyperplanes without changing the induced partition of P. Define $\mathcal{P}_+$ (resp. $\mathcal{P}_-$, $\mathcal{P}_H$) to be the partition of P_+ (resp. P_-, P_H) with respect to $H_1, ..., H_{m-1}$. Applying the induction hypothesis several times yields

$$
\begin{aligned}
P &= P_+ + P_- - P_H \\
P_\pm &= \sum_{Q \in \mathcal{P}^\partial_\pm} (-1)^{\operatorname{codim}(Q \subseteq P_\pm)} \cdot Q \\
P_H &= \sum_{Q \in \mathcal{P}^\partial_H} (-1)^{\operatorname{codim}(Q \subseteq P_H)} \cdot Q.
\end{aligned}
\tag{4.2}
$$

Because of the boundary condition, we have a disjoint decomposition

$$\mathcal{P}^\partial = \mathcal{P}^\partial_+ \amalg \mathcal{P}^\partial_- \amalg \mathcal{P}^\partial_H,$$

which immediately implies the desired equation together with (4.2).

Now let $\mathcal{P}$ be an arbitrary partition of P. Let $\mathcal{H}$ be the set of those hyperplanes in V which contain a $(\dim(V) - 1)$-dimensional polytope of $\mathcal{P}$. Let $\mathcal{Q}$ be the partition of P with respect to $\mathcal{H}$. We can think of $\mathcal{Q}$ as obtained from $\mathcal{P}$ by extending the codimension 1 polytopes of $\mathcal{P}$ through P, see Fig. 4.1.

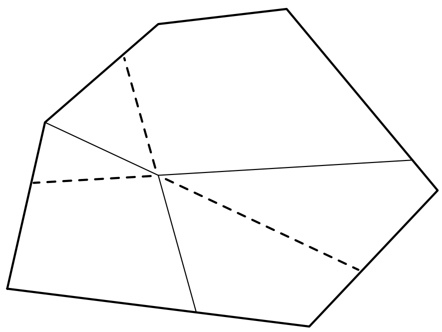

Figure 4.1: If the straight lines indicate $\mathcal{P}$, then the straight and dashed lines together indicate $\mathcal{Q}$.

Given some $S \in \mathcal{P}^\partial$ let $\mathcal{Q}_S$ be the partition of S with respect to $\mathcal{H}$. By the first part we have

$$P = \sum_{Q \in \mathcal{Q}^\partial} (-1)^{\operatorname{codim}(Q \subseteq P)} \cdot Q \tag{4.3}$$

$$S = \sum_{Q \in \mathcal{Q}^\partial_S} (-1)^{\operatorname{codim}(Q \subseteq S)} \cdot Q. \tag{4.4}$$

It is straightforward to check that there is a disjoint decomposition

$$\mathcal{Q}^\partial = \coprod_{S \in \mathcal{P}^\partial} \mathcal{Q}^\partial_S. \tag{4.5}$$

Combining (4.3), (4.4), and (4.5) gives

$$\begin{aligned}
P &= \sum_{Q \in \mathcal{Q}^\partial} (-1)^{\operatorname{codim}(Q \subseteq P)} \cdot Q \\
&= \sum_{S \in \mathcal{P}^\partial} \sum_{Q \in \mathcal{Q}^\partial_S} (-1)^{\operatorname{codim}(Q \subseteq P)} \cdot Q \\
&= \sum_{S \in \mathcal{P}^\partial} (-1)^{\operatorname{codim}(S \subseteq P)} \cdot \sum_{Q \in \mathcal{Q}^\partial_S} (-1)^{\operatorname{codim}(Q \subseteq S)} \cdot Q \\
&= \sum_{S \in \mathcal{P}^\partial} (-1)^{\operatorname{codim}(S \subseteq P)} \cdot S.
\end{aligned}$$

□

Remark 4.13. If $P \subseteq H \otimes_{\mathbb{Z}} \mathbb{R}$ and all of the polytopes in $\mathcal{P}$ are integral, then $\mathcal{Q}$ as in the final step of the proof will in general not be integral. Nevertheless, the final equation contains only elements of the subgroup $\mathcal{P}(H) \subseteq \mathcal{P}(H \otimes_{\mathbb{Z}} \mathbb{R})$.

We often want to cut an integral polytope P along a flat hyperplane H into two halves and apply the cutting relation of Lemma 4.9. In general, however, the intersection $P \cap H$ and thus the two halves P_+ and P_- will not be integral again, so the cutting relation will not be an equation in the *integral* polytope group. In order to circumvent this problem, we can first stretch the polytope as explained in the following lemma. For this, let for $h \in \mathbb{R}$

$$c_h \colon \mathbb{R}^n \to \mathbb{R}^n,\ (x_1, ..., x_n) \mapsto (x_1, ..., x_{n-1}, h),$$

which we can think of as compressing the vector space to a flat hyperplane.

Lemma 4.14 (Vertical stretching). *Let $H = \{x \in \mathbb{R}^n \mid x_n = h\}$ be a flat hyperplane. Then for every integral polytope $P \subseteq \mathbb{R}^n$ of dimension n there exists an integer $k \geq 0$ such that for $Q = P + k \cdot (\mathcal{Z} + *\mathcal{Z})$ we have the equation*

$$Q \cap H = c_h(Q).$$

In particular, the intersection $Q \cap H$ is an integral polytope.

Proof. We take $k = \max\{|p_n - h| \mid p \in P\}$.

The inclusion $\subseteq$ is obvious since we have $c_h(q) = q$ for $q \in Q \cap H$.

For the reverse inclusion, let $q \in c_h(Q)$. Since $c_h(Q) = c_h(P)$, we can write $q = c_h(p)$ for some $p \in P$. It remains to show that $q \in Q$. By the choice of k, the elements $p + k \cdot \boldsymbol{z}$ and $p - k \cdot \boldsymbol{z}$ lie in different halves of Q with respect to H. But $c_h(p)$ is a convex combination of these two elements and lies therefore itself in Q. □

We will also need the following lemma.

Lemma 4.15 (Vertical gluing). *Let $H = \{x \in \mathbb{R}^n \mid x_n = h\}$ be a flat hyperplane. If $P, Q \subseteq \mathbb{R}^n$ are two (integral) polytopes such that*

$$P \cap H = c_h(P) = c_h(Q) = Q \cap H, \tag{4.6}$$

then the set $P_+ \cup Q_-$ is a (integral) polytope, where P_+ denote the upper half of P and Q_- denotes the lower half of Q with respect to H.

If additionally $h = 0$, i.e. $H = \boldsymbol{z}^\perp$, then we have:

(1) $(P + *P) \cap H = (P \cap H) + (*P \cap H)$*;*

(2) $(P + *P)_+ = P_+ + *(P_-)$;

(3) $(P + *P)_- = P_- + *(P_+)$.

Proof. Denote the vertex sets of P_+ resp. Q_- by $V(P_+)$ resp. $V(Q_-)$. We will show

$$P_+ \cup Q_- = \mathrm{hull}(V(P_+) \cup V(Q_-)),$$

where the inclusion $\subseteq$ is obvious.

For the reverse inclusion, it suffices to show that $P_+ \cup Q_-$ is convex. Let $p \in P_+$ and $q \in Q_-$, and take a convex combination $x = t \cdot p + (1-t) \cdot q$. Since P_+ and Q_- are convex, we may assume that $x \in H$ (and deal with other convex combinations inside P_+ and Q_- individually). We can also write $x = t \cdot c_h(p) + (1-t) \cdot c_h(q)$. Assumption (4.6) then implies that $x \in P \cap H = Q \cap H \subseteq P_+ \cup Q_-$. This finishes the proof of the first statement.

In the equalities (1), (2) (3), the inclusion $\supseteq$ is true irrespective of the assumption that $P \cap H = c_h(P)$.

To prove $\subseteq$ in (1), let $p \in P$, $q \in *P$ with $p_n + q_n = 0$. Then $p + q = c_0(p+q) = c_0(p) + c_0(q)$ which lies in $(P \cap H) + (*P \cap H)$ since by assumption $c_0(P) = P \cap H$ and thus $c_0(*P) = *c_0(P) = *(P \cap H) = *P \cap H$.

To prove $\subseteq$ in (2), let $p \in P$, $q \in *P$ with $p_n + q_n \geq 0$. If $p_n, q_n \geq 0$, then $p \in P_+$ and $q \in *(P_-)$ and we are done. If $p_n \geq 0$ and $q_n \leq 0$, then take $p' = p + q_n \cdot \boldsymbol{z}$ and $q' = q - q_n \cdot \boldsymbol{z} = c_0(q)$. We have $p' \in P_+$ since it is a convex combination of p and $c_0(p) \in P$, and we have $q' \in *(P_-)$ since $c_0(*P) = *P \cap H \subseteq *(P_-)$. Thus $p + q = p' + q' \in P_+ + *(P_-)$.

The third claim is proved similarly. $\square$

4.2 Symmetric and antisymmetric elements

We are already in a position to prove parts (1) and (2) of Theorem 4.1.

Proof of Theorem 4.1 (1). As we have noted before the inclusion $\ker(\mathrm{id} - *) \supseteq \mathrm{im}(\mathrm{id} + *)$ is obvious. We prove the reverse inclusion by induction on the rank of the free-abelian group $H \cong \mathbb{Z}^n$. If $n = 0$, then there is nothing to prove.

If $n \geq 1$, let $P - Q \in \ker(\mathrm{id} - *)$, i.e., $P + *Q = *P + Q$. If we prove that the symmetric polytope $P + *Q$ lies in $\mathrm{im}(\mathrm{id} + *)$, then this also holds for $P - Q$ since they differ by the summand $Q + *Q$.

So let $A = P + *Q$. By the vertical stretching technique of Lemma 4.14, we can assume that the intersection $A \cap \boldsymbol{z}^\perp$ is integral. The cutting relation of Lemma 4.9 then gives us an equation in $\mathcal{P}(\mathbb{Z}^n)$

$$A_+ + A_- = A + (A \cap \boldsymbol{z}^\perp)$$

for the two halves of A with respect to $\boldsymbol{z}^\perp$. Since A is symmetric, we have $A_- = *(A_+)$. Moreover $A \cap \boldsymbol{z}^\perp$ is a symmetric polytope in the polytope group $\mathcal{P}(\mathbb{Z}^{n-1})$, so by induction hypothesis there exists an element $x \in \mathcal{P}(\mathbb{Z}^{n-1})$ such that $A \cap \boldsymbol{z}^\perp = x + *x$. Hence we have

$$A = A_+ + *(A_+) - (x + *x) \in \mathrm{im}(\mathrm{id} + *)$$

and we are done. $\square$

Proof of Theorem 4.1 (2). We deal with $\mathcal{P}(H)$ first. Again the inclusion $\supseteq$ in the claim $\ker(\mathrm{id}+*) = \mathrm{im}(\mathrm{id}-*)$ is obvious. For the opposite inclusion, we proceed again by induction on the rank of $H \cong \mathbb{Z}^n$. If $n = 0$, then there is once more nothing to prove.

Let $P - Q \in \ker(\mathrm{id} + *)$, so
$$P + *P = Q + *Q. \tag{4.7}$$
After vertical stretching (see Lemma 4.14), we may assume
$$P \cap H = c_0(P) \quad \text{and} \quad Q \cap H = c_0(Q), \tag{4.8}$$
where here and henceforth we let $H = \boldsymbol{z}^\perp$. Then Lemma 4.15 (1) together with (4.7) implies
$$(P \cap H) + (*P \cap H) = (Q \cap H) + (*Q \cap H).$$
We may therefore apply the induction hypothesis to $(P \cap H) - (Q \cap H)$ and obtain an integral polytope R contained in H such that
$$(P \cap H) + *R = (Q \cap H) + R. \tag{4.9}$$
Clearly $P + *R - (Q + R) \in \ker(\mathrm{id} + *)$, and it suffices to prove that this element lies in $\mathrm{im}(\mathrm{id} - *)$. To ease notation, put $A = P + *R$ and $B = Q + R$. We see from (4.8), (4.9) and the fact that R lies in H the equalities
$$G := c_0(A) = A \cap H = (P \cap H) + *R = (Q \cap H) + R = B \cap H = c_0(B).$$
We are therefore in the situation of Lemma 4.15 so that the two halves A_+ and B_- (with respect to H) can be glued together to give a polytope $S := A_+ \cup B_-$. Moreover, Lemma 4.15 (3) gives
$$A_- + *(A_+) = (A + *A)_- = (B + *B)_- = B_- + *(B_+). \tag{4.10}$$

If we put $T = S - B$, then several applications of the cutting relation (see Lemma 4.9) yield
$$\begin{aligned}
T - *T &= S - *S - B + *B \\
&= (A_+ + B_- - G) - (*A_+ + *B_- - *G) - (B_+ + B_- - G) + (*B_+ + *B_- - *G) \\
&= A_+ + B_- + *B_+ - *A_+ - B_+ - B_- \\
&\overset{(4.10)}{=} A_+ + A_- + *A_+ - *A_+ - B_+ - B_- \\
&= (A_+ + A_- - G) - (B_+ + B_- - G) \\
&= A - B,
\end{aligned}$$
which completes the proof for $\mathcal{P}(H)$.

We deduce the statement for the quotient $\mathcal{P}_T(H)$ as follows. The map
$$\mathrm{sym}\colon \mathcal{P}_T(H) \to \mathcal{P}(H), \quad P - Q \mapsto P + *P - (Q + *Q)$$
is well-defined and fits into the commutative diagram
$$\begin{tikzcd}
\mathcal{P}(H) \arrow[r, "\mathrm{id}+*"] \arrow[d] & \mathcal{P}(H) \arrow[d] \\
\mathcal{P}_T(H) \arrow[r, "\mathrm{id}+*"] \arrow[ur, "\mathrm{sym}"] & \mathcal{P}_T(H),
\end{tikzcd}$$

where the vertical maps are the projections. Since $\operatorname{sym}(x)$ is a difference of two polytopes which are symmetric about the origin, $\operatorname{sym}(x)$ is a point if and only if it is zero. This implies

$$\ker\big(\mathrm{id} + *\colon \mathcal{P}_T(H) \to \mathcal{P}_T(H)\big) = \ker\big(\operatorname{sym}\colon \mathcal{P}_T(H) \to \mathcal{P}(H)\big).$$

Because of the commutative diagram above, any preimage of an element $x \in \ker\big(\operatorname{sym}\colon \mathcal{P}_T(H) \to \mathcal{P}(H)\big)$ in $\mathcal{P}(H)$ will lie in

$$\ker\big(\mathrm{id} + *\colon \mathcal{P}(H) \to \mathcal{P}(H)\big) = \operatorname{im}\big(\mathrm{id} - *\colon \mathcal{P}(H) \to \mathcal{P}(H)\big).$$

Thus

$$\ker\big(\operatorname{sym}\colon \mathcal{P}_T(H) \to \mathcal{P}(H)\big) \subseteq \operatorname{im}\big(\mathrm{id} - *\colon \mathcal{P}_T(H) \to \mathcal{P}_T(H)\big)$$

and the reverse inclusion is obvious. This finishes the proof of Theorem 4.1 (2). □

Theorem 4.1 (2) will be used in the proof of Proposition 5.19, where we put restrictions on the possible form of the L^2-torsion polytope of amenable groups. We still owe here the relevant connection to the norm maps of Section 3.7.4. Namely, it is proved in [FL16b, Lemma 3.8] that the map

$$\mathcal{P}(H) \to \prod_{\varphi \in \operatorname{Hom}(H,\mathbb{Z})} \mathcal{P}(\mathbb{Z}),\ x \mapsto (\mathcal{P}(\varphi)(x))_\varphi \tag{4.11}$$

is injective. This becomes false if we pass to the quotients in the target to obtain maps

$$\xi\colon \mathcal{P}(H) \to \prod_{\varphi \in \operatorname{Hom}(H,\mathbb{Z})} \mathcal{P}_T(\mathbb{Z}),\ x \mapsto (\mathcal{P}_T(\varphi)(x))_\varphi;$$
$$\xi_T\colon \mathcal{P}_T(H) \to \prod_{\varphi \in \operatorname{Hom}(H,\mathbb{Z})} \mathcal{P}_T(\mathbb{Z}),\ x \mapsto (\mathcal{P}_T(\varphi)(x))_\varphi,$$

rather we have the following.

Lemma 4.16. *Let H be a finitely generated free-abelian group. Then we have*

$$\begin{aligned}\ker(\xi) &= \ker\big(\mathfrak{N}\colon \mathcal{P}(H) \to \operatorname{Map}(\operatorname{Hom}(H,\mathbb{R}),\mathbb{R}))\big) \\ &= \ker\big(\mathrm{id} + *\colon \mathcal{P}(H) \to \mathcal{P}(H)\big)\end{aligned} \tag{4.12}$$

and

$$\begin{aligned}\ker(\xi_T) &= \ker\big(\mathfrak{N}\colon \mathcal{P}_T(H) \to \operatorname{Map}(\operatorname{Hom}(H,\mathbb{R}),\mathbb{R}))\big) \\ &= \ker\big(\mathrm{id} + *\colon \mathcal{P}_T(H) \to \mathcal{P}_T(H)\big)\end{aligned} \tag{4.13}$$

Proof. Recall that any $\varphi \in \operatorname{Hom}(H,\mathbb{Z})$ induces maps $\mathcal{P}_T(\varphi)\colon \mathcal{P}(H) \to \mathcal{P}_T(\mathbb{Z}) \cong \mathbb{Z}$ and $\mathcal{P}_T(\varphi)\colon \mathcal{P}_T(H) \to \mathcal{P}_T(\mathbb{Z}) \cong \mathbb{Z}$. Unraveling the definitions gives

$$\mathfrak{N}(P)(\varphi) = \|\varphi\|_P = \mathcal{P}_T(\varphi)(P)$$

for any polytope $P \in \mathfrak{P}(H)$ (regardless of whether we view P as a class in $\mathcal{P}(H)$ or $\mathcal{P}_T(H)$). This implies the first equality in (4.12) and (4.13).

It is shown in [FL16b, Section 3.7] that two integral polytopes $P, Q \in \mathfrak{P}(H)$ satisfy $P + *P = Q + *Q$ if and only if $\mathfrak{N}(P) = \mathfrak{N}(Q)$. This is precisely the second equality in both (4.12) and (4.13). □

4.3 Algebra vs. geometry II: Shadows

In this section, we introduce another set of techniques that will be needed in the proof of the remaining two parts of Theorem 4.1.

4.3.1 Shadow maps.

Definition 4.17 (Height and height maps)**.** Given a subset $S \subseteq \mathbb{R}^n$, we call

$$h(S) = \min\{x_n \mid x \in S\} \tag{4.14}$$

the *height* of S. Since $h(S+T) = h(S) + h(T)$, we obtain an induced homomorphism

$$h\colon \mathcal{P}(\mathbb{R}^n) \to \mathbb{R}, \;\; P \mapsto h(P)$$

called *height map.*

Recall that for $h \in \mathbb{R}$ we have defined the map

$$c_h\colon \mathbb{R}^n \to \mathbb{R}^n, \; (x_1, ..., x_n) \mapsto (x_1, ..., x_{n-1}, h).$$

Definition 4.18 (Shadows and shadow maps)**.** The *(lower) shadow* of a subset $S \subseteq \mathbb{R}^n$ is defined as

$$\mathrm{Sh}(S) = \mathrm{hull}(S \cup c_{h(S)}(S)).$$

In analogy with the previous definitions, we would like to define the *(lower) shadow map* as the group homomorphism

$$\mathrm{Sh}\colon \mathcal{P}(\mathbb{R}^n) \to \mathcal{P}(\mathbb{R}^n), \;\; P \mapsto \mathrm{Sh}(P).$$

The next lemma shows that this is indeed possible.

Lemma 4.19. *Let $P, Q \subseteq \mathbb{R}^n$ be two (integral) polytopes. Then $\mathrm{Sh}(P)$ is a (integral) polytope and we have*

$$\mathrm{Sh}(P+Q) = \mathrm{Sh}(P) + \mathrm{Sh}(Q).$$

Proof. Denote the vertex set of P by V. It is easy to check that $\mathrm{Sh}(P)$ is the convex hull of the finite set $V \cup c_{h(P)}(V)$. This shows that $\mathrm{Sh}(P)$ is indeed a polytope which is integral provided that P is integral.

In order to prove additivity, we recall that for any subsets $S, T \subseteq \mathbb{R}^n$ we have

$$\mathrm{hull}(S+T) = \mathrm{hull}(S) + \mathrm{hull}(T).$$

Hence it suffices to show that

$$\mathrm{hull}((P+Q) \cup c_{h(P+Q)}(P+Q)) = \mathrm{hull}((P \cup c_{h(P)}(P)) + (Q \cup c_{h(Q)}(Q))). \tag{4.15}$$

Since $h(P+Q) = h(P) + h(Q)$, the inclusion $\subseteq$ already follows from the inclusion of the underlying sets

$$(P+Q) \cup c_{h(P+Q)}(P+Q) \subseteq (P \cup c_{h(P)}(P)) + (Q \cup c_{h(Q)}(Q)). \tag{4.16}$$

For the inclusion $\supseteq$, let $p \in P \cup c_{h(P)}(P)$ and $q \in Q \cup c_{h(Q)}(Q)$, and we will show that $p+q$ is contained in the left-hand side of (4.15). This is obvious if $(p,q) \in P \times Q$ or $(p,q) \in c_{h(P)}(P) \times c_{h(Q)}(Q)$.

Let us now assume that $p \in P$ and $q \in c_{h(Q)}(Q)$. Write $q = c_{h(Q)}(q')$ for some $q' \in Q$. Then $p+q$ lies on the convex hull of the points $p+q'$ and $c_{h(P)}(p)+q = c_{h(P+Q)}(p+q')$. By inclusion (4.16), these latter points lie in

$$\mathrm{hull}((P \cup c_{h(P)}(P)) + (Q \cup c_{h(Q)}(Q)))$$

and hence so does $p+q$. The case $p \in c_{h(P)}(P)$ and $q \in Q$ is completely analogous. □

Remark 4.20. The choice of min instead of max in Definition 4.18 is arbitrary. Completely analogously, we may define an *upper height map*

$$h^+ \colon \mathcal{P}(\mathbb{R}^n) \to \mathbb{R}, \;\; P \mapsto \max\{x_n \mid x \in P\}$$

and an *upper shadow map*

$$\mathrm{Sh}^+ \colon \mathcal{P}(\mathbb{R}^n) \to \mathcal{P}(\mathbb{R}^n), \;\; P \mapsto \mathrm{hull}(P \cup c_{h^+(P)}(P)).$$

Then the equations

$$h^+(*P) = -h(P) \quad \text{and} \quad \mathrm{Sh}^+(*P) = *\mathrm{Sh}(P)$$

are easy to verify.

The shadow of a polytope allows us to increase the dimension in a simple controlled way. It will be our main tool in the construction of a basis for $\mathcal{P}(\mathbb{Z}^n)$ out of one for $\mathcal{P}(\mathbb{Z}^{n-1})$. It is crucial in this process that taking shadows preserves the algebraic structure, as shown by the previous lemma.

It is straightforward to see that both the face maps and shadow maps induce maps $\mathcal{P}(\mathbb{Z}^n)$, $\mathcal{P}_T(\mathbb{R}^n)$, and $\mathcal{P}_T(\mathbb{Z}^n)$.

4.3.2 The shadow partition.

Definition 4.21 (Types of codimension 1 faces)**.** Let $P \subseteq \mathbb{R}^n$ be a polytope with $\dim(P) = n$ and let $F \subseteq P$ be a codimension 1 face. Then there is up to positive scalar a unique $\varphi \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R})$ with $F_\varphi(P) = F$. The face F will be called *bottom, vertical*, or *top* face depending on whether $\varphi(\boldsymbol{z}) < 0$, $\varphi(\boldsymbol{z}) = 0$, or $\varphi(\boldsymbol{z}) > 0$.

A face F of P is a bottom (resp. vertical, top) face if and only if the face $*F$ of $*P$ is a top (resp. vertical, bottom) face.

Definition 4.22 (Grounded polytopes, pillars, and almost-pillars)**.** Let $P \subseteq \mathbb{R}^n$ be a polytope with $\dim(P) = n$.

(1) P is *grounded* if it has only one bottom face and this bottom face is flat. This unique bottom face will be referred to as the *ground*.

(2) P is a *pillar* if there is a flat polytope Q and a $k > 0$ such that $P = Q + k \cdot \mathcal{Z}$.

(3) P is an *almost-pillar* if it has a unique bottom face and a unique top face.

We record the following properties.

Lemma 4.23. *(1) Let $P \subseteq \mathbb{R}^n$ be a grounded polytope with $\dim(P) = n$ whose ground G is contained in the hyperplane $H = \{x \in \mathbb{R}^n \mid x_n = h\}$. Then the image of the* grounding map

$$g\colon P \to \mathbb{R}^n,\ (x_1, ..., x_n) \mapsto (x_1, ..., x_{n-1}, h)$$

is G.

(2) Every pillar is an almost-pillar.

(3) For any polytope $P \subseteq \mathbb{R}^n$ such that $\dim(\mathrm{Sh}(P)) = n$, $\mathrm{Sh}(P)$ is grounded.

(4) If $P \subseteq \mathbb{R}^n$ is contained in a hyperplane which is not flat and $\dim(\mathrm{Sh}(P)) = n$, then $\mathrm{Sh}(P)$ is a grounded almost-pillar.

Proof. The last three statements are obvious.

For part (1) only the inclusion $G \supseteq g(P)$ is non-trivial. Let $G = F_1, ..., F_m \subseteq P$ be the set of codimension 1 faces of P and let $\varphi_i \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R})$ be such that $F_i = F_{\varphi_i}(P)$. With $c_i = \varphi_i(F_i)$ we get

$$P = \{x \in \mathbb{R}^n \mid \varphi_i(x) \leq c_i \text{ for all } 1 \leq i \leq m\}.$$

Given $p \in P$, we verify these inequalities for $g(p)$, which implies $g(p) \in P \cap H = G$. Since $F_1 = G$ is flat, we have up to positive scalar $\varphi_1(x) = -x_n$ and $c_1 = -h$. The first inequality is therefore automatically satisfied for $x = g(p)$. For the remaining inequalities, we prove $\varphi_i(g(p)) \leq \varphi_i(p)$. Since $g(p) - p$ is a negative multiple of $\boldsymbol{z}$, it suffices to show $\varphi_i(\boldsymbol{z}) \geq 0$. But P is grounded, so none of these remaining faces is a bottom face and $\varphi_i(\boldsymbol{z}) \geq 0$ follows. □

We also record the following simple consequence needed later.

Lemma 4.24. *Let P be a polytope such that $*P$ is a grounded almost-pillar. Let F be the unique bottom face of P. Then there exists a pillar Q and a grounded almost-pillar S such that in $\mathcal{P}(\mathbb{R}^n)$ we have*

$$P = Q + F - S.$$

Proof. We leave the easy case where P is a pillar to the reader. If P is not a pillar, then F is not flat and so $S = \mathrm{Sh}(F)$ is a grounded almost-pillar by the previous lemma. The union $Q := S \cup P$ is a pillar (see also Fig. 4.2), and cutting Q along $F = P \cap S$ yields

$$Q = P + S - F$$

by the cutting relation (see Lemma 4.9). □

The following proposition will be one of the main tools for building a basis of the integral polytope group. This is because it tells us how grounded polytopes can be decomposed into smaller canonical pieces. We can then invoke the partition relation (see Proposition 4.12) to turn this decomposition into a group-theoretic relation.

Denote for a polytope P its set of faces by $\mathcal{F}(P)$.

Proposition 4.25 (Shadow partition)**.** *Let $P \subseteq \mathbb{R}^n$ be a grounded polytope. For every top face $F \subseteq P$, let*

$$P(F) := \mathrm{Sh}(F) + (h(F) - h(P)) \cdot *\mathcal{Z}. \tag{4.17}$$

Then the set

$$\mathcal{P} = \bigcup_{\substack{F \subseteq P \\ \textit{top face}}} \mathcal{F}(P(F))$$

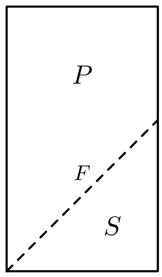

Figure 4.2: Taking the union of P and $S = \mathrm{Sh}(F)$ produces the pillar Q.

is a partition of P *(see also Figure 4.3) that will be referred to as the* shadow partition *of* P. *If* P *is integral, then the shadow partition contains only integral polytopes.*

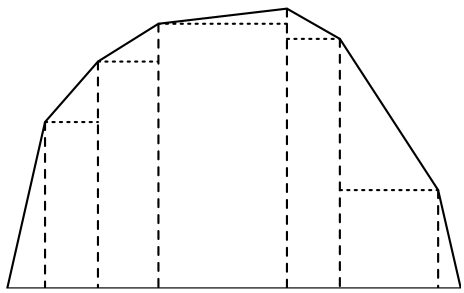

Figure 4.3: The dashed vertical lines indicate the shadow partition of a 2-dimensional grounded polytope. Within each $P(F)$ as in (4.17), the dotted horizontal line is the ground of $\mathrm{Sh}(F)$.

Proof. The second condition on a partition, namely that faces of elements in $\mathcal{P}$ are themselves in $\mathcal{P}$ (see Definition 4.10), is clear.

Next we prove $P = \bigcup_{Q \in \mathcal{P}} Q$. Let $g \colon P \to G$ be the grounding map of Lemma 4.23 (1). For any point $p \in P$ there exists a top face F and $f \in F$ such that $g(p) = g(f)$. Then f and $g(p)$ are contained in $P(F)$. Since p is a convex combination of f and $g(p)$ and $P(F)$ is convex, we see $p \in P(F)$. Hence $P \subseteq \bigcup_{Q \in \mathcal{P}} Q$. For the reverse inclusion, we observe

$$h(P(F)) = h(\mathrm{Sh}(F)) - h(F) + h(P) = h(P)$$

from which the inclusion $P(F) \subseteq P$ follows since P is grounded.

We finally need to show that for any $Q, Q' \in \mathcal{P}$ the intersection $Q \cap Q'$ is empty or a face in both of them. It suffices to do this for elements in $\mathcal{P}$ of dimension n. If $Q = P(F)$, $Q' = P(F')$, then $Q \cap Q' = \emptyset$ if $F \cap F' = \emptyset$. Otherwise $F \cap F'$ is a face in both F and F'. Hence

$$Q \cap Q' = \mathrm{Sh}(F \cap F') + (h(F \cap F') - h(P)) \cdot *\mathcal{Z}$$

is a face in Q and Q'.

If P is integral, then for all top faces F of P the shadow $\mathrm{Sh}(F)$ is integral, and $h(F)$ and $h(P)$ are integers. Thus $P(F)$ is integral. $\square$

4.4 A basis for the integral polytope group

In this section we prove Theorem 4.1 (3) in two steps. In a first subinduction step we explain how to construct the set $\mathcal{B}_n$ provided that the sets $\mathcal{B}_i$ for $1 \leq i \leq n-1$ are already available. This will in the second step be used in order to construct a basis for $\mathcal{P}_T(H)$ by induction on the rank of H.

4.4.1 The subinduction step: Increasing the dimension of the polytopes. In this section we construct an explicit basis for $\mathcal{P}(\mathbb{Z}^n)$, built from bases of the various subgroups of $\mathbb{Z}^n$. Roughly speaking, we throw together all these bases and their images under the shadow map.

Recall that a subgroup $G \subseteq H$ is *pure* if there is a linear subspace $U \subseteq H \otimes_{\mathbb{Z}} \mathbb{R}$ such that $G = H \cap U$, and that we denote by $\mathcal{P}_T^m(G)$ the subgroup of $\mathcal{P}_T(G)$ generated by the polytopes of dimension at most m.

Proposition 4.26 (Adding the last dimension)**.** *Assume that there are sets $\mathcal{B}_1 \subseteq \mathcal{B}_2 \subseteq \ldots \subseteq \mathcal{B}_{n-1} \subseteq \mathcal{P}_T(\mathbb{Z}^n)$ such that*

(1) $\mathcal{B}_m \cap \mathcal{P}_T(G)$ is a basis for $\mathcal{P}_T^m(G)$ for every pure subgroup $G \subseteq \mathbb{Z}^n$ and $1 \leq m \leq n-1$;

(2) $\mathcal{B}_m \setminus \mathcal{B}_{m-1}$ contains only polytopes of dimension m.

Then there is a set $\mathcal{C}_n \subseteq \mathcal{P}_T(\mathbb{Z}^n)$ containing only polytopes of dimension n such that $\mathcal{B}_{n-1} \cup \mathcal{C}_n$ is a basis for $\mathcal{P}_T(\mathbb{Z}^n)$.

Proof. Let

$$\mathcal{C}_n = \{\, \mathrm{Sh}(B) \mid B \in \mathcal{B}_{n-1},\ \mathrm{Sh}(B) \text{ is } n\text{-dimensional} \,\}.$$

We first prove that $\mathcal{B}_n := \mathcal{B}_{n-1} \cup \mathcal{C}_n$ is a generating set for $\mathcal{P}_T(\mathbb{Z}^n)$. Let $\langle S \rangle \subseteq \mathcal{P}_T(\mathbb{Z}^n)$ denote the subgroup generated by a subset S.

Let $P \subseteq \mathbb{R}^n$ be an integral polytope. By condition (1) all 1-dimensional polytopes of length 1 are contained in $\langle \mathcal{B}_1 \rangle$. This implies that the unit n-cube lies in $\langle \mathcal{B}_1 \rangle$ and hence in $\langle \mathcal{B}_n \rangle$. After possibly adding the unit n-cube, we may therefore assume without loss of generality that P is n-dimensional.

Note that in $\mathcal{P}_T(\mathbb{Z}^n)$ we have $\mathcal{Z} = *\mathcal{Z}$. By vertical stretching (see Lemma 4.14), there exists $k \in \mathbb{Z}$ such that $P + k \cdot \mathcal{Z}$ intersects $\boldsymbol{z}^\perp$ in an integral polytope P' and cutting along this intersection produces a grounded half P_+ and a half P_- such that $*P_-$ is grounded. By the cutting relation (see Lemma 4.9), we have in $\mathcal{P}_T(\mathbb{Z}^n)$

$$P = P_+ + P_- - P' - k \cdot \mathcal{Z}.$$

Now P' and $\mathcal{Z}$ lie in $\mathcal{P}^{n-1}(\mathbb{Z}^n) = \langle \mathcal{B}_{n-1} \rangle$. Hence it suffices to show that P_+ and P_- lie in $\langle \mathcal{B}_n \rangle$.

First we take care of P_+. This is a grounded polytope with ground P'. Let $\mathcal{P}$ be the shadow partition of P_+ of Lemma 4.25. All polytopes in $\mathcal{P}$ of dimension at most $n-1$ lie in $\langle \mathcal{B}_{n-1} \rangle$. The remaining elements of $\mathcal{P}$ are of the form

$$P(F) = \mathrm{Sh}(F) + (h(F) - h(P)) \cdot \mathcal{Z}.$$

where $F \subseteq P_+$ is a top face. If we show that the polytopes $P(F)$ lie in $\langle \mathcal{B}_n \rangle$, then the partition relation (see Proposition 4.12) implies that $P_+ \in \langle \mathcal{B}_n \rangle$.

By assumption there are $B_i \in \mathcal{B}_{n-1}$ and $\lambda_i \in \mathbb{Z}$ ($1 \leq i \leq k$) such that $F = \sum_{i=1}^k \lambda_i \cdot B_i$. By Lemma 4.19 the shadow map is a group homomorphism, so we have

$$\operatorname{Sh}(F) = \sum_{i=1}^{k} \lambda_i \cdot \operatorname{Sh}(B_i). \tag{4.18}$$

If $\operatorname{Sh}(B_i)$ is n-dimensional, then $\operatorname{Sh}(B_i) \in \mathcal{C}_n \subseteq \mathcal{B}_n$. Otherwise $\operatorname{Sh}(B_i) \in \langle \mathcal{B}_{n-1} \rangle$. Thus from (4.18) we see that $\operatorname{Sh}(F)$ and also $P(F)$ lie in $\langle \mathcal{B}_n \rangle$.

In order to deal with P_-, it suffices to show that $\langle \mathcal{B}_n \rangle$ is closed under the involution. Let $B \in \mathcal{B}_n$. Again by assumption, there is nothing to prove if $B \in \mathcal{B}_{n-1}$, so let $B \in \mathcal{C}_n$. Then B is a grounded almost-pillar by Lemma 4.23. Lemma 4.24 applies to produce a pillar Q and a grounded almost-pillar S such that

$$*B = Q + *F - S,$$

where F is the top face of B. We have $Q, *F \in \langle \mathcal{B}_{n-1} \rangle$. Since S is a grounded polytope, we may proceed with it as with P_+ to verify $S \in \langle \mathcal{B}_n \rangle$, and so $*B \in \langle \mathcal{B}_n \rangle$. This completes the proof that $\langle \mathcal{B}_n \rangle = \mathcal{P}_T(\mathbb{Z}^n)$.

Now we show that $\mathcal{B}_n$ is linearly independent. Let us assume that there are pairwise distinct elements $P_j^i \in \mathcal{B}_i \setminus \mathcal{B}_{i-1}$ and $\lambda_j^i \in \mathbb{Z}$ for $1 \leq i \leq n$ and $1 \leq j \leq s_i$ such that

$$\sum_{i=1}^{n} \sum_{j=1}^{s_i} \lambda_j^i \cdot P_j^i = 0. \tag{4.19}$$

Since $\mathcal{B}_{n-1}$ is linearly independent, it suffices to show that $\lambda_k^n = 0$ for all $1 \leq k \leq s_n$. For this we first need an auxiliary step.

Claim: If $P_k^{n-1} \in \mathcal{B}_{n-1}$ such that $\operatorname{Sh}(P_k^{n-1})$ is n-dimensional, then $\lambda_k^{n-1} = 0$.

Let $H \subseteq \mathbb{R}^n$ be the hyperplane containing P_k^{n-1} and consider the pure subgroup $G = H \cap \mathbb{Z}^n$. Since $\operatorname{Sh}(P_k^{n-1})$ is n-dimensional, i.e., P_k^{n-1} is not flat, there is $\varphi \in \operatorname{Hom}(\mathbb{R}^n, \mathbb{R})$ and $c \in \mathbb{R}$ such that $H = \{x \in \mathbb{R}^n \mid \varphi(x) = c\}$ and $\varphi(\boldsymbol{z}) < 0$. Applying the face map in φ-direction to (4.19) yields the equation

$$\sum_{i=1}^{n} \sum_{j=1}^{s_i} \lambda_j^i \cdot F_\varphi(P_j^i) = 0 \tag{4.20}$$

in $\mathcal{P}_T(G)$. We claim that $F_\varphi(P_j^i)$ has dimension $n-1$ if and only if $i = n-1$ and $P_j^i \in \mathcal{P}_T(G)$. The 'if'-part is clear. The 'only if'-part is obvious except for the full-dimensional P_j^n, $1 \leq j \leq s_n$. By Lemma 4.23 (3) P_j^n is grounded with ground A, say. Since $\varphi(\boldsymbol{z}) < 0$, we have $F_\varphi(P_j^n) = F_\varphi(A)$ and this is a proper face of A because A is flat. Thus $F_\varphi(P_j^n)$ is at most $(n-2)$-dimensional.

This means that (4.20) breaks up into a sum x of $(n-1)$-dimensional elements in $\mathcal{B}_{n-1} \cap \mathcal{P}_T(G)$ and a sum y in $\mathcal{P}_T^{n-2}(G)$. Since the basis $\mathcal{B}_{n-1} \cap \mathcal{P}_T(G)$ of $\mathcal{P}_T^{n-1}(G)$ extends the basis $\mathcal{B}_{n-2} \cap \mathcal{P}_T(G)$ of $\mathcal{P}_T^{n-2}(G)$, this can only happen if $x = y = 0$. Hence $\lambda_j^{n-1} = 0$ for all j such that $P_j^{n-1} \in \mathcal{P}_T(G)$ which includes in particular $j = k$. This proves the claim, which brings us to the original goal.

Claim: For all $1 \leq k \leq s_n$ we have $\lambda_k^n = 0$.

Write $P_k^n = \mathrm{Sh}(B)$ for some $B \in \mathcal{B}_{n-1}$, and let H denote the hyperplane containing B. Take a covector ψ with $H = \{x \in \mathbb{R}^n \mid \psi(x) = c\}$ and $\psi(\boldsymbol{z}) > 0$. Then

$$F_\psi(P_k^n) = F_\psi(B) = B,$$

but the previous claim ensures that $\lambda_k^{n-1} = 0$ if $P_k^{n-1} = B$. Thus the summands in

$$\sum_{i=1}^{n} \sum_{j=1}^{s_i} \lambda_j^i \cdot F_\psi(P_j^i) = 0$$

are distinct elements of $\mathcal{B}_{n-1} \cap \mathcal{P}_T(G)$ and elements lying in $\mathcal{P}_T^{n-2}(G)$, where $G = \mathbb{Z}^n \cap H$. By the same argument as in the previous claim we deduce $\lambda_k^n = 0$. □

4.4.2 The induction step: Increasing the rank. Now we can recall and prove the statement of Theorem 4.1 (3).

Theorem 4.27 (Basis for the integral polytope group)**.** *There are sets $\mathcal{B}_1 \subseteq \mathcal{B}_2 \subseteq ... \subseteq \mathcal{B}_n \subseteq \mathcal{P}_T(\mathbb{Z}^n)$ such that:*

(1) $\mathcal{B}_m \cap \mathcal{P}_T(G)$ is a basis for $\mathcal{P}_T^m(G)$ for every pure subgroup $G \subseteq \mathbb{Z}^n$ and $1 \leq m \leq n$;

(2) $\mathcal{B}_m \setminus \mathcal{B}_{m-1}$ contains only polytopes of dimension m.

In particular, $\mathcal{B}_n$ is a basis for $\mathcal{P}_T(\mathbb{Z}^n)$.

Moreover, if $\mathcal{A} \subseteq \mathbb{Z}^n$ denotes a basis of $\mathbb{Z}^n$ and $\mathcal{B}_n' \subseteq \mathcal{P}(\mathbb{Z}^n)$ is a set of representatives for $\mathcal{B}_n \subseteq \mathcal{P}_T(\mathbb{Z}^n)$, then $\mathcal{A} \cup \mathcal{B}_n'$ is a basis for $\mathcal{P}(\mathbb{Z}^n)$.

Proof. We use induction on m. For the base case $m = 1$ we let $\mathcal{B}_1$ be the set of (translation classes of) 1-dimensional polytopes in $\mathcal{P}_T(\mathbb{Z}^n)$ which are not a proper multiple of another (translation class of a) 1-dimensional polytope in $\mathcal{P}_T(\mathbb{Z}^n)$. Clearly, $\mathcal{B}_1 \cap \mathcal{P}_T(G)$ is a generating set for $\mathcal{P}_T^1(G)$ provided that $G \subseteq \mathbb{Z}^n$ is a *pure* subgroup. On the other hand, using the additivity of the face map it is easy to make the following observation: Take any pairwise distinct $P_1, ..., P_k \in \mathcal{B}_1$ and some $\lambda_1, ..., \lambda_k \in \mathbb{Z}$. Given any $Q \in \mathcal{B}_1$ and $\mu \in \mathbb{Z}$, the polytope $\sum_{i=1}^k \lambda_i \cdot P_i$ possesses $\mu \cdot Q$ as a 1-dimensional face (up to translation) if and only if there exists an index j such that $P_j = Q$ and $\lambda_j = \mu$. This readily implies that $\mathcal{B}_1$ is linearly independent.

For the induction step from $m-1$ to m, we suppose that the sets $\mathcal{B}_1 \subseteq ... \subseteq \mathcal{B}_{m-1}$ have been constructed. Let

$$\mathcal{U}_m = \{U \subseteq \mathbb{Z}^n \mid U \text{ is a pure subgroup of rank } m\}.$$

For any $U \in \mathcal{U}_m$, Proposition 4.26 allows us to extend $\mathcal{B}_{m-1} \cap \mathcal{P}_T(U)$ to a basis $\mathcal{B}_m^U$ of $\mathcal{P}_T(U)$. Now we let

$$\mathcal{B}_m = \bigcup_{U \in \mathcal{U}_m} \mathcal{B}_m^U.$$

It is clear that $\mathcal{B}_m \setminus \mathcal{B}_{m-1}$ contains only polytopes of dimension m. We need to verify that $\mathcal{B}_m \cap \mathcal{P}_T(G)$ is a basis for $\mathcal{P}_T^m(G)$ for every pure subgroup $G \subseteq \mathbb{Z}^n$. For $\mathrm{rank}(G) \leq m-1$ this follows from the induction hypothesis since $\mathcal{B}_m \cap \mathcal{P}_T(G) = \mathcal{B}_{m-1} \cap \mathcal{P}_T(G)$ and $\mathcal{P}_T^m(G) = \mathcal{P}_T^{m-1}(G)$. If $\mathrm{rank}(G) = m$, then $G \in \mathcal{U}_m$, and $\mathcal{B}_m \cap \mathcal{P}_T(G) = \mathcal{B}_m^G$ is by construction a basis of $\mathcal{P}_T^m(G) = \mathcal{P}_T(G)$. In the last case that $\mathrm{rank}(G) > m$ we consider the set

$$\mathcal{U}_m^G = \{U \subseteq G \mid U \text{ is a pure subgroup of rank } m\} \subseteq \mathcal{U}_m.$$

Then $\mathcal{P}_T^m(G)$ is generated by the union of all $\mathcal{P}_T(U)$ with $U \in \mathcal{U}_m^G$. On the other hand, each such $\mathcal{P}_T(U)$ is generated by $\mathcal{B}_m^U \subseteq \mathcal{B}_m$. This shows that $\mathcal{B}_m \cap \mathcal{P}_T(G)$ generates $\mathcal{P}_T^m(G)$. It remains to prove that $\mathcal{B}_m$ is linearly independent. This is in very much the same spirit as the corresponding proof of Proposition 4.26.

Let $P_i \in \mathcal{B}_m$ be pairwise distinct elements and $\lambda_i \in \mathbb{Z}$ $(1 \leq i \leq k)$ such that

$$\sum_{i=1}^{k} \lambda_i \cdot P_i = 0.$$

Again it suffices to prove $\lambda_i = 0$ for all i such that $P_i \in \mathcal{B}_m \setminus \mathcal{B}_{m-1}$ since $\mathcal{B}_{m-1}$ is linearly independent by induction hypothesis. For a fixed $P_j \in \mathcal{B}_m \setminus \mathcal{B}_{m-1}$, let $U \in \mathcal{U}_m$ be such that $P_j \in \mathcal{P}_T(U)$. Let $H \subseteq \mathbb{R}^n$ be a hyperplane such that

$$U \cap U' = H \cap U' \tag{4.21}$$

for every $U' \in \mathcal{U}_m$ such that there exists an index $1 \leq i \leq k$ with $P_i \in \mathcal{P}_T(U') \cap (\mathcal{B}_m \setminus \mathcal{B}_{m-1})$. Pick $\varphi \in \operatorname{Hom}(\mathbb{R}^n, \mathbb{R})$ with $H = \ker \varphi$. Applying the face map induces the equation

$$\sum_{i=1}^{k} \lambda_i \cdot F_\varphi(P_i) = 0.$$

But because of (4.21), $F_\varphi(P_i)$ is m-dimensional if and only if P_i is m-dimensional and $P_i \in \mathcal{P}_T(U)$, or in other words $P_i \in \mathcal{B}_m^U \setminus \mathcal{B}_{m-1}$, and the remaining summands lie in $\mathcal{P}_T^{m-1}(U)$. Since $\mathcal{B}_m^U$ extends the basis $\mathcal{B}_{m-1} \cap \mathcal{P}_T(U)$ of $\mathcal{P}_T^{m-1}(U)$, we must have $\lambda_i = 0$ for all i such that $F_\varphi(P_i)$ is m-dimensional. In particular $\lambda_j = 0$ and the proof is complete.

The 'moreover'-part follows directly from the split exactness of the sequence

$$0 \to H \to \mathcal{P}(H) \to \mathcal{P}_T(H) \to 0$$

which was first proved in [FL16b, Lemma 3.8 (2)], but follows now also from the fact that $\mathcal{P}_T(H)$ is free-abelian. $\square$

Remark 4.28. The above construction applies also to produce a basis for the real vector space $\mathcal{P}_T(\mathbb{R}^n)$. The only wording that needs to be replaced is *pure subgroup* with *linear subspace.*

4.5 The involution as face Euler characteristic

In this last section we will prove Theorem 4.1 (4). First we recall the main player and the statement.

Definition 4.29 (Face Euler characteristic)**.** Given a polytope $P \subseteq \mathbb{R}^n$ denote by $\mathcal{F}(P)$ the set of faces of P, including P itself. We call

$$\chi_{\mathcal{F}}(P) = \sum_{F \in \mathcal{F}(P)} (-1)^{\dim(F)} \cdot F \in \mathcal{P}(\mathbb{R}^n)$$

the *face Euler characteristic* of P.

Theorem 4.30 (Involution as face Euler characteristic)**.** *For any polytope $P \subseteq \mathbb{R}^n$ we have in $\mathcal{P}(\mathbb{R}^n)$*

$$*P = -\chi_{\mathcal{F}}(P).$$

The relation of involutions and Euler-type relations has a long history in polytope theory [Sal68, McM77, McM89, KP92, Kho97, Kla99], just to mention a few. In fact, Theorem 4.30 is a corollary of [McM89, Theorem 2] by virtue of the isomorphism given in [McM89, Theorem 9]. However, we thought it worthwhile giving a proof of Theorem 4.30 by completely elementary geometric means.

The proof of Theorem 4.1 (3) provides us with the following strategy: We show a partition relation for face Euler characteristics (see Proposition 4.32), prove the statement for shadows (see Lemma 4.33), and combine these two facts to obtain the claim for any grounded polytope. The general case follows easily from this special case.

Lemma 4.31 (Cutting relation for face Euler characteristics)**.** *Let $P \subseteq \mathbb{R}^n$ be a polytope and $H \subseteq \mathbb{R}^n$ be a hyperplane. Denote the two halves of P with respect to H by P_1 and P_2. Then*

$$\chi_{\mathcal{F}}(P) + \chi_{\mathcal{F}}(P \cap H) = \chi_{\mathcal{F}}(P_1) + \chi_{\mathcal{F}}(P_2). \tag{4.22}$$

If Theorem 4.30 holds for three polytopes among $P, P_1, P_2, P \cap H$, then also for the fourth.

Proof. We distinguish four cases as to how H cuts a face $F \in \mathcal{F}(P)$:

(1) If $F \cap H = \emptyset$, then F is a face of one of the P_i and contributes $(-1)^{\dim(F)} \cdot F$ to both sides of (4.22).

(2) If $F \cap H = F$, then F is a face of P_1, P_2 and $P \cap H$, and it contributes $(-1)^{\dim(F)} \cdot 2F$ to both sides.

(3) If $F \cap H \neq F$ and $F \cap H$ is a face of F, then F is a face in exactly one P_i and contributes $(-1)^{\dim(F)} \cdot F$ to both sides. (Note that $F \cap H$ will itself then fall into case (2).)

(4) Otherwise, the cutting relation (see Lemma 4.9) yields

$$F + (F \cap H) = F_1 + F_2 \tag{4.23}$$

for the two halves of F with respect to H. Then $F \cap H$ is also a face in P_1, P_2 and $P \cap H$ which is not covered by the other cases. This means that F contributes $(-1)^{\dim(F)} \cdot (F - (F \cap H))$ to the left-hand side and $(-1)^{\dim(F)} \cdot (F_1 + F_2 - 2 \cdot (F \cap H))$ to the right-hand side. These two values coincide by (4.23).

Every summand of the face Euler characteristics has now been counted exactly once, so that the desired equation (4.22) follows.

The last statement follows by comparing this with the involution of the cutting relation $*P + *(P \cap H) = *P_1 + *P_2$ (see Lemma 4.9). ☐

Proposition 4.32 (Partition relation for face Euler characteristics)**.** *Let $P \subseteq \mathbb{R}^n$ be a polytope and $\mathcal{P}$ be a partition of P. Then we have in $\mathcal{P}(\mathbb{R}^n)$ the equation*

$$\chi_{\mathcal{F}}(P) = \sum_{Q \in \mathcal{P}^\partial} (-1)^{\operatorname{codim}(Q \subseteq P)} \cdot \chi_{\mathcal{F}}(Q).$$

Proof. This follows from Lemma 4.31 in exactly the same way as Proposition 4.12 (the partition relation) follows from Lemma 4.9 (the cutting relation). ☐

Next we show that Theorem 4.30 is true for the pieces in a shadow partition (see Proposition 4.25).

Lemma 4.33 (Face Euler characteristics of shadows)**.** *Assume that Theorem 4.30 is known for polytopes of dimension at most $n-1$.*

(1) If $P \subseteq \mathbb{R}^n$ is a polytope of dimension at most $n-1$, then we have

$$*\mathrm{Sh}(P) = -\chi_{\mathcal{F}}(\mathrm{Sh}(P));$$

(2) If $P \subseteq \mathbb{R}^n$ is a polytope of dimension at most $n-1$, then we have

$$*(P + \mathcal{Z}) = -\chi_{\mathcal{F}}(P + \mathcal{Z});$$

(3) Let $P \subseteq \mathbb{R}^n$ be a polytope. Then Theorem 4.30 holds for P if and only if it holds for $P + \mathcal{Z}$ (or equivalently $P + *\mathcal{Z}$).

Proof. (1) We may assume that $\mathrm{Sh}(P)$ is of dimension n. Recall that $\mathrm{Sh}(P)$ is grounded by Lemma 4.23. Let $G \subseteq \mathrm{Sh}(P)$ be its ground and $g : \mathrm{Sh}(P) \to G$ be the grounding map. Every face $F \subseteq P$ such that $F \neq g(F)$ induces the following faces of $\mathrm{Sh}(P)$:

(i) F itself;

(ii) $g(F)$, which has the same dimension as F;

(iii) The intermediate face $\mathrm{Sh}(F) + (h(F) - h(P)) \cdot *\mathcal{Z}$, which has dimension $\dim(F) + 1$.

Next we argue that we can discard the case that $F = g(F)$. Namely, then F only produces the single face F in $\mathrm{Sh}(P)$. However, if we were to count the three polytopes in (i) – (iii) together, then

$$F + g(F) - (\mathrm{Sh}(F) + (h(F) - h(P)) \cdot *\mathcal{Z}) = F + F - F = F.$$

Hence we may as well take the three summands above instead of F in the following calculations. In this way we avoid a case analysis and notational overload.

The subsets of $\mathcal{F}(\mathrm{Sh}(P))$ corresponding to faces of type (i) and (ii) are $\mathcal{F}(P)$ and $\mathcal{F}(G)$, respectively. By assumption we have $*P = -\chi_{\mathcal{F}}(P)$ and $*G = -\chi_{\mathcal{F}}(G)$ since these are polytopes of dimension $n-1$. Now we calculate using the additivity of the shadow map (see Lemma 4.19)

$$\begin{aligned}
&\chi_{\mathcal{F}}(\mathrm{Sh}(P)) \\
={}& \sum_{F \in \mathcal{F}(\mathrm{Sh}(P))} (-1)^{\dim(F)} \cdot F \\
={}& \sum_{F \in \mathcal{F}(P)} (-1)^{\dim(F)} \cdot F + \sum_{F \in \mathcal{F}(G)} (-1)^{\dim(F)} \cdot F \\
&+ \sum_{F \in \mathcal{F}(P)} (-1)^{\dim(F)+1} \cdot (\mathrm{Sh}(F) + (h(F) - h(P)) \cdot *\mathcal{Z}) \\
={}& \chi_{\mathcal{F}}(P) + \chi_{\mathcal{F}}(G) - \mathrm{Sh}(\chi_{\mathcal{F}}(P)) - \sum_{F \in \mathcal{F}(P)} (-1)^{\dim(F)} \cdot (h(F) - h(P)) \cdot *\mathcal{Z} \\
={}& -*P - *G + \mathrm{Sh}(*P) - h \cdot *\mathcal{Z},
\end{aligned} \tag{4.24}$$

where we put

$$h = \sum_{F \in \mathcal{F}(P)} (-1)^{\dim(F)} \cdot (h(F) - h(P)).$$

Note that since the faces of P determine a cell structure on P, we have

$$\sum_{F\in\mathcal{F}(P)} (-1)^{\dim(F)} = \chi(P) = 1$$

and hence

$$\begin{aligned} h &= \sum_{F\in\mathcal{F}(P)} (-1)^{\dim(F)} \cdot (h(F) - h(P)) \\ &= -h(P) \cdot \sum_{F\in\mathcal{F}(P)} (-1)^{\dim(F)} + \sum_{F\in\mathcal{F}(P)} (-1)^{\dim(F)} \cdot h(F) \\ &= -h(P) + h(\chi_{\mathcal{F}}(P)) \\ &= -h(P) - h(*P). \end{aligned} \tag{4.25}$$

Recall from Remark 4.20 that we may define a height and shadow map in the opposite direction

$$h^+ \colon \mathcal{P}(\mathbb{R}^n) \to \mathbb{R}^n \quad \text{and} \quad \mathrm{Sh}^+ \colon \mathcal{P}(\mathbb{R}^n) \to \mathcal{P}(\mathbb{R}^n)$$

satisfying the equations

$$h^+(*P) = -h(P) \tag{4.26}$$

and

$$\mathrm{Sh}^+(*P) = *\mathrm{Sh}(P). \tag{4.27}$$

Now consider the pillar $\mathrm{Sh}(*P) \cup \mathrm{Sh}^+(*P) = *G + (h^+(P) - h(P)) \cdot *\mathcal{Z}$. By (4.25), (4.26) and (4.27) we have

$$\mathrm{Sh}(*P) \cup *\mathrm{Sh}(P) = \mathrm{Sh}(*P) \cup \mathrm{Sh}^+(*P) = *G + (h^+(P) - h(P)) \cdot *\mathcal{Z} = *G + h \cdot *\mathcal{Z}.$$

By the cutting relation (see Lemma 4.9), cutting this pillar along $*P$ gives

$$\mathrm{Sh}(*P) + *\mathrm{Sh}(P) = *G + h \cdot *\mathcal{Z} + *P.$$

We conclude by comparing this with equation (4.24)

$$*\mathrm{Sh}(P) = *G + h \cdot *\mathcal{Z} + *P - \mathrm{Sh}(*P) = -\chi_{\mathcal{F}}(\mathrm{Sh}(P)).$$

(2) This part is similar to the first one, but easier. The face analysis, which we leave to the reader, yields in this case

$$\begin{aligned} \chi_{\mathcal{F}}(P + \mathcal{Z}) &= \chi_{\mathcal{F}}(P) + \chi_{\mathcal{F}}(P + \boldsymbol{z}) + \sum_{F\in\mathcal{F}(P)} (-1)^{\dim(F)+1} \cdot (F + \mathcal{Z}) \\ &= 2 \cdot \chi_{\mathcal{F}}(P) + \chi(P) \cdot \boldsymbol{z} - \chi_{\mathcal{F}}(P) - \chi(P) \cdot \mathcal{Z} \\ &= \chi_{\mathcal{F}}(P) + \boldsymbol{z} - \mathcal{Z} \\ &= - * P - *\mathcal{Z}. \end{aligned}$$

(3) Assume that $\dim(P) = n$. There is a partition of $P + \mathcal{Z}$ that has the pieces P and $F + \mathcal{Z}$ for all top faces $F \subseteq P$, see Fig. 4.4.

By part (2), Theorem 4.30 holds for all elements of this partition except possibly for P. Thus comparing the partition relation of polytopes (see Proposition 4.12)

$$*(P + \mathcal{Z}) = *P + \sum_{Q\in\mathcal{P}^{\partial}, Q\neq P} (-1)^{\mathrm{codim}(Q\subseteq P)} \cdot *Q$$

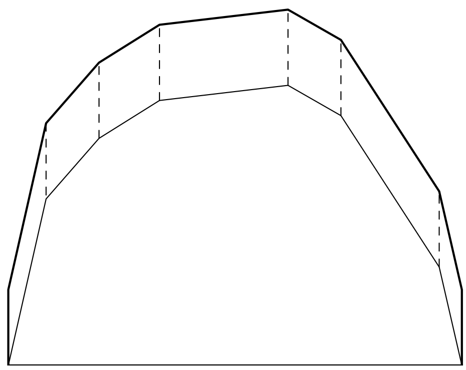

Figure 4.4: A partition of $P + \mathcal{Z}$ with pieces P and $F + \mathcal{Z}$ for all top faces $F \subseteq P$.

with the partition relation for face Euler characteristics (see Proposition 4.32)

$$\chi_{\mathcal{F}}(P + \mathcal{Z}) = \chi_{\mathcal{F}}(P) + \sum_{Q \in \mathcal{P}^{\partial}, Q \neq P} (-1)^{\operatorname{codim}(Q \subseteq P)} \cdot \chi_{\mathcal{F}}(Q)$$

implies

$$*P = -\chi_{\mathcal{F}}(P) \quad \text{if and only if} \quad *(P + \mathcal{Z}) = -\chi_{\mathcal{F}}(P + \mathcal{Z}).$$

□

For completeness we record the following trivial observation.

Lemma 4.34. *For any polytope Q we have $\mathcal{F}(*Q) = *\mathcal{F}(Q)$ and Theorem 4.30 is true for Q if and only if it is true for $*Q$.*

We are now ready to prove the main theorem of this section.

Proof of Theorem 4.30. Let P be an arbitrary polytope. We prove the claim by induction on the dimension of P. If $\dim(P) = 0$, there is nothing to prove.

Let now $\dim(P) = n$. By vertical stretching (Lemma 4.14) and Lemma 4.33 (3) we may assume that P can be cut along an integral codimension 1 polytope into a grounded half P_+ and a half P_- such that $*P_-$ is grounded. By the cutting relation for face Euler characteristics (see Lemma 4.31), Lemma 4.34, and the induction hypothesis, it suffices to prove the claim for grounded polytopes.

Let P be grounded and consider the shadow partition $\mathcal{P}$ of P (see Proposition 4.25). Theorem 4.30 is true for all elements in $\mathcal{P}$ by the induction hypothesis and Lemma 4.33 (1) and (3). The two partition relations of Proposition 4.12 and Proposition 4.32 then imply it for P. □

5 The L^2-torsion Polytope of Amenable Groups

This chapter is dedicated to partial solutions of the following conjecture proposed by Friedl-Lück-Tillmann [FLT16, Conjecture 6.4].

Conjecture 5.1 (Vanishing of the L^2-torsion polytope of amenable groups)**.** *Let $G \neq \mathbb{Z}$ be an amenable group satisfying the Atiyah Conjecture. Suppose that G is of type F and that* $\mathrm{Wh}(G) = 0$*. Then we have*

$$P(G) = 0.$$

Recall that the L^2-torsion polytope of groups was introduced in Definition 3.49. We first comment on the conditions appearing above and then on indications that Conjecture 5.1 might be true.

In comparison with the original formulation, we replaced *not virtually* $\mathbb{Z}$ with *not isomorphic to* $\mathbb{Z}$ since any torsion-free virtually $\mathbb{Z}$ group is in fact isomorphic to $\mathbb{Z}$. Since $P(\mathbb{Z})$ is the negative of an interval of length one, infinite cyclic groups need to be excluded from the statement of the conjecture. Infinite amenable groups are L^2-acyclic (see [Lüc02, Corollary 6.75]) which is why we can drop this assumption in the conjecture above. Recall from Lemma 3.23 that any torsion-free elementary amenable group satisfies the Atiyah Conjecture. Among these, all torsion-free virtually solvable groups are known to have trivial Whitehead group since they satisfy the K-theoretic Farrell-Jones Conjecture, as proved by Wegner [Weg15].

In the context of L^2-invariants and related fields, infinite amenable groups stand out as a class of groups satisfying strong vanishing results. An infinite amenable G has

- vanishing L^2-Betti numbers, see [Lüc02, Corollary 6.75], or [Lüc02, Theorem 7.2 (1) and (2)] for a strengthening of this statement;
- vanishing L^2-torsion (provided that G is of type F), see [LT14, Theorem 1.3];
- vanishing rank gradient and vanishing K-homology gradients with respect to a normal chain with trivial intersection (provided that G is finitely generated), see [AN12, Theorem 3];
- vanishing rank gradient and vanishing K-homology gradients with respect to *any* chain (provided that G is finitely presented), see [AJZN11, Theorem 1];
- fixed price 1 in the theory of cost of groups, see [OW80, Theorem 6] combined with [Gab00, Théorème 3].
- vanishing simplicial volume (provided that G is the fundamental group of a closed connected orientable manifold), see [Gro83, Section 3.1, Corollary (C)].

Special attention deserves in the setting of this thesis the following result due to Wegner [Weg00, Theorem 5.4.5]: If G is a group of type F which is of det $\geq$ 1-class (i.e., $\det_{\mathcal{N}(G)}(r_A) \geq 1$ for all matrices $A \in M_{m,n}(\mathbb{Z}G)$) and which contains a non-trivial elementary amenable normal subgroup, then G is L^2-acyclic and has vanishing L^2-torsion. In particular, the L^2-torsion of any infinite elementary amenable group of type F vanishes. This result was slightly generalized in [Weg09, Theorem 1]. If true, Conjecture 5.1 would extend this long list of vanishing results and in particular provide a polytope analogue for Wegner's result.

In this chapter we introduce the notion of groups *of* $P \geq 0$*-class* and even stronger *of polytope class* by virtue of the polytope homomorphism. This notion is a polytope analogue of the det $\geq$ 1-class mentioned above. Then we show that all infinite amenable groups satisfying the Atiyah Conjecture possess these properties. We then adapt Wegner's proof in order to show that a group of type F which is of $P \geq 0$-class and contains a non-abelian elementary amenable normal subgroup has vanishing L^2-torsion polytope. In particular, the L^2-torsion polytope of any infinite elementary amenable group of type F vanishes. Finally we provide some evidence for Conjecture 5.1 beyond elementary amenability. A self-contained presentation of the results in this chapter can be found in [Fun17].

5.1 Groups of $P \geq 0$-class

In this section we introduce a polytope analogue of the notion det $\geq$ 1-class concerning the Fuglede-Kadison determinant (see Subsection 3.1.3). First we need a partial order on polytope groups.

Definition 5.2 (Partial order on polytope groups)**.** Let H be a finitely generated free-abelian group. We define a partial order on $\mathcal{P}(H)$ by declaring

$$P - Q \leq P' - Q' \text{ if and only if } P + Q' \subseteq P' + Q.$$

Likewise, we define a partial order on the translation quotient $\mathcal{P}_T(H)$ by declaring

$$P - Q \leq P' - Q' \text{ if and only if } P + Q' \subseteq P' + Q \text{ up to translation.}$$

It is easy to see that this definition does not depend on the choice of representatives.

Recall that we denote by $[r_A] \in K_1^w(\mathbb{Z}G)$ the class of the map $r_A \colon \mathbb{Z}G^n \to \mathbb{Z}G^n$ given by right multiplication with a matrix $A \in M_{n,n}(\mathbb{Z}G)$ which becomes invertible over $\mathcal{D}(G)$.

Definition 5.3 ($P \geq 0$-class and polytope class)**.** A group G is *of* $P \geq 0$*-class* if it is torsion-free, satisfies the Atiyah Conjecture, $b_1(G) < \infty$, and we have for any matrix $A \in M_{n,n}(\mathbb{Z}G)$ which becomes invertible over $\mathcal{D}(G)$ that

$$\mathbb{P}\big([r_A \colon \mathbb{Z}G^n \to \mathbb{Z}G^n]\big) \geq 0$$

in $\mathcal{P}_T(H_1(G)_f)$. We call G *of polytope class* if $\mathbb{P}\big([r_A \colon \mathbb{Z}G^n \to \mathbb{Z}G^n]\big)$ is even represented by a polytope, i.e., it lies in the image of the inclusion $\mathfrak{P}_T(H_1(G)_f) \to \mathcal{P}_T(H_1(G)_f)$ of the monoid of integral polytopes up to translation.

Example 5.4. (1) A finitely generated free-abelian group H is of polytope class since the Dieudonné determinant $\det_{\mathcal{D}(H)}(A)$ coincides with the determinant $\det_{\mathbb{Z}H}(A)$ over the commutative ring $\mathbb{Z}H$ and is therefore represented by an element in $\mathbb{Z}H$. Hence $\mathbb{P}\big([r_A \colon \mathbb{Z}H^n \to \mathbb{Z}H^n]\big)$ is represented by a polytope.

(2) If G is a torsion-free group satisfying the Atiyah Conjecture and $b_1(G) \leq 1$, then G is of polytope class. Namely, let $\mathcal{D}(K)[u^{\pm}] \subseteq \mathcal{D}(G)$ be a subring determined by a generator of $\operatorname{Hom}(G, \mathbb{Z})$. Then it follows by virtue of the Euclidean function

on $\mathcal{D}(K)[u^{\pm}]$ given by the degree that $\det_{\mathcal{D}(G)}(A)$ is represented by an element in $\mathcal{D}(K)[u^{\pm}]$. (This argument was also used in the proof of Theorem 3.52.) Thus $\mathbb{P}\big([r_A\colon \mathbb{Z}G^n \to \mathbb{Z}G^n]\big)$ is represented by an interval.

We know from Theorem 3.45 (1) that the L^2-torsion polytope is a simple homotopy invariant of free finite L^2-acyclic G-CW-complexes. This can be strengthened if G is of $P \geq 0$-class.

Lemma 5.5. *Let G be a group of $P \geq 0$-class. Then the composition*

$$\mathrm{Wh}(G) \xrightarrow{\zeta} \mathrm{Wh}^w(G) \xrightarrow{\mathbb{P}} \mathcal{P}_T(H_1(G)_f)$$

is trivial. Moreover, the L^2-torsion polytope is a homotopy invariant of free finite L^2-acyclic G-CW-complexes.

Proof. An element in the image of ζ is of the form $[r_A\colon \mathbb{Z}G^n \to \mathbb{Z}G^n]$ for a matrix $A \in M_{n,n}(\mathbb{Z}G)$ which has an inverse $A^{-1} \in M_{n,n}(\mathbb{Z}G)$. Since G is of $P \geq 0$-class, we have

$$0 = \mathbb{P}([\mathrm{id}]) = \mathbb{P}\big([r_A]\big) + \mathbb{P}\big([r_{A^{-1}}]\big) \geq 0,$$

and hence $\mathbb{P}\big([r_A]\big) = 0$. The 'moreover' part immediately follows from this because of Theorem 3.45 (1). ☐

Remark 5.6 (Extension of $P(G)$ to groups of $P \geq 0$-class)**.** Lemma 5.5 allows us to drop $\mathrm{Wh}(G) = 0$ from the list of conditions in the definition of the L^2-torsion polytope $P(G)$ of groups (see Definition 3.49), provided that G is of $P \geq 0$-class. Put differently, we can extend the definition of $P(G)$ to groups G which are of type F and of $P \geq 0$-class. We will take this into account in the formulations for the rest of this chapter.

5.2 Polytope class and amenability

The goal of this section is to prove the following result.

Theorem 5.7 (Polytope class and amenability)**.** *Let G be a torsion-free amenable group with $b_1(G) < \infty$ satisfying the Atiyah Conjecture. Then G is of polytope class.*

Its proof requires some preparation. The first lemma is possibly well-known in polytope theory, but we were not able to find the statement nor an implicit proof in the literature. In any case, it might as well be helpful in other situations.

Lemma 5.8 (Detecting polytopes by their faces)**.** *Let H be a finitely generated free-abelian group. Then $x \in \mathcal{P}(H)$ is represented by a polytope if and only if for every $\varphi \in \mathrm{Hom}(H, \mathbb{Z})$ the class $F_\varphi(x) \in \mathcal{P}(H)$ is represented by a polytope.*

Proof. It suffices to prove this for $H = \mathbb{Z}^n$. Equip $V_H = \mathbb{R}^n$ with the standard inner product. The forward direction of the lemma is obvious.

For the backwards direction write $x = P - Q$ for integral polytopes P and Q. By assumption $F_\varphi(x) = F_\varphi(P) - F_\varphi(Q)$ is an integral polytope for any $\varphi \in \mathrm{Hom}(H, \mathbb{Z})$, say S^φ, so $F_\varphi(P) = F_\varphi(Q) + S^\varphi$. We can write

$$P = \{x \in V_H \mid \psi_i(x) \leq c_i\}$$

for certain $\psi_i \in \operatorname{Hom}(H, \mathbb{Z}) \subseteq \operatorname{Hom}_{\mathbb{R}}(V_H, \mathbb{R})$ and $c_i \in \mathbb{R}$ $(i = 1, ..., k)$ such that no inequality is redundant. Then

$$S = \operatorname{hull}\left(\bigcup_{i=1}^{k} S^{\psi_i}\right)$$

is an integral polytope satisfying $P \subseteq Q + S$. The remainder of the proof will be occupied with proving $Q + S \subseteq P$. Although this is intuitively clear, its proof requires a number of steps. In the following, Greek letters will always denote elements in $\operatorname{Hom}(H, \mathbb{Z})$ without explicitly saying this. Moreover, given a compact subset $A \subseteq \mathbb{R}^n$ and φ, we will use the shorthand notations

$$A_\varphi = F_\varphi(A);$$
$$\varphi(A) = \max\{\varphi(a) \mid a \in A\}.$$

First note that we have

$$F_\varphi(P_\psi) = P_\varphi \cap P_\psi = F_\psi(P_\varphi)$$

provided that the intersection in the middle is non-trivial, and likewise for Q.

Step 1: If φ, ψ are such that $P_\varphi \cap P_\psi$ is non-empty, then $Q_\varphi \cap Q_\psi$ and $S^\varphi \cap S^\psi$ are non-empty, and we have

$$P_\varphi \cap P_\psi = \left(Q_\varphi \cap Q_\psi\right) + \left(S^\varphi \cap S^\psi\right).$$

We first argue that $Q_\varphi \cap Q_\psi$ is non-empty. Pick a vertex $p \in P_\varphi \cap P_\psi$, and let α be such that $P_\alpha = p$. Then $p = P_\alpha = Q_\alpha + S^\alpha$, hence $Q_\alpha = q$ and $S^\alpha = s$ are just points. After translating Q, we may assume that $s = 0$ and $p = q$. Then for every β such that P_β contains p we have $Q_\beta \subseteq P_\beta$ and $p \in Q_\beta$. This applies in particular to φ and ψ, hence $p \in Q_\varphi \cap Q_\psi$.

Now we compute

$$F_\varphi(S^\psi) = F_\varphi(P_\psi) - F_\varphi(Q_\psi) = F_\psi(P_\varphi) - F_\psi(Q_\varphi) = F_\psi(S^\varphi),$$

hence $F_\varphi(S^\psi) \subseteq S^\varphi \cap S^\psi$ and $S^\varphi \cap S^\psi$ is non-empty. We also have

$$\begin{aligned}\left(S^\varphi \cap S^\psi\right) + F_\varphi(Q_\psi) &= \left(S^\varphi \cap S^\psi\right) + \left(Q_\varphi \cap Q_\psi\right)\\ &\subseteq \left(P_\varphi \cap P_\psi\right)\\ &= F_\varphi(P_\psi).\end{aligned}$$

From this it follows that $S^\varphi \cap S^\psi \subseteq F_\varphi(S^\psi)$. Thus we proved $F_\varphi(S^\psi) = S^\varphi \cap S^\psi$. Now we conclude

$$\begin{aligned}P_\varphi \cap P_\psi &= F_\varphi(P_\psi)\\ &= F_\varphi(Q_\psi) + F_\varphi(S^\psi)\\ &= \left(Q_\varphi \cap Q_\psi\right) + \left(S^\varphi \cap S^\psi\right).\end{aligned}$$

Step 2: Let $v_0, v_1, v_2 \in S^{n-1}$ be such that v_1 lies on a geodesic path of length at most π from v_0 to v_2 in S^{n-1}. Write $\varphi_i = \langle v_i, \cdot \rangle \colon \mathbb{R}^n \to \mathbb{R}$. If P is any polytope such that $P_{\varphi_1} \cap P_{\varphi_2}$ is non-trivial, then we have

$$\varphi_0(P_{\varphi_2}) = \varphi_0(P_{\varphi_1} \cap P_{\varphi_2}).$$

Pick an element $x \in P_{\varphi_1} \cap P_{\varphi_2}$ attaining the maximum on the right. Assume that we have

$$\varphi_0(P_{\varphi_2}) > \varphi_0(P_{\varphi_1} \cap P_{\varphi_2}).$$

Then there exists $y \in P_{\varphi_2}$ such that $\varphi_0(y) > \varphi_0(x)$, $\varphi_1(y) < \varphi_1(x)$, and $\varphi_2(y) = \varphi_2(x)$. In other words,

$$\begin{aligned}
\langle y - x, v_0 \rangle &> 0; \\
\langle y - x, v_1 \rangle &< 0; \\
\langle y - x, v_2 \rangle &= 0
\end{aligned}$$

which cannot happen if v_1 lies on a geodesic path of length at most π from v_0 to v_2.

Step 3: We have $S^\varphi = S_\varphi$.

Let φ, ψ be arbitrary and write (up to scalar) $\varphi = \langle v, \cdot \rangle$ and $\psi = \langle w, \cdot \rangle$ for unit vectors v, w. There is a sequence of unit vectors $v = v_0, v_1, ..., v_m = w$ running along a geodesic path of length at most π from v to w in S^{n-1} such that $P_{\varphi_i} \cap P_{\varphi_{i+1}}$ is non-trivial for all $0 \leq i \leq m-1$. For brevity write from now on $P_i = P_{\varphi_i}$, $Q_i = Q_{\varphi_i}$, and $S^i = S^{\varphi_i}$. Then we have by Step 1

$$P_i \cap P_{i+1} = \big(Q_i \cap Q_{i+1}\big) + \big(S^i \cap S^{i+1}\big)$$

and by Step 2

$$\begin{aligned}
\varphi(P_{i+1}) &= \varphi(P_i \cap P_{i+1}); \\
\varphi(Q_{i+1}) &= \varphi(Q_i \cap Q_{i+1}).
\end{aligned}$$

This implies

$$\begin{aligned}
\varphi(S^{i+1}) &= \varphi(P_{i+1}) - \varphi(Q_{i+1}) \\
&= \varphi(P_i \cap P_{i+1}) - \varphi(Q_i \cap Q_{i+1}) \\
&= \varphi(S^i \cap S^{i+1}) \\
&\leq \varphi(S^i).
\end{aligned}$$

Since this is true for all $i = 0, ..., m-1$, we conclude $\varphi(S^\psi) \leq \varphi(S^\varphi)$ and hence $S^\varphi = S_\varphi$.

Step 4: We have $Q + S \subseteq P = \{x \in \mathbb{R}^n \mid \psi_i(x) \leq c_i\}$.

Pick arbitrary $s \in S$ and $q \in Q$. With the aid of Step 3 we can calculate

$$\begin{aligned}
\psi_i(q + s) &= \psi_i(q) + \psi_i(s) \\
&\leq \psi_i(Q_{\psi_i}) + \psi_i(S_{\psi_i}) \\
&= \psi_i(P_{\psi_i}) = c_i
\end{aligned}$$

and hence $q + s \in P$. $\square$

We also need the following auxiliary gadget.

Definition 5.9. Let H be a finitely generated free-abelian group and $G \subseteq H$ a subgroup. We consider $\mathfrak{P}_T(G)$ as a submonoid of $\mathfrak{P}_T(H)$. Then we let $\mathcal{P}_T(H, G)$ be the submonoid of $\mathcal{P}_T(H)$ containing all elements that can be written as a difference $P - Q$ for some $P \in \mathfrak{P}_T(H)$ and $Q \in \mathfrak{P}_T(G)$.

Example 5.10. (1) For any subgroup $G \subseteq H$ one has

$$\mathfrak{P}_T(H) = \mathcal{P}_T(H, 0) \subseteq \mathcal{P}_T(H, G) \subseteq \mathcal{P}_T(H, H) = \mathcal{P}_T(H).$$

We can interpret $\mathcal{P}_T(H, G)$ as interpolating between the monoid of integral polytopes and the integral polytope group.

(2) Let H be of rank 2 and let G_1, G_2 be two subgroups of rank 1. If $G_i \cap G_j = 0$, then $\mathcal{P}_T(H, G_1) \cap \mathcal{P}_T(H, G_2) = \mathfrak{P}_T(H)$.

Motivated by the last example we propose the following problem.

Question 5.11. *Let H be a finitely generated free-abelian group and G_1, G_2 be two subgroups. Do we always have*

$$\mathcal{P}_T(H, G_1) \cap \mathcal{P}_T(H, G_2) = \mathcal{P}_T(H, G_1 \cap G_2)?$$

If this question has an affirmative answer, then the next lemma, for which we provide a different argument, would immediately follow.

Lemma 5.12. *Let H be a finitely generated free-abelian group. Then*

$$\bigcap_{\varphi \in \operatorname{Hom}(H,\mathbb{Z})} \mathcal{P}_T(H, \ker\varphi) = \mathfrak{P}_T(H).$$

Proof. We prove the statement by induction on the rank of H. The rank 1 case is obvious.

For the higher rank case, pick an element x in the above intersection. For any homomorphism $\varphi\colon H \to \mathbb{Z}$ we can find $P_\varphi \in \mathfrak{P}_T(H)$ and $Q_\varphi \in \mathfrak{P}_T(\ker\varphi)$ such that $x = P_\varphi - Q_\varphi$. Fix some homomorphism $\alpha\colon H \to \mathbb{Z}$. Then

$$F_\alpha(x) = F_\alpha(P_\varphi) - F_\alpha(Q_\varphi) \in \mathcal{P}_T(\ker\alpha, \ker\alpha \cap \ker\varphi).$$

Since φ was arbitrary, we conclude

$$F_\alpha(x) \in \bigcap_{\varphi \in \operatorname{Hom}(H,\mathbb{Z})} \mathcal{P}_T(\ker\alpha, \ker\alpha \cap \ker\varphi) = \bigcap_{\psi \in \operatorname{Hom}(\ker\alpha,\mathbb{Z})} \mathcal{P}_T(\ker\alpha, \ker\psi).$$

From the induction hypothesis we conclude $F_\alpha(x) \in \mathfrak{P}_T(\ker\alpha)$. As this holds for every homomorphism $\alpha\colon H \to \mathbb{Z}$, we may apply the previous Lemma 5.8 to deduce that $x \in \mathfrak{P}_T(H)$. □

For the rest of this chapter we will use the following notation. Given a non-trivial $x \in \mathbb{Z}G$ we denote by $P(x) \in \mathcal{P}_T(H_1(G)_f)$ the image of the class of the $\mathbb{Z}G$-map $r_x\colon \mathbb{Z}G \to \mathbb{Z}G$ under $\mathbb{P}\colon K_1^w(\mathbb{Z}G) \to \mathcal{P}_T(H_1(G)_f)$. Recall from Section 3.7.2 that this is easily computable, namely for the kernel $K = \ker(G \to H_1(G)_f)$ we have an isomorphism $\mathbb{Z}G \cong \mathbb{Z}K * H_1(G)_f$ and $P(x)$ is the class of $\operatorname{hull}(\operatorname{supp}(x))$ in $\mathcal{P}_T(H_1(G)_f)$.

Proof of Theorem 5.7. Recall from Lemma 3.23 (2) that $\mathbb{Z}G$ satisfies the Ore condition with respect to $T = \mathbb{Z}G \setminus \{0\}$ and the inclusion induces an isomorphism $T^{-1}\mathbb{Z}G \xrightarrow{\cong} \mathcal{D}(G)$.

Let $A \in M_{n,n}(\mathbb{Z}G)$ be a matrix which becomes invertible over $\mathcal{D}(G)$. If $b_1(G) = 0$, then there is nothing to prove. Otherwise let us fix some epimorphism $\varphi\colon G \to \mathbb{Z}$ and denote its kernel by K. Consider the associated twisted Laurent polynomial ring $\mathcal{D}(K)_t[u^\pm] \subseteq \mathcal{D}(G)$ as in Theorem 3.24 (1). The Euclidean function on $\mathcal{D}(K)_t[u^\pm]$ given by the degree allows us to transform A to a triangular matrix T over $\mathcal{D}(K)_t[u^\pm]$ by using the operations

- Permute rows or columns;
- Multiply a row on the right or a column on the left with an element of the form $y \cdot u^m$ for some non-trivial $y \in \mathcal{D}(K)$ and $m \in \mathbb{Z}$;
- Add a right $\mathcal{D}(K)_t[u^{\pm}]$-multiple of one row (resp. column) to another row (resp. column).

These operations do not change the class $[A] \in K_1(\mathcal{D}(G))$. Since $\mathcal{D}(K) = (\mathbb{Z}K \setminus \{0\})^{-1}\mathbb{Z}K$, we may then multiply T with suitable elements in $\mathbb{Z}K$ to obtain a matrix over $\mathbb{Z}K_t[u^{\pm}] = \mathbb{Z}G$. This implies that there are elements $x \in \mathbb{Z}G$ and $y \in \mathbb{Z}K \setminus \{0\}$ such that we have in $K_1(\mathcal{D}(G))$

$$[A] = [T] = [x \cdot y^{-1}].$$

This implies

$$\mathbb{P}\big([r_A \colon \mathbb{Z}G^n \to \mathbb{Z}G^n]\big) = P(x) - P(y) \in \mathcal{P}_T(H_1(G)_f, \ker \overline{\varphi})$$

for the epimorphism $\overline{\varphi} \colon H_1(G)_f \to \mathbb{Z}$ induced by φ. Since φ was arbitrary, we have

$$\mathbb{P}\big([r_A \colon \mathbb{Z}G^n \to \mathbb{Z}G^n]\big) \in \bigcap_{\substack{\varphi \in \mathrm{Hom}(G,\mathbb{Z}) \\ \text{surjective}}} \mathcal{P}_T(H_1(G)_f, \ker \overline{\varphi}).$$

By Lemma 5.12, this intersection is equal to $\mathfrak{P}_T(H_1(G)_f)$. □

5.3 Polytope class and the L^2-torsion polytope

In this section we adapt Wegner's strategy built in [Weg00, Weg09] in the setting of the L^2-torsion polytope. Together with the knowledge that torsion-free amenable groups are of polytope class, one of its applications will be the vanishing of the L^2-torsion polytope of every elementary amenable group of type F. In order to motivate our first lemma we give a rough idea of the argument:

Instead of localizing the group ring $\mathbb{Z}G$ at $\mathbb{Z}G \setminus \{0\}$ in order to obtain $\mathcal{D}(G)$, we localize at a much smaller set $S \subseteq \mathbb{Z}G$ in order to obtain an intermediate ring $\mathbb{Z}G \subseteq S^{-1}\mathbb{Z}G \subseteq \mathcal{D}(G)$. This set is small enough so that the polytopes of invertible matrices over $S^{-1}\mathbb{Z}G$ still satisfy $P \geq 0$, but it is large enough so that the localized cellular chain complex $S^{-1}C_*(EG)$ is already contractible. Combining these two facts makes the image of the Whitehead torsion of $S^{-1}C_*(EG)$ under an adjusted polytope homomorphism $K_1(S^{-1}\mathbb{Z}G) \to \mathcal{P}_T(H_1(G)_f)$ computable. But this image coincides with the negative of the L^2-torsion polytope $P(G)$.

Lemma 5.13. *Let G be a group of type F which satisfies the Atiyah Conjecture and $b_1(G) < \infty$. Suppose that G contains a non-trivial abelian normal subgroup $A \subseteq G$ such that $A \cap \ker(\mathrm{pr} \colon G \to H_1(G)_f) \neq 0$. Then*

$$S = \{x \in \mathbb{Z}A \setminus \{0\} \mid P(x) = 0 \ \text{ in } \ \mathcal{P}_T(H_1(G)_f)\}.$$

is a multiplicatively closed subset with respect to which $\mathbb{Z}G$ satisfies the Ore condition and such that $S^{-1}\mathbb{Z} = 0$ for the trivial $\mathbb{Z}G$-module $\mathbb{Z}$.

Proof. Since for any two elements $x, y \in \mathbb{Z}G$ we have $P(x \cdot y) = P(x) + P(y)$, it is clear that S is multiplicatively closed. The proof for the left and right Ore condition follows as in [Weg00, Proof of Theorem 5.4.5, Step 2 and 3], see also [Lüc02, Lemma 3.119]. We include the argument here for the sake of completeness. Note that the canonical involution on $\mathbb{Z}G$ respects S, so it suffices to prove the right Ore condition.

Let $r \in \mathbb{Z}G, s \in S$ and fix a set of representatives $\{g_i \mid i \in I\}$ for the cosets $Ag \in A\backslash G$. Write $r = \sum_{i\in I} a_i g_i$ for certain $a_i \in \mathbb{Z}A$, where almost all a_i vanish. Put $I' = \{i \in I \mid a_i \neq 0\}$. The element $s_i = g_i s g_i^{-1}$ lies in $\mathbb{Z}A$ since A is normal and $P(s_i) = P(s) = 0$. These two facts imply $s_i \in S$.

Define $s' = \prod_{i\in I'} s_i \in \mathbb{Z}A$, $x_i = s'/s_i \in \mathbb{Z}A$, and $r' = \sum_{i\in I'} x_i a_i g_i \in \mathbb{Z}G$. Then we compute

$$\begin{aligned} s' \cdot r &= \sum_{i\in I'} s' a_i g_i = \sum_{i\in I'} x_i s_i a_i g_i = \sum_{i\in I'} x_i a_i s_i g_i \\ &= \sum_{i\in I'} x_i a_i g_i s g_i^{-1} g_i = \sum_{i\in I'} x_i a_i g_i s = r' \cdot s \end{aligned}$$

Finally we prove $S^{-1}\mathbb{Z} = 0$. Pick some non-trivial $a \in A \cap \ker(\mathrm{pr}\colon G \to H_1(G)_f) \neq 0$ (this is the only part where we need this assumption). Then $P(1-a) = 0$ in $\mathcal{P}_T(H_1(G)_f)$, so $1-a$ lies in S. Since $1-a$ acts by multiplication with 0 on $\mathbb{Z}$, we conclude $S^{-1}\mathbb{Z} = 0$. $\square$

In the following proof we denote the Whitehead torsion of a finite based free contractible R-chain complex by $\tau(C_*)$.

Lemma 5.14. *Let G be a group of $P \geq 0$-class. Let $S \subseteq \mathbb{Z}G$ be a multiplicatively closed subset with respect to which $\mathbb{Z}G$ satisfies the Ore condition. Suppose that $P(s) = 0$ in $\mathcal{P}_T(H_1(G)_f)$ for all $s \in S$.*

If X is a free finite L^2-acyclic G-CW-complex such that $S^{-1}H_n(X) = 0$, then

$$P(X; G) = 0.$$

Proof. This is based on ideas appearing in [Weg00, Proof of Theorem 5.4.5, Step 4 and 5], see also [Lüc02, Lemma 3.114].

First we consider the following commutative diagram

$$\begin{tikzcd} \widetilde{K}_1^w(\mathbb{Z}G) \arrow[dr, "i"] \arrow[rrr, bend left, "\mathbb{P}"] & & & \\ & \widetilde{K}_1(\mathcal{D}(G)) \arrow[r, "\det_{\mathcal{D}(G)}"] & \mathcal{D}(G)^\times_{\mathrm{ab}}/\{\pm 1\} \arrow[r, "P"] & \mathcal{P}_T(H_1(G)_f) \\ \widetilde{K}_1(S^{-1}\mathbb{Z}G) \arrow[ur, "j"] \arrow[urrr, bend right, "\mathbb{P}'"] & & & \end{tikzcd}$$

Here i and j denote the obvious maps, $\det_{\mathcal{D}(G)}$ is the Dieudonné determinant, P is the map defined in (3.12), $\mathbb{P}$ denotes the composition of the top row (which is essentially the polytope homomorphism), and $\mathbb{P}'$ denotes the composition of the bottom row.

Let A be an invertible $S^{-1}\mathbb{Z}G$-matrix. By multiplying A with a suitable $s \in S$ we obtain a $\mathbb{Z}G$-matrix B which is invertible over $S^{-1}\mathbb{Z}G$ and thus also over $\mathcal{D}(G)$. Then we have $[A] = [B] - [s]$ in $\widetilde{K}_1(S^{-1}\mathbb{Z}G)$ and $\mathbb{P}'([B]) = \mathbb{P}([B])$. We assume that $P(s) = 0$ and that G is of $P \geq 0$-class, so we have

$$\mathbb{P}'([A]) = \mathbb{P}'([B]) - \mathbb{P}'([s]) = \mathbb{P}'([B]) - P(s) = \mathbb{P}([B]) \geq 0. \tag{5.1}$$

Since the same reasoning applies to A^{-1}, we have $\mathbb{P}'([A]) = 0$ and thus $\mathbb{P}' = 0$.

Denote by $C_* = C_*(X)$ the cellular $\mathbb{Z}G$-chain complex of X equipped with some choice of cellular basis. The $\mathcal{D}(G)$-chain complex $\mathcal{D}(G) \otimes_{\mathbb{Z}G} C_*$ is contractible by Lemma 3.39 and we have

$$i(\rho_u^{(2)}(C_*;\mathcal{N}(G))) = \tau(\mathcal{D}(G) \otimes_{\mathbb{Z}G} C_*).$$

Since localization is flat and $S^{-1}H_n(X) = 0$, the $S^{-1}\mathbb{Z}G$-chain complex $S^{-1}C_* = S^{-1}\mathbb{Z}G \otimes_{\mathbb{Z}G} C_*$ is also contractible, and we have

$$\begin{aligned} j(\tau(S^{-1}C_*)) &= \tau(\mathcal{D}(G) \otimes_{S^{-1}\mathbb{Z}G} S^{-1}C_*) \\ &= \tau(\mathcal{D}(G) \otimes_{S^{-1}\mathbb{Z}G} S^{-1}\mathbb{Z}G \otimes_{\mathbb{Z}G} C_*) \\ &= \tau(\mathcal{D}(G) \otimes_{\mathbb{Z}G} C_*) \\ &= i(\rho_u^{(2)}(C_*;\mathcal{N}(G))). \end{aligned}$$

Thus we see

$$\mathbb{P}(\rho_u^{(2)}(C_*;\mathcal{N}(G))) = \mathbb{P}'(\tau(S^{-1}C_*)) = 0. \qquad \square$$

Theorem 5.15 (The L^2-torsion polytope and elementary amenability)**.** *Let G be a group of type F which is of $P \geq 0$-class. Suppose that G contains a non-abelian elementary amenable normal subgroup. Then G is L^2-acyclic and we have*

$$P(G) = 0.$$

Proof. The group G is L^2-acyclic by [Lüc02, Theorem 1.44]. Let E be the non-abelian elementary amenable subgroup.

Case 1: E is not virtually abelian. It follows from the proof of [HL92, Corollary 2] that E is virtually solvable. Let $F \subseteq E$ be a maximal solvable normal subgroup of finite index in E. Since we assume that E is not virtually abelian, F is not abelian. Hence the lowest non-trivial subgroup A in the derived series of F is abelian and contained in $[F, F] \subseteq [G, G]$. In particular, $A \cap \ker(\mathrm{pr}\colon G \to H_1(G)_f) \neq 0$. Since A is characteristic in E, A is normal in G.

Case 2: E is virtually abelian. Let A be a normal abelian subgroup of finite index. Since E is not abelian, $\ker(\mathrm{pr}\colon E \to H_1(E)_f)$ is non-trivial and hence infinite as G is torsion-free. But any infinite subgroup of E must intersect A non-trivially. Thus in particular, $A \cap \ker(\mathrm{pr}\colon G \to H_1(G)_f) \neq 0$.

In both cases we may apply Lemma 5.13. This provides us with a subset $S \subseteq \mathbb{Z}G$ satisfying the assumptions of Lemma 5.14 for $X = EG$. Hence $P(G) = 0$. $\square$

The following is the main result of this chapter.

Corollary 5.16 (The L^2-torsion polytope of elementary amenable groups vanishes)**.** *Let G be an amenable group of type F satisfying the Atiyah Conjecture. If G contains a non-abelian elementary amenable normal subgroup, then*

$$P(G) = 0.$$

In particular, the L^2-torsion polytope of an elementary amenable group of type F vanishes.

Proof. By Theorem 5.7 an amenable group G of type F satisfying the Atiyah Conjecture is of polytope class. Hence the first statement follows directly from Theorem 5.15.

For the second statement, recall from Lemma 3.23 that an elementary amenable group G of type F satisfies the Atiyah Conjecture. Hence $P(G) = 0$ follows from the previous statement provided that G is non-abelian. If G is abelian, then G must be finitely generated free-abelian, so $P(G) = 0$ follows from $\rho_u^{(2)}(G) = 0$ as seen in [FL16b, Example 2.7]. $\square$

We emphasize the following remark that was also used in the proof of Theorem 5.15.

Remark 5.17. An elementary amenable group of type F (or more generally, with finite cohomological dimension) is in fact virtually solvable by a result of Hillman-Linnell [HL92, Corollary 1].

Remark 5.18 (Generalization to the universal L^2-torsion)**.** The proof of Corollary 5.16 crucially relies on the existence of a partial order on polytope groups even though the original statement does not involve them. One difficulty in proving the corresponding statement for the universal L^2-torsion $\rho_u^{(2)}(G)$ lies in the structural deficit of $\mathrm{Wh}^w(G)$ that it does not carry a meaningful partial order.

5.4 Evidence for non-elementary amenable groups

In this short final section, we offer concluding evidence for the validity of Conjecture 5.1 for amenable groups that are not elementary amenable. This computation is to a great extent based on known results.

Proposition 5.19 (L^2-torsion polytope and amenability)**.** *Let $G \neq \mathbb{Z}$ be an amenable group of type F satisfying the Atiyah Conjecture. Then $P(G)$ lies in the kernel of the norm homomorphism $\mathfrak{N}\colon \mathcal{P}_T(H_1(G)_f) \to \mathrm{Map}(H^1(G;\mathbb{R}),\mathbb{R})$ and there is an integral polytope $P \in \mathfrak{P}_T(H_1(G)_f)$ such that*

$$P(G) = P - *P.$$

Proof. Let $\mathrm{pr}\colon G \to H_1(G)_f = H$ be the obvious projection. Suppose that $H \neq 0$ since there is nothing to prove otherwise. Let $\varphi\colon H \to \mathbb{Z}$ be an epimorphism, and put $K = \ker(\varphi \circ \mathrm{pr}\colon G \to \mathbb{Z})$. Then we have by Theorem 3.52 and Lemma 3.17

$$\begin{aligned}
\mathfrak{N}(P(G))(\varphi) &= -\chi^{(2)}(EG;\mathcal{N}(G);\varphi \circ \mathrm{pr}) \\
&= -\chi^{(2)}(i^*EG;\mathcal{N}(K)) \\
&= -\chi^{(2)}(EK;N(K)).
\end{aligned}$$

As a subgroup of an amenable group, K is itself amenable. Since $G \neq \mathbb{Z}$, K must be infinite. Since infinite amenable groups are L^2-acyclic by [Lüc02, Corollary 6.75], we have $\chi^{(2)}(EK;N(K)) = 0$. (Note that for this argument it is irrelevant that $i^*EG = EK$ is not a *finite* K-CW-complex.) Thus we have

$$\mathfrak{N}(P(G))(\varphi) = 0$$

for all surjective homomorphisms $\varphi\colon H \to \mathbb{Z}$. This generalizes to homomorphisms $\varphi\colon H \to \mathbb{Q}$ since $\mathfrak{N}(P(G))$ is homogeneous, and to homomorphisms $\varphi\colon H \to \mathbb{R}$ since $\mathfrak{N}(P(G))$ is continuous. Hence

$$P(G) \in \ker\big(\mathfrak{N}\colon \mathcal{P}_T(H) \to \mathrm{Map}(\mathrm{Hom}(H,\mathbb{R}),\mathbb{R})\big).$$

We have seen in Theorem 4.1 (2) and Lemma 4.16 that

$$\ker\big(\mathfrak{N}\colon \mathcal{P}_T(H) \to \mathrm{Map}(\mathrm{Hom}(H,\mathbb{R}),\mathbb{R})\big) = \mathrm{im}\,\big(\mathrm{id} - *\colon \mathcal{P}_T(H) \to \mathcal{P}_T(H)\big).$$

Hence there exists an integral polytope $P \subseteq H \otimes_{\mathbb{Z}} \mathbb{R}$ such that

$$P(G) = P - *P. \qquad \square$$

6 The L^2-torsion Polytope of Free Group HNN Extensions

In this final chapter we shift our view from amenable groups towards the other side of the universe of groups, namely to free groups. The theory of universal L^2-torsion and the L^2-torsion polytope of groups can be applied to any endomorphism of a group with finite classifying space by considering the corresponding HNN extension. There are (at least) three reasons why it is reasonable to apply this first to endomorphisms of finitely generated free groups. First, free groups have the simplest possible classifying space and so computations seem within reach; second, HNN extensions of free groups can be thought of as being close to 3-manifold groups and so promising results might be expected; third, the outer automorphism group $\mathrm{Out}(F_n)$ of free groups is an important object in geometric group theory, but notoriously hard to handle. L^2-torsion invariants have the potential to carry a significant amount of information for these automorphisms.

We begin with several general observations about the universal L^2-torsion of free group endomorphisms before we concentrate on a class of free group automorphisms called *unipotent polynomially growing*. For these we present an inductive procedure to fully compute the universal L^2-torsion, see Theorem 6.15. This will imply a strong relation between the L^2-torsion polytope and the BNS-invariant introduced in Section 2.4, see Corollary 6.20. These results overlap in part with [FK16]. We also obtain the equality of the twisted L^2-Euler characteristic and the degree of L^2-torsion functions for all polynomially growing automorphisms, see Corollary 6.21.

6.1 Free group HNN extensions

Definition 6.1. Let A be a group and $\alpha\colon A \to A$ be an endomorphism. Then the *HNN extension associated to* α is the group with presentation

$$A{*_\alpha} = \langle\, A, t \mid \text{ for all } a \in A\colon\ t^{-1}at = \alpha(a)\,\rangle.$$

It is *ascending* if α is injective. The *canonical epimorphism* associated to the HNN extension is the map $A{*_\alpha} \to \mathbb{Z}$ determined by mapping t to 1 and A to 0. A group homomorphism $\mu\colon A{*_\alpha} \to G$ is *admissible* if the canonical epimorphism factors over μ. If $A = F_n$ is a finitely generated free group (of rank n), then we simply refer to $F_n{*_\alpha}$ as a *free group HNN extension*.

Classifying spaces for HNN extensions are well understood, we include the following lemma for later reference.

Lemma 6.2. *Let A be a group and $\alpha\colon A \to A$ be a monomorphism. Take a model for the classifying space BA and choose a realization $B\alpha\colon BA \to BA$ of α. Then we have:*

(1) The mapping torus $T_{B\alpha}$ of $B\alpha$ is a model for the classifying space of the HNN extension $A*_\alpha$. If BA is finite, then so is $T_{B\alpha}$.

(2) The simple homotopy type of $T_{B\alpha}$ does not depend on the choice of BA or $B\alpha$.

(3) Let $\mu\colon \pi_1(T_{B\alpha}) = A*_\alpha \to G$ be an admissible group homomorphism. Then the G-covering $\overline{T_{B\alpha}} \to T_{B\alpha}$ associated to μ is L^2-acyclic.

Proof. (1) is easy to check.

(2) See [Coh73, (22.1)].

(3) is a result of Lück [Lüc94b, Theorem 2.1]. □

Remark 6.3. We also mention that the Whitehead group of a free group HNN extension F_n*_α vanishes by a theorem of Waldhausen [Wal78, Theorem 19.4].

Definition 6.4. Let A be a group of type F, $\alpha\colon A \to A$ a monomorphism, and $\mu\colon A*_\alpha = \pi_1(T_{B\alpha}) \to G$ an admissible homomorphism. Let $\overline{T_{B\alpha}} \to T_{B\alpha}$ denote the G-covering associated to μ as in Lemma 6.2 (3). Then the *universal L^2-torsion* of the pair (α, μ) is defined as
$$\rho_u^{(2)}(\alpha;\mu) = \rho_u^{(2)}(\overline{T_{B\alpha}};\mathcal{N}(G)) \in \mathrm{Wh}^w(G).$$
This is well-defined in view of Theorem 3.45 (1) and Lemma 6.2 (2). If G is torsion-free, satisfies the Atiyah Conjecture and has finite first Betti number, then we define the *L^2-torsion polytope* of (α,μ) to be
$$P(\alpha;\mu) = \mathbb{P}(-\rho_u^{(2)}(\alpha;\mu)) \in \mathcal{P}(H_1(G)_f).$$
If $\mu = \mathrm{id}$, then we simply write $\rho_u^{(2)}(\alpha)$ and $P(\alpha)$.

In order to deal with the universal L^2-torsion of free group HNN extensions in practice, we quickly recall Fox calculus [Fox53, Fox54].

Remark 6.5. Let F be a free group with generating set $s_1, ..., s_n$. Then the *Fox derivatives* $\frac{\partial}{\partial s_i}\colon \mathbb{Z}F \to \mathbb{Z}F$ are the $\mathbb{Z}$-linear maps determined by the following properties:
$$\frac{\partial 1}{\partial s_i} = 0;$$
$$\frac{\partial s_j}{\partial s_i} = \delta_{ij};$$
$$\frac{\partial vw}{\partial s_i} = \frac{\partial v}{\partial s_i} + v \cdot \frac{\partial w}{\partial s_i}$$
for any $v, w \in F$. If $F \to G$ is an epimorphism, then we denote the induced map $\frac{\partial}{\partial s_i}\colon \mathbb{Z}F \to \mathbb{Z}G$ by the same symbol.

Definition 6.6 (Fox matrices)**.** Given a finite group presentation $G = \langle s_1, ..., s_n \mid r_1, ..., r_m\rangle$, its *Fox matrix* is the matrix $\left(\frac{\partial r_i}{\partial s_j}\right) \in M_{m,n}(\mathbb{Z}G)$. We will sometimes denote it by $F(G)$ although it depends on the presentation rather than the group.

Let F be a free group with generating set $s_1, ..., s_n$ and $\alpha\colon F \to F$ be an endomorphism. Then the *Fox matrix* of α is the matrix
$$F(\alpha) = \left(\frac{\partial \alpha(s_i)}{\partial s_j}\right) \in M_{n,n}(\mathbb{Z}F).$$

Lemma 6.7 (Fox matrix of a free group HNN extension)**.** *Let $\alpha\colon F_n \to F_n$ be an endomorphism of the free group F_n. Take for the HNN extension $\pi_\alpha = F_n *_\alpha$ the finite presentation*

$$\pi_\alpha = \langle\, s_1, ..., s_n, t \mid s_i t \alpha(s_i)^{-1} t^{-1} \text{ for all } i \,\rangle.$$

Then the Fox matrix of this presentation is given by

$$F(\pi_\alpha) = \left(\begin{array}{cc} & s_1 - 1 \\ \operatorname{Id} - t \cdot F(\alpha) & \vdots \\ & s_n - 1 \end{array} \right) \in \mathbb{Z}\pi_\alpha^{n \times (n+1)}.$$

Proof. If we let $r_i = s_i t \alpha(s_i)^{-1} t^{-1}$, then this follows from the computations

$$\begin{aligned} \frac{\partial r_i}{\partial s_j} &= \delta_{ij} + s_i t \left(\frac{\partial \alpha(s_i)^{-1}}{\partial s_j} + \alpha(s_i)^{-1} \cdot \frac{\partial t^{-1}}{\partial s_j} \right) \\ &= \delta_{ij} - s_i t \alpha(s_i)^{-1} \cdot \frac{\partial \alpha(s_i)}{\partial s_j} \\ &= \delta_{ij} - t \cdot \frac{\partial \alpha(s_i)}{\partial s_j} \\ \frac{\partial r_i}{\partial t} &= s_i - s_i t \alpha(s_j)^{-1} t^{-1} = s_i - 1. \end{aligned}$$

□

The following lemma is the obvious analogue of [FL16a, Theorem 5.1] for the universal L^2-torsion rather than L^2-Euler characteristics and for free group HNN extensions instead of 3-manifolds.

Lemma 6.8 (Universal L^2-torsion of free group HNN extensions)**.** *Let $\alpha\colon F_n \to F_n$ be a monomorphism of the free group F_n, and put $\pi_\alpha = F_n *_\alpha$. Let $A \in M_{n,n+1}(\mathbb{Z}\pi_\alpha)$ be the Fox matrix of π_α as in Lemma 6.7. Let $\mu\colon \pi_\alpha \to G$ be an admissible homomorphism. Pick $1 \le i \le n$ such that $\mu(s_i)$ has infinite order. Let A_i be the $n \times n$-matrix obtained from A by deleting the i-th column. Then we have in* $\operatorname{Wh}^w(G)$

$$\rho_u^{(2)}(\alpha;\mu) = [r_{\mu(s_i)-1}\colon \mathbb{Z}G \to \mathbb{Z}G] - [r_{\mu(A_i)}\colon \mathbb{Z}G^n \to \mathbb{Z}G^n].$$

Likewise, we have

$$\rho_u^{(2)}(\alpha;\mu) = [r_{\mu(t)-1}\colon \mathbb{Z}G \to \mathbb{Z}G] - [r_{\mu(A_{n+1})}\colon \mathbb{Z}G^n \to \mathbb{Z}G^n].$$

Proof. Let $X \to T_{B\alpha}$ denote the G-covering associated to μ. The cellular $\mathbb{Z}G$-chain complex of $C_*(X)$ looks like

$$... \to 0 \to \mathbb{Z}G^n \xrightarrow{\mu(A)} \mathbb{Z}G^{n+1} \xrightarrow{\bigoplus \mu(s_i)-1} \mathbb{Z}G \to 0,$$

where A is the matrix occurring in the statement of the lemma. Since $\mu(s_i)$ has infinite order, the chain complex $\operatorname{el}(r_{\mu(s_i)-1})$ is L^2-acyclic. From the exact sequence

$$0 \to \operatorname{el}(r_{\mu(s_i)-1}) \to C_*(X) \to \Sigma \operatorname{el}(r_{\mu(A_i)}) \to 0$$

we deduce that $\Sigma \operatorname{el}(r_{A_i})$ is also L^2-acyclic. Moreover, since $\rho_u^{(2)}$ is an (additive) L^2-torsion invariant, we have

$$\rho_u^{(2)}(\alpha;\mu) = \rho_u^{(2)}(X; \mathcal{N}(G)) = [r_{\mu(s_i)-1}] - [r_{\mu(A_i)}].$$

The proof for t instead of s_i is analogous, but since μ is admissible the order of $\mu(t)$ is automatically infinite. $\square$

Remark 6.9. Lemma 6.8 implies that the Alexander polynomial of free group HNN extensions can be reinterpreted in terms of the universal L^2-torsion as follows. Let $\pi_\alpha = F_n *_\alpha$, let $\mathrm{pr}\colon \pi_\alpha \to H_1(\pi_\alpha)_f = H$ denote the projection, and let $X \to B\pi_\alpha$ be the H-covering associated to pr. A $\mathbb{Z}H$-presentation of the Alexander module A_{π_α} is given by

$$\ldots \to 0 \to \mathbb{Z}H^n \xrightarrow{\mathrm{pr}(A)} \mathbb{Z}H^{n+1} \to A_{\pi_\alpha} \to 0$$

By the same argument as in the proof of [McM02, Theorem 5.1] one sees that

$$\Delta_{\pi_\alpha} = \begin{cases} \det(\mathrm{pr}(A_i))/(\mathrm{pr}(s_i)-1) & \text{if } b_1(\pi_\alpha) \geq 2 \text{ and } \mathrm{pr}(s_i) \neq 0; \\ \det(\mathrm{pr}(A_{n+1})) & \text{if } b_1(\pi_\alpha) = 1. \end{cases}$$

Upon comparing this with Lemma 6.8, we see that Δ_{π_α} essentially corresponds to $\rho_u^{(2)}(X;\mathcal{N}(H))$ under the isomorphism

$$\mathrm{Wh}^w(H) \cong K_1(\mathcal{D}(H))/\{\pm h\} \cong \mathcal{D}(H)^\times/\{\pm h\}$$

given in Theorem 3.43 and by the (Dieudonné) determinant.

6.2 Norms on the first cohomology of free group HNN extensions

Recall from Theorem 3.53 that for 3-manifolds the image of the L^2-torsion polytope under the norm homomorphims $\mathfrak{N}$ is precisely the Thurston norm. In this chapter, we will show that for free group HNN extensions the L^2-torsion polytope also induces a seminorm on its first cohomology. Most of the work is already done by the following theorem.

Theorem 6.10. *Let k be a skew-field and H a finitely generated free-abelian group. Let $k * H$ be some crossed product and $Q = T^{-1}(k*H)$ its quotient field. If $x \in K_1(Q)$ is represented by an element $A \in M_{n,n}(k*H)$, then the image of x under the composition*

$$K_1(Q) \xrightarrow{\det_Q} Q_{\mathrm{ab}}^\times \xrightarrow{P} \mathcal{P}(H) \xrightarrow{\mathfrak{N}} \mathrm{Map}(\mathrm{Hom}(H,\mathbb{R}),\mathbb{R})$$

(where P is defined as in Section 3.7.2) is a seminorm.

Proof. This is due to Friedl-Harvey [FH07, Theorem 2.2]. $\square$

Theorem 6.11. *Let $\alpha\colon F_n \to F_n$ be a monomorphism of the free group F_n, and put $\pi_\alpha = F_n *_\alpha$. Let $\mu\colon \pi_\alpha \to G$ be an admissible homomorphism to a torsion-free group satisfying the Atiyah Conjecture and with finite first Betti number.*

Then the image of $-\rho_u^{(2)}(\alpha;\mu)$ under the composition

$$\mathrm{Wh}^w(G) \xrightarrow{\mathbb{P}} \mathcal{P}(H_1(G)_f) \xrightarrow{\mathfrak{N}} \mathrm{Map}(H^1(G;\mathbb{R}),\mathbb{R})$$

is a seminorm.

Definition 6.12. In the sequel this seminorm will be denoted by

$$\delta(\alpha,\mu)\colon H^1(G;\mathbb{R}) \to \mathbb{R}$$

in analogy with Section 2.3, and we abbreviate

$$\delta(\alpha) = \delta(\alpha, \mathrm{id}).$$

Remark 6.13. Recall from Theorem 3.52 that we have

$$\delta(\alpha, \mu)(\varphi) = -\chi^{(2)}(\overline{T_{B\alpha}}; \mathcal{N}(G), \varphi)$$

for the G-covering $\overline{T_{B\alpha}} \to T_{B\alpha}$ associated to μ.

Proof of Theorem 6.11. In this proof, let $\|\cdot\| = \mathfrak{N}(\mathbb{P}(-\rho_u^{(2)}(\alpha;\mu)))$. Since $\|\cdot\|$ is in the image of $\mathfrak{N}$ and hence a difference of seminorms, $\|\cdot\|$ is continuous and satisfies $\|r\cdot\varphi\| = |r|\cdot\|\varphi\|$.

Now fix some $\varphi \in H^1(G, \mathbb{R})$. By the same argument as in the proof of [FL16a, Theorem 5.5], we can find a generating set $s_1, ..., s_n$ of F_n such that $\mu(s_1) \neq 0$ and $\varphi \circ \mu(s_1) = 0$. Denote the Fox matrix of π_α with respect to the presentation coming from this generating set by A (see Lemma 6.7). Let A_1 be the $n \times n$-matrix obtained from A by deleting the first column. Then we have by Lemma 6.8

$$\rho_u^{(2)}(\alpha;\mu) = -[r_{\mu(A_1)} \colon \mathbb{Z}G^n \to \mathbb{Z}G^n] + [r_{\mu(s_1)-1} \colon \mathbb{Z}G \to \mathbb{Z}G].$$

Now put

$$\|\cdot\|_1 = \mathfrak{N}\big(\mathbb{P}([r_{\mu(A_1)}])\big) \quad \text{and} \quad \|\cdot\|_2 = \mathfrak{N}\big(\mathbb{P}(-[r_{\mu(s_1)-1}])\big).$$

Then $\|\cdot\|_1$ is a seminorm by Theorem 6.10 and we note that $\|\varphi\|_2 = \varphi \circ \mu(s_1) = 0$. This already shows $\|\varphi\| \geq 0$. Given any $\psi \in H^1(G;\mathbb{R})$ we calculate

$$\begin{aligned} \|\varphi + \psi\| &= \|\varphi+\psi\|_1 - \|\varphi+\psi\|_2 \\ &= \|\varphi+\psi\|_1 - (\varphi+\psi)\circ\mu(s_1) \\ &\leq \|\varphi\|_1 + \|\psi\|_1 - \psi\circ\mu(s_1) \\ &= \|\varphi\| + \|\psi\|, \end{aligned}$$

which completes the proof. □

6.3 L^2-torsion invariants of UPG automorphisms

In this section we compute the universal L^2-torsion of a special class of free group automorphisms called *unipotent polynomially growing*. As a corollary we can determine all L^2-torsion invariants of polynomially growing automorphisms.

6.3.1 Universal L^2-torsion of UPG automorphisms.

Definition 6.14 (Unipotent polynomially growing automorphisms)**.** Let d be a word metric on a finitely generated free group F_n. An automorphism $\alpha\colon F_n \to F_n$ is *polynomially growing* if for every $g \in F_n$ the quantity $d(1, \alpha^k(g))$ grows at most polynomially in k . If, additionally, the image of α under the projection $\mathrm{Aut}(F_n) \to \mathrm{GL}(n,\mathbb{Z})$ is unipotent, then α is *unipotent polynomially growing*, short *UPG*.

We will prove

Theorem 6.15 (Universal L^2-torsion of UPG automorphisms)**.** *Let* $\pi_\alpha = F_n \rtimes_\alpha \mathbb{Z}$ *with* $n \geq 1$ *and* $\alpha\colon F_n \to F_n$ *a UPG automorphism. Then there are elements* $g_1, ..., g_{n-1} \in$

$\pi_\alpha \setminus F_n$ (which can be chosen to coincide with those of Theorem 6.16 below) such that for any admissible homomorphism $\mu\colon \pi_\alpha \to G$ to a torsion-free group G, we have $\mu(g_i) \neq 0$ and

$$\rho_u^{(2)}(\alpha;\mu) = -\sum_{i=1}^{n-1} [r_{\mu(1-g_i)}\colon \mathbb{Z}G \to \mathbb{Z}G].$$

The proof of Theorem 6.15 is motivated by the following computation of the BNS invariant of HNN extensions along polynomially growing automorphisms due to Cashen-Levitt [CL16, Theorem 5.2].

Theorem 6.16 (BNS invariant of polynomially growing automorphisms)**.** *Let $\pi_\alpha = F_n \rtimes_\alpha \mathbb{Z}$ with $n \geq 2$ and $\alpha\colon F_n \to F_n$ a polynomially growing automorphism. Then there are elements $g_1, ..., g_{n-1} \in \pi_\alpha \setminus F_n$ such that*

$$\Sigma(\pi_\alpha) = -\Sigma(\pi_\alpha) = \{[\varphi] \in S(\pi_\alpha) \mid \varphi(g_i) \neq 0 \text{ for all } 1 \leq i \leq n-1\}.$$

Both Cashen-Levitt's theorem and our result are based on the following lemma.

Lemma 6.17. *For $n \geq 2$ and a UPG automorphism $\alpha \in \mathrm{Aut}(F_n)$, there exists $\beta \in \mathrm{Aut}(F_n)$ representing the same outer automorphism class as α such that either*

(1) there is a non-trivial β-invariant splitting $F_n = B_1 * B_2, \beta = \beta_1 * \beta_2$; or

(2) there is a splitting $F_n = B_1 * \langle x\rangle$ such that B_1 is β-invariant and $\beta(x) = xu$ for some $u \in B_1$.

Proof. This is [CL16, Proposition 5.9] and its proof relies on Bestvina–Feighn–Handel's train track theory [BFH00]. □

We also mention

Lemma 6.18. *Every polynomially growing automorphism $\alpha\colon F_n \to F_n$ has a power that is UPG.*

Proof. This is the content of [BFH00, Corollary 5.7.6]. □

Proof of Theorem 6.15. We use induction on the rank n of the free base group F_n.

For the base case $n = 1$ we have $\alpha = \pm\mathrm{id}_{\mathbb{Z}}$. But $-\mathrm{id}\colon \mathbb{Z} \to \mathbb{Z}$ is not unipotent, so we must have $\alpha = \mathrm{id}$. Hence $\pi_\alpha = \mathbb{Z} \rtimes_{\mathrm{id}} \mathbb{Z} = \mathbb{Z}^2$, for which we have seen in Example 3.46 that $\rho_u^{(2)}(\mathbb{Z}^2;\mu) = 0$. Since $n-1 = 0$, the set of elements $g_1, ..., g_{n-1}$ is empty and so the sum appearing in Theorem 6.15 vanishes as well. For the base case in Theorem 6.16 one recalls $\Sigma(\mathbb{Z}^2) = S(\mathbb{Z}^2)$ from Example 2.10.

For $n \geq 2$ we first invoke Lemma 6.17. As the isomorphism class of $F_n \rtimes_\alpha \mathbb{Z}$ only depends on the outer automorphism class of α, we can assume that α itself admits a splitting as in Lemma 6.17. The two cases appearing there will now be dealt with separately.

Case 1: There is an α-invariant splitting $F_n = B_1 * B_2$, $\alpha = \alpha_1 * \alpha_2$. Put $\pi_i = B_i \rtimes_{\alpha_i} \mathbb{Z}$ and denote the stable letter in both products by t. Then we have

$$\pi_\alpha = F_n \rtimes_\alpha \mathbb{Z} \cong \pi_1 *_{\langle t\rangle} \pi_2. \tag{6.1}$$

For $i = 1, 2$ let r_i denote the rank of B_i. The induction hypothesis applied to π_i gives elements

$$g_1^{(i)}, \ldots, g_{r_i-1}^{(i)} \in \pi_i \setminus B_i$$

such that $\mu(g_j^{(i)}) \neq 0$ for all $i = 1, 2$ and $1 \leq j \leq r_i - 1$, and we have

$$\rho_u^{(2)}(\alpha_i; \mu|_{\pi_i}) = -\sum_{j=1}^{r_i-1} [\mu(1 - g_j^{(i)})].$$

As in the proof of Theorem 6.16 we take $g_1, ..., g_{n-1}$ to be the union of the $g_j^{(i)}$ ($i = 1, 2$, $1 \leq j \leq r_i - 1$) and the generator t of the edge group of the splitting (6.1). Since μ is admissible, we have $\mu(t) \neq 0$. Also notice that $r_1 + r_2 = n$ and if we take free generating sets of B_1 and B_2, then the Fox matrix of α with respect to their union is of the form

$$F(\alpha) = \begin{pmatrix} F(\alpha_1) & 0 \\ 0 & F(\alpha_2) \end{pmatrix}.$$

Now Lemma 6.8 allows us to compute in $\mathrm{Wh}^w(G)$

$$\begin{aligned}
\rho_u^{(2)}(\alpha; \mu) &= \big[\mu(t-1)\big] - \big[\mu(\mathrm{Id} - t \cdot F(\alpha))\big] \\
&= \big[\mu(t-1)\big] - \big[\mu(\mathrm{Id} - t \cdot F(\alpha_1))\big] - \big[\mu(\mathrm{Id} - t \cdot F(\alpha_2))\big] \\
&= \quad \rho_u^{(2)}(\alpha_1; \mu|_{\pi_1}) + \rho_u^{(2)}(\alpha_2; \mu|_{\pi_2}) - \big[\mu(t-1)\big] \\
&= -\sum_{j=1}^{r_1-1} \big[\mu(1 - g_j^{(1)})\big] - \sum_{j=1}^{r_2-1} \big[\mu(1 - g_j^{(2)})\big] - \big[\mu(t-1)\big] \\
&= -\sum_{j=1}^{n-1} \big[\mu(1 - g_j)\big].
\end{aligned}$$

Case 2: There is a splitting $F_n = B_1 * \langle x \rangle$ such that B_1 is α-invariant and $\alpha(x) = xu$ for some $u \in B_1$. In this case, let $\alpha_1 = \alpha|_{B_1}$ and $\pi_1 = B_1 \rtimes_{\alpha_1} \mathbb{Z}$. Denote the stable letter of π_1 and π_α by t. First we notice

$$\begin{aligned}
\pi_\alpha &= \langle F_n, t \mid t^{-1} y t = \alpha_1(y) \text{ for all } y \in F_n \rangle \\
&= \langle B_1, x, t \mid t^{-1} b t = \alpha_1(b) \text{ for all } b \in B_1, t^{-1} x t = xu \rangle \\
&= \langle \pi_1, x \mid x^{-1} t x = t u^{-1} \rangle \\
&= \pi_1 *_{\alpha'},
\end{aligned} \tag{6.2}$$

where $\alpha' \colon \langle t \rangle \to \langle t u^{-1} \rangle$ maps t to $t u^{-1}$.

From the induction hypothesis applied to π_1 we get elements $g_1, ..., g_{n-2} \in \pi_1 \setminus B_1$ such that $\mu(g_j) \neq 0$ for all $1 \leq j \leq n-2$, and

$$\rho_u^{(2)}(\alpha_1; \mu|_{\pi_1}) = -\sum_{j=1}^{n-2} [\mu(1 - g_j)].$$

As in the proof of Theorem 6.16 we add to this set the generator $g_{n-1} = t$ of the edge group of the splitting (6.2). Since μ is admissible, we have $\mu(t) \neq 0$.

If we take as free generating set for F_n the union of a free generating set of B_1 and $\{x\}$, then the Fox matrix of α is of the form

$$F(\alpha) = \begin{pmatrix} F(\alpha_1) & 0 \\ * & 1 \end{pmatrix}$$

Now Lemma 6.8 allows us to compute in $\mathrm{Wh}^w(G)$

$$\begin{aligned}
\rho_u^{(2)}(\alpha;\mu) &= \big[\mu(t-1)\big] - \big[\mu(\mathrm{Id} - t\cdot F(\alpha))\big] \\
&= \big[\mu(t-1)\big] - \big[\mu(\mathrm{Id} - t\cdot F(\alpha_1))\big] - \big[\mu(1-t)\big] \\
&= \rho_u^{(2)}(\alpha_1;\mu|_{\pi_1}) - \big[\mu(1-t)\big] \\
&= -\sum_{j=1}^{n-2} \big[\mu(1-g_j)\big] - \big[\mu(1-t)\big] \\
&= -\sum_{j=1}^{n-1} \big[\mu(1-g_j)\big].
\end{aligned} \tag{6.3}$$

This finishes the proof of Theorem 6.15 $\square$

Remark 6.19 (Extension to polynomially growing automorphisms)**.** We suspect Theorem 6.15 to hold as well for polynomially growing automorphisms. However, Lemma 6.18 is not sufficient for this. In order to reduce Theorem 6.15 for polynomially growing automorphisms to the case of UPG automorphisms, one also needs a better understanding of the restriction homomorphism

$$i^*\colon \mathrm{Wh}^w(F_n \rtimes_\alpha \mathbb{Z}) \to \mathrm{Wh}^w(F_n \rtimes_{\alpha^k} \mathbb{Z})$$

(induced by the obvious inclusion $i\colon F_n \rtimes_{\alpha^k} \mathbb{Z} \to F_n \rtimes_\alpha \mathbb{Z}$) since it maps $\rho_u^{(2)}(F_n \rtimes_\alpha \mathbb{Z})$ to $\rho_u^{(2)}(F_n \rtimes_{\alpha^k} \mathbb{Z})$, see Theorem 3.45 (5).

6.3.2 L^2-torsion polytope, L^2-Euler characteristics and L^2-torsion functions for UPG automorphisms. Our first corollary is an analogue of Friedl-Tillmann's [FT15, Theorem 1.1]. For this recall the BNS-invariant $\Sigma(G)$ of a finitely generated group G as introduced in Section 2.4.

Corollary 6.20 (L^2-torsion polytope determines BNS invariant for UPG automorphisms)**.** *Let $\pi_\alpha = F_n \rtimes_\alpha \mathbb{Z}$ with $n \geq 2$ and $\alpha\colon F_n \to F_n$ a UPG automorphism. Then:*

(1) For any large epimorphism $\mu\colon \pi_\alpha \to G$ onto a torsion-free group satisfying the Atiyah Conjecture we have

$$P(\alpha;\mu) = P(\alpha)$$

and this element is represented by a symmetric polytope. In particular, all higher-order Alexander norms agree with $\delta(\alpha)$.

(2) For any $\varphi \in H^1(\pi_\alpha;\mathbb{R})$ we have

$$[\varphi] \in \Sigma(\pi_\alpha) \text{ if and only if } F_\varphi(P(\alpha)) = 0 \text{ in } \mathcal{P}_T(H_1(\pi_\alpha)_f),$$

i.e., if and only if φ maximizes $P(\alpha)$ in a single vertex.

Proof. The first part follows directly from Theorem 6.15, namely we compute for the elements $g_1, ..., g_{n-1}$ appearing there

$$P(\alpha;\mu) = \mathbb{P}\big(-\rho_u^{(2)}(\alpha;\mu)\big) = \mathbb{P}\left(\sum_{i=1}^{n-1} [r_{\mu(1-g_i)}\colon \mathbb{Z}G \to \mathbb{Z}G]\right) = \sum_{i=1}^{n-1} P(1-g_i) = P(\alpha).$$

Since the higher-order Alexander norms are determined by the L^2-torsion polytopes $P(\alpha;\mu)$ (see Corollary 3.29 and Theorem 3.52), the 'in particular' part follows immediately.

We also deduce that any one-dimensional face of $P(\alpha)$ contains a translate of $P(1-g_i)$ for some $1 \leq i \leq n-1$. Thus we obtain the following list of equivalences.

$$\begin{aligned} F_\varphi(P(\alpha)) \neq 0 &\Leftrightarrow F_\varphi(P(\alpha)) \text{ is not a vertex} \\ &\Leftrightarrow F_\varphi(P(\alpha)) \text{ contains a one-dimensional face} \\ &\Leftrightarrow F_\varphi(P(\alpha)) \text{ contains a translate of } P(1-g_i) \text{ for some } i \\ &\Leftrightarrow \varphi(g_i) = 0 \text{ for some } i \\ &\Leftrightarrow [\varphi] \notin \Sigma(\pi_\alpha) \end{aligned}$$

where the last equivalence is precisely Cashen-Levitt's Theorem 6.16. □

Recall from Theorem 3.31 that for 3-manifolds and free group HNN extensions the degree of L^2-torsion function is in general an upper bound for the corresponding twisted L^2-Euler characteristic. Our second corollary strengthens this for polynomially growing automorphisms.

Corollary 6.21 (Equality of L^2-Euler characteristic and degree of L^2-torsion function)**.** *Let $\pi_\alpha = F_n \rtimes_\alpha \mathbb{Z}$ with $n \geq 1$ and $\alpha\colon F_n \to F_n$ a polynomially growing automorphism. Then there are elements $g_1, ..., g_{n-1} \in \pi_\alpha \smallsetminus F_n$ and a positive integer k such that for any $\varphi \in H^1(\pi_\alpha;\mathbb{R})$:*

(1) We have

$$\delta(\alpha)(\varphi) = -\chi^{(2)}(\widetilde{T_{B\alpha}}; \mathcal{N}(\pi_\alpha), \varphi) = \frac{1}{k} \cdot \sum_{i=1}^{n-1} |\varphi(g_i)|,$$

where $\widetilde{T_{B\alpha}} \to T_{B\alpha}$ denotes the universal cover.

(2) The φ-twisted L^2-torsion function is given by

$$\rho^{(2)}(\widetilde{T_{B\alpha}}; \varphi)(t) \doteq \frac{1}{k} \cdot \begin{cases} \sum_{\varphi(g_i)<0} \varphi(g_i) \cdot \log(t) & \text{if } t \leq 1; \\ \sum_{\varphi(g_i)>0} \varphi(g_i) \cdot \log(t) & \text{if } t \geq 1. \end{cases}$$

In particular,

$$\deg(\rho^{(2)}(\widetilde{T_{B\alpha}}; \varphi)) = \frac{1}{k} \cdot \sum_{i=1}^{n-1} |\varphi(g_i)|.$$

In particular, we have the equalities

$$\delta(\alpha)(\varphi) = -\chi^{(2)}(\widetilde{T_{B\alpha}}; \mathcal{N}(\pi_\alpha), \varphi) = \deg(\rho^{(2)}(\widetilde{T_{B\alpha}}; \varphi))$$

Proof. (1) The first equality is Theorem 3.52. By Lemma 6.18 α admits a power that is UPG, say α^k. We view $\pi_{\alpha^k} = F_n \rtimes_{\alpha^k} \mathbb{Z}$ as an index k subgroup in π_α and denote the inclusion by i. Then we have by the restriction formula of Theorem 3.16 (4)

$$\chi^{(2)}(\widetilde{T_{B\alpha}}; \mathcal{N}(\pi_\alpha), \varphi) = \frac{1}{k} \cdot \chi^{(2)}(\widetilde{T_{B\alpha}}; \mathcal{N}(\pi_{\alpha^k}), \varphi \circ i).$$

Since α^k is UPG, Theorem 6.15 provides elements $g_1, ..., g_{n-1} \in \pi_{\alpha^k} \smallsetminus F_n$ such that

$$\begin{aligned}
\delta(\alpha^k)(\varphi \circ i) &= \mathfrak{N}(\mathbb{P}(-\rho_u^{(2)}(\alpha^k)))(\varphi \circ i) \\
&= \mathfrak{N}\left(\sum_{i=1}^{n-1} P(1-g_i)\right)(\varphi \circ i) \\
&= \sum_{i=1}^{n-1} |\varphi(g_i)|.
\end{aligned}$$

Thus we may take the same g_i for α instead of α^k in order to deduce the desired statement.

(2) With the aid of Remark 3.1 and Theorem 6.15 we calculate the $(\varphi \circ i)$-twisted L^2-torsion function of α^k to be

$$\begin{aligned}
\rho^{(2)}(\widetilde{T_{B\alpha}}; \varphi \circ i)(t) &= \sum_{i=1}^{n-1} \rho^{(2)}(\mathrm{el}(r_{1-g_i}); \varphi \circ i)(t) \\
&= \sum_{i=1}^{n-1} \rho^{(2)}(\mathrm{el}(r_{1-t^{\varphi(g_i)}\cdot g_i})) \\
&= \begin{cases} \sum_{\varphi(g_i)<0} \varphi(g_i) \cdot \log(t) & \text{if } t \leq 1; \\ \sum_{\varphi(g_i)>0} \varphi(g_i) \cdot \log(t) & \text{if } t \geq 1. \end{cases}
\end{aligned}$$

Now apply the restriction formula of Theorem 3.10 (5) to $i\colon \pi_{\alpha^k} \to \pi_\alpha$. $\square$

Remark 6.22 (Rank of the fiber)**.** If $\varphi\colon \pi_\alpha \to \mathbb{Z}$ is an epimorphism with finitely generated kernel $K = \ker(\varphi)$, it is well-known that K is free itself [GMSW01, Theorem 2.6 and Remark 2.7]. If α is polynomially growing, then Cashen-Levitt [CL16, Theorem 6.1] compute the rank of this kernel to be

$$\operatorname{rank}(K) = 1 + \frac{1}{l} \cdot \sum_{i=1}^{n-1} |\varphi(g_i)|,$$

where l is the least positive integer such that α^l is UPG and g_i are the elements appearing in Theorem 6.15 for α^l. We can easily derive this rank computation from Corollary 6.21 (1) with the aid of Lemma 3.17. Namely, if we let $k\colon K \to \pi_\alpha$ be the inclusion, then

$$\begin{aligned}
\frac{1}{l} \cdot \sum_{i=1}^{n-1} |\varphi(g_i)| &= -\chi^{(2)}(\widetilde{T_{B\alpha}}; \mathcal{N}(\pi_\alpha), \varphi) \\
&= -\chi^{(2)}(k^*\widetilde{T_{B\alpha}}; \mathcal{N}(K)) \\
&= -\chi^{(2)}(K) \\
&= \operatorname{rank}(K) - 1.
\end{aligned}$$

7 References

[AFW15] M. Aschenbrenner, S. Friedl, and H. Wilton, *3-manifold groups*, EMS Series of Lectures in Mathematics (2015).

[Ago08] I. Agol, *Criteria for virtual fibering*, J. Topol. **1** (2008), no. 2, 269–284.

[Ago13] ______, *The virtual Haken conjecture*, Doc. Math. **18** (2013), 1045–1087. With an appendix by Agol, Daniel Groves, and Jason Manning.

[AJZN11] M. Abért, A. Jaikin-Zapirain, and N. Nikolov, *The rank gradient from a combinatorial viewpoint*, Groups Geom. Dyn. **5** (2011), 213–230.

[Ale28] J. W. Alexander, *Topological invariants of knots and links*, Trans. Amer. Math. Soc. **30** (1928), no. 2, 275–306.

[AN12] M. Abért and N. Nikolov, *Rank gradient, cost of groups and the rank versus Heegaard genus problem*, J. European Math. Soc. **14** (2012), no. 5, 1657–1677.

[BA16] F. Ben Aribi, *The L^2-Alexander invariant detects the unknot*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), vol. XV (2016), 683–708.

[BFH00] M. Bestvina, M. Feighn, and M. Handel, *The Tits alternative for* $\mathrm{Out}(F_n)$*. II. A Kolchin type theorem*, Ann. of Math. (2) **151** (2000), no. 2, 517–623.

[BNS87] R. Bieri, W. D. Neumann, and R. Strebel, *A geometric invariant of discrete groups*, Invent. Math. **90** (1987), no. 3, 451–477.

[BP92] R. Benedetti and C. Petronio, *Lectures on hyperbolic geometry*, Universitext, Springer, Berlin, 1992.

[BS81] R. Bieri and R. Strebel, *A geometric invariant for modules over an abelian group*, J. Reine Angew. Math. **322** (1981), 170–189.

[But07] J. O. Button, *Mapping tori with first Betti number at least two*, J. Math. Soc. Japan **59** (2007), no. 2, 351–370. MR2325689 (2008i:57002)

[CFF17] J. C. Cha, S. Friedl, and F. Funke, *The Grothendieck group of polytopes and norms*, Münster J. Math. **10** (2017), 75–81.

[CL16] C. H. Cashen and G. Levitt, *Mapping tori of free group automorphisms, and the Bieri–Neumann–Strebel invariant of graphs of groups*, J. Group Theory **19** (2016), no. 2, 191–216.

[Cla17] M. Clay, *ℓ^2-torsion of free-by-cyclic groups*, Quarterly J. Math **68** (2017), 617–634.

[Cla99] B. Clair, *Residual amenability and the approximation of L^2-invariants*, Michigan Math. J. **46** (1999), 331–346.

[Coc04] T. Cochran, *Noncommutative knot theory*, Alg. Geom. Topol. **4** (2004), 347–398.

[Coh73] M. M. Cohen, *A course in simple-homotopy theory*, Graduate Texts in Mathematics, vol. 10, Springer-Verlag, New York, 1973.

[COT03] T. Cochran, K. Orr, and P. Teichner, *Knot concordance, Whitney towers and L^2-signatures*, Ann. of Math. (2) **157** (2003), 433–519.

[DFL15a] J. Dubois, S. Friedl, and W. Lück, *The L^2-Alexander torsion is symmetric*, Alg. Geom. Top. **15** (2015), no. 6, 3599–3612.

[DFL15b] ———, *Three flavors of twisted invariants of knots*, Introduction to Modern Mathematics, Advanced Lectures in Mathematics **33** (2015), 143–170.

[DFL16] ———, *The L^2-Alexander torsion of 3-manifolds*, J. Topology **9** (2016), no. 3, 889–926.

[Die43] J. Dieudonné, *Les déterminants sur un corps non commutatif*, Bull. Soc. Math. France **71** (1943), 27–45.

[Dun01] N. M. Dunfield, *Alexander and Thurston norms of fibered 3-manifolds*, Pac. J. Math. **200** (2001), no. 1, 43–58.

[DW10] J. Dubois and C. Wegner, *L^2-Alexander invariant for torus knots*, C. R. Math. Acad. Sci. Paris **348** (2010), no. 21-22, 1185–1189.

[DW15] ———, *Weighted L^2-invariants and applications to knot theory*, Commun. Contemp. Math. **17** (2015), no. 1.

[ES05] G. Elek and E. Szabó, *Hyperlinearity, essentially free actions and L^2-invariants*, Math. Ann. **332** (2005), 421—441.

[FH07] S. Friedl and S. Harvey, *Non-commutative Multivariable Reidemeister Torsion and the Thurston Norm*, Alg. Geom. Top. **7** (2007), 755–777.

[FK06] S. Friedl and T. Kim, *The Thurston norm, fibered manifolds and twisted Alexander polynomials*, Topology **45** (2006), no. 6, 929 –953.

[FK08] ———, *Twisted Alexander norms give lower bounds on the Thurston norm*, Trans. Amer. Math. Soc. **360** (2008), 4597–4618.

[FK16] F. Funke and D. Kielak, *Alexander and Thurston norms, and the Bieri-Neumann-Strebel invariants for free-by-cyclic groups*, arXiv:1605.09067 (2016).

[FL15] S. Friedl and W. Lück, *The L^2-torsion function and the Thurston norm of 3-manifolds*, arXiv:1510.00264 (2015).

[FL16a] ———, *L^2-Euler characteristics and the Thurston norm*, arXiv:1609.07805 (2016).

[FL16b] ———, *Universal L^2-torsion, polytopes and applications to 3-manifolds*, arXiv:1609.07809 (2016).

[FLT16] S. Friedl, W. Lück, and S. Tillmann, *Groups and polytopes*, arXiv:1609.07805 (2016).

[Fox53] R. H. Fox, *Free differential calculus. I. Derivation in the free group ring*, Ann. of Math. (2) **57** (1953), 547–560.

[Fox54] ———, *Free differential calculus. II. The isomorphism problem of groups*, Ann. of Math. (2) **59** (1954), 196–210.

[Fri07] S. Friedl, *Reidemeister torsion, the Thurston norm and Harvey's invariants*, Pac. J. Math. **230** (2007), 271–296.

[FSW16] S. Friedl, D. S. Silver, and S. G. Williams, *The Turaev and Thurston norms*, Pacific J. Math. **284** (2016), no. 2, 365–382.

[FT15] S. Friedl and S. Tillmann, *Two-generator one-relator groups and marked polytopes*, arXiv:1501.03489 (2015).

[Fun16] F. Funke, *The integral polytope group*, arXiv:1605.01217 (2016).

[Fun17] ———, *The L^2-torsion polytope of amenable groups*, arXiv:1704.07164 (2017).

[FV15] S. Friedl and S. Vidussi, *The Thurston norm and twisted Alexander polynomials*, J. reine angew. Math. **707** (2015), 87–102.

[Gab00] D. Gaboriau, *Coût des relations d'équivalence et des groupes*, Invent. math. **139** (2000), no. 1, 41–98.

[Gab83] D. Gabai, *Foliations and the topology of 3-manifolds*, J. Differential Geom. **18** (1983), no. 3, 445–503.

[GMSW01] R. Geoghegan, M. L. Mihalik, M. Sapir, and D. T. Wise, *Ascending HNN extensions of finitely generated free groups are Hopfian*, Bull. London Math. Soc. **33** (2001), no. 3, 292–298.

[Gro83] M. Gromov, *Volume and bounded cohomology*, Inst. Hautes Etudes Sci. Publ. Math. **56** (1983), 5–99.

[Har05] S. L. Harvey, *Higher-order polynomial invariants of 3-manifolds giving lower bounds for the Thurston norm*, Topology **44** (2005), 895–945.

[Har06] ———, *Monotonicity of degrees of generalized Alexander polynomials of groups and 3-manifolds*, Math. Proc. Cambridge Philos. Soc. **140** (2006), 431–450.

[Her16] G. Herrmann, *The L^2-Alexander torsion for Seifert fiber spaces*, arXiv:1602.08768 (2016).

[HL92] J.A. Hillman and P.A. Linnell, *Elementary amenable groups of finite Hirsch length are locally-finite by virtually solvable*, J. Austral. Math. Soc. (Series A) **52** (1992), 237–241.

[HS98] E. Hess and T. Schick, *L^2-torsion of hyperbolic manifolds*, manuscripta mathematica **97** (1998), no. 3, 329–334.

[Kho97] A.G. Khovanskii, *Consistent partitions of polytopes and polynomial measures*, Topics in Singularity Theory: V.I. Arnold's 60th Anniversary Collection, Amer. Math. Soc. Transl., Ser. 2, Vol. 180, Amer. Math. Soc., Providence, RI, 1997.

[Kla99] D. A. Klain, *An Euler relation for valuations of polytopes*, Advances in Math. **147** (1999), no. 1, 1–34.

[KLM88] P. H. Kropholler, P. A. Linnell, and J. A. Moody, *Applications of a new K-theoretic theorem to soluble group rings*, Proc. Amer. Math. Soc. **104** (1988), no. 3, 675–6843.

[KP92] A.G. Khovanskii and A. Pukhlikov, *Finitely additive measures of virtual polytopes*, Algebra and Analysis **4** (1992), no. 2, 161–185; translation in St. Petersburg Math. J. **4** (1993), no. 2, 337–356.

[KS69] R. C. Kirby and L. C. Siebenmann, *On the triangulation of manifolds and the Hauptvermutung*, Bull. Amer. Math. Soc. **75** (1969), 742–749.

[Lin06] P. A. Linnell, *Noncommutative localization in group rings*, Non-commutative localization in algebra and topology, 2006.

[Lin93] ———, *Division rings and group von Neumann algebras*, Forum Math. **5** (1993), no. 6, 561–576.

[Liu13] Y. Liu, *Virtual cubulation of nonpositively curved graph manifolds*, J. of Topology **6** (2013), 793–822.

[Liu17] ———, *Degree of L^2-Alexander torsion for 3-manifolds*, Invent. math. **207** (2017), 981–1030.

[LL16] P. Linnell and W. Lück, *Localization, Whitehead groups, and the Atiyah Conjecture*, arXiv:1602.06906 (2016).

[LR05] W. Lück and H. Reich, *The Baum-Connes and the Farrell-Jones Conjectures in K- and L-theory*, Handbook of K-theory, Volume 2, Springer, Berlin, 2005.

[LS99] W. Lück and T. Schick, *L^2-torsion of hyperbolic manifolds of finite volume*, Geom. Func. Anal. **9** (1999), 518–567.

[LT14] H. Li and A. Thom, *Entropy, Determinants, and L^2-Torsion*, J. Amer. Math. Soc. **27** (2014), no. 1, 239–292.

[Lüc02] W. Lück, *L^2-invariants: theory and applications to geometry and K-theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 44, Springer-Verlag, Berlin, 2002.

[Lüc15] ———, *Twisting L^2-invariants with finite-dimensional representations*, arXiv:1510.00057 (2015).

[Lüc94a] ———, *Approximating L^2-invariants by their finite-dimensional analogues*, Geom. Funct. Anal. **4** (1994), 455–481.

[Lüc94b] ———, *L^2-Betti numbers of mapping tori and groups*, Topology **33** (1994), 203–214.

[LZ06a] W. Li and W. Zhang, *An L^2-Alexander-Conway invariant for knots and the volume conjecture,* Differential geometry and physics, Nankai Tracts Math., vol. 10, World Sci. Publ., Hackensack, NJ, 2006.

[LZ06b] ———, *An L^2-Alexander invariant for knots*, Commun. Contemp. Math. **8** (2006), 167–187.

[LZ08] ———, *Twisted L^2-Alexander-Conway invariants for knots,* Topology and physics, Nankai Tracts Math., vol. 12, World Sci. Publ., Hackensack, NJ, 2008.

[McM00] C. T. McMullen, *Polynomial invariants for fibered 3-manifolds and Teichmüller geodesics for foliations*, Ann. Sci. École Norm. Sup. (4) **33** (2000), no. 4, 519–560.

[McM02] ———, *The Alexander polynomial of a 3-manifold and the Thurston norm on cohomology*, Ann. Sci. École Norm. Sup. (4) **35** (2002), no. 2, 153–171.

[McM77] P. McMullen, *Valuations and Euler-type relations on certain classes of convex polytopes*, Proc. London Math. Soc. **35** (1977), 113–135.

[McM89] ———, *The polytope algebra*, Adv. Math. **78** (1989), no. 1, 76–130.

[Mil62] J. Milnor, *A Duality theorem for Reidemeister Torsion*, Ann. of Math. **76** (1962), 134–147.

[OW80] D. S. Ornstein and B. Weiss, *Ergodic theory of amenable group actions. I: The Rohlin lemma*, Bull. Amer. Math. Soc. **2** (1980), no. 1, 161–164.

[PW12] P. Przytycki and D. T. Wise, *Mixed* 3*-manifolds are virtually special*, arXiv:1205.6742 (2012).

[PW14] ———, *Graph manifolds with boundary are virtually special*, J. of Topology **7** (2014), 419–435.

[Ros94] J. Rosenberg, *Algebraic K-Theory and Its Applications*, Graduate Text in Mathematics, vol. 147, Springer, New York, 1994.

[Sal68] G. T. Sallee, *Polytopes, valuations, and the Euler relation,* Canad. J. Math. **20** (1968), 1412 –1424.

[Sch01] T. Schick, *L^2-determinant class and approximation of L^2-Betti numbers*, Trans. Amer. Math. Soc. **353** (2001), 3247–3265.

[Sch93] R. Schneider, *Convex bodies: the Brunn-Minkowski theory*, Cambridge Univ. Press, Cambridge, 1993.

[Sil81] R. J. Silvester, *Introduction to algebraic K-theory*, Chapman & Hall, London, 1981.

[Spe50] E. Specker, *Additive Gruppen von Folgen ganzer Zahlen*, Port. Math. **9** (1950), 131–140.

[Ste75] B. Stenström, *Rings of quotients*, Die Grundlehren der Mathematischen Wissenschaften, Band 217, An introduction to methods of ring theory, Springer, New York, 1975.

[Str12] R. Strebel, *Notes on the Sigma invariants*, arXiv:1204.0214, 2012.

[Thu86] W. P. Thurston, *A norm for the homology of* 3*-manifolds*, Mem. Amer. Math. Soc. **59** (1986), no. 339, i–vi and 99–130.

[Tur02] V. G. Turaev, *A homological estimate for the Thurston norm*, arXiv:math. GT (2002).

[Tur75] ———, *The Alexander polynomial of a three-dimensional manifold*, Math. USSR Sbornik **26** (1975), 313–329.

[Wal78] F. Waldhausen, *Algebraic K-Theory of Generalized Free Products, Part 2*, Ann. of Math. **108** (1978), no. 2, 205–256.

[Weg00] C. Wegner, *L^2-invariants of finite aspherical CW-complexes with fundamental group containing a non-trivial elementary amenable normal subgroup*, Ph.D. thesis, Münster (2000).

[Weg09] ———, *L^2-invariants of finite aspherical CW-complexes*, manuscripta math. **128** (2009), no. 4, 469–481.

[Weg15] ———, *The Farrell-Jones Conjecture for virtually solvable groups*, J. Topology **8** (2015), no. 4, 975–1016.

[Wis12a] D. T. Wise, *From riches to RAAGs: 3-manifolds, right–angled Artin groups, and cubical geometry*, CBMS Regional Conference Series in Mathematics, vol. 117, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 2012.

[Wis12b] ———, *The structure of groups with a quasiconvex hierarchy*, Preprint (2012).

Zusammenfassung

Die Dissertation “The L^2-Torsion Polytope of Groups and the Integral Polytope Group” beschäftigt sich mit Torsions-Invarianten von freien endlichen G-CW-Komplexen, die L^2-*azyklisch* sind, d.h. dessen L^2-Bettizahlen gänzlich verschwinden. Ein Großteil dieser Invarianten wurde intensiv für 3-Mannigfaltigkeiten untersucht, vor allem in Verbindung mit der *Thurston-Norm*. Dagegen liegt der Fokus dieser Arbeit auf der *universellen L^2-Torsion* und dem *L^2-Torsions-Polytop* von Gruppen. Diese beiden Invarianten wurden kürzlich von Friedl-Lück [FL16b] konstruiert.

Für eine Gruppe G ist das L^2-Torsions-Polytop $P(G)$ ein Element der *integralen Polytop-Gruppe* $\mathcal{P}(H_1(G)_f)$ der freien ersten Homologie $H_1(G)_f$ von G. Da die integrale Polytop-Gruppe erst in diesem Zusammenhang größere Aufmerksamkeit erhielt, sind Ergebnisse über die Struktur dieser Gruppe bisher nur vereinzelt vorhanden. Andererseits verspricht eine detaillierte Untersuchung von $\mathcal{P}(H)$ rückwirkend auch Informationen über $P(G)$. In unserem ersten Hauptergebnis Theorem 4.1 führen wir eine solche Untersuchung aus. Wir konstruieren dort unter anderem eine geometrisch fassbare Basis für $\mathcal{P}(H)$ und interpretieren die Involution auf $\mathcal{P}(H)$, die durch Spiegelung am Ursprung induziert ist, als eine Art Euler-Charakteristik. Diese Involution weist außerdem zwei Untergruppen von $\mathcal{P}(H)$ aus, nämlich die der symmetrischen und die der asymmetrischen Elemente, die wir konkret bestimmen.

Danach widmen wir uns einer Analyse des L^2-Torsions-Polytop von zwei sehr unterschiedlichen Typen von Gruppen, nämlich auf der einen Seite unendlich amenablen Gruppen und auf der anderen Seite HNN-Erweiterungen von endlich erzeugten freien Gruppen. Aufbauend auf Wegners Beweis [Weg00] für das Verschwinden der klassischen L^2-Torsion von Gruppen, die einen nicht-trivialen elementar amenablen Normalteiler enthalten, führen wir den Begriff einer *Gruppe von $P \geq 0$-Klasse* ein. Wir zeigen in Theorem 5.7, dass unendlich amenable Gruppen, die die Atiyah-Vermutung erfüllen, diese Eigenschaft besitzen. Als Nebeneffekt erhalten wir damit die Homotopie-Invarianz des L^2-Torsions-Polytops über unendlich amenablen Gruppen. In Theorem 5.15 zeigen wir dann, dass jede Gruppe von $P \geq 0$-Klasse, die einen nicht-abelschen elementar amenablen Normalteiler enthält, verschwindendes L^2-Torsions-Polytop hat. Dies bestätigt insbesondere eine Vermutung von Friedl-Lück-Tillmann [FLT16] für den Fall einer elementar amenablen Gruppe. Als Anwendung der Untersuchung der integralen Polytop-Gruppe liefern wir in Proposition 5.19 auch einen Hinweis für diese Vermutung im Falle nicht-elementar amenabler Gruppen.

Während also amenable Gruppen aus Sicht des L^2-Torsions-Polytops gewissermaßen unsichtbar sind, trifft dies auf HNN-Erweiterung von nicht-abelschen freien Gruppen nie zu. Für die Klasse der UPG-Automorphismen einer freien Gruppe berechnen wir in Theorem 6.15 explizit die universelle L^2-Torsion. Dies erlaubt uns in einem zweiten Schritt zu zeigen, dass die Bieri-Neumann-Strebel-Invariante der assoziierten HNN-Erweiterung aus dem L^2-Torsions-Polytop abgelesen werden kann, was ein Analogon eines Theorems von Friedl-Tillmann [FT15] darstellt.