Graded and Generalized Geometry Methods for Gravity

by

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Abstract

We explore a comprehensive framework for gauge and gravity theories, based on a combination of methods from graded symplectic geometry and from generalized geometry. We reformulate some well-established gravitational theories in this language and establish a relation to gauge theories. The models of gravity we consider range from the type II effective string action for the bosonic fields with NS-NS boundary conditions and other actions with T-dual stringy fluxes, to General Relativity in the Palatini formulation with frame fields. A sketch of the technique for reconstructing non-abelian gauge theories is also given.

The general idea is the following: On the symplectic geometry side, we implement a grading up to degree 2 that enlarges the set of coordinates in the local chart for the manifold, so as to naturally support the geometric data of a metric tensor in terms of a graded symplectic structure. This is essential, since the (pseudo-Riemannian) metric is the fundamental field for a gravity theory. Then we implement interactions with gauge and other fields, the metric counted among them, by deforming the canonical Poisson brackets of the graded phase space coordinates. We do not deform the Hamiltonian but rather retain the free one. The relation to gauge theory is obtained via a graded version of Moser lemma: The deformation can be undone by a change of local phase space coordinates. The graded diffeomorphisms that carry this change of coordinates are parametrized by a gauge field and are not unique. The freedom is a gauge symmetry. Differential graded manifolds of the type studied here are classified by higher algebraic structures, such as Lie and Courant algebroids, which encode the symmetries of the bundle of generalized geometry. The correspondence stems from derived brackets with the Hamiltonian homological vector field. In our case the latter is left unchanged by the deformation and all the novelty in the algebroid can be tracked back to the deformed graded Poisson brackets. Furthermore, we present a new formulation of generalized differential geometry that together with the algebroid brackets, enable us to characterize an affine connection, as well as torsion and curvature tensors on the generalized bundle. In terms of the corresponding Ricci scalar the generalized gravity actions are obtained in almost the same fashion as the Hilbert-Einstein action.

Keywords: *graded geometry, graded Poisson structures, generalized geometry, Courant algebroid, Hamiltonian, gravity, supergravity, connection, curvature, torsion, gauge theory, gauge symmetry.*

Contents

Chapter 1

Introduction

More than a century after Einstein's formulation of General Relativity, from the theoretical viewpoint gravity still leaves more intriguing questions than satisfactory answers. The observations at our energy scales and at solar system or galactic distances confirms that GR works perfectly fine. Even the most elusive manifestation of the graviton, the propagation of gravitational waves, remnants of some very energetic phenomena of coalescence and merging of some compact objects, were finally discover by the LIGO-Virgo collaboration in 2017. Nevertheless the nature of gravity, entirely geometrical rather than particle-like, made it difficult to perpetuate the quantization program applied to the other gauge theories of the fundamental forces of the Standard Model. By power-counting, GR is non-renormalizable being its coupling constant (Newton's gravitational constant G_N) dimensionful, hence it would necessitate of an infinite number of counterterms to cure the divergencies that appear from the loop expansion in the Feynman diagrams for the self-interactions of a spin-2 particle. An infinite number of counterterms corresponds to an equal number of parameters due to which the theory hence loses its predictivity. The coupling to fermions and spin is still debatable on the theoretical basis, and requires increasingly complex theories. These open questions are better tackled through the implementation of supersymmetry among the other symmetries and the exploitation of superspaces in the place of ordinary spaces: this culminated in the high energy gravitational theory known as Supergravity. Supergravity is also the effective limit of the various string theories, which in fact contain a massless spin-2 particle in the spectrum and moreover are quantizable. Another reasonable setting where the unsolved matters are better accommodated is a gauge theory of gravity, the Lorentz group playing an important role in it. A further advance of the strings-inspired geometrical and algebraic methods are *differential graded manifolds* (dg-manifolds) combined together with *algebroids*, yielding also a graded Poisson algebra for the gauge theory side. Here we will use this mathematical setup to provide a unified treatment of a few gravitational theories, starting from NS-NS Supergravity to end up with (speculations on) some non-UV completed gravitational theories like gravity with spin connections. It is essentially new to apply and tune these mathematical foundations to rewrite (part of) some known actions. The method shows its power in making various gravitational theories at different regimes descend from the same derivation. A simple and convincing argument comes (as usual!) from relativistic electromagnetism: a Poisson algebra and a Hamiltonian are a good alternative to the Lagrangian formalism for the charged particle interacting with an external electromagnetic field. In fact, shifting the canonical momenta by the electromagnetic potential manages to reproduce the same result in the Hamiltonian theory. The Poisson algebra is deformed as a non-zero $p-p$ Poisson bracket shows up in this way. A more elegant treatment of monopoles is allowed via the Hamiltonian theory too. Being gravity for many aspects quite similar to electromagnetism, it seems legitimate to extend this reasoning to the former. But a metric cannot be endowed in the standard symplectic form because it is a symmetric tensor, hence graded Darboux coordinates come to the rescue.

1.1 Problems addressed and structure of the thesis

Some decades ago it was noticed that it is useful to describe target space with graded Grassmannian variables. $\mathbb{Z}_2$ grading is special in this sense because of its handiness for supersymmetry, however in our work we were concerned above all with gradings in $\mathbb{N}$ or at most $\mathbb{Z}$. A differential calculus adapted to the presence of commuting and anticommuting d-dimensional coordinates is also available [1]. Graded phase space i.e. $T^*\mathcal{M}$, for $\mathcal{M}$ some graded manifold, can host a graded symplectic structure, as much as the standard counterpart T^*M is a Poisson manifold. A Hamiltonian function Θ is also present. The differential originated by Θ and expressed as a Hamiltonian vector field by means of the brackets, $\mathrm{Q} = \{\cdot, \Theta\}$, $\mathrm{Q}^2 = 0$, describes a cohomology which is a twisted de Rham cohomology. In the degree-2 case, the combination $g + B$, where g is a non-degenerate bilinear symmetric form (most of the times a Riemannian metric) and B is a 2-form, can be used to map some of the Darboux coordinates of degree 1, which in the present situation are isomorphic to inner products with vector fields ι_{∂_i} and differential forms dx^i, to a non-canonical chart. The map is a diffeomorphism of $T^*\mathcal{M}$ and sends $\omega \mapsto \omega + d\alpha =: \omega'$, coherently with Moser lemma [2]; moreover the Hamiltonian is unchanged under the map. The geometrical data of the metric g and the 3-form $H = dB$ constitute part of the graviton multiplet in closed string theory. Anyway in this kind of geometric description interpretations are still open for the scalar dilaton ϕ that completes the multiplet. Some authors prefer to include it through the volume form when they focus on sigma-models and T-duality. We found at least one way how to incorporate it in the finite map that pulls back ω' to the canonical ω. We were able to show this in the first paper [3] where we had a non-canonical graded Poisson algebra for the degree-2 case. This number is special because of the equivalence with Courant algebroids on $TM \oplus T^*M$ pointed out by Ševera [4] and elucidated by Roytenberg [5]. The algebroid bracket, pairing and anchor and their relations are determined via derived brackets of the graded Poisson brackets with the Hamiltonian. Through the introduction of a further piece of geometrical data, namely a skew-symmetric Lie-like bracket, it was possible to retain from the whole construction a connection on $TM \oplus T^*M$. The physically relevant setting was obtained by restricting to TM. Its Ricci tensor, integrated against the non-symmetric combination $g - B$, yielded the SUGRA action in 10 dimensions for the NS-NS fields, which is the same as the low energy action of type II strings. However the biggest achievement we see in our work is the fact that we got the suitable non-canonical graded Poisson structure for this: it can be quantized straightforwardly, and as a first type of quantization we will probably see the rise of Weyl or Clifford algebras, because of the metric being a quadratic form. It is the first time that degree-2 graded symplectic geometry is applied in the context of Supergravity actions and in the background of non-geometric Q[1] and R tensors, despite being already known to be equivalent to a Courant algebroid. The overall construction is not completely based on that type of geometry, however, but needs some further differential geometry elements of $TM \oplus T^*M$. Differential graded manifolds, in which some of the conditions are relaxed (e.g. $\mathrm{Q}^2 \approx 0$, in a weaker sense), have instead proven themselves more adapted

[1] Note that Q refers to the homological vector field, while the italic Q refers to the mixed symmetry tensor.

to the task of rephrasing the statement of T-duality [6]. Drawing some inspiration from this, the second area where we applied our alternative description was "dual" gravity in the background of Q and R fields T-dual to the Neveu-Schwarz field $H \in \Omega^3(M)$, where $Q \in \Gamma(\Lambda^2 TM \otimes T^*M)$ and $R \in \mathfrak{X}^3(M)$. Via an educated guess, we provided the Poisson structure on the same degree-2 symplectic manifold and the respective Courant algebroid connection. The latter, constrained respectively to TM and T^*M reproduced, through its curvature invariants, the NS-NS type II string effective action (with no scalar dilaton) *and* GR for an inverse metric ($G^{-1} : \Gamma(T^*M) \vee \Gamma(T^*M) \mapsto C^\infty(M)$) with R^2, $Q \lrcorner R$ and Q^2 terms in the background. This is where our model starts becoming predictive, too. The latter action in fact did not exist before in the literature, despite enjoying the correct gauge symmetries. In the end evidences of the inclusiveness of the model will be given, as it will be applied to reproduce GR and gravity with torsion, both in the metric and in the vielbein formulation.

The rest of the introduction will present and contextualize the three areas in which the language of graded Poisson algebras, through dg-manifolds and/or Courant algebroids was deployed: gravitation, mostly gauge theories, Supergravity for the NS-NS bosonic fields and gravity with stringy non-geometric fluxes. A fundamental lemma that validates our whole construction is enunciated at the end of the section. In the body of the text we will review generalized geometry 2.1 and graded geometry 2.4, while section 3 will contain a detailed exposition of the results which are collected in the manuscripts [3] and [7]. After some Conclusions and Outlook 4, the reader can find a short appendix 4.2 with the derivations of some results used in the main text.

1.2 Gravity: Einstein-Cartan theory and Poincaré gauge theory

Einstein's theory is based on Riemannian geometry. The connection is required to be metric compatible and symmetric (i.e. torsionless), thus it is determined uniquely by the metric $g_{\mu\nu}$. The Levi-Civita theorem assures its uniqueness. As such, the only dynamical field is the symmetric tensor $g_{\mu\nu}$, which has 10 independent components in 4 dimensions. There is a unique action with these ingredients that is invariant under diffeomorphisms, the Hilbert-Einstein action:

$$S_{\mathrm{HE}} = \int_M R[g] \sqrt{-\det g}\, d^4x.$$

As a 4-dimensional field theory for the massless spin-2 field $g_{\mu\nu}$, i.e. a representation for the double cover of $SO(1,3)$, it propagates just two degrees of freedom: 4 dof's are fixed by diffeomorphism invariance, and other 4 by the residual gauge invariance.

Riemann-Cartan geometry, a geometry which consists in the data of a manifold with a metric and an affine connection independent from each other, but with the requirement that the covariant derivative of the metric vanishes, can again host the GR action. It is always possible to write it down upon implementation of a zero torsion constraint: in this case the action takes the name of Palatini action. However now the 10 components of the metric are due to an orthonormal frame for TM $\{e_a{}^\nu \partial_\nu\}$, a 4×4 matrix, which transforms under the Lorentz group $SO(1,3)$, therefore the 6 dof's of this symmetry must be gauged away. Riemann-Cartan geometry however allows for more possibilities: in fact the spin connection $\omega_\mu^{ab} dx^\mu$, a tensor-valued form, antisymmetric in the Latin indices a, b which transforms as well under the Lorentz group, does not need to be uniquely determined by $\{e_a\}$, the fiber of the frame bundle.

Einstein-Cartan and Poincaré gauge theory achieve the purpose of making gravity more similar to a gauge theory, building on the Palatini formulation. More specifically the gauge group is the Poincaré group $\mathbb{R}^4 \rtimes SO(1,3)$ of spacetime symmetries. Then the field strength for local translations is the torsion tensor and that for local Lorentz transformations is the curvature; moreover the connection needs not be symmetric. Poincaré gauge theory is more general than Einstein-Cartan theory as the connection is just affine. Nice exhaustive reviews are [8] and [9].

Following Trautman [10] and using coframes 1-forms θ^ρ and the language of tensor-valued differential forms, the connection is the spin connection $\omega^\mu{}_\nu = \Gamma^\mu{}_{\nu\rho}\theta^\rho$ (all indices are spacetime indices, and the $SO(1,3)$-indices are omitted, the relation being $\theta^\mu \equiv \theta^{\mu a} e_a$). The curvature $\Omega^\mu{}_\nu$ and the torsion Θ^μ are respectively:

$$\Omega^\mu{}_\nu = d\omega^\mu{}_\nu + \omega^\mu{}_\rho \wedge \omega^\rho{}_\nu, \quad \Theta^\mu = d\theta^\mu + \omega^\mu{}_\nu \wedge \theta^\nu.$$

If $n_{\mu\nu} := \star(\theta_\mu \wedge \theta_\nu)$, then the Einstein-Cartan action is

$$S_{\mathrm{EC}} = \frac{1}{16\pi} \int g^{\nu\rho} n_{\mu\rho} \wedge \Omega^\mu{}_\nu, \tag{1.1}$$

and the quantities with respect to which the variations are computed and successively Noether theorem is applied are (the matter fields,) the coframe θ and the connection ω, *independently*, in the same fashion as in the GR action the metric and the Christoffel symbols are treated independently. EC theory and Poincaré gauge theory offer some conceptual advantages: aside from the already mentioned fact that gravity has now a closer resemblance to the other forces in the Standard Model, it is also evident that dealing with matter spinors can be done more neatly, and then that the canonical energy-momentum tensor (which is the Noether current of the translational symmetry) and its symmetric definition have a more clear understanding. Recall that the other Noether current (for the Lorentz generators) is the spin density tensor.

As a side comment, notice that the torsion tensor is a non-dynamical field (i.e. the variational principle returns an algebric equation for it). One can nevertheless build a kinematical term for it, a Lagrangian of Yang-Mills type, see for example the nicely written reference [11]

$$\mathcal{L} = -\frac{1}{8}\chi^{\mu\nu}{}_a{}^{\rho\sigma}{}_b \Theta^a{}_{\mu\nu}\Theta^b{}_{\rho\sigma}\, n,$$

where χ is the constitutive tensor, and n is the Hodge dual of the unit (hence the volume form), $n \equiv \star 1$. A connection for which just $\Theta^\mu \neq 0$, while the curvature $\Omega^\mu{}_\nu = 0$, is called Weitzenböck connection [12], and arises in the context of gauge theories for the translational group, T_4 in 4 dimensions (whose group manifold is $\mathbb{R}^4$).

In the main text we will be interested in using graded geometry and a graded Poisson algebra instead of the Riemann or Riemann-Cartan geometry as the setup of the gravitational theory. An attempt in this sense can already be found in [13] where the authors considered $T^*[1]M$, in which M has doubled local coordinates $\{x^\mu, \zeta_\nu\}$. We will instead prefer to work with standard coordinates only. Another result that relied on an additional grading worth to be mentioned is [14], where the authors proved that the Maurer-Cartan equation of a graded Lie algebra (modulo automorphisms) is equivalent to the vacuum Einstein equations.

1.3 Supergravity in type II effective string theory

Supergravity is a gravitational theory which enjoys local supersymmetry [15]. It stands on its own right, but it can also be seen as the low-energy effective action of the 5 string theories, since in the $\alpha' \to 0$ limit of large string tension, the massive modes become very heavy thus not observable anymore, while the massless graviton is kept in the spectrum. Under these premises, it is therefore a good approximation to replace string theory by a supergravity theory. In the rest of the thesis we will be giving more geometrical and algebraic derivations of these actions, in particular the type II strings common sector; for the moment let us present the general theory following the textbook [16]. Type IIA string theory has a M-theory origin: the dual theory under S-duality (which roughly speaking inverts the string coupling constant) of M-theory compactified on a circle is 10-dimensional type IIA string theory. Hence their supergravity limit are related by dimensional reduction. The fermionic fields are a pair of gravitini and a pair of dilatini, instead the bosonic fields are a 10-dimensional metric, a 2-form B (the Kalb-Ramond field), a dilaton, a 1-form A_1 and a 3-form A_3. The bosonic action includes three distinct types of terms, a NS-NS fields action, a R-R fields action and a Chern-Simon term.

Type IIB supergravity cannot be reconstructed from dualities and reductions, but rather through the principles of supersymmetry and gauge invariance. The field content is the same as in type IIA, apart from the gauge potentials of the R-R fields, which are instead a 0-, 2-, and 4-form C_0, C_2, C_4. The action splits again in three terms as before, where the action for the R-R fields and the Chern-Simons term look completely different. However the action below, for the fields with worldsheet NS boundary conditions, is a constant presence in both supergravities:

$$S = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{-\det g}\, e^{-2\phi} \left(R + 4\partial_\mu \phi \partial^\mu \phi - \frac{1}{12} H^2 \right). \tag{1.2}$$

Generalized complex geometry (GCG), as initiated by Hitchin and Gualtieri, has shown since the beginning to be a promising framework for getting some useful insights in the context of strings and supersymmetric sigma models. The correct generalization of a Lie algebra to this setting is the concept of Courant algebroid, therefore the differential geometry (exterior derivative, Lie derivative, etc.) is based on the algebroid bracket in such situation. Moreover in GCG, complex geometry and symplectic geometry are contained as limiting cases. Proving some properties of generalized Kähler structures and D-branes that for some manifolds could not have been proven otherwise were among the first achievements of GCG [17], [18].

Aside from gauge theoretical derivations of (1.2), a Generalized Geometry derivation of the Supergravity action of the NS-NS fields and of the R-R fields has been foreseen quite soon. Notice that the complex structure is not needed for this task. Since the fundamental fields mediating the gravitational interactions are the metric alongside with a 2-form field and a scalar field, a bundle $E \cong TM \oplus T^*M$ fits perfectly for this aim. It has a natural 1-gerbe structure (this concept will be explained right before section 2.3) that dictates how the B-field is patched over the base manifold. The transition functions are a collection of 1-forms that enjoy themselves a gauge symmetry.

Reference [19] was the first to show that T-duality invariance, diffeomorphism invariance, B-field gauge transformations and local Lorentz-invariance of spacetime could be combined in an action obtained from the Ricci curvature tensor of a connection, for a reduction of the $O(d,d) \times \mathbb{R}$ structure group of the extended tangent bundle to $(SO(d-1,1) \times SO(1,d-1)) \times \mathbb{R}$. This yields a spacetime with Lorentzian signature, while the $\mathbb{R}$ element is a conformal factor proportional to the scalar dilaton. They asked the connection to be compatible with the $SO(d-1,1) \times SO(1,d-1)$-invariant generalized metric $\mathbf{G}$

$$\mathbf{G} = \begin{pmatrix} g - Bg^{-1}B & -Bg^{-1} \\ g^{-1}B & g^{-1} \end{pmatrix}, \quad g \text{ Riemannian metric}, B \in \Lambda^2 T^*M, \tag{1.3}$$

and with the dilaton factor. In the local chart, its torsion was also required to be zero. The connection coefficients are built in terms of natural pairing of vector fields and 1-forms, and $\mathbf{G}$, however the set of requests did not completely fix the gauge freedom, therefore some undetermined tensors appeared along the way. By fixing the freedom through some arbitrary choices, in the end the Ricci curvature tensor is computed and successively contracted with $\mathbf{G}$. Such action reproduces the SUGRA Lagrangian of (g, H, ϕ), as the generalized analogue of the Einstein-Hilbert action.

Waldram et al. in [20] completed the picture by discussing the lift of the connection to the Spin bundle that hosts the supersymmetric partners (gravitino, dilatino etc.), therefore implementing supersymmetry in the action and completing the previous action with the R-R fields contribution. The latter, being sections of the Spin bundle, could be incorporated and contributed to the generalized Ricci tensor in a $\mathrm{Spin}(d-1,1)\times\mathrm{Spin}(1,d-1)$-covariant manner, by adding the Mukai pairing[2] of them. The generalized differential geometry of $(TM \oplus T^*M)\times\mathbb{R}$ was developed in detail, and to get a unique (and non-null) Ricci tensor (for the NS-NS fields) its entries were accurately chosen to belong to one of the two eigenbundles of $\mathbf{G}$, in an alternate fashion. The result coincided with the sum of the beta-functions of g and B, as computed from the worldsheet theory. Using the gamma matrices for the Clifford algebra, a curvature scalar could also be furnished, and this reproduced the type II effective Lagrangian for g, H, ϕ (1.2) and that for the R-R fields. In a later work [21] the authors applied these techniques of generalized differential geometry to heterotic string theory and M-theory too.

In these successful approaches there were however some troubles, caused by the arbitrariness left even after the implementation of all the symmetry conditions and by the failure of the standard definitions of torsion and curvature tensors to be tensors for the generalized tangent bundle. This led the community to look for various further improvements. Ševera and Valach [22] and separately Garçia-Fernandez [23] bypassed the problem of defining a consistent Riemann curvature tensor by focusing directly on the Ricci tensor, built with the generalized metric (equivalent to having a subbundle of E) and a divergence operator, $div : \Gamma(E) \mapsto C^\infty(M)$, $div fe = f div\, e + \rho(e)f$, for $f \in C^\infty(M)$ and $\rho : \Gamma(E) \mapsto \Gamma(TM)$. Then the type II supergravity action is retrieved as the Laplacian acting on a half-density which depends on the dilaton.

In the literature cited so far, the dilaton field is always taken into account via the volume form $e^{-2\phi}\sqrt{-\det g}$ or declinations of this idea. An alternative proposal regarding how to incorporate it, in the context of effective strings action but easily extended to Poisson-Lie T-duality, was made in [24]: there, together with few more conditions omitted here, the differential of ϕ shall satisfy:

$$\langle d\phi, Z\rangle = \langle \nabla_{\rho^*(h_{\mathbf{G}}^{-1}(e_k))}\rho^*(e^k), h_{\mathbf{G}}^{-1}(Z)\rangle_E,$$

where ∇ belongs to the class of Levi-Civita connection for E, $\{e^k\}$ local frame, $\rho^* : \Gamma(T^*M) \mapsto \Gamma(E^*)$ and $h_{\mathbf{G}}$ is the following metric on 1-forms:

$$h_{\mathbf{G}}(\zeta, \sigma) := \mathbf{G}(\rho^*(\zeta), \rho^*(\sigma)), \quad \zeta, \sigma \in \Omega^1(M).$$

Inspired by the achievements and the open questions on this topic, we also built upon these results and managed to ascribe the fields of the common sector of type II Supergravities in

[2]For spinors $\Psi, \Psi' \in \Gamma(S)$, the Mukai pairing $\langle , \rangle$ is

$$\langle \Psi, \Psi' \rangle = \sum_n (-)^{[(n+1)/2]}\, \Psi^{(d-n)} \wedge \Psi'^{(n)}$$

degree-2 graded Poisson brackets, whose derived brackets are Courant algebroid brackets. How we did so will be clear from the paragraph 3.2. Our findings do not rely just on graded Poisson algebra, as we employed also some further differential (generalized) geometry objects along the way.

1.4 Gravity and non-geometric fluxes

Another action that we considered interesting to formulate in the language of graded Poisson algebra, deploying the equivalence of the latter with generalized geometry and Courant algebroid, is some action for the gravitational field in the background of non-geometric fluxes. Before reviewing the origins and motivations and the state of the art of such a theory, for a better understanding there are two main aspects that need a slightly deeper explanation: T-duality and Double Field Theory (DFT).

1.4.1 T-duality

T-duality is one of the most striking symmetries in string theory. Its existence is a purely stringy effect, in contrast to the non-extended nature of the point particle. For a first understanding of T-duality a rough explanation can be the following: a closed string theory compactified over a circle is equivalent (T-dual) to another closed string theory where the compactification radius is the inverse of the previous one. T-duality was first noticed by Buscher [25] for the associated sigma model. His derivation can be summarized in this way: consider the following model with a metric and a 2-form,

$$S=\frac{1}{4\pi\alpha'}\int_\Sigma d\tau d\sigma\, g_{ij}dX^i(\tau,\sigma)\wedge\star dX^j(\tau,\sigma)+B_{ij}dX^i(\tau,\sigma)\wedge dX^j(\tau,\sigma), \tag{1.4}$$

where Σ is a Riemann surface and X is a smooth differentiable map $X:\Sigma\mapsto M$, for M d-dimensional manifold, and assume that the metric has an isometry in some directions (i.e. a Killing vector field). Then there is a set of transformations that shuffles the components of the metric and those of the 2-form to give another different metric and B field while keeping the action fixed. If there is a curved metric on Σ, and $R^{(2)}$ is the Ricci scalar of the Levi-Civita connection, the scalar dilaton can become part of the model through addition of

$$\frac{1}{4\pi}\int_\Sigma d\tau d\sigma\, \phi R^{(2)}\star 1,$$

and covariantization of (1.4). Topologically, T-duality exchanges the first Chern class of the bundle with the fiberwise integral of H, see [26].

Performing successive T-duality on these fields shortly became a very intriguing research question among physicists because of phenomenological reasons. Shelton-Taylor-Wecht in their comprehensive paper [27] discussed the chain of T-dualities for the NS-NS flux H

$$H_{ijk}\xrightarrow{T^k}f_{ij}{}^k\xrightarrow{T^j}Q_i{}^{jk}\xrightarrow{T^i}R^{ijk}, \tag{1.5}$$

as well as that for the R-R fields. Already after a second T-duality the geometry is not globally well defined since the transition functions now must include also T-duality transformations (i.e. the patching is possible just if one glues together coordinates related by a T-duality). The third T-duality, which is just formal because there are no isometric directions left and hence Buscher rules do not apply, but instead one must resort to a doubling of the coordinates, gives even more bizarre results dubbed non-geometric spaces. Despite this, the fluxes can be fundamental in the stabilization of string theory vacua.

Intuitively, for example in the case of a $\mathbb{T}_3$ torus, two successive T-dualities return a $\mathbb{T}_2$ fibration over a circle, see the review [28], section 6.1.

In the rest of the thesis we will not use anything more than what described here. Let us just mention that nowadays there are other types of T-dualities under study, like those due to non-abelian isometries and of Poisson-Lie type. Generalized geometry and GCG are again the perfect setting for them: T-dualities are $O(d,d;\mathbb{Z})$-transformations, and the $O(d,d;\mathbb{R})$ group is often taken as structure group of the CA $TM \oplus T^*M$. [29] proved that T-duality is an isomorphism of Courant structures over different base manifolds. Among the others, [30], [24] and [22] are other good example of how this string theory property has a more mathematical generalized geometry formulation. How the NS-NS fluxes sit in generalized geometry is explained by Ellwood in [31].

1.4.2 Double Field Theory

Already in the process of obtaining the R-flux (1.5) it was convenient to double the coordinates of the base manifold. But doubling the number of dimensions shows its potential also in the formulation of a field theory for the generalized metric and the dilaton, manifestly invariant under $O(d,d)$-transformations and in a hidden way invariant under doubled diffeomorphisms, as done by Hohm, Hull and Zwiebach in [32]. To follow their conventions, where in a local chart the coordinates are $X^M = \left(\tilde{x}_i, x^i\right)$, $M = 1, \dots 2d$, $i = 1, \dots d$, let us call $\mathcal{H}$ the generalized metric $\mathbf{G}$ (1.3) in which the rows are swapped with the columns in this sense:

$$\mathcal{H}_{MN} = \begin{pmatrix} g^{ij} & -B_{ik}g^{kj} \\ g^{ik}B_{kj} & g_{ij} - B_{ik}g^{kl}B_{lj} \end{pmatrix}.$$

Let d(X) be a scalar field. Then the action is

$$S_{\mathrm{DFT}} = \int dx d\tilde{x}\, e^{-2\mathrm{d}} \bigg(\frac{1}{8}\mathcal{H}^{MN}\partial_M \mathcal{H}^{KL}\partial_N \mathcal{H}_{KL} - \frac{1}{2}\mathcal{H}^{MN}\partial_N \mathcal{H}^{KL}\partial_L \mathcal{H}_{MK} \\ - 2\partial_M \mathrm{d}\partial_N \mathcal{H}^{MN} + 4\mathcal{H}^{MN}\partial_M \mathrm{d}\partial_N \mathrm{d} \bigg). \tag{1.6}$$

(Be aware that the indices are always raised with the pairing, not with $\mathcal{H}$.) Compared to other prior formulations [33] in terms of the combination $g + B$, that transforms non-linearly under the $O(d,d)$ group, here it is possible to check the invariance at a first sight, since $e^{-2\mathrm{d}}$ transforms as a scalar and $\mathcal{H}$ is a $O(d,d)$-tensor. The gauge invariance (diffeomorphisms wrt the doubled coordinates) is then proven by noticing that a generalized Lie derivative can be formed with the help of the rather obvious extension of the skew-symmetric Courant bracket to DFT, dubbed C-bracket $[,]_{C-br}$:

$$[\xi_1, \xi_2]^M_{C-br} = \xi^N_{[1}\partial_N \xi^M_{2]} - \frac{1}{2}\xi^P_{[1}\partial^M \xi_{2]\,P}, \quad \xi_1, \xi_2 \in \Gamma(TM \oplus T^*M).$$

Therefore

$$\delta_\xi \mathcal{H}^{MN} = \xi^P \partial_P \mathcal{H}^{MN} + \left(\partial^M \xi_P - \partial_P \xi^M\right)\mathcal{H}^{PN} + \left(\partial^N \xi_P - \partial_P \xi^N\right)\mathcal{H}^{MP} =: \mathcal{L}_\xi \mathcal{H}^{MN},$$

while $e^{-2\mathrm{d}}$ transform as a density.

The theory is extremely useful for countless applications in string theory. Anyway its enlarged space cannot be physical and DFT cannot live on its own: some other conditions have to be imposed on top in order to reconstruct the real physical space, like the "strong constraint" $\partial_M \tilde{\partial}^M = 0$. Apart from these issues, since the introduction of DFT and the recovery of a consistent action (1.6), many authors developed the geometrical and algebraic structures associated with it, see among the others the earlier work [34] and the later work [35].

For what concerns us, we will just exploit the observation that in some low-energy limits and after solving the strong constraint, the DFT action (1.6) describes an action for a "dual" graviton (for the moment not better specified) in the background of the non-geometric fluxes Q and R (1.5).

In the body of this thesis we will encounter an extensive explanation of how dg-manifolds of degree 2 and their corresponding CA enter the formulation of a Hilbert-Einstein Lagrangian for a metric $G^{-1} \in \vee^2 TM$ together with a bivector $\Pi \in \mathfrak{X}^2(M)$. This was inspired by the results of the companion papers [36] and [37] and the Lie algebroid derivation in [38]. As already mentioned, the authors found the supergravity limit of the DFT action: the fields of their theory are given by the combination $G^{-1} + \Pi$, and the covariant calculus on the doubled space stems from the derivative operator $\tilde{D}^i$

$$\tilde{D}^i := \tilde{\partial}^i - \Pi^{ij}\partial_j.$$

The action for $G^{-1} + \Pi$ was obtained as the following field redefinition:

$$(g, B, \phi) \mapsto \left(G^{-1}, \Pi, \tilde{\phi}\right),$$

which can be motivated by T-duality but anyway it is argued to lack a solid mathematical ground. [38] repaired to this deficiency through a more rigorous formulation via non-geometric frames for a Lie algebroid on T^*M, where the manifold M is not doubled. The action they obtained was formed through invariant combinations of the pullback to T^*M of the NS-NS tensors g and $H = dB$, and the Levi-Civita connection. Contrary to the other papers [36] and [37], this time there was no connection $\nabla : \Gamma(E) \mapsto \Gamma(T^*M) \otimes \Gamma(E)$, where $E = T^*M$ for M either a regular manifold or with doubled coordinates, whose Ricci scalar could be the constitutive Lagrangian. We will encompass both the situations by giving a CA and graded Poisson algebra formulation of a Hilbert-Einstein action of gravity in the background of non-geometric fields Q and R.

1.5 Moser lemma

A crucial passage in our results is uncannily Moser lemma. It gives the chance to put the gauge fields, either abelian and non-abelian, in a prominent position in the Hamiltonian theory, which otherwise will be the free theory. Alternatively they are expressed through Legendre transform of the interacting terms in the Lagrangian. The Hamiltonian approach is complementary to the Lagrangian description but it is more handful for first quantization aspects, that the physics' community is nowadays eager to find for the gravitational field.

For the version of the lemma we are going to use[3], a good intuition of the result is the following: for a symplectic manifold M, consider the smooth 1-parameter family of symplectic forms ω_t and a symplectic form of reference ω_0, where $t \in \mathbb{R}, t \in [0,1]$, $\omega_{t=0} \equiv \omega_0$. They differ by an exact 1-parameter dependent 2-form:

$$\omega_t = \omega_0 + dA_t,$$

[3]There is a version for the volume form, but we will not discuss it.

where it is understood that $dA_{t=0} = 0$. Hence they can be connected via a smooth family of diffeomorphisms $\varphi_t : M \mapsto M$, which acts on differential forms by pullback. For each t, φ_t relates them in this way:

$$\varphi_t^* \omega_t = \omega_0.$$

The technical formulation can be found in the mathematical preliminaries, section 2.4. The generators are hence changes of phase space coordinates. Clearly, opportune coordinates can highlight some aspects and hide others. dA_t is an exact 2-form and thus A_t and $A_t + da_t$, which belong to the same cohomology class, yield the same symplectic form. Hence, if a Hamiltonian function is selected, $a_t \in \Omega^1(M)$ is a gauge symmetry of the system. We see that depending on A_t, various gauge theories can be implemented through a deformation of the canonical symplectic form: usually, the momenta p_i are shifted by the vector field corresponding to the deformation. Our main concerns will be a $Sp(2d) \times O(d,d)$ graded structure where the coordinates $\{\xi_\alpha\}$ labeling the $O(d,d)$ part will be rescaled and/or shifted: in this case, the vector field for the deformation of the symplectic structure does not depend on the standard conjugate momenta p. We will later look at other graded symplectic structures such as $T^*[1]M$ or $T^*[2]M \oplus T[1]M$. The net effect will always be to introduce interactions with a gravitational field and other tensor fields, such as the Neveu-Schwarz field H, the dilaton ϕ and the non-geometric fluxes Q and R.

Before moving to the next section it is worth pointing out that this version of Moser's trick finds a straightforward application in *Darboux theorem*: in a neighborhood of any point on the symplectic manifold M the symplectic form restricted to the neighborhood is symplectomorphic to the canonical symplectic form of $\mathbb{R}^{2n} \equiv M$.

Chapter 2

Mathematical Preliminaries

2.1 Generalized Geometry and algebroids

This first section of fundamental mathematical preliminaries aims at making the reader more familiar with generalized geometry, as initiated by Nigel Hitchin [39] and his doctoral students Gualtieri and Cavalcanti [17]. This kind of geometry unifies aspects of symplectic, complex and Riemannian geometry, but complex geometry will not be contemplated in the research questions pursued during the Ph.D. Algebroids, the generalization of the concept of an algebra to a pointwise algebra, will also be important for the rest of the thesis. Most of the theoretical work is not original and is well-know among the community; anyway there will be also a few personal contributions. Although the concepts might be sometimes quite technical, we will try to keep the exposition accessible to a physicist, and in the meantime to not lack mathematical rigor.

2.1.1 Algebroids

In order to endow the sections of a vector bundle E over a smooth manifold M with an algebra, the first big challenge is to make possible that the $\mathbb{R}$-linear product of sections, the bracket $[,] : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$, could respect the Leibniz rule. This requires the introduction of a linear map of E-sections into vector fields, $\rho : \Gamma(E) \mapsto \Gamma(TM)$.

Definition 2.1.1. Lie algebroid. [40] A *Lie algebroid* is a vector bundle $E \xrightarrow{\pi} M$, a binary operation on sections $[\cdot,\cdot]$ and a linear map $\rho : \Gamma(E) \mapsto \Gamma(TM)$ called *anchor* such that:

1. $[e_1, [e_2, e_3]] = [[e_1, e_2], e_3] + [e_2, [e_1, e_3]]$, for $e_i \in \Gamma(E)$;
2. $[e_1, fe_2] = f[e_1, e_2] + \rho(e_1)f\, e_2$, for $f \in C^\infty(M)$;
3. $[e_1, e_2] = -[e_2, e_1]$.

Notice that the anchor is a homomorphism of the Lie algebroid bracket with the Lie bracket of vector fields: $\rho([e_1, e_2]) = [\rho(e_1), \rho(e_2)]$. It follows from compatibility between the Jacobi identity and the Leibniz rule.

The tangent bundle, taking as anchor map any automorphism $\rho : \Gamma(TM) \mapsto \Gamma(TM)$ and as bracket $[,]$ the Lie bracket of vector fields, is a Lie algebroid over a point. The cotangent bundle T^*M constitutes another example [41]. It becomes a Lie algebroid whenever a bivector $\beta \in \mathfrak{X}^2M$ is assigned, leading to the natural anchor $\rho : \Gamma(T^*M) \mapsto$

$\Gamma(TM)$, $\rho \equiv \beta$, as the bivector is responsible for raising an index up, and when the bracket on elements of $\Gamma(T^*M)$ is the following

$$[\sigma, \kappa]_{\text{Koszul}} = \mathcal{L}_{\beta(\sigma)}\kappa - \iota_{\beta(\kappa)} d\sigma.$$

The Jacobi identity holds iff β is a Poisson bivector, alternatively stated as $\{\beta, \beta\} = 0$ by Jacobi identity of the Poisson bracket.

With the Lie algebroid bracket and the anchor map the notion of a differential d_E can be shaped analogously to the standard differential geometry one: $d_E : \Omega^\bullet(E) \mapsto \Omega^{\bullet+1}(E)$,

$$\begin{aligned} d_E\varpi(U_0, \ldots, U_k) = &\sum_i (-)^i \rho(U_i)\varpi(U_0, \ldots \hat{U}_i, \ldots U_k) \\ &+ \sum_{i<j} \varpi([U_i, U_j], \ldots, \hat{U}_i, \ldots, \hat{U}_j, \ldots, U_k), \end{aligned} \tag{2.1}$$

where hatted sections are omitted from the expression.

For most applications it will turn out useful to have a metric on the E-sections, and at the same time to dismiss the antisymmetry of the bracket. Upon imposing a couple of more conditions regulating the properties of metric and bracket together, the resulting algebroid is called Courant algebroid.

Definition 2.1.2. Courant algebroid. [42] A *Courant algebroid* (CA) is a Lie algebroid in which the antisymmetry of the bracket is dropped, and furthermore there is a fiber-wise $C^\infty(M)$-linear symmetric form $\langle \cdot, \cdot \rangle$ which respects these compatibility conditions:

1. $\rho(e_1)\langle e_2, e_3\rangle = \langle [e_1, e_2], e_3\rangle + \langle e_2, [e_1, e_3]\rangle$;
2. $\rho(e_1)\langle e_2, e_3\rangle = \langle e_1, [e_2, e_3] + [e_3, e_2]\rangle$.

An example of consistent bracket for this algebroid is the so called Dorfman bracket (see (2.7)).

The first of these conditions can be rephrased as the invariance of the symmetric bilinear form $\langle , \rangle$ under the adjoint action. If $\mathcal{L}_{e_1}$ is the derivation corresponding to a Lie derivative built with the CA bracket, and $g_E := \langle , \rangle : \Gamma(E) \vee \Gamma(E) \mapsto C^\infty(M)$ it can be alternatively stated as the Killing vector condition $\mathcal{L}_{e_1} g_E = 0$. It is surely interesting that each generalized vector is Killing for the CA metric. In standard Riemannian geometry, in fact, the Killing vector equation is solved only if the metric has an isometry in the direction of that vector field.

The last axiom measures the failure of the bracket to be antisymmetric. It can be alternatively stated with the help of the coboundary operator $\mathcal{D} : C^\infty(M) \mapsto \Gamma(E)$,

$$\mathcal{D} := g_E^{-1} \circ \rho^* \circ d, \quad \rho^* : \Gamma(T^*M) \mapsto \Gamma(E^*), \quad g_E : \Gamma(E) \mapsto \Gamma(E^*).$$

Its action on a function is more clearly seen through the following fully contracted formula:

$$\langle \mathcal{D}f, e\rangle = \rho(e)f, \; \forall\, f \in C^\infty(M). \tag{2.2}$$

Then item 2 in definition 2.1.2 becomes

$$[e, e] = \frac{1}{2}\mathcal{D}\langle e, e\rangle. \tag{2.3}$$

If however e_2, e_3 belong to an *isotropic* subbundle $L \subset E$, which by definition for any generic element l, l', $\langle l, l'\rangle = 0$, then the LHS of the axiom is identically zero and antisymmetry persists, when restricted to the isotropic subbundle.

Interestingly, for the CA bracket the behavior of the first entry under $C^\infty(M)$-multiplication can be extracted with the help of $\mathcal{D}$, the lack of antisymmetry for $[,]$ (item 2 in definition 2.1.2), and the transformation property for the second slot, which is known from the Lie algebroid definition 2.1.1:

$$\begin{aligned}[fe_1, e_2] &= -[e_2, fe_1] + \mathcal{D}\langle e_2, fe_1\rangle \\ &= -f[e_2, e_1] - \rho(e_2)(f)e_1 + \mathcal{D}f\,\langle e_1, e_2\rangle + f\,\mathcal{D}\langle e_1, e_2\rangle \\ &= f[e_1, e_2] - \rho(e_2)(f)e_1 + \mathcal{D}f\,\langle e_1, e_2\rangle \end{aligned} \tag{2.4}$$

Other remarkable properties that concern the coboundary operator are listed below:

$$\begin{aligned}&[e, \mathcal{D}f] = \mathcal{D}\langle e, \mathcal{D}f\rangle, \\ &[\mathcal{D}f, e] = 0, \\ &\langle \mathcal{D}g, \mathcal{D}f\rangle = 0, \qquad\qquad \text{for } \textit{exact} \text{ CA, (2.5)}\end{aligned}$$

The first can be proven contracting the resulting section with another section and then applying the first axiom listed in 2.1.2 and the homomorphism property of the anchor. The second descends directly from the failure of antisymmetry of the CA bracket and the previous result. The third ensures that for the coboundary there is no obstruction to the cohomology, when (2.5) holds.

Another unexpected feature is that the Leibniz rule for definition 2.1.2 can actually be omitted: it descends from the first axiom there. Kosmann-Schwarzbach showed this in [43]. This is not usually remarked in the literature, and therefore the algebraic derivation will be pointed out here. Ultimately, Leibniz rule for $[,]$ is a plain consequence of the product rule for the derivation $\rho(e)$:

$$\rho(e_1)\langle fe_2, e_3\rangle = (\rho(e_1)f)\,\langle e_2, e_3\rangle + f\rho(e_1)\langle e_2, e_3\rangle.$$

Applying the first axiom on both sides of the equality, then, yields:

$$\langle [e_1, fe_2], e_3\rangle + f\langle e_2, [e_1, e_3]\rangle = (\rho(e_1)f)\,\langle e_2, e_3\rangle + f\langle [e_1, e_2], e_3\rangle + f\langle e_2, [e_1, e_3]\rangle;$$

an elementary rearrangement of the above concludes the proof.

As an example of Courant algebroid it is worth mentioning the double of a *Lie bialgebroid* [44]. A Lie bialgebroid is built up with a pair of Lie algebroids: $(E, \rho, [,]_E)$ and the Lie algebroid involving the dual vector bundle $(E^*, \rho^*, [,]_{E^*})$. Together they produce a Lie bialgebroid when the differential d_{E^*} (2.1) induced by $[,]_{E^*}$, is a derivation of $[,]_E$. A Courant algebroid arises considering as total space $\mathbf{E} \equiv E \oplus E^*$, for sections the ordered pairs $(e, \varepsilon) \in \Gamma(E \oplus E^*)$, as pairing the natural contraction

$$\langle (e, \varepsilon), (e', \varepsilon')\rangle = \varepsilon'(e) + \varepsilon(e'),$$

and the following $[,]_\mathbf{E}$ as bracket:

$$[(e, \varepsilon), (e', \varepsilon')]_\mathbf{E} := \left([e, e']_E + \mathcal{L}^{E^*}_\varepsilon e' - \iota_{\varepsilon'} d_{E^*} e, [\varepsilon, \varepsilon']_{E^*} + \mathcal{L}^E_e \varepsilon' - \iota_{e'} d_E \varepsilon\right),$$

while the anchor is $\rho_\mathbf{E}(e, \varepsilon) = \rho(e) + \rho^*(\varepsilon)$. The axioms of definition 2.1.2 are verified for $(\mathbf{E}, [,]_\mathbf{E}, \langle,\rangle, \rho_\mathbf{E})$ in [44]. The converse is true too, and relies on the notion of *Dirac structures*. Dirac structures $\hat{L}$ are isotropic subbundles which are of maximal dimension (half of the total dimension of the upper space in the bundle) and involutive under $[,]_\mathbf{E}$ (i.e. $[\hat{L}, \hat{L}]_\mathbf{E} \subset \hat{L}$). Thus given $(\mathbf{E}, [,]_\mathbf{E}, \langle,\rangle, \rho_\mathbf{E})$, two complementary Dirac structure E_1

and E_2 are necessarily isomorphic to their respective dual, $E_1^* \cong E_2$, and can be naturally given the structure of a Lie bialgebroid, using the already existing objects.

An important notion is that of *exact* Courant algebroids E. These algebroids stand in the short exact sequence

$$0 \to T^*M \xrightarrow{j} E \xrightarrow{\rho} TM \to 0. \tag{2.5}$$

Hence j is an injective map while ρ is surjective, and $\mathrm{Ker}\rho = \mathrm{Im}j$. Notice that generally $\mathrm{Im}j \equiv \mathrm{Im}(g_E^{-1} \circ \rho^*) \subseteq \mathrm{Ker}\rho$ always. Every exact CA is thus isomorphic to a CA on the generalized tangent bundle $TM \oplus T^*M$. The classification of exact CAs was carried out by Ševera in his famous letters [4], who found that they differ just for a representative in the $[H]$-class, $H \in H^3(M, \mathbb{R})$. We can briefly recall how an element of the third cohomology class can "label" an exact Courant algebroid.

The image of the inclusion map j from the sequence (2.5) is isotropic because, as $\rho = j^* \circ g_E$,

$$\langle j(\gamma), j(\sigma)\rangle_E = \langle \gamma, \rho(j(\sigma))\rangle = 0,$$

where the pairing before the last equality is the canonical pairing of vector fields with forms. However $\mathrm{Im}j$ is not just isotropic: it is maximally isotropic and involutive, $[j(\gamma), j(\sigma)] = 0 \in j(T^*M)$, thus it is a Dirac subbundle. This, at least for exact 1-forms, descends from $0 = [\mathcal{D}f, \mathcal{D}f'] = [j(df), j(df')]$. Thus it is always possible to define an isotropic splitting (i.e. a connection) $s : \Gamma(TM) \mapsto \Gamma(E)$, $\rho \circ s = \mathrm{id}_{TM}$, $\langle s(X), s(Y)\rangle = 0$. Then $s + j : \Gamma(TM \oplus T^*M) \xrightarrow{\sim} E$, and thus we can compare the bracket of E with the homomorphic bracket of $TM \oplus T^*M$. But then using 1) the axioms of the CA, 2) the observation that the image of s is transverse to the image of j (i.e. $\langle s(X), j(\sigma)\rangle = \iota_X\sigma$), and 3) the remaining properties of $\mathrm{Im}s$ and $\mathrm{Im}j$ previously mentioned, it can be proven that the difference can just be a 3-tensor, which is also completely skew-symmetric.

Hence if $U, V \in \Gamma(E), U = X+\gamma, V = Y+\sigma$, where $X, Y \in \Gamma(TM)$ and $\gamma, \sigma \in \Gamma(T^*M)$, a generic bracket with some $H \in H^3(M, \mathbb{R})$ that respects all the CA axioms is the *H-twisted Dorfman bracket*:

$$[U, V]_{\mathrm{D},H} = \mathcal{L}_X V - \iota_Y d\tau + H(X, Y, \cdot). \tag{2.6}$$

In the following we will refer to the untwisted version as $[,]_{\mathrm{D}}$:

$$[U, V]_{\mathrm{D}} = [X, Y] + \mathcal{L}_X\sigma - \iota_Y d\gamma. \tag{2.7}$$

Exact CAs clearly do not exhaust all the physically important possibilities for a Courant algebroid. If no such sequence as (2.5) can be found, but instead the anchor ρ is surjective, the Courant algebroid is said to be *transitive*, the topic being analyzed extensively in [45]. E can nevertheless be separated into complementary subbundles, the isotropic Q such that $Q \cap \mathrm{Ker}\rho = \{0\}$, and the isotropic $C \subset \mathrm{Ker}\rho$, $C \cap \mathrm{Im}j = \{0\}$, in this way:

$$E = \mathrm{Im}j \oplus Q \oplus C.$$

They are the appropriate setting for heterotic string theory, which is best described by $\mathrm{Im}j \equiv T^*M$, $Q \equiv TM$ and the Lie algebra $\mathfrak{g}$ in place of C: $E \cong TM \oplus \mathfrak{g} \oplus T^*M$.

The skew-symmetry of the bracket can be recovered by dismissing another feature: the new bracket shall not respect the Jacobi identity. Failure to respect it is measured by the exterior derivative of the Nijenhuis tensor, taken with the coboundary $\mathcal{D}$. There are a few equivalent definitions of this tensor. It measures the obstruction of an almost complex structure, $\mathcal{J} : TM \mapsto T^*M, \mathcal{J}^2 = -\mathbb{1}$, to be originated by a complex structure, i.e. an integrable $\mathcal{J}$ (global). We will recur to a different qualification.

Definition 2.1.3. Nijenhuis operator. [17] The Nijenhuis operator $N : \Gamma(E)\times\Gamma(E) \mapsto \Gamma(E)$ on generalized vectors $e_1, e_2, e_3 \in \Gamma(E)$ is

$$\langle N(e_1, e_2), e_3\rangle = \frac{1}{3}\langle [e_1, e_2], e_3\rangle + \mathrm{cyclic}(e_1, e_2, e_3),$$

where the bracket employed is the antisymmetric one.

In [17], proposition 3.16, a neat proof that the Jacobiator,

$$\mathrm{Jac}(e_1, e_2, e_3) := [e_1, [e_2, e_3]] + [e_2, [e_3, e_1]] + [e_3, [e_1, e_2]],$$

is not zero for a skew-symmetric CA bracket, but rather it is given by $\mathcal{D}\langle N(e_1, e_2), e_3\rangle = \mathrm{Jac}(e_1, e_2, e_3)$, can be consulted. Then an example of the bracket that we sought goes under the name of *Courant bracket* $[,]_{\mathrm{C}}$:

$$[U, V]_{\mathrm{C}} = [X, Y]_{\mathrm{Lie}} + \mathcal{L}_X\sigma - \mathcal{L}_Y\gamma - \frac{1}{2}d\left(\iota_X\sigma - \iota_Y\gamma\right). \tag{2.8}$$

This is nothing but the antisymmetrization of the Dorfman bracket. Under multiplication with a function f, the bracket transforms to

$$[U, fV]_{\mathrm{C}} = \rho(U)fV - \frac{1}{2}\langle U, V\rangle\,\mathcal{D}f + f[U, V]_{\mathrm{C}}. \tag{2.9}$$

It goes without saying that also the axioms, item 1 and item 2 of definition 2.1.2 do not look the same for the skew-symmetric version of the bracket. In particular, the former is replaced by

$$\rho(e_1)\langle e_2, e_3\rangle = \langle [e_1, e_2] + \mathcal{D}\langle e_1, e_2\rangle, e_3\rangle + \langle [e_1, e_3] + \mathcal{D}\langle e_1, e_3\rangle, e_2\rangle,$$

while the latter is obviously identically zero (in fact, it tells us that a skew-symmetric version of the CA bracket must be given by the bracket with no such symmetry, minus $\mathcal{D}\langle\cdot,\cdot\rangle$).

2.1.2 Generalized metric

The more ordinary choice for a CA metric, in the isomorphism of an exact CA with the Whitney sum of tangent and cotangent space $E \cong TM \oplus T^*M$, is the natural pairing of vector fields with 1-forms, η,

$$\eta = \begin{pmatrix} 0 & \mathbb{1}_d \\ \mathbb{1}_d & 0 \end{pmatrix}.$$

The isometries of this off-diagonal block matrix construct the (special) real orthogonal group of split signature,

$$O(d, d) = \{O \in \mathrm{Mat}_{2d} \,|\, O^T\eta O = \eta\}.$$

These matrices can only have determinant $\det O = \pm 1$. Although well established, it is nevertheless worth to recall what the generators for the part connected to the identity of this Lie group are. By virtue of the split into a vector space and its dual, the generator can be represented as a block matrix with 4 $d \times d$ dimensional blocks. The most general generator o, which satisfies the condition $o^T\eta = -\eta o$, is the following block matrix:

$$o = \begin{pmatrix} a & \beta \\ B & -a^T \end{pmatrix}, \quad a \in \mathrm{End}(TM), B \in \Lambda^2 T^*M, \beta \in \Lambda^2 TM.$$

Hence $\mathfrak{o}(d,d) = \mathrm{End}(TM) \oplus \Lambda^2 TM \oplus \Lambda^2 T^*M$. Exponentiation of an element from each of the classes yields respectively the group elements:

$$\mathbf{A} = \begin{pmatrix} A & 0 \\ 0 & A^{-T} \end{pmatrix}, \quad \exp\beta = \begin{pmatrix} \mathbb{1} & \beta \\ 0 & \mathbb{1} \end{pmatrix}, \quad \exp B = \begin{pmatrix} \mathbb{1} & 0 \\ B & \mathbb{1} \end{pmatrix}, \tag{2.10}$$

where $A(x) \equiv \exp a \in \mathrm{Aut}(TM)$. All the considerations on the symmetry group and algebra are valid as well for a generic vector space and its dual. An important Weyl group of the $O(d,d)$ group is $O(d,d;\mathbb{Z})$, i.e. the orthogonal group of split signature with integer coefficients: it is the group for T-duality transformations among different stringy backgrounds with isometries (section 1.4.1).

Anyway, many other instances of a CA metric are comprised in definition 2.1.2: there are no constraints which force us to choose the η pairing. Thus the symmetry group for the metric varies accordingly, and depending on the purposes, one or the other will be more convenient. We shall in fact employ some more or less slight deviation to the $O(d,d)$-invariant pairing.

Another very useful concept for physicists, aside from the algebroid metric, is that of the *generalized metric*. This is a further geometrical structure originated from the reduction of $O(d,d)$ to its maximal compact subgroup $O(d) \times O(d)$. It is also equivalent to the choice of a d-dimensional subbundle E_+ positive definite w.r.t. η. So its existence assumes the presence of the algebroid standard pairing. On the other hand, splitting E into E_+ and its orthogonal complement E_-, grants the presence of a positive definite metric τ,

$$\tau(U,V) = \eta(U_+,V_+) - \eta(U_-,V_-), \tag{2.11}$$

where the subscripts denote to which complementary subbundle the vectors belong. Another characterization can be $E_+ = \mathrm{Ker}(\mathrm{id} - \tau)$. τ is naturally seen as the linear map $\tau : \Gamma(E) \mapsto \Gamma(E)$, and from the above expression (2.11) it is not difficult to be convinced that $\tau^2(U) = U$. Thus τ is technically an *involution*. This last statement is also equivalent to express E_+ as the graph of $(g + B)$, for $g \in \vee^2 T^*M$ and $B \in \Lambda^2 T^*M$. Let us briefly discuss the argument in favor of this observation: $E_\pm$ (the positive and negative definite subbundles) are maximal, thus of dimension d; but E is also $E \cong TM \oplus T^*M$, $\eta(TM,TM) = 0 = \eta(T^*M,T^*M)$, implying that neither TM nor T^*M have a non-null intersection with E_+. This suggests that E_+ comes from the graph of an element of $\mathrm{Hom}(TM,T^*M)$, which has a unique decomposition into symmetric and antisymmetric part, g and B. Hence $E_+ \ni U_+ = X + (g+B)(X)$, and it can be checked that $\eta(U_+,U_+) = 2g(X,X)$, so g must be positive ($\eta_{|E_+}$ is). The generic element of E_-, being $\eta(E_+,E_-) = 0$, must then be $U_- = X - (g-B)(X)$. Vice versa, namely that the presence of g and B leaves $E_\pm$ defined, can also be shown to hold true.

From now on, with "generalized metric" we will mostly address a more convenient object for computations. Since E_+ induces g as metric on the tangent space, $\underline{g}$ as below is the metric on $TM \oplus T^*M$:

$$\underline{g} := \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix},$$

and the B-field will enter as a B-transform to construct the (g,B)-dependent metric $\mathbf{G} = (\exp B)^T \circ \underline{g} \circ \exp B$.

Definition 2.1.4. Generalized metric. The symmetric bilinear form $\mathbf{G} : \Gamma(E) \times \Gamma(E) \mapsto C^\infty(M)$,

$$\mathbf{G} = \begin{pmatrix} g - Bg^{-1}B & -Bg^{-1} \\ g^{-1}B & g^{-1} \end{pmatrix}. \tag{2.12}$$

Notice that $\mathbf{G}$ already appeared previously in the introduction. Given that they descend from the B-transform which is an $O(d,d)$ element, the set of all generalized metrics $\mathbf{G}$ is the coset $O(d,d)/\left[O(d)\times O(d)\right]$. This is explained by the $O(d,d)$-action being transitive on the set of $\mathbf{G}$, and by $O(d)\times O(d)$ being the stabilizer (see also lemma 3.9.2 and 3.9.3 in [46]). Be reminded that the generalized metric in general does not serve as the $\langle,\rangle$ for the Courant algebroid: further restrictions must be applied in order for $\mathbf{G}$ to be used for that purpose.

Let us mention that one could as well choose an element of $\mathrm{Hom}(T^*M,TM)$ to yield the decomposition $E=E_+\oplus E_-$. This map will have a symmetric part, $G^{-1}:\Gamma(T^*M)\vee\Gamma(T^*M)\mapsto C^\infty(M)$, and an antisymmetric one, $\Pi\in\mathfrak{X}^2M$. Thus $(g+B)$ and $G^{-1}+\Pi$ should be related according to the *closed-open strings relations* [47]:

$$(g+B)^{-1}=G^{-1}+\Pi. \tag{2.13}$$

As before, the generalized metric in terms of G^{-1} and Π can just be computed from the β-transform (with Π as bivector):

$$\begin{aligned}\mathbf{G}&=\begin{pmatrix}\mathbb{1}&0\\ \Pi&\mathbb{1}\end{pmatrix}\begin{pmatrix}G&0\\0&G^{-1}\end{pmatrix}\begin{pmatrix}\mathbb{1}&-\Pi\\0&\mathbb{1}\end{pmatrix}\\&=\begin{pmatrix}G&-G\Pi\\ \Pi G&G^{-1}-\Pi G\Pi\end{pmatrix}.\end{aligned}$$

The presence of vector bundles with a metric brings us to the natural question of setting up a connection and thus transporting generalized vector fields and taking covariant derivative of the tensors, i.e. the natural question of the differential geometry of the structures seen so far. This is the topic of the next subsection.

2.2 Differential generalized geometry

In the current part and in the subsequent ones we are going to clarify some differential geometry objects and their properties, in the realm of Generalized Geometry. Some of the most well-grounded definitions will be given and discussed, however we will mostly focus on the equivalent (or, for some instances, completely new) denotations provided in the personal research papers [3] and [7]. There, we introduced new definitions for a differential (2.14), a skew-symmetric bracket on sections $\Gamma(E)$ 2.2.2, a torsion tensor (2.3.1) and curvature tensors (2.25), (2.26). They all shared a closer resemblance to their counterparts in Riemannian geometry.

A coboundary operator $\mathcal{D}:=g_E^{-1}\circ\rho^*\circ d$, where d is the de Rham differential, was already employed previously with the purpose of making some of the axioms for the Courant algebroid more transparent. In the course of the thesis we will instead prefer to resort to a differential d_ρ (anchor map dependent), whose action on the functions coincides with that of $\mathcal{D}$, but it is extended to $\Omega^\bullet(E)$. It is firstly defined to act on the functions $\Omega^0(E)$ in a local patch.

Definition 2.2.1. Differential d_ρ. A *differential* $d_\rho:\Omega^0(E)\cong C^\infty(M)\mapsto\Omega^1(E)$, $d_\rho^2=0$, is given in a local coordinate patch with $\xi_\alpha\cong\left(\iota_{\partial_i},dx^i\right)$ by:

$$d_\rho:=\tilde{\xi}^\alpha\rho(\xi_\alpha), \tag{2.14}$$

where $\tilde{\xi}^\alpha$ is the dual coordinate ($\tilde{\xi}^\alpha:=g_E^{\alpha\beta}\xi_\beta$). According to the scopes, it can also be intended as $d_\rho=\xi_\alpha\tilde{\rho}\left(\tilde{\xi}^\alpha\right)$, $\tilde{\rho}:\Gamma(E^*)\mapsto\Gamma(TM)$ (however when acting on functions it then yields a section of E by duality, rather than a 1-form on E).

As anticipated, the differential acts on functions $f \in C^\infty(M)$ in the same way as $\mathcal{D}$ does (see (2.2)):

$$\langle d_\rho f, e\rangle = \rho(e)f, \quad e \equiv e^\alpha \xi_\alpha.$$

On the preferred holonomic basis $\{\xi_\alpha\}$, $d_\rho \xi_\alpha$ vanishes. It squares to zero if the anchor map is global, e.g. it is a constant projector. Later at the end of section 2.6 we will give an alternative proof (from graded symplectic geometry) that (2.14) truly yields $d_\rho^2 = 0$; it stems from an isomorphism with a cohomological vector field for the graded Poisson structure. In the wish to extend the action of d_ρ to generalized k-forms in the most obvious way, the generalization of the Lie bracket of vector fields would be the ideal candidate (section 4 of [3]):

Definition 2.2.2. Generalized Lie bracket. A *generalized Lie bracket* $[\![,]\!]$ is a bilinear operation which is antisymmetric and $\mathbb{R}$-linear: $[\![,]\!] : \Gamma(E) \times \Gamma(E) \mapsto \Gamma(E)$,

$$[\![U, V]\!] = -[\![V, U]\!], \qquad [\![U, fV]\!] = \rho(U)f\, V + f[\![U, V]\!], \quad \forall f \in C^\infty(M),$$

where $\rho : \Gamma(E) \mapsto \Gamma(TM)$.

An example of generalized Lie bracket can be constructed first of all in a coordinate-dependent fashion. Consider the holonomic basis $\{\xi_\alpha\}$ for the fiber of $E \cong T^*[1]M \oplus T[1]M$[1] generalized tangent bundle and set $[\![\xi_\alpha, \xi_\beta]\!] = 0$, pretty much as the Lie bracket of tangent vectors is null on the partials. Then on two sections $U, V \in \Gamma(E)$ written in this basis, a local expression for $[\![,]\!]$ with arguments $\Gamma(E) \ni U = U^\alpha(x)\xi_\alpha, V = V^\alpha(x)\xi_\alpha$ can be:

$$[\![U, V]\!] = (\rho(U)V^\alpha(x) - \rho(V)U^\alpha(x))\ \xi_\alpha. \tag{2.15}$$

On a different, anholonomic basis, the Lie-like bracket will be modified according to the Leibniz property. Its expression will hence differ from that in the holonomic basis (2.15) by an antisymmetric 2-tensor. The local expression can be defined globally if in the overlapping patches (covering the whole manifold) one forces its defining properties of definition 2.2.2, applying Leibniz rule to the transition functions. The mechanism is consistent as long as in the triple overlaps ρ is not defined just patch-wise, but globally (which is always true if ρ is the anchor for an algebroid).

Another way to find a coordinate-independent expression for the Lie-like bracket is also available. Let $\varsigma : E \xrightarrow{\sim} E'$ be the tautological 1-form, or soldering form. Then the ρ-differential on ς, which itself is a 1-form in $\Omega^1(E)$, can produce a 2-form provided that a generalized Lie bracket $[\![,]\!]$ (2.2.2) is employed:

$$d_\rho\varsigma(U, V) = \rho(U)\varsigma(V) - \rho(V)\varsigma(U) - \varsigma([\![U, V]\!]). \tag{2.16}$$

In the local coordinate chart the soldering form ς has a canonical formula:

$$\varsigma = dx^i \otimes \partial_i \oplus \partial_i \otimes dx^i,$$

therefore $d_\rho\varsigma = 0$ and thus from (2.16) the expression for $[\![,]\!]$ in holonomic coordinates (2.15) is recovered, as $[\![\xi_\alpha, \xi_\beta]\!] = 0$.

When the bracket $[\![,]\!]$ is independently given, rather than determined from the ρ-differential, d_ρ would differentiate a form $\varpi \in \Omega^{k-1}(E)$ according to:

$$\begin{aligned} d_\rho\varpi(U_1, \dots, U_k) = &\sum_i (-)^i \rho(U_i)\varpi\left(U_1, \dots, \hat{U}_i, \dots U_k\right) \\ &+ \sum_{i<j} \varpi\left([\![U_i, U_j]\!], \dots \hat{U}_i, \dots, \hat{U}_j, \dots U_k\right), \end{aligned}$$

[1] In section 2.4 we will see that the number in the square brackets denotes odd parity of the fibers: a minus sign appears when commuting a pair of coordinates.

see the companion formula for a Lie algebroid (2.1). $d_\rho^2 = 0$ iff the anchor is a homomorphism with the Lie bracket of vector fields and iff $[\![,]\!]$ respects the Jacobi identity; for example the bracket (2.15), in the holonomic basis, respects this condition already when ρ is required to be a homomorphism with the Lie algebra of $\mathfrak{X}^1(M)$.

Example 2.2.3. The standard instance of Dorfman bracket, i.e. the bracket for the CA $(TM \oplus T^*M, \eta, \mathrm{pr}_{TM}, [,]_{\mathrm{D}})$:

$$[U, V]_{\mathrm{D}} = [X, Y]_{\mathrm{Lie}} + \mathcal{L}_X \sigma - \iota_Y d\sigma,$$

is partly given by $[\![,]\!]$. In fact, setting $\eta := \langle , \rangle$, one can easily check that

$$[U, V]_{\mathrm{D}} - [\![U, V]\!] = \langle d_\rho U, V \rangle \,, \tag{2.17}$$

where actually $d_\rho U \in \Omega^1(E)$ (so it is not a 2-form as one may think), as it is given by $d_\rho U = \rho^*(dU^\alpha(x))\xi_\alpha$, with ρ^* that in this case (where $\rho = \mathrm{pr}_{TM}$) just embeds $dU^\alpha(x)$ in E^*. Then the pairing works as the musical isomorphism to lower the index on V^β. This leaves a section of the dual E^* defined on the RHS. The sides of the equation match when on the LHS we contract with $\Gamma(E^*) \ni \langle \cdot, W \rangle$, and on the RHS we plug in W, so that on both sides we are left with a function.

Of the two main notions of differentiation on a smooth manifold, the Lie derivative and the covariant derivative, we already hinted a bit at the extension of the former to Courant algebroids. The Dorfman derivative is formally a Lie derivative but built with the non-skewsymmetric CA bracket. Despite the lack of antisymmetry, it fulfills

$$\mathcal{L}_U \mathcal{L}_V W - \mathcal{L}_V \mathcal{L}_U W = \mathcal{L}_{[U,V]_{\mathrm{D}}} W.$$

This is, in fact, an equivalent version of the Jacobi identity (in the form given in item 1, common to definition 2.1.1 and definition 2.1.2). Conversely, in order to get a covariant derivative a connection is needed. On a Courant algebroid a connection $\nabla : \Gamma(E) \mapsto \Gamma(E^*) \otimes \Gamma(E)$ is specified by the properties

$$\begin{aligned} \nabla_{fU} V &= f \nabla_U V, \\ \nabla_U f V &= \rho(U) f\, V + f \nabla_U V, \end{aligned} \tag{2.18}$$

of any affine connection on a vector bundle. A common request is usually to ask for compatibility with the pairing:

$$\rho(U)\langle V, W \rangle = \langle \nabla_U V, W \rangle + \langle V, \nabla_U W \rangle.$$

In the next part we will see that this is not an independent requirement but it is rather related to one of the axiom for a CA, and ultimately is a consequence of having a graded Poisson algebra associated with the Courant algebroid. The set of all affine connections compatible with $\langle , \rangle$ has the structure of an affine space modeled on $\Gamma\left(E^* \otimes \mathfrak{o}(E)\right)$, with $\mathfrak{o}(E)$ bundle of skew-symmetric endomorphisms of E with respect to $\langle , \rangle$. Another additional feature that could be implemented is certainly the requirement for the generalized metric to be covariantly constant, $\nabla \mathbf{G} = 0$. This is obviously equivalent to say that $\nabla : \Gamma(E_\pm) \mapsto \Gamma(E^* \otimes E_\pm)$, i.e. the eigenbundles are preserved. From a graded Poisson algebra viewpoint $\nabla \mathbf{G} = 0$ is also an independent requirement. We will see that to get our results these properties of the connection will be imposed in a different way.

It would be intriguing if the Generalized Geometry analogue of the Levi-Civita theorem could hold. This is not the case, however. Torsion-free connections (compatible with the generalized metric) are proven to exist (see proposition 3.3 in [23]), but there is plenty of them. According to the just cited reference, connections with a fixed torsion (not necessarily zero) form an affine space modeled on the space of mixed symmetry 3-tensors Σ,

$$\begin{aligned}
&\Sigma = \Gamma\left(E^{\otimes^3}\right) \cap \Sigma,\\
&\text{where } \Sigma := \Gamma(E) \oplus \Sigma_0, \quad \Gamma(E) \ni e \text{ with } \sigma^e(W,U,V) = \langle W,V\rangle\langle e,U\rangle - \langle W,U\rangle\langle e,V\rangle,\\
&\qquad \Sigma_0 := \left\{\sigma \in \Sigma \,|\, \textstyle\sum_{i=1}^{rank\,E} \sigma(\xi_i, \tilde{\xi}^i, \cdot) = 0\right\}.
\end{aligned} \tag{2.19}$$

As an E^*-valued endomorphism of E, σ^e is reminiscent of the 1-*form valued endomorphisms* in pseudo-Riemannian geometry, due to the variation of a metric connection with fixed torsion upon a conformal change of the metric. Later, we will have $e = \begin{pmatrix} 0 \\ -(1/3)d\phi(x) \end{pmatrix}$, $\phi(x) \in C^\infty(M)$, in section 3.2, and as just explained it cannot change the torsion: this claim will also be explicitly verified. Our connection will actually not belong to the space of torsion-free generalized connections.

However notice that the Levi-Civita theorem, despite being invalid for Generalized Geometry which is a mixture of complex, symplectic and Riemannian geometry, holds instead for para-hermitian geometry [48]. This kind of geometry is studied in relation with DFT (see section 1.4.2) because it can potentially solve many ambiguities of the latter concerning global aspects. Let us give a short and rather approximate definition. Para-hermitian manifolds are triples (M, K, η) where M is a manifold of even dimension endowed with a $(1,1)$-tensor field $K \in \mathrm{End}(TM)$ such that $K^2 = \mathbb{1}$, η is the $O(d,d)$ pairing and is compatible with K in the following sense: $K^T \eta K = -\eta$, or equivalently $\eta(K(X), Y) + \eta(X, K(Y)) = 0$. It thus induces a non-degenerate 2-form $\omega = \eta(K(X), Y)$, generally not closed. Also, the subbundles $L_\pm$, $\Gamma(L_\pm) \ni \frac{1}{2}(\mathbb{1} \pm K)(X)$, are required to be integrable. The topic was recently reviewed in [49].

A nice example of a connection frequently employed in Generalized Geometry is the Bismut connection. Remind that $(g+B)$ and $-(g-B)$ are the applications whose graph gives rise to E_+ and E_- respectively. Call $\mathbf{C} : E_\pm \mapsto E_\mp$ the operator that switches the two eigenbundles among themselves. Given that the generalized vector $U = X + \gamma \in \Gamma(E)$ decomposes in $E_+ \oplus E_-$ as

$$X + \gamma = \frac{1}{2}\Big[\, (X_+ + X_-) + \big(g^{-1}(\gamma)_+ - g^{-1}(\gamma)_-\big) - \big(g^{-1}B(X)_+ - g^{-1}B(X)_-\big)\,\Big],$$

where the subscripts underline to which of the mutually exclusive subspaces $E_\pm$ the vectors belong, then the operator $\mathbf{C}$ applied to it yields:

$$\mathbf{C}(X + \gamma) = X - \gamma + 2B(X).$$

Definition 2.2.4. Generalized Bismut connection. [18] The connection ∇ given by

$$\nabla_W U = [W_+, U_-]_{\mathrm{D},(-)} + [W_-, U_+]_{\mathrm{D},(+)} + [\mathbf{C}(W_+), U_+]_{\mathrm{D},(+)} + [\mathbf{C}(W_-), U_-]_{\mathrm{D},(-)}.$$

Interestingly, one can show that the applications $(g+B)$ and $(-g+B)$, if invertible, are a homomorphism of the Bismut connection with either one of the two regular connections $\nabla^\pm : \Gamma(TM) \mapsto \Gamma(T^*M \otimes TM)$,

$$\nabla^\pm_X Y = \nabla^{\mathrm{L.C.}}_X Y \mp \frac{1}{2} g^{-1} H(X, Y, \cdot).$$

This connection can be taken as starting point to compute the low-energy string effective action with no dilaton (depicted in section 1.3).

Often we will recur to the possibility to employ a splitting of the CA exact sequence. Let us collect here some definitions for further reference.

A *splitting* of the sequence (2.5) is a map $s : \Gamma(TM) \mapsto \Gamma(E)$ such that $\rho(s(X)) = X$. When $\langle s(X), s(Y)\rangle = 0$, it was possible to infer that exact CA are equivalent up to a choice of a closed 3-form (an element in the third cohomology class). However, non-isotropic splitting will be relevant too, since they leave us with an induced metric g_{TM} on regular tangent vector fields:

$$\langle s(X), s(Y)\rangle =: g_{TM}(X, Y).$$

When a splitting is present, by means of the CA metric and the canonical pairing of vector fields and forms the dual map is available too, $s^* : \Gamma(E) \mapsto \Gamma(T^*M)$

$$\langle s(X), U\rangle_E = \langle X, s^*(U)\rangle;$$

on the RHS the pairing is the canonical one of vector fields and 1-forms. With s^* one can also check whether $s^*(j(\gamma)) = \gamma$, but whether this may hold or not is a rather accessory feature for the maps.

Dualizing the sequence will also lead to interesting results. It corresponds to consider this sequence

$$0 \to TM \xrightarrow{\Delta} E \xrightarrow{\Delta^*} T^*M \to 0. \tag{2.20}$$

The procedure is orchestrated in this way: one starts with the usual short exact sequence and retains the $s \equiv \Delta$ splitting from it, then builds Δ^* such that $\mathrm{Im}\Delta = \mathrm{Ker}\Delta^*$, and eventually works out a cosplitting $r : \Gamma(T^*M) \mapsto \Gamma(E)$, either isotropic or not. In the latter case, there is an induced metric on forms $g_{(T^*M)}$:

$$\langle r(\gamma), r(\sigma)\rangle =: g_{(T^*M)}(\gamma, \sigma).$$

Furthermore, the generalized tangent bundle has a 1-*gerbe structure*. This is the higher geometric analogue of a circle bundle, i.e. a principal S^1-bundle with $U(1)$ as structure group. The exposition follows [50] but in terms of gauge fields in replacement of transition functions. The 1-form $U(1)$-field A in the overlap of two patches $U_{(i)} \cap U_{(j)} \neq \{0\}$ behaves as follows

$$A_{(i)} = A_{(j)} - df_{(ij)}, \quad f = e^{2\pi i\varphi} \in C^\infty(M).$$

The set of functions, on triple overlaps of open sets $U_i \cap U_j \cap U_k$, shall respect a cocycle condition:

$$f_{(ij)} + f_{(jk)} + f_{(ki)} = 1.$$

This structure generalizes to line bundles over the intersection of two patches, to constitute a so-called 1-gerbe structure. The gauge field is now a 2-form B field for the CA $E \cong TM \oplus T^*M$. On the overlapping local patches $U_i \cap U_j$ with non-empty intersection,

$$B_{(i)} = B_{(j)} - dA_{(ij)}.$$

d here is the de Rham differential. Now on intersecting overlaps of a triple of open sets, $U_i \cap U_j \cap U_k \neq \{0\}$ a cocycle condition needs to be fulfilled by the cohomological forms A themselves:

$$A_{(ij)} + A_{(jk)} + A_{(ki)} = (dh)_{(ijk)}.$$

h are smooth functions with values in $U(1)$. Later the 1-gerbe structure will play an important role when constructing scalar curvatures for the generalized tangent bundle, because they will be naturally invariant under the gauge symmetry of the B-field (see 3.2).

We just saw that having a gerbe is also equivalent to possess a cohomology class, in these cases $H^2(U,\mathbb{Z})$ and $H^3(U,\mathbb{Z})$ (we encountered $H^3(U,\mathbb{R})$, but this is just a matter of rescaling with π factors). Looking at the generalized tangent space in the short exact sequence (2.20), it is actually more natural to have a homology class $H_3(U,\mathbb{Z})$, since there TM is fibered over T^*M and not vice versa. The gauge field is the bivector Π, being locally $R = d_\rho\Pi$ for $\rho(U) = \beta(\gamma)$, $\beta : \Gamma(T^*M) \mapsto \Gamma(TM)$ and $R \in \mathfrak{X}^3(M)$ representative of the third homology class. As previously, on double overlapping open sets, Π patches according to $\Pi_{(i)} = \Pi_{(j)} - d_\beta\alpha_{(ij)}$, where $\alpha \in \Lambda(TM)$. The cocycle condition in the intersection of three open sets holds for α:

$$\alpha_{(ij)} + \alpha_{(jk)} + \alpha_{(ki)} = (d_\beta\nu)_{(ijk)}\,.$$

Invariance under the gauge symmetry of the Π field will be present in curvature scalars for the generalized tangent bundle (see section 3.3).

2.3 Torsion and curvature tensors

Having a CA bracket which is either non-antisymmetric (Dorfman bracket) or that does not fulfill the Jacobi identity (Courant bracket $[,]_{\mathrm{C}}$) puts a serious obstruction to the definition of the torsion tensor and the curvature tensor, borrowed from standard differential geometry. For the torsion, an alternative definition appeared in [18]:

$$T_{\mathrm{C}}(W,V,U) = \langle \nabla_W V - \nabla_V W - [W,V]_{\mathrm{C}}\,, U\rangle + \frac{1}{2}\left(\langle \nabla_U W, V\rangle - \langle \nabla_U V, W\rangle\right). \tag{2.21}$$

Tensoriality in the first slot is easily checked by means of (2.9), then in the second slot follows from antisymmetry and in the third is immediate. A similar definition of torsion tensor can however be furnished with the Dorfman bracket in the place of the Courant bracket [51]:

$$T_{\mathrm{D}}(W,V,U) = \langle \nabla_W V - \nabla_V W - [W,V]_{\mathrm{D}}, U\rangle + \langle \nabla_U W, V\rangle. \tag{2.22}$$

Again, checking that T_{D} is a tensor takes one line of computation and uses the result in (2.4), antisymmetry among the first two entries, and manifest tensoriality of the third slot. T_{C} is evidently totally antisymmetric, but it is also quite straightforward to show that also $T_{\mathrm{D}} \in \Lambda^3E^*$: it is sufficient to use one time the axioms 1 and 2 in the definition 2.1.2, which in plain English tell us that the bracket is a derivation of the pairing and that it fails to be antisymmetric by $\mathcal{D}$ (or d_ρ) of the pairing.

However if a generalized Lie-bracket is at our disposal a new possibility for the torsion tensor opens up. This is what we used in [3] and its follow-up paper [7].

Definition 2.3.1. Torsion tensor. A *torsion tensor* for E-vectors $T : \Lambda^2E^* \mapsto E$, $T(fW,V) = fT(W,V)$ and $T(W,V) = -T(V,W)$, can be defined by the following expression:

$$T(W,V) = \nabla_W V - \nabla_V W - [\![W,V]\!]. \tag{2.23}$$

This torsion is more general than (2.21) and (2.22) since the bracket employed here fulfills a fewer number of conditions. In the following we will sometimes slightly abuse the notation and refer with T to the fully contracted expression, $T : \Lambda^2 E^* \otimes E^* \mapsto C^\infty(M)$, $T(U, V, W) = \langle T(U, V), W\rangle$. It should not cause confusion. We shall now see that if the connection depends on the Dorfman bracket for the CA and on the Lie-like bracket, the torsion T_{D} coincides with T. That is a side remark descending from an observation, reported here in the form of a proposition. It was proven by ourselves in [3].

Proposition 2.3.2 (Mutual dependence of brackets and connection)**.** *Given a non-degenerate symmetric bilinear form $\langle,\rangle$ on the sections of a vector bundle E and an anchor $\rho : E \mapsto TM$, then a generalized Lie-bracket $[\![,]\!]$ on E-sections (as in definition 2.2.2), an affine connection $\nabla : \Gamma(E) \mapsto \Gamma(E^*) \otimes \Gamma(E)$ which is also metric and has totally antisymmetric torsion $T \in \Gamma(\Lambda^3 E^*)$, and a bracket $[,]$ as in definition 2.1.2, but for which the Jacobi identity in Leibniz form (i.e. as in item 1 of definition 2.1.1) does not necessarily hold, depend upon each other in this way:*

$$\langle [U, V] - [\![U, V]\!], W\rangle = \langle \nabla_W U, V\rangle. \tag{2.24}$$

Proof. (i) Assume to be given $[,]$ and $[\![,]\!]$ under the hypotheses of the proposition. Then their difference shall satisfy the properties of an affine connection (2.18), be metric for $\langle,\rangle$ and have fully antisymmetric torsion. Tensoriality for W is obvious. Multiplication of V with a function still transforms the expression as a tensor because the CA bracket $[,]$ and the Lie-like bracket change in a way that they compensate each other, see the second axiom of definition 2.1.1, and definition 2.2.2. Then sending U in fU leads to

$$\begin{aligned}\langle \nabla_W fU, V\rangle &= \langle -\rho(V) f\, U + d_\rho f \langle U, V\rangle + f[U, V] - f[\![U, V]\!] + \rho(V) f\, U, W\rangle \\ &= f\langle [U, V] - [\![U, V]\!], W\rangle + \rho(W) f\langle U, V\rangle = f\langle \nabla_W U, V\rangle + \rho(W) f\langle U, V\rangle,\end{aligned}$$

as expected from an affine connection $\nabla_W U$. Metricity follows from the antisymmetry of $[\![,]\!]$ and the second axiom for the bracket in definition 2.1.2. The torsion of the connection is completely antisymmetric because

$$\langle T(U, V), V\rangle = \langle [V, V], U\rangle - \langle [U, V], V\rangle = \frac{1}{2}\rho(U)\langle V, V\rangle - \frac{1}{2}\rho(U)\langle V, V\rangle = 0.$$

(ii) Assume now to be given $[\![,]\!]$ and the metric connection with skew torsion under the hypotheses of the proposition. Fulfillment of the Leibniz rule for the second entry of the resulting $[,]$ is a plain consequence of the Leibniz rule for the Lie-like bracket. The first axiom is fulfilled because of metricity and thanks to $\langle T(V, U), W\rangle = -\langle T(W, U), V\rangle$. The second axiom is then a consequence of ∇ being metric. (iii) Starting with the non-skewsymmetric bracket and the connection, their difference generates a generalized Lie bracket because it is antisymmetric, since the lack of antisymmetry of $[,]$ is compensated by the metric preserving connection, while fulfillment of the Leibniz rule is a direct consequence of the same condition on $[,]$.

□

Proposition 2.3.2 is certainly interesting on its own right. However there is opportunity for some quite curious consequences, which present themselves whenever we relax some of the demands for $[,]$ or ∇. As pointed out below definition 2.1.2, the first axiom (item 1 in 2.1.2) is a bit counterintuitive in usual geometry. Shall you not be interested in requiring it for the binary operation $[,]$, while keeping the other two conditions (i.e. the Leibniz rule for the second entry and that a coboundary regulates the failure of antisymmetry for $[,]$), then the connection is less restricted. It is still affine (via Leibniz rule of $[,]$ and $[\![,]\!]$) and metric (this property is associated with the second axiom, item 2 of 2.1.2) but its torsion is not forcefully skewsymmetric. Instead for the Lie-like bracket, switching from the maximal to the minimal set of assumptions for $[,]$ or ∇, does not provide any change to it. Jacobi identity for the CA bracket is supposedly related to the curvature of the connection, but further investigations are still pending. These remarks, for a connection on vector fields arising as the difference of an $\exp \mathcal{G}$ deformation of the bracket and the bracket itself, $\mathcal{G} \in S^2T^*M$, were already presented in [52] and [53].

One immediate striking consequence of the proposition is that for the affine connections which are the difference between a Dorfman bracket and a generalized Lie bracket the associated torsion tensor (2.22) is equivalent to the contraction of the one of definition 2.3.1 with a third generalized vector:

$$T_{\mathrm{D}}(U,V,W) = \langle [V,W]_{\mathrm{D}} - [\![V,W]\!], U\rangle - \langle [U,W]_{\mathrm{D}} - [\![U,W]\!], V\rangle - \langle [\![U,V]\!], W\rangle \equiv T(U,V,W).$$

This is not surprising as both T_{D} and T belong to $\Lambda^3 E^*$.

Notice that example 2.2.3, which was meant to show the presence of the Lie-like bracket $[\![,]\!]$ in the standard Dorfman bracket (2.7), falls also into the proposition (2.24): one should see $\langle d_\rho U, V\rangle \in \Omega^1(E)$ in (2.17) as the contraction of the *flat* connection $\partial U \in \Gamma(E^* \otimes E)$ with the generalized vector $\Gamma(E^*) \ni \langle \cdot, V\rangle$, which leaves with a section of the dual to E. The connection due to these choices has no non-trivial connection symbols and zero curvature.

Although (2.24) will mostly be used to determine a connection on E-sections, it is essential to remark once again that starting with a given connection and one of the two types of brackets, that relation leads to the determination of the other bracket, *globally*. Showing that the bracket $[,]$ on E-sections could automatically respect the Jacobi identity, using (2.24), is still work in progress.

Defining a curvature tensor with the CA-bracket is troubling too. The Riemann curvature of the connection, when mirroring the standard definition of Riemannian geometry,

$$Riem(W,V)U = \nabla_W \nabla_V U - \nabla_V \nabla_W U - \nabla_{[W,V]_{\mathrm{D}}} U, \quad \text{“(wrong)”},$$

misses anyway a very important point: it fails to be a tensor. This is clear from the covariant derivative of the commutator, as the latter is given by the Dorfman bracket, that is known to behave in different ways under $C^\infty(M)$-multiplication in the first (2.4) or second slot (axiom 2 of 2.1.1). Employing $[,]_{\mathrm{C}}$ (2.8) in place of the Dorfman bracket does not improve the situation either: due to the transformation property under $C^\infty(M)$-multiplication (2.9), *Riem* with $[,]_{\mathrm{C}}$ would not be a tensor too.

With the Lie-like bracket this task is much more handful.

Definition 2.3.3. Riemann curvature tensor. Given a generalized Lie-bracket $[\![,]\!]$ and a CA connection ∇, the *Riemann curvature tensor* for that connection $\mathbf{Riem} \in \Gamma\left(\bigotimes^3 E\right) \mapsto \Gamma(E)$ is the commutator of covariant derivatives minus the covariant derivative of the Lie-like bracket:

$$\mathbf{Riem}(W,V,U) = [\nabla_W, \nabla_V]\, U - \nabla_{[\![W,V]\!]} U. \tag{2.25}$$

Riem is immediately seen to be $C^\infty(M)$-linear thanks to the properties of $[\![,]\!]$ and ∇.

The Ricci curvature tensor $\mathbf{Ric} \in \Gamma(E^* \otimes E^*)$ is then, as in standard Riemannian geometry, the partial trace of $\mathbf{Riem}$, seen as a section of $\bigotimes^3 E^* \otimes E$:

$$\mathbf{Ric}(V,U) = \sum_{\alpha}^{2d} \langle \mathbf{Riem}(\xi_\alpha, V, U), \tilde{\xi}^\alpha \rangle. \tag{2.26}$$

Another smart solution to the puzzle of the Riemann curvature tensor in Generalized Geometry, suggested almost independently by Ševera, Valach [54] and Garçia-Fernandez [23], consists in bypassing the latter and instead focusing directly on the Ricci curvature tensor built by virtue of a *divergence* operator, div : $\Gamma(E) \mapsto C^\infty(M)$ that satisfies the Leibniz rule:

$$\mathrm{div} f\, e = \rho(e) f + f\, \mathrm{div} e, \quad \forall\, e \in \Gamma(E),\ f \in C^\infty(M).$$

Thus a good choice can be for example

$$\mathrm{div} e = \mathrm{tr} \nabla e, \tag{2.27}$$

where tr denotes the trace. Then the generalized Ricci tensor GRic, referring to E_+, with $\langle , \rangle_{|E_+}$ non-degenerate (positive-definiteness is not a necessary condition here), as a pseudo-generalized metric, is defined by the aforementioned authors as:

$$\begin{aligned}
&\mathrm{GRic} : \Gamma(E_+) \times \Gamma(E_-) \to C^\infty(M), \\
&\mathrm{GRic}(U_+, V_-) := \mathrm{div}[V_-, U_+]_{\mathrm{D},(+)} - \rho(V_-) \mathrm{div}\, U_+ - \mathrm{tr}_{E_+}[[\cdot, V_-]_{\mathrm{D},(-)}, U_+]_{\mathrm{D},(+)}.
\end{aligned}$$

The subscripts on the Dorfman bracket indicate projection onto the subspaces of the positive or negative eigenvalues for $\langle , \rangle$. The proof of $C^\infty(M)$-linearity in both entries is the object of proposition 3.2 of [22]. Essentially, it follows from the Leibniz rule for the divergence and from

$$\mathrm{tr}_{E_+}[[\cdot, fV_-]_{\mathrm{D},(-)}, U_+]_{\mathrm{D},(+)} = f \mathrm{tr}_{E_+}[[\cdot, V_-]_{\mathrm{D},(-)}, U_+]_{\mathrm{D},(+)} + \rho\left([V_-, U_+]_{\mathrm{D},(+)}\right) f;$$

then for U_+ one must work out the expression for $\mathrm{GRic}(V_-, U_+)$ and use the previous result. The above formula for GRic is subsequently proven to be equivalent to the Ricci tensor of a connection metric w.r.t. a Riemannian metric g and with torsion $H \in \Omega^3(M)$, evaluated on the anchored vector fields.

The first half of mathematical background knowledge needed for the comprehension of the main body of this thesis finishes here. The next half will remind the reader about graded spaces and graded Poisson algebras.

2.4 Graded spaces and Poisson algebras

This section illustrates the second topic of mathematical preliminaries on which our research work was based: graded geometry and Poisson algebras. The review is mostly based on [55], [56], [57] and [5]. Let us start with some useful definitions about graded geometry.

2.4.1 Graded Geometry

A first notion to recall is that of graded vector space.

Definition 2.4.1. Graded vector space. A vector space V which is the direct sum of vector spaces V_i over a ring of characteristic zero, $V = \oplus_{i\in\mathbb{Z}} V_i$.

The degree of an element $v \in V$ will be denoted by $|v|$. If for example $v = v_{i_1} \in V_{i_1}$, where $i_1 = 1 \in \mathbb{Z}$, then the degree (or grading) of v is $|v| = 1$. In the course of the thesis we will largely employ the notation $V[n]$ to refer to degree shifting by $n \in \mathbb{Z}$ of a regular vector space V. The symmetric and exterior algebra, $S(V)$ and $\Lambda(V)$ respectively, are easily defined as in the non-graded case upon careful assignation of the degree-dependent signs. If the tensor algebra is

$$T(V) = \oplus_{n\geq 0} V^{\otimes^n} = \mathbb{1} \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \dots,$$

and we consider the ideals

$$I_S := \left\{ v, w \in V \mid v \otimes w - (-1)^{|v|\,|w|}\, w \otimes v \right\},$$
$$I_\Lambda := \left\{ v, w \in V \mid v \otimes w + (-1)^{|v|\,|w|}\, w \otimes v \right\},$$

then the symmetric algebra and the exterior algebra are the quotients:

$$S(V) := T(V)/I_S, \quad \Lambda(V) := T(V)/I_\Lambda.$$

Aside from vector spaces, we wish to set up a grading to manifolds as well, so that we could use some graded differential calculus. This is not too complicated, and requires the notion of graded algebra.

Definition 2.4.2. Graded algebra. $(A, \cdot)$ is a graded algebra if A is a graded vector space and the bilinear associative product $\cdot : A \otimes A \mapsto A$ has degree zero. Graded commutativity of the algebra is the following condition on the product:

$$a \cdot b = (-1)^{|a|\,|b|}\, b \cdot a, \quad a, b \in A.$$

Definition 2.4.3. Graded manifold. A manifold M and a graded vector space V such that the graded algebra of polynomial functions $\mathcal{O}(U)$ of a contractible open set U of M is isomorphic to $C^\infty(U) \otimes \mathcal{S}(V)$, where $\mathcal{S}V = \otimes_{i\in 2\mathbb{N}} S(V_i) \otimes \Lambda(V_{i-1})$.

Graded manifolds will be denoted with curly Latin letters such as $\mathcal{M}$. In order to know the grading of an element of the algebra $\mathcal{O}(\mathcal{M}) = \oplus_{i\in\mathbb{N}} \mathcal{O}_i(\mathcal{M})$ it is useful to recur to the Euler vector field[2] ϵ:

$$\epsilon = |x^i| x^i \frac{\partial}{\partial x^i},$$

where x^i are local coordinates on $\mathcal{M}$, i.e. $|x^i| = 0$ if x^i is a coordinate for U, $|x^i| = j$ if $x^i \in V_j$. The graded manifold is also said to be technically a *N-manifold* since the algebra of polynomial functions $\mathcal{O}(\mathcal{M})$ is graded commutative according to definition 2.4.2, which roughly speaking means that even coordinates commute, while odd coordinates anticommute. This terminology will be used from time to time in the thesis.

Notice that if the degrees are $0, 1 \in \mathbb{Z}_2$, one talks about *supermanifolds*: the algebra of $\mathbb{Z}_2$-graded polynomial functions $\mathcal{O}(U)$ is isomorphic to $C^\infty(U) \otimes \Lambda V_1^*$, where the super vector space ($\mathbb{Z}_2$-graded) decomposes as $V = V_0 \oplus V_1$.

A straightforward generalization of the concept of graded manifold is that of graded vector bundle, $E[k]$: in fact if $E \xrightarrow{\pi} M$ is a vector bundle, its graded version is merely a graded manifold with base M and with algebra of functions $\mathcal{O}(E[k])$ given by

$$\mathcal{O}(E[k]) = \begin{cases} \Gamma(\Lambda E^*), & \text{for odd}\, k \\ \Gamma(SE^*), & \text{for even}\, k; \end{cases}$$

For all our purposes, we will pick up graded manifolds which are graded vector bundles.

2.5 Graded Poisson structures

As mentioned in the introduction, Poisson algebras will play a relevant role in the thesis. The most familiar instance where a Poisson algebra appears in a context of physical interest is classical Hamiltonian mechanics: the phase space of a particle, combining the positions and momenta the particle is allowed to take, possesses a natural symplectic structure. Hamiltonian functions (energy of the system) can be given and the dynamics is then unveiled via Poisson brackets of the conjugate coordinates with the Hamiltonian.

Applications of Poisson/symplectic structures, above all in the graded case, are very essential for some other very relevant topics in physics, as illustrated in the remaining subsections of this section: quantization (BRST quantization and à la Batalin-Vilkovisky) and sigma models. Let us go through the fundamental preliminaries first.

Caveat: unless explicitly specified, we will always assume that the Poisson manifold is also symplectic, i.e. that the Poisson bivector is non-degenerate.

First of all, some objects which are well-known in the ungraded situation will be extended to the graded picture here.

Definition 2.5.1. Graded symplectic form. $\omega \in \Lambda^2 T^*\mathcal{M}$, non-degenerate and closed with respect to the de Rham differential ($d\omega = 0$), is graded with degree n if $\mathcal{L}_\epsilon \omega = n$ (see footnote[2]).

Note that in particular, for $n = 2$ in d dimensions, the putative symmetry group $Sp(4d)$, due to the graded commutativity of the coordinates, is actually $Sp\,(2d) \times SO\,(d, d)$.

[2] the Euler vector field is the infinitesimal generator of the *multiplicative action* Φ of $\mathbb{R}$ on the vector bundle E:

$$\Phi : (t, e) \in \mathbb{R} \times \Gamma(E) \mapsto \Phi_t(e) := \exp(t) \cdot e \in \Gamma(E)$$
$$\epsilon := \frac{d}{dt}_{|t=0} \Phi_t(e).$$

The Euler vector field naturally gives rise to the notion of Euler Lie derivative $\mathcal{L}_\epsilon$.

A graded symplectic manifold has a graded ω of maximal degree. Inversion of the symplectic form yields a Poisson bracket $\{,\}$ of opposite degree, i.e. $|\{,\}| = -n$. Hence $\{\mathcal{O}_k(\mathcal{M}), \mathcal{O}_l(\mathcal{M})\} \subset \mathcal{O}_{k+l-n}(\mathcal{M})$. Let us also briefly show the definition of a graded Poisson algebra:

Definition 2.5.2. Graded Poisson algebra. A triple $(A, \cdot, \{,\})$ where A is a graded vector space, the product is bilinear and associative and carries no degree, while $\{,\} : A \times A \mapsto A$ has degree $-n$, is graded skew-symmetric and fulfills the graded Jacobi identity, in order:

$$\begin{aligned} \{a, b\} &= -(-1)^{(|a|-n)(|b|-n)}\{b, a\}, \\ \{a, \{b, c\}\} &= \{\{a, b\}, c\} + (-1)^{(|a|-n)(|b|-n)}\{b, \{a, c\}\}, \quad a, b, c \in A. \end{aligned}$$

The bracket is also a biderivation of the product:

$$\{a, b \cdot c\} = \{a, b\} \cdot c + (-1)^{|b|(|a|-n)} b \cdot \{a, c\}. \tag{2.28}$$

As a rule of thumb, for n even (resp. odd), the additional minus sign is "activated" just when both the involved elements of the algebra are odd (resp. even).

A differential structure on graded symplectic manifolds $\mathcal{M}$ is provided by a homological vector field.

Definition 2.5.3. Homological vector field Q. An element $\mathrm{Q} \in \mathfrak{X}(\mathcal{M})$ of odd degree $m < n$ is a *homological vector field* if $\mathcal{L}_{\mathrm{Q}}\omega = 0$.

Technically, the presence of Q turns a N-manifold into a NQ-manifold. Other names after which these differential graded symplectic manifolds are called are also *dg-symplectic manifolds* and *QP-manifolds*.

Some unexpected results about graded symplectic manifolds are contained in [5] (lemma 2.2 therein):

1. Any symplectic form of degree $n \geq 1$ is necessarily exact:

$$\omega = d\left(\frac{1}{n}\iota_\epsilon \omega\right). \tag{2.29}$$

   (Could be easily proven by graded Cartan identities with the Euler Lie derivative $\mathcal{L}_\epsilon$.)

2. Any vector field X of degree $m > -n$ preserving ω is Hamiltonian:

$$\iota_X \omega = \pm\left(\frac{1}{m+n}\iota_X \iota_\epsilon \omega\right).$$

   Proof: using $[\epsilon, X] = mX$, the identity $[\mathcal{L}_\epsilon, \mathcal{L}_X] = \mathcal{L}_{[\epsilon, X]}$ and the observation that $[\mathcal{L}_\epsilon, d]\omega = 0$ (which follows from $[d, d] = d^2 = 0$),

$$\begin{aligned} [\mathcal{L}_\epsilon, \mathcal{L}_X]\omega &= \mathcal{L}_\epsilon d\left(\iota_X \omega\right) - \mathcal{L}_X n\omega = \mp d\left(\mathcal{L}_\epsilon \iota_X \omega\right) - d\iota_X n\omega \\ \mathcal{L}_{[\epsilon, X]}\omega &= \mathcal{L}_X m\omega \end{aligned}$$

   Then the statement follows directly from $d\left((m+n)\iota_X \omega\right) = \pm d\left(d\iota_X \iota_\epsilon \omega\right)$.

$\square$

Therefore we can assert that a homological vector field Q by definition induces a *Hamiltonian* function Θ of degree $n+m$,

$$\mathrm{Q} = \{\cdot, \Theta\},$$

which then, since $\mathrm{Q}^2 = 0$, satisfies the *classical master equation* by graded Jacobi identity:

$$\{\Theta, \Theta\} = 0. \tag{2.30}$$

We will later see that working in a local open set with Darboux coordinates, the vector Q and our differential d_ρ (2.14) are related.

2.5.1 Classification of degree 1 and 2 symplectic manifolds

Reference [5] nicely illustrates what the symplectic structures are for the degree 1 and 2 cases. Here we present the classification of that paper in a more accessible and less technical way, mostly by focusing on the local coordinate description.

Degree 1. When the maximal degree is 1, the graded symplectic manifold $\mathcal{M}$ is a vector bundle $E \xrightarrow{\pi} M$ where locally the coordinates on the base $\{x^i\}$ have degree 0, while the fibers χ_i have degree 1, and are anticommuting. Then the Poisson brackets $(|\{,\}| = -1)$, by degree counting, can just be

$$\{x^i, x^j\} \equiv 0, \quad \{x^i, \chi_j\} = \delta^i_j = -\{\chi_j, x^i\}, \quad \{\chi_i, \chi_j\} = -\{\chi_j, \chi_i\} = 0 \subset \mathcal{O}_1(\mathcal{M}).$$

The symplectic form is thus $\omega = \delta_i{}^j dx^i \wedge d\chi_j$. The classification of this type of graded manifolds is fully exhausted by the shifted cotangent bundle $T^*[1]M$. Other results concerning the Hamiltonian and the associated derived bracket (defined in 2.6.1) are postponed to section 2.6.

Degree 2. Assigning the second non-trivial degree gives rise to a fibration on $\mathcal{M}$

$$M_2 \to M_1 \to M_0,$$

where the subscript refers to the maximal degree there. In a local chart for the manifold M_0 the coordinates are degree-0 $\{x^i\}$. Then M_1 is locally the trivialization of a vector bundle, with coordinates $\{x^i, \chi_j\}$, where the degree-1 $\{\chi_j\}$ are anticommuting. M_2 is hence the symplectic realization of the latter: basically it "completes the puzzle with the missing pieces", i.e. the conjugate coordinates $\{p_i, \theta^j\}$, respectively of degree 2 and degree 1, so that the symplectic form can have the maximal degree, as prescribed. The Poisson brackets are thus of even degree -2. Let us write them down for the algebra of functions for better convenience:

$$\begin{array}{ll}
\{\mathcal{O}_0(\mathcal{M}), \mathcal{O}_0(\mathcal{M})\} \equiv 0 \equiv \{\mathcal{O}_0(\mathcal{M}), \mathcal{O}_1(\mathcal{M})\}, & \{\mathcal{O}_1(\mathcal{M}), \mathcal{O}_1(\mathcal{M})\} \subset \mathcal{O}_0(\mathcal{M}), \\
\{\mathcal{O}_2(\mathcal{M}), \mathcal{O}_1(\mathcal{M})\} \subset \mathcal{O}_1(\mathcal{M}), & \{\mathcal{O}_2(\mathcal{M}), \mathcal{O}_0(\mathcal{M})\} \subset \mathcal{O}_0(\mathcal{M}), \\
\{\mathcal{O}_2(\mathcal{M}), \mathcal{O}_2(\mathcal{M})\} \subset \mathcal{O}_2(\mathcal{M}), & \{\mathcal{O}_2(\mathcal{M}), \mathcal{O}_1(\mathcal{M})\} \subset \mathcal{O}_1(\mathcal{M}).
\end{array}$$

Of these, the only non-trivial bracket which is even under parity is thus the first in the right column: its output must hence be a (non-degenerate) symmetric bilinear form, i.e. a metric. For the $p - p$ bracket (the last on the left), the degree counting hints at the fact that $\mathcal{O}_2$ shall be a Lie algebra; anyway the Leibniz rule applied to the last bracket in the right column and the graded Jacobi identity show first of all that the action of $\mathcal{O}_2$ on $\mathcal{O}_1$ is that of an anchor map ρ and eventually the former is a Lie algebroid, $\mathbb{A}$ (see definition 2.1.1). Furthermore, since actually $\mathcal{O}_1\mathcal{O}_1 = \Gamma(\Lambda^2 M_1^*)$ acts trivially on the polynomial functions of degree 0, as it can be noticed via Leibniz rule, $\mathbb{A}$ and $\Lambda^2 M_1^*$ sit in the short exact sequence:

$$0 \to \Lambda^2 M_1^* \to \mathbb{A} \xrightarrow{\rho} TM_0 \to 0.$$

Thus thanks to this sequence we can infer that the graded symplectic structure for the degree 2 case is completely determined by the vector bundle M_1 with metric. The canonical symplectic form is

$$\omega = \delta_i^j dp_j \wedge dx^i + \frac{1}{2}\delta_j^i d\chi_i \wedge d\theta^j. \tag{2.31}$$

On the other hand, the opposite implication is also admissible: if a vector bundle with a metric, $(E, \langle , \rangle)$, is given, then a degree-2 symplectic manifold can always be constructed quite naturally. Let us sketch the procedure. First, shift the degree of E to build the graded bundle $E[1]$. This is a Poisson manifold according to the explanation in the classification of degree 1 symplectic manifolds. Then consider the cotangent space to $E[1]$, $T^*[2]E[1]$: this is the natural symplectic manifold of degree 2 for the Whitney sum of $E[1]$ with its dual. A projector $p : T^*[2]E[1] \mapsto E[1] \oplus E^*[1]$ can always be defined. If $\iota : E[1] \hookrightarrow E[1] \oplus E^*[1]$ embeds $E[1]$ into $E[1] \oplus E^*[1]$ via $\langle , \rangle : E[1] \mapsto E^*[1]$, pulling back $T^*[2]E[1]$ with $\iota^* := \iota_{\mathcal{M}}$ leads to the following commuting diagram:

$$\begin{tikzcd}
\mathcal{M} \arrow[r, "\iota_{\mathcal{M}}"] \arrow[d, "\tilde{p}"] & T^*[2]E[1] \arrow[d, "p"] \\
E[1] \arrow[r, "\iota"] & (E \oplus E^*)[1]
\end{tikzcd}$$

Thus the minimal symplectic realization of $E[1]$ is the fiber product:

$$\mathcal{M} = T^*[2]E[1] \times_{(E \oplus E^*)[1]} E[1].$$

There are other instances of degree-2 dg-manifolds worth mentioning. They are constructed from a vector bundle with a pseudo-Euclidean metric and a connection, and as such the symplectic form depends on how the connection splits the tangent bundle to E into vertical and horizontal subspace. This is enunciated in a proposition proven by Rothstein in [58].

Proposition 2.5.4. *For a vector bundle E, endowed with g pseudo-Euclidean metric, and ∇ metric connection, then $T^*[2]M \oplus E[1]$ is a graded symplectic manifold with an* exact *(g, ∇)-dependent symplectic form $\omega = d\nu$,*

$$\nu = \pi_1^* \nu_0 + \pi_2^* \vartheta.$$

Here π_1 (resp. π_2) projects onto $T^*[2]M$ (resp. $E[1]$), ν_0 is the canonical 1-form (the symplectic potential) and $\vartheta \in \Omega^1(E[1])$ annihilates on the horizontal subspace of $TE[1]$, and on a vertical vector v, $\vartheta(v) = \frac{1}{2} g(v, \cdot)$.

Hence the canonical ω, labeling ξ_a the local coordinates for the fibers and η_{ab} the constant metric, is

$$\omega = \delta_i^j dx^i \wedge dp_j + \frac{1}{2}\eta_{ab}\, d\xi_a \wedge d\xi_b. \tag{2.32}$$

Also $E[1] \oplus E^*[1]$ with the duality pairing, and the connection induced from a connection on E, falls into the hypotheses of the proposition, thus $T^*[2]M \oplus (E[1] \oplus E^*[1])$ is another degree-2 symplectic manifold.

The presentation here concludes the analysis on Poisson manifolds of degree 2; we hope to have made it clear, although some further mathematical rigor could have been used in some parts. Later both $T^*[2]E[1]$ and $T^*[2]M \oplus E[1]$ will be the starting setup for our considerations in the "Results" chapter. In particular $E[1]$ is going to be the shifted tangent space to a manifold M; moreover we will furnish a non-canonical Poisson algebra there, as allowed by Moser lemma, by explicitly finding the coordinate transformation that leads to another non-degenerate and closed 2-form ω', or by directly suggesting a ω' under the conditions of validity for the lemma. The differentiable Q-structure and the relations to other interesting geometrical objects are faced later in section 2.6. As for now, let us focus on Moser lemma and its version for graded variables.

2.5.2 Moser lemma for graded manifolds

This lemma is really crucial for our personal novel contributions. In the original version [2], Jürgen Moser focused mostly on global aspects, using cycles and cocycles in the proof. Instead we will just need the local version, with exact and closed forms. In fact, as we are based on graded geometry, there is going to be some small sign differences in the graded version of the lemma. Since we are mostly concerned with degree-2 manifolds, and knowing that they are of type $T^*[2]E[1] \equiv \mathcal{M}$, the lemma will be proven for this instance of dg-manifold. It will show that any non-degenerate and closed form of degree 2 is symplectic. Be reminded that the de Rham differential d

$$d : \Gamma\left(\Lambda^k T^*\mathcal{M}\right) \mapsto \Gamma\left(\Lambda^{k+1} T^*\mathcal{M}\right),$$

is ungraded, and therefore preserves the total degree.

Suppose hence that a symplectic form is given. Moser lemma justifies to consider deformations of the assigned structure we begin with. They are labeled by the diffeomorphisms of $\mathcal{M}$.

Theorem 2.5.5. ***Moser lemma***
Consider $(T^*[2]E[1], \omega_0)$ graded symplectic manifold with ω_0 symplectic form of degree 2. Take a family of other closed and non-degenerate 2-forms labeled by a real parameter $t \in [0,1]$, ω_t, such that locally $\omega_t - \omega_0 = dA_t$, for A_t 1-form. Then at fixed t, ω_t is the pullback of ω_0 by some degree-preserving diffeomorphism φ_t of $T^*[2]E[1] \equiv \mathcal{M}$, $\varphi : [0,1] \times \mathcal{M} \mapsto \mathcal{M}$:

$$\varphi_t^* \omega_t = \omega_0. \tag{2.33}$$

Proof: The key of the proof is to check whether the diffeomorphism φ_t can really relate the two forms ω_t and ω for each t. This is equivalent to check whether φ_t can be the flow of a vector field X_t:

$$\frac{d}{dt}\varphi_t = X_t \circ \varphi_t.$$

Differentiating ω_0 in (2.33) with respect to t yields:

$$0 = \frac{d}{dt}\varphi_t^* \omega_t = \varphi_t^* \left(\mathcal{L}_{X_t}\omega_t + \frac{d}{dt}\omega_t\right). \tag{2.34}$$

But since $\omega_t = \omega_0 + dA_t$, and employing the graded Cartan's identity $\mathcal{L}_{X_t} = \iota_{X_t} d + (-1)^{|X_t|} d\iota_{X_t}$, the above chain of relations reduces to

$$(-1)^{|X_t|}\,\iota_{X_t}\omega_t = \dot{A}_t, \tag{2.35}$$

where the dot above A_t obviously denotes derivative w.r.t. t. By non-degeneracy of ω_t (2.35) can always be inverted, implying that a vector field X_t always exists.

□

This lemma allows hence the construction, by means of a diffeomorphism, of a non-canonical symplectic form (and hence its Poisson bivector) whose associated vector bundle will be an exact Courant algebroid. The relation between $(T^*[2]E[1]M, \omega)$ and CA necessitate of the homological vector field Q and will be illustrated in a while. In virtue of the correspondence, also the local expressions of the structures associated with the CA will be given in a general basis different than the coordinate basis.

2.6 Derived brackets

In this section all the aspects concerning the differentiable structure of the graded symplectic manifolds (provided by a homological vector field Q) will be carefully explained. The most general Q vectors and the Hamiltonian functions will also be given. As such the section complements 2.5.

Graded manifolds of degree 1 and 2 and the Poisson algebra of the functions on them have interesting relations with non-graded manifolds and other kinds of algebra or algebroid there. The relation is enclosed in the derived brackets. In this part of the thesis we will review this aspect following closely Y. Kosmann-Schwarzbach [59] and eventually expanding her presentation in some points.

The general definition of a derived bracket is the following:

Definition 2.6.1. Derived bracket. Let $(\mathfrak{U}, [,], \mathrm{D})$ be a differential graded Lie algebra, $|[,]| = -n$. Then $[,]_{(\mathrm{D})} : \mathfrak{U} \times \mathfrak{U} \mapsto \mathfrak{U}$ defined as

$$[a, b]_{(\mathrm{D})} := (-1)^{n+|a|+1}[\mathrm{D}a, b]$$

is a non-antisymmetric bracket of degree $n + 1$ and fulfills a restricted version of Jacobi identity, which turns it into what is technically known as a Leibniz bracket[3]. An odd (resp. even) graded Lie algebra bracket has an even (resp. odd) derived bracket.

Most often what is used in the derived bracket definition is an element of the algebra $h \in \mathfrak{U}$, instead than the interior derivation D, with the requirement that $[h, h] = 0$:

$$[[a, h], b] \equiv [a, b]_{(\mathrm{D})}. \tag{2.36}$$

h is then a Hamiltonian function. The attentive reader shall have noticed that this is completely similar to the homological vector field (definition 2.5.3) and the Hamiltonian (2.30) introduced in the context of graded manifold with a symplectic structure.

Example 2.6.2. With the celebrated Cartan identities

$$[\mathcal{L}_X, \iota_Y] = [[\iota_X, d], \iota_Y] = \iota_{[X,Y]},$$

it is immediate to notice that the Lie bracket of vector fields on the RHS is a derived bracket with the de Rham differential, on the space of endomorphisms of $\Omega^\bullet(M)$.

Some general results relate even or odd Poisson brackets on supermanifolds to Poisson or Schouten brackets on graded symplectic manifolds. They are stated in the form of two theorems, due to Voronov [60]:

[3] there is also a corresponding algebroid, the Leibniz algebroid, which is a Lie algebroid (definition 2.1.1) whose bracket, as already stressed, does not have a definite symmetry.

Theorem 2.6.3. *For any odd Poisson bracket* $[,]$ *on a supermanifold* $\mathcal{M}$ *there exists a quadratic Hamiltonian* S *on* $T^*\mathcal{M}$ *by which*

$$[f,g] = \{\{f,S\},g\}, \quad \forall f,g \in C^\infty(\mathcal{M}).$$

On the RHS f,g *are considered as functions of* $T^*\mathcal{M}$ *that are constant on the fibers.*

For local coordinates (x^A, p_A)[4] on $T^*\mathcal{M}$, the quadratic Hamiltonian S is $S = \frac{1}{2}S^{AB}(x)p_A p_B$, where $S^{AB}(x) = [x^A, x^B]$.

A similar result holds for the even Poisson bracket; the symplectic manifold of reference will be $T^*\Pi\mathcal{M}$, where Π here means ulterior parity inversion (so if $\mathcal{M}$ has even (odd) coordinates, the conjugate coordinates will be anticommuting (commuting)). Beforehand, however, let us give a quick look at the Schouten-Nijenhuis bracket on multivector fields. It is the standard bracket for a Gerstenhaber algebra, but we will not go further with the latter.

Definition 2.6.4. (Schouten bracket.) Let $a_1 a_2 \dots a_n$ and $b_1 b_2 \dots b_m$ be elements of $\mathfrak{X}^\bullet(M)$ (the exterior algebra $\Lambda^\bullet TM$). Then the Schouten bracket $[,]_{\mathrm{SN}}$ is

$$[a_1 a_2 \dots a_n, b_1 b_2 \dots b_m]_{\mathrm{SN}} := \sum_{l,j} (-1)^{l+j} [a_l, b_j] \, a_1 \dots a_{l-1} a_{l+1} \dots a_n b_1 \dots b_{j-1} b_{j+1} \dots b_m, \tag{2.37}$$

where the bracket on the RHS is the Lie bracket of vector fields.

Remark 1. The Schouten-Nijenhuis bracket of multivector fields is a Poisson bracket of degree -1 on $T^*[1]M$. In fact, $[,]_{\mathrm{SN}} : \mathfrak{X}^m \times \mathfrak{X}^n \mapsto \mathfrak{X}^{m+n-1}$, and retains both the identity (2.28), where the product is associative, and the Jacobi identity from the Lie bracket. For the details about $T^*[1]M$ being a Poisson manifold please review section 2.5.1.

Remark 2. example 2.6.2 can be extended to multivectors, showing that the Schouten-Nijenhuis bracket $[,]_{\mathrm{SN}}$ is a derived bracket as well.

Theorem 2.6.5. *For any even Poisson bracket* $\{,\}$ *on* $\mathcal{M}$ *there exists a quadratic Hamiltonian* P *on* $T^*\Pi\mathcal{M}$ *such that*

$$\{f,g\} = [[f,P]_{SN}, g]_{SN}, \quad \forall f,g \in C^\infty(\mathcal{M}).$$

f,g *on the right side of the equation are seen as functions of* $T^*\Pi\mathcal{M}$ *which are constant on the fibers, and* $[,]_{SN}$ *is the Schouten-Nijenhuis bracket on multivector fields.*

In the local coordinates $(x^A, \tilde{\pi}_A)$ on $T^*\Pi\mathcal{M}$ the Hamiltonian is the Poisson bivector $P = \frac{1}{2}P^{AB}(x)\tilde{\pi}_A\tilde{\pi}_B$.

In the following we will actually employ the theorems the other way round, i.e. starting with the symplectic manifolds $T^*\mathcal{M}$ or $T^*\Pi\mathcal{M}$, we will seek a Hamiltonian function, and hence compute the derived bracket.

The outlined theorems give rise to many interesting relations: first of all the odd Poisson brackets on $\mathcal{M}$ of theorem 2.6.3 are good constituting elements for Lie algebras. Let us discuss them in an example.

[4]The use of capital letters, from now on, aims at mirroring that $\mathcal{M}$ can have various sets of coordinates.

Example 2.6.6. Consider the graded space $E^*[1]$ with local coordinates $\{\tilde{e}_i\}$. The cotangent space to $E^*[1]$ is equipped with the canonically conjugated pair $\{\tilde{e}_i, \tilde{x}^i\}$. Respectively, they have degree 1 and -1, so $|\{,\}| = 0$ and $\{\tilde{e}_i, \tilde{x}^k\} = \delta_i^k = -\{\tilde{x}^k, \tilde{e}_i\}$. The most general Hamiltonian function $H \in C^\infty(T^*E^*[1])$ and homological vector field Q on $E[1]$ are

$$H = \frac{1}{2} C_{ij}^k \tilde{e}_k \tilde{x}^i \tilde{x}^j, \quad \mathrm{Q} = \frac{1}{2} \tilde{x}^i \tilde{x}^j C_{ij}^k \frac{\partial}{\partial \tilde{x}^k}.$$

$\{H, H\} = 0$ provided that $C^k_{(ij)} = 0$.

But then if $X(\tilde{e}), Y(\tilde{e}) \in C^\infty(E^*[1]) = \Gamma(\Lambda^\bullet E)$,

$$\begin{aligned} \{\{X(\tilde{e}), H\}, Y(\tilde{e})\} &= -\, \tilde{e}_k \left\{ \frac{\partial X(\tilde{e})}{\partial \tilde{e}_i} C^k_{[ij]} \tilde{x}^j, Y(\tilde{e}) \right\} \\ &= \tilde{e}_k \frac{\partial X(\tilde{e})}{\partial \tilde{e}_i} C^k_{[ij]} \frac{\partial Y(\tilde{e})}{\partial \tilde{e}_j} \equiv [X, Y]_{\mathrm{SN}}\,, \quad \text{see (2.37)}. \end{aligned}$$

Thus, as theorem 2.6.3 states, E is a Lie algebra with a basis $\{e_i\}$ and structure constants C_{ij}^k, and the derived bracket has yielded the Schouten bracket of multivector fields. Remarkably, the construction can be generalized to the product manifold $X \times T^*E^*[1]$, where $\omega_{|X} = dx \wedge dp$, and Hamiltonian

$$H = \tilde{x}^k \mathcal{A}_k^i p_i + \frac{1}{2} C_{ij}^k \tilde{e}_k \tilde{x}^i \tilde{x}^j.$$

On the other hand, considering the ungraded $T^*E^* \ni \left(e^i, p_k\right)$ and the hamiltonian μ, $\mu(e_i, e_j) := C_{ij}^k e_k$, theorem 2.6.5 applies and thus the Poisson bracket on $f, g \in C^\infty(E^*)$ is the derived bracket

$$\{f, g\}_\mu = [[f, \mu]_{\mathrm{SN}}, g]_{\mathrm{SN}}. \tag{2.38}$$

On the right side of the equation f, g are interpreted as functions of T^*E^* which are constant on the fibers (no p-dependence).

This last example paves the way for discussing the slightly more complicated generalization of a pointwise Lie algebra structure.

Example 2.6.7. Consider $T^*A^*[1]$, where in a local trivialization for $A^*[1]$ the coordinates are $(x^i, \tilde{\alpha}_a)$, and $|x| = 0$, $|\tilde{\alpha}| = 1$. The symplectic manifold is very close to the degree 2 case analyzed previously, however here the parity of the canonically conjugated variables will be the same of their partner, while the degree will be reverted in sign, so that the Poisson bracket will be neutral. This situation is more adapted for a particular derived structure. Later in this same section we will go back to $T^*[2]T[1]M$. For now the consistent conjugated pair that completes the local description of $T^*A^*[1]$ is built with p_i, with grading 0, and $\tilde{\beta}^b$, with grading -1. The canonical Poisson structure is $\{x^i, p_j\} = \delta_j^i = -\{p_j, x^i\}$, $\{\tilde{\alpha}_a, \tilde{\beta}^b\} = \delta_a^b = \{\tilde{\beta}^b, \tilde{\alpha}_a\}$. The most general Hamiltonian H of total degree -1 and Poisson square 0 is:

$$H = \mathcal{A}_a^i(x) p_i \tilde{\beta}^a + \frac{1}{2} \tilde{\alpha}_c C_{ab}^c(x) \tilde{\beta}^a \tilde{\beta}^b, \quad \{H, H\} = 0.$$

The differential equation $\{H, H\} = 0$ is fulfilled if

$$\mathcal{A}_{[b|}{}^i(x) \partial_i \mathcal{A}_{|c]}{}^j(x) = -\mathcal{A}_a{}^j(x) C^a{}_{bc}(x), \quad C^l{}_{[m|n}(x) C^n{}_{|bc]}(x) + \mathcal{A}_{[m|}{}^i(x) \partial_i C^l{}_{|bc]}(x) = 0.$$

It is not difficult to infer the homological vector field Q:

$$\begin{aligned} \mathrm{Q} =& \tilde{\beta}^a \mathcal{A}_a^i(x) \frac{\partial}{\partial x^i} - \partial_k \mathcal{A}_a^i(x) p_i \tilde{\beta}^a \frac{\partial}{\partial p_k} + \mathcal{A}_a^i(x) p_i \frac{\partial}{\partial \tilde{\alpha}_a} \\ &+ \frac{1}{2} \tilde{\beta}^a \tilde{\beta}^b C_{ab}^c(x) \frac{\partial}{\partial \tilde{\beta}^c} + \frac{1}{2} \tilde{\alpha}_c \partial_k C_{ab}^c(x) \tilde{\beta}^a \tilde{\beta}^b \frac{\partial}{\partial p_k}. \end{aligned}$$

The derived bracket of the Hamiltonian with $u, v \in C^\infty(T^*\Pi A^*[1])$ which do not depend on $p, \tilde{\beta}$ and for the sake of simplicity let us assume that they are linear in $\tilde{\alpha}$ corresponds to:

$$\begin{aligned} \{\{u(x,\tilde{\alpha}), H\}, v(x,\tilde{\alpha})\} =& \left\{ -\partial_i u \, \mathcal{A}_a^i(x) \tilde{\beta}^a + \frac{\partial u}{\partial \tilde{\alpha}_a} \mathcal{A}_a^i(x) p_i - \frac{1}{2} \tilde{\alpha}_c C_{ab}^c(x) \left[\frac{\partial u}{\partial \tilde{\alpha}_a} \tilde{\beta}^b - \tilde{\beta}^a \frac{\partial u}{\partial \tilde{\alpha}_b} \right], v \right\} \\ =& - \partial_i u \, \mathcal{A}_a^i(x) \frac{\partial v}{\partial \tilde{\alpha}_a} + \partial_i v \, \mathcal{A}_a^i(x) \frac{\partial u}{\partial \tilde{\alpha}_a} - \frac{1}{2} \tilde{\alpha}_c C_{ab}^c(x) \left[\frac{\partial u}{\partial \tilde{\alpha}_a} \frac{\partial v}{\partial \tilde{\alpha}_b} - \frac{\partial v}{\partial \tilde{\alpha}_a} \frac{\partial u}{\partial \tilde{\alpha}_b} \right]. \end{aligned} \tag{2.39}$$

As granted by theorem 2.6.3, the derived bracket is hence the Schouten bracket (actually, in this case where the multivectors are just vector fields, it is the Lie bracket) of $u, v \in C^\infty(A^*[1]) = \Gamma(\Lambda^\bullet A)$ for the Lie algebroid $A \xrightarrow{\pi} M$, where (x^i) are coordinates on M, and (x^i, e_a) parametrize a trivialization for A. This is is true if

$$[e_a, e_b] = C_{ab}^c(x) e_c. \quad [e_a, f(x)] = \mathcal{A}_i^a(x) \partial_i f.$$

Moreover the properties of $[,]_{\mathrm{SN}}$ for a Lie algebroid, antisymmetry and its behavior under $C^\infty(M)$ multiplication, are immediate to test. Jacobi identity stems from $C^c_{[a|l}(x) C^l_{|bd]}(x) + \mathcal{A}_{[a|}{}^i(x) \partial_i C^c{}_{|bd]}(x) = 0$.

Example 2.6.8. On the other hand, in the same conditions as in the previous example, $A^* \ni \left(x^i, \alpha_j\right)$ hosts a natural Poisson bivector

$$P = \mathcal{A}_a^i(x) \frac{\partial}{\partial x^i} \frac{\partial}{\partial \alpha_a} + \frac{1}{2} \alpha_c C_{ab}^c(x) \frac{\partial}{\partial \alpha_b} \frac{\partial}{\partial \alpha_a}.$$

Then by theorem 2.6.5, if $\psi, \varsigma \in \Gamma(\Lambda^\bullet A)$ (actually taken of degree 0) the derived bracket of the Schouten bracket with P is the Poisson bracket of ψ, ς, now seen as functions on A^*:

$$[[\psi, P]_{\mathrm{SN}}, \varsigma]_{\mathrm{SN}} = \{\psi, \varsigma\}. \tag{2.40}$$

The situation analyzed so far was based on supermanifolds, i.e. coordinates of degree $0, 1$ and their canonical conjugates of degree 0 and -1, but in the next section 3 it will be more convenient to focus on graded manifolds, so the coordinates will have grading in $\mathbb{N}$, at most $\mathbb{Z}$. Upon slight changes in the signs and in the total degrees allowed to the functions, the derived bracket construction will be pretty much alike.

We are about to close our ascent to increasingly complex algebraic and differential structures by analyzing, as anticipated, the NQ-manifold $T^*[2]T[1]M \ni (x^i, \theta^a, \chi_a, p_i)$ with respective degree $(0, 1, 1, 2)$. The best reference for this is certainly [5]. For simplicity, call $\xi_\alpha := (\theta^a, \chi_a)$, implying that Greek indices will range on a space of doubled dimension. The most general Hamiltonian shall have total degree $3 = |\mathrm{Q}| + 2$,

$$\Theta = \xi_\alpha \tilde{\rho}^{\alpha i}(x) p_i + \frac{1}{3!} C^{\alpha\beta\gamma}(x) \xi_\alpha \xi_\beta \xi_\gamma. \tag{2.41}$$

The functions $\tilde{\rho}$ and C in the Hamiltonian are linear maps, $\tilde{\rho} : (T^*[1]M \oplus T[1]M)^* \mapsto T^*[2]M$, $\tilde{\rho}(\eta^{\alpha\beta}\xi_\beta) = \tilde{\rho}^{\alpha i}p_i$ and $C \in \left(\otimes^3(T^*[1]M \oplus T[1]M)\right)$. It is indeed the case that Θ is Hamiltonian as we shall verify through the associated homological vector field Q:

$$\begin{aligned} \mathrm{Q} =& \tilde{\rho}^{\alpha i}(x)p_i\eta_{\alpha\beta}\frac{\partial}{\partial\xi_\beta} + \xi_\alpha\partial_k\tilde{\rho}^{\alpha i}(x)p_i\frac{\partial}{\partial p_k} - \xi_\alpha\tilde{\rho}^{\alpha i}(x)\frac{\partial}{\partial x^i} \\ &+ \frac{1}{6}\partial_k C^{\alpha\beta\gamma}(x)\xi_\alpha\xi_\beta\xi_\gamma\frac{\partial}{\partial p_k} + \frac{1}{2}C^{\alpha\beta\gamma}(x)\xi_\beta\xi_\gamma\eta_{\alpha\delta}\frac{\partial}{\partial\xi_\delta}. \end{aligned} \tag{2.42}$$

It is certainly true that $\iota_{\mathrm{Q}}\omega = d\Theta$ for Q in (2.42) and Θ in (2.41) and the canonical symplectic form, as it can be shown with a straightforward computation paying attention to the sign switch when the differential d needs to pass over a degree-1 coordinate.

Now we can check the derived bracket, with the given Hamiltonian, of a pair of functions $U, V \in \mathcal{O}_1(T^*[2]T[1]M) \cong \Gamma(TM \oplus T^*M)$, so that $U^\alpha(x)\xi_\alpha = X^i\chi_i + \gamma_i\theta^i \cong X^i\partial_i + \gamma_i dx^i$ (and $V^\alpha\xi_\alpha = Y^k\chi_k + \sigma_k\theta^k$). First let us look at the simpler instance of $\tilde{\rho} \equiv \eta$ and $C \equiv 0$ and :

$$\{\{U(x), \Theta\}, V(x)\} = X^k\partial_k V(x) - Y^k\partial_k U(x) + \langle\theta^k\partial_k U(x), V(x)\rangle. \tag{2.43}$$

From Cartan's magic formula the coordinate-free expression of (2.43) is immediate:

$$[X, Y]_{\mathrm{Lie}} + \mathcal{L}_X\sigma - \iota_Y d\gamma \equiv [U, V]_{\mathrm{D}},$$

by comparison with (2.7). The pairing is naturally assigned via $\{\xi_\alpha, \xi_\beta\} = \eta_{\alpha\beta}$. Extraction of the anchor map for the bundle $TM \oplus T^*M$ is performed via the derived bracket with a function in $\mathcal{O}_1$, U, and one in $\mathcal{O}_0$, f. To see this, assume to have a more general ρ in the Hamiltonian:

$$\{\{U, \Theta\}, f\} = -\{\{f, \Theta\}, U\} = \{\xi_\alpha\tilde{\rho}^{\alpha i}\partial_i f, U^\beta\xi_\beta\} = \rho(U)^i\partial_i f.$$

If otherwise ρ was fixed to be the projector, in the last equality $X^i\partial_i f$ would had appeared. Turning the C tensors on yields:

$$\begin{aligned} \{\{U(x), \Theta\}, V(x)\} =& U^\beta(x)\rho_\beta{}^i(x)\partial_i V(x) + \eta\left(\xi_\alpha\tilde{\rho}^{\alpha i}(x)\partial_i U(x), V(x)\right) - V^\beta(x)\rho_\beta{}^i(x)\partial_i U(x) \\ &- C^{\alpha\beta\gamma}(x)U^\mu(x)V^\nu(x)\eta_{\alpha\mu}\eta_{\beta\nu}\xi_\gamma = [U, V]_{\mathrm{D}} + C(U, V, \cdot). \end{aligned} \tag{2.44}$$

It comes with no surprise that it coincides with the Dorfman brackets of generalized vectors, twisted by C. The derived structure can be shown to fully coincide with that of an exact CA. The condition of being exact, i.e. that $\rho^*(T[1]M)$ is isotropic subspace of $T^*[1]M \oplus T[1]M$, was derived in section 4 of [5], where the author also specified what the master equation $\{\Theta, \Theta\} = 0$ means in terms of the CA axioms. From that, he recovered the exactness, as already mentioned, as well as that ρ is a homomorphism between $T^*[1]M \oplus T[1]M$ and TM, and that if $C_{\alpha\beta\gamma} := \langle[\xi_\alpha, \xi_\beta], \xi_\gamma\rangle$ the Jacobi identity is fulfilled. The concise derivation of this result is not reproduced here: we will instead analyze the master equation with non-canonical Poisson brackets in the next chapter 3.

Alternatively, the CA bracket is also the derived bracket given by the commutator of endomorphisms of $\Omega^\bullet(M)$ with differential the de Rham differential:

$$[V, W]_{\mathrm{D}} = [[V, d], W]. \tag{2.45}$$

This is a simple computation but nevertheless it can be examined in the appendix 4.2. The above correspondence holds also when the Dorfman bracket is twisted with a closed 3-form H, i.e.

$$[V, W]_{\mathrm{D}} = [Y, Z] + \mathcal{L}_Y\kappa - \iota_Z d\sigma - H(Y, Z, \cdot),$$

upon replacement of d with the operator $d - H\wedge$ in the commutator of endomorphisms of $\Omega^\bullet(M)$. Notice that $[d - H\wedge, d - H\wedge] = 0$ just because $dH = 0$.

Now that the local Darboux chart is at our disposal and the most general Q for the degree-2 manifold $T^*[2]T[1]M$ is given in (2.42), an issue that still needs some further explanation concerns the question whether the ρ-differential of the previous section 2.2, in local coordinates (2.14), is related to Q, and hence square to zero because Q does. The differential is modeled over (2.41) when $C \equiv 0$. On the functions in $C^\infty(M)$ the action of Q and d_ρ is exactly the same[5]:

$$\mathrm{Q}f = \{f, \Theta\} = -\tilde{\xi}^\alpha \rho(\xi_\alpha) f = -d_\rho f.$$

Only the third term of Q in (2.42) differentiate $f(x)$. We know that d_ρ could be defined to act on forms if a Lie-like bracket is *independently* provided. Q in the local Darboux chart for the degree 2 case, is the differential for the cohomology $(\mathcal{O}(\mathcal{M}), \mathrm{Q})$, but does not generally build a differential for the exterior algebra of the degree 1 functions $\Omega^\bullet(E) \cong \Gamma(\Lambda^\bullet E^*)$. One can surely exploit the biderivation rule for the Poisson bracket (2.28) to compute $\{\varpi, \Theta\}$, $\varpi \in \Omega^k(E)$ but the operation does not close in $\Omega^\bullet(E)$: terms containing p arise. Hence it would be necessary to ask for the elements due to a p factor to be identically zero, so that $\mathrm{Q}\varpi \in \Omega^{k+1}(E)$. In this way, also every contraction $\langle \xi_\alpha, \xi_\beta \rangle$ is avoided. Otherwise one can start implementing the latter ($\tilde{\xi}$ in the Hamiltonian being in the orthogonal complement to every ξ in ϖ), so no p term will be found. It is easier to visualize this with an example:

$$\varpi \in \Omega^2(E) \cong \Gamma(\Lambda^2 E^*), \quad \varpi = \frac{1}{2}\varpi_{ij}\theta^i\theta^j, \qquad \tilde{\rho}(\chi_i) = p_i.$$

For these choices, $\{\varpi, \Theta\} = -\dfrac{1}{3!}\theta^i\theta^j\theta^k\partial_i\varpi_{jk} \in \Omega^3(E)$. But if instead the 2-form is chosen as

$$\varpi = \frac{1}{2}\varpi^{ij}\chi_i\chi_j,$$

then we will see the appearance of $\varpi^{ij}\chi_i p_j$ terms.

We hence believe that, under these assumptions, one could attempt to interpret d_ρ (2.14), modulo signs, as the $\dfrac{\partial}{\partial x^i}$ component of Q, that from now on we shall call Q_0:

$$d_\rho \cong \mathrm{Q}_0 := -\xi_\alpha \tilde{\rho}^{\alpha i}(x)\partial_i. \tag{2.46}$$

Despite having the right behavior on the exterior algebra of the degree 1 functions, Q_0 is not a genuine differential as $\mathrm{Q}_0^2 \neq 0$. This can also be seen from $\mathcal{L}_{\mathrm{Q}_0}\omega \neq 0$, i.e. the vector is not Hamiltonian. What might be tried instead is to consider the degenerate graded Poisson structure:

$$\{p_i, x^j\} = \delta_i^j, \quad \{\xi_\alpha, \xi_\beta\} = 0, \tag{2.47}$$

where all the remaining Poisson brackets are zero. Then Q_0 (2.46) is homological if the anchor $\tilde{\rho}$ is actually global (i.e. it is not point-wise dependent).

A degenerate graded Poisson structure as in (2.47) opens up the possibility to furnish another interpretation for the generalized Lie bracket. Another homological vector field $\check{\mathrm{Q}}$ (2.48) that can be assigned to the degenerate Poisson structure can be easily found in:

$$\check{\mathrm{Q}} := -{\rho_\alpha}^i(x) p_i \frac{\partial}{\partial \xi_\alpha}. \tag{2.48}$$

[5] The minus sign is due to the convention on the Poisson bracket, here $\{p_i, x^j\} = {\delta_i}^j$.

The vector field induces a flat connection on the space of degree 1 linear functions, which are equivalent to sections of a vector bundle E. Then the generalized Lie bracket (2.15) has the nice interpretation of a derived bracket:

$$\{\check{Q}U, V\} - \{\check{Q}V, U\} = [\![U, V]\!], \quad U, V \in \Gamma(E). \tag{2.49}$$

The same expression of the generalized Lie bracket as a derived bracket holds for the canonical Poisson structure (2.31) too, as stressed in our work [61]. However then the degree-1 vector field is not Hamiltonian but just nilpotent.

Caveat: the above considerations regard a differential geometry object defined in our personal work [3], [7] thus they are not compared with the existing literature yet and are susceptible to a deeper investigation for the moment being.

2.6.1 Gauge symmetries

To implement a gauge symmetry, we look for some infinitesimal transformations that leave the master equation invariant at first order. Thus the derived algebroid bracket will be invariant under the symmetry and fulfillment of the defining axioms is ensured. It is not difficult to see that whichever operator δ_α that acts on any function $F \in C^\infty(M)$ via Poisson bracket

$$\delta_\alpha F = \{\alpha, F\}, \quad \alpha \in C^\infty(\mathcal{M}),$$

generates the symmetry, leaving the Poisson structure unchanged. In fact δ_α is a graded derivation, might it be even or odd, and a consequence of this is that $\{,\}$ keeps satisfying the graded Jacobi identity then. The product rule of the derivation δ_α on $\{\Theta, \Theta\}$ on shell, i.e. using the classical master equation, yields:

$$\begin{cases} \delta_\alpha\{\Theta,\Theta\} = \{\alpha,\{\Theta,\Theta\}\}, \\ \{\delta_\alpha\Theta,\Theta\} \pm \{\Theta,\delta_\alpha\Theta\} = \{\{\alpha,\Theta\},\Theta\} \pm \{\Theta,\{\alpha,\Theta\}\}, \end{cases} \qquad \Longrightarrow \{\delta_\alpha\Theta,\Theta\} \pm \{\Theta,\delta_\alpha\Theta\} = \{\alpha,\{\Theta,\Theta\}\},$$

where the plus (minus) sign is due to an even (odd) degree for δ_α. So if for Θ the master equation holds, it holds for $\Theta + \delta_\alpha\Theta$ too at first order in α (i.e. we do *not* require that $\{\delta_\alpha\Theta, \delta_\alpha\Theta\} = 0$, which is not a priori true). The algebra closes irrespective to the total degree of α, $|\alpha| = |\beta|$:

$$\delta_\alpha\delta_\beta - (-)^{|\beta|}\delta_\beta\delta_\alpha = \{\alpha,\{\beta,\cdot\}\} - (-)^{|\beta|}\{\beta,\{\alpha,\cdot\}\} = \{\{\alpha,\beta\},\cdot\} = \delta_{\{\alpha,\beta\}}.$$

A symmetry for the derived brackets, instead, must respect the stronger condition that the Hamiltonian itself should not be transformed, i.e. $\delta_\alpha\Theta \overset{!}{=} 0$ [62]. Hence α shall be the Poisson bracket of the Hamiltonian with a function ϱ of degree $|\varrho| = |\alpha| - |\Theta| - |\{,\}|$,

$$\alpha = \{\varrho, \Theta\} \tag{2.50}$$

The total degree is important for classification of the gauge symmetries implemented by δ_α. They are first of all distinguished between degree-preserving (i.e. $|\alpha| + |\{,\}| = 0$), and degree-changing. Here we will just comment on the former. They are the infinitesimal canonical transformations of the Poisson structure. For dg symplectic manifolds of degree 2, the most general expression of a degree-2 function is [5]

$$\alpha = \varsigma^i(x)p_i + \frac{1}{2}M^{\alpha\beta}(x)\xi_\alpha\xi_\beta. \tag{2.51}$$

It is easy to be convinced that $\varsigma^i(x)p_i$ generates diffeomorphisms, while the second term generates $O(d,d)$-transformations. $M^{\alpha\beta}(x)\xi_\alpha\xi_\beta$ decomposes as $B_{ij}(x)\theta^i\theta^j + \beta^{ij}(x)\chi_i\chi_j + 2A^i{}_j\chi_i\theta^j$, where B and β are antisymmetric tensors. In fact:

$$\begin{aligned}
\delta_\alpha x^i &= \{x^i, \alpha\} = -\varsigma(x)^i, \\
\delta_\alpha \xi_\gamma &= M^{\alpha\beta}(x)\eta_{\gamma\alpha}\xi_\beta, \\
\delta_\alpha p_i &= \partial_i \varsigma^j(x)p_j + \frac{1}{2}\partial_i M^{\alpha\beta}(x)\xi_\alpha\xi_\beta.
\end{aligned}$$

The above system of equations, which shows that the canonical transformations of a 2-graded Poisson algebra are the elements of $\mathfrak{gl}(d)$ and the algebra $\mathfrak{o}(d,d)$, is well known. For example one can consult [5], and the system of equations (3.3) in there. However it is worth stressing something that perhaps is not common knowledge on the matter. When in the Hamiltonian the coanchor is $\tilde{\rho}(\theta^i) = \eta^{ij}p_j$, and of all the infinitesimal canonical transformations we consider the following $\mathfrak{o}(d,d)$ element:

$$\beta^{lm}(x) + \eta^{ln}\partial_n b^m(x), \tag{2.52}$$

then in the canonical Poisson brackets, the Hamiltonian does not vary under the above $\mathfrak{o}(d,d)$-transformation. In fact $\eta^{ij}\partial_j\beta^{lm}(x) = 0$, as β is closed, and

$$\left\{\chi_i\eta^{ij}p_j, \frac{1}{2}\chi_l\chi_m\eta^{ln}\partial_n b^m(x)\right\} = \frac{1}{2}\chi_i\eta^{ij}\chi_l\eta^{ln}\partial_j\partial_n b^m(x) = 0, \tag{2.53}$$

since $\chi_i\chi_l = -\chi_l\chi_i$. An alternative way to see this is by noticing that the second term in (2.52) is actually given by a degree-1 function ϱ, $\{\varrho, \Theta\}$:

$$\frac{1}{2}\chi_i\chi_j\eta^{il}\partial_l b^j(x) = \{\chi_j b^j(x), \chi_i\eta^{il}p_l\}.$$

Thus (2.52) describes a gauge transformed β. Gauge symmetries of the bivector β are hence genuine symmetries of the derived bracket too. Another equivalent way to express this is by saying that the element $\chi_l\eta^{ln}\partial_n b^m(x)\chi_m$ is in the cohomology of β with differential Q. The symmetry is still present at the level of the non-canonical Poisson brackets, implemented by $\varphi \in \mathrm{Diff}(T^*[2]T[1]M)$, upon the consequent changes $\Theta(\xi,p) \mapsto \Theta(\varphi(\xi,p))$ and $\varphi^*(d_\rho b)$, where $b \in \mathfrak{X}^1(M)$ is the same than in (2.52), and $d_\rho b \in \mathfrak{X}^2(M)$ is also informally "$\delta_{\delta_b}\Theta = 0$". This observation will be recovered in the end of section 3.3 in slightly different and specific circumstances.

Example 2.6.9 (Taken and readapted from [63])**.** The generalized metric $\mathcal{H}$ of generalized geometry defined in 2.1.4 is also retrieved from a canonical transformation involving the degree-1 coordinates only, provided by the generating function $F(\theta, \underline{\theta})$:

$$F(\theta,\underline{\theta}) = \theta g\underline{\theta} + \frac{1}{2}\underline{\theta}B\underline{\theta} - \frac{1}{2}\theta B\theta.$$

In fact, being

$$\underline{\chi} = -\frac{\partial F}{\partial\underline{\theta}}, \quad \chi = \frac{\partial F}{\partial\theta},$$

the conjugate new and old coordinate $\underline{\chi}$ and χ, direct computation yields the coordinate redefinition:

$$\underline{\chi} = \underline{\theta}B + \theta g, \quad \chi = \underline{\theta}g + \theta B$$

which is eventually solved for $\underline{\theta}$ and $\underline{\chi}$:

$$\underline{\theta} = \chi g^{-1} - \theta B g^{-1}, \quad \underline{\chi} = \chi g^{-1}B - \theta B g^{-1}B + \theta g.$$

It can be compared to $\mathcal{H}$ and seen to agree.

2.7 AKSZ sigma models

We would now like to shortly comment upon a fruitful topic for applications of the theory of QP-manifolds. All the graded symplectic and differentiable structures analyzed previously, such as the symplectic form ω, the homological vector field Q and the Hamiltonian Θ, will have a nice interpretation as, respectively, Poisson (anti-)brackets, some second order nilpotent operator and action functionals for the fields in $\mathrm{Map}(\Sigma, M)$. Also, the grading will become form degree.

Let us start with one of most relevant application of N-manifolds in physics, which is found in the notorious "AKSZ sigma models" [64]. The idea is to provide the space $\mathrm{Map}(\Sigma, M)$ with a symplectic structure, with M, Σ some manifolds of various dimensions, $dim\,\Sigma < dim\,M$ and in particular M can be naturally given a NQ-structure. The general AKSZ action functional, of which we will make sense in the rest of the section, is

$$S_{\mathrm{AKSZ}} = \int \omega_{AB}\varphi^A \wedge d\varphi^B + \frac{1}{2}\varphi^* P,$$

for ω symplectic form on M, $\varphi \in$ Map and P Hamiltonian of ω.

As a first example pick up for target space M the N-manifold of degree 1, $T[1]N$, with Poisson bivector $P = \frac{1}{2}P^{ij}(x)\frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}$. Now, promote the local coordinates x to functions $\varphi : \Sigma_2 \mapsto T[1]N$ ($dim\,\Sigma_2 = 2$) and the conjugate anticommuting variables χ to the connection 1-forms $\kappa \in \Omega^1(\Sigma_2, \varphi^* T[1]N)$. The action functional corresponding for these fields goes under the name of Poisson sigma-model:

$$S_{\mathrm{Poisson}} = \int_{\Sigma_2} \kappa_i \wedge d\varphi^i + \frac{1}{2}\varphi^* P(\kappa). \tag{2.54}$$

The model has the following gauge symmetry with gauge parameter ϵ:

$$\delta\varphi^i = -P^{ij}\epsilon_j, \quad \delta\kappa_i = d\epsilon_i + P^{lm}{}_{,i}\kappa_l\epsilon_m.$$

The comma refers to partial derivation w.r.t. φ. The gauge algebra structure is encoded in a Lie algebroid [65], with structure functions $f^{ij}{}_k = P^{ij}{}_{,k}$ and anchor map $\rho^{ij} = P^{ij}$.

Curiously, 2-dimensional Euclidean R^2-gravity, where R is the Ricci scalar for the Levi-Civita connection,

$$S_{R^2} = \frac{1}{4}\int_{\Sigma_2} d^2x\,\sqrt{\det g}\,\left(\frac{1}{4}R^2 + 1\right)$$

falls into this case of Poisson sigma models [66]. The Poisson structure is a quadratic modification (in one of the fields) of

$$\{\varphi^i, \varphi^j\} = \sum_{k=1}^{3} \epsilon^{ijk}\varphi^k.$$

The triplet of fields is provided by the zweibeins e^1, e^2, and the spin connection. Locally the metric g can be shown to be of generalized Schwarzschild form with mass corresponding to the Casimir function of $\{,\}$.

Often (2.54) is modified by the addition of a Wess-Zumino term, which twists the model by a closed 3-form $H \in \Omega^3(M)$ pulled back to D_3, $\partial D_3 = \Sigma_2$ [67]:

$$S_{\mathrm{HPSM}} = S_{\mathrm{Poisson}} + \int_{D_3} \varphi^* H.$$

To keep the theory topological, the Poisson condition for the Poisson bivector is relaxed to

$$\frac{1}{2}\left[P,P\right]_{\mathrm{SN}} = \iota_{P\otimes P\otimes P}H.$$

To know more about the very tight relation between string actions, Poisson sigma models with Wess-Zumino terms and T-duality the interested reader can consult [68].

As a second instance consider instead the target space to be a N-manifold of degree 2 such as $T^*[2]T[1]M =: E$. All the coordinates are now seen as forms of different form degree: the coordinates for the base become the fields $\varphi : \Sigma_3 \mapsto E$, $dim\,\Sigma_3 = 3$, what would be the coordinates $\xi_\alpha \in T^*[1]M \oplus T[1]M$ are instead promoted to be the connection 1-forms $A_\alpha \in \Omega^1(\Sigma_3, \varphi^* E)$, and finally the even momenta p become 2-forms $F_i \in \Omega^2(\Sigma_3, \varphi^* T^* M)$:

$$S_{\text{Courant}} = \int_{\Sigma_3} F_i \wedge d\varphi^i + \frac{1}{2}\eta^{\alpha\beta} A_\alpha \wedge dA_\beta + A_\alpha \tilde{\rho}^{\alpha i} \wedge F_i + \frac{1}{6} C^{\alpha\beta\gamma} A_\alpha \wedge A_\beta \wedge A_\gamma.$$

As one can easily suspect from parallelism with the Poisson sigma model, the conditions for gauge invariance and on-shell closure of the gauge transformations for the Courant sigma model were shown to correspond to the axioms of the Courant algebroid [69].

In recent years also sigma models with target space of dimension higher than 3 have been built. Their set of conditions defines then the higher structure of a Lie algebroid up to homotopy. The underlying physics behind such sigma model was argued in [70] to be that of the $SL(5)$ M-theory fluxes and their Bianchi identity.

To conclude, in this short section we discussed how to the geometric and algebraic data of dg-symplectic manifolds of lowest degree (1 and 2), as well as to the related Lie algebroids and Courant algebroids, it is possible to attach a topological field theory. However we did not make sense of the homological vector field in this setup yet: it will emerge in the realm of BV or BFV quantization of classical field theories which are degenerate because of a gauge symmetry.

2.7.1 BV-BFV quantization

When the action functional is degenerate due to gauge symmetries, path integral quantization cannot work: it will sum over gauge equivalent configurations. Perturbation theory breaks down. A way out of this problem is suggested by BV quantization. Roughly speaking, BV quantization is a prescription for quantization which consists, first of all, in enlarging the space of fields with the antifields and the ghosts (and antighosts), which are scalar or vector fields with an integer number ("the ghost") attached to them, and that are in the cohomology of a order 2 nilpotent operator. The original space of fields is embedded in the new space, and the latter is given an odd symplectic structure ω. Then the action functional is extended to a new action functional $\mathcal{S}$, with fields, antifields, ghosts, antighosts, ghosts for ghosts etc... so that $\iota_{\mathrm{Q}}\omega = d\mathcal{S}$ and hence the classical master equation

$$\{\mathcal{S}, \mathcal{S}\} = 0,$$

is solved by $\mathcal{S}$. Q is the nilpotent operator, of ghost number 1. The classical master equation is just the first approximation of the quantum master equation:

$$\frac{1}{2}\{\mathcal{S}, \mathcal{S}\} - i\hbar\,\Delta\mathcal{S} = 0,$$

where Δ is the BV Laplacian.

By contrast, the BFV formalism concerns instead the first order formulation of the theory, i.e. it is intrinsically about the Hamiltonian function rather than the Lagrangian.

The BFV action hence contains information about the first-class constraints, i.e. a Lagrangian submanifold U ($\omega_{|_U} = 0$): the cohomology of Q in ghost number zero yields the reduced phase space of the Hamiltonian formulation.

We can briefly see how BV quantization works effectively applying the technique to the so called abelian BF model. The example is taken from [71]. The BF model is a very simple instance of Poisson sigma model. In the most simple case, it is based on a 2-dimensional differentiable manifold M with gauge group G (abelian Lie group), whose dynamical fields are a scalar field φ and a connection 1-form A with values in $\mathfrak{g}$, and associated field strength $F = dA$:

$$S_{BF} = -\frac{1}{2}\int_M \varphi\, F = \int_M A_i(x) \star d\varphi^i(x)$$

where $\star$ is the Hodge star operator (recall that the model is 2-dimensional), constructed with the completely antisymmetric pseudotensor (Levi-Civita symbol). The action functional enjoys $U(1)$ gauge symmetry, $\delta A_i = d\epsilon_i$, $\delta\varphi^i = 0$. In the quantization scheme, the gauge parameter is replaced by the ghost c_i, a scalar with odd parity, set to be $\delta c = 0$. Then we must introduce an antifield for each field of the theory, $A^*, \varphi^*, c^* =: \Psi^*$. Their ghost numbers are assigned by the equation

$$\mathrm{gh}\Psi + \mathrm{gh}\Psi^* = -1.$$

Next, the odd Poisson structure is defined by requiring that for each type of field, Ψ should be the respective canonically conjugated coordinate to Ψ^*. Finally the BV action functional is constructed for the ghost 0 classical action S_0:

$$\mathcal{S} = S_0 + (-)^{\mathrm{gh}\Psi}\int_M \Psi^*\delta\Psi + O\left({\Psi^*}^2\right); \tag{2.55}$$

for the present case of the abelian BF theory the new piece is just made up of

$$\mathcal{S} = S_{BF} + \int_M A^* \wedge dc. \tag{2.56}$$

The gauge transformations on the missing fields can be worked out with the homological vector field $\delta := \{\mathcal{S}, \cdot\}$, for $\mathcal{S}$ in (2.56):

$$\delta A_i(x) = dc_i, \quad \delta A^{*i}(x) = \star\, d\varphi^i, \quad \delta\varphi_i^*(x) = -\star\, dA_i(x), \quad \delta c^{*i} = -\mathrm{div}A^{*i},$$

where div is the divergence operator.

We will not go deeper in the subject. It is instead time to move to the “Results” chapter.

Chapter 3

Results

After having settled the fundamental background knowledge for the understanding of the research done during the Ph.D. studies, and furthermore having discussed some more topics of general interest than those strictly necessary for the comprehension of the rest of the thesis, we are finally ready to present the personal results contained in [3] and [7]. On the way we will also extend the presentation with some other more less known interesting results which are not original. The discussion about preexisting literature and other comments are postponed to chapter 4.

3.1 Non-canonical NQ-$T^*[2]T[1]M$

The first inquiry concerned the possibility to equip the generalized tangent space with a non-canonical graded Poisson algebra that could host the degree of freedom of the dilaton. We then drove our attention to the implementation of the closed-open string relations (2.13) in the graded symplectic structure. Thus our starting point was the NQ-manifold $T^*[2]T[1]M$.

3.1.1 Deformation via vielbein

The graded NQ-manifold $T^*[2]T[1]M$ was in this case given a non-canonical symplectic form by means of a local invertible map $\mathcal{E}(x) : T^*[1]M \oplus T[1]M \mapsto T^*[1]M \oplus T[1]M$. Such application is also a vielbein, i.e. a section of the associated frame bundle $\mathcal{F}M$ (where a doubling of vector fields with forms occurs too), in particular it is the frame in which the metric on the fibers is some curved generalization of the $O(d,d)$-invariant pairing η:

$$\mathcal{G} = \mathcal{E}^T \eta \mathcal{E}. \tag{3.1}$$

The linear invertible transformation $\mathcal{E}(x)$ is hence a well-behaved diffeomorphism of $T^*[2]T[1]M$ therefore Moser lemma applies; the generating vector field is given in the appendix 4.2. A non-canonical symplectic form ω' arises as the closed non-degenerate 2-form which is pulled back via $\mathcal{E}(x)$ to the canonical one ω_0, i.e. $\omega_0 = \mathcal{E}^*\omega'$:

$$\omega' = dx^i \wedge dp_i + d\left[\xi_\alpha \left(\mathcal{E}^{-1}\right)^\alpha{}_\beta\right] \eta^{\beta\gamma} \wedge d\left[\xi_\delta \left(\mathcal{E}^{-1}\right)^\delta{}_\gamma\right] \tag{3.2}$$

$$\begin{aligned} &= dx^i \wedge dp_i + d\xi_\alpha \left[\left(\mathcal{E}^{-1}\right)^\alpha{}_\beta \eta^{\beta\gamma} \left(\mathcal{E}^{-T}\right)_\gamma{}^\delta\right] \wedge d\xi_\delta + dx^j \partial_j \left(\mathcal{E}^{-T}\right)_\beta{}^\alpha \left[\xi_\alpha \eta^{\beta\gamma} \left(\mathcal{E}^{-T}\right)_\gamma{}^\delta\right] \wedge d\xi_\delta \\ &\quad + dx^j \left[\partial_j \left(\mathcal{E}^{-T}\right)_\beta{}^\alpha \left[\xi_\alpha \eta^{\beta\gamma} \xi_\delta\right] \partial_k \left(\mathcal{E}^{-1}\right)^\delta{}_\gamma\right] \wedge dx^k. \end{aligned} \tag{3.3}$$

Defining

$$\Gamma_{i\alpha}{}^{\beta}(x) := \left(\mathcal{E}^{-1}\right)^{\beta}{}_{\gamma}(x)\partial_i \mathcal{E}^{\gamma}{}_{\alpha}(x) \tag{3.4}$$

and using the expression in components of $\mathcal{G}$ from (3.1), we can write down the Poisson brackets, deformed w.r.t. the Darboux chart in use. They are displayed below, together with their index-free counterpart, where $\upsilon(x), \varsigma(x)$ are linear functions of p, $U(x)$ and $V(x) := V^{\alpha}(x)\xi_{\alpha}$ in $\mathcal{O}_1$ and $f(x) \in C^{\infty}(M)$:

$$\begin{array}{ll} \{p_i, x^j\}' = \delta_i{}^j, & \{\upsilon(x), f(x)\}' = \upsilon.f, \\ \{\xi_\alpha, \xi_\beta\}' = \mathcal{G}_{\alpha\beta}(x), & \{U(x), V(x)\}' = \mathcal{G}(U, V), \\ \{p_i, \xi_\alpha\}' = \Gamma_{i\alpha}{}^{\beta}\xi_\beta(x), & \{\upsilon(x), U(x)\}' = \nabla_\upsilon U, \\ \{p_i, p_j\}' = 0, & \{\upsilon(x), \varsigma(x)\}' = [\upsilon, \varsigma]_{\text{Lie}}. \end{array} \tag{3.5}$$

An affine connection with null curvature (due to the null $p - p$ bracket) appears. It can be checked to behave as required under $C^{\infty}(M)$-multiplication thanks to the Leibniz rule for the Poisson bracket. The connection is metric for $\mathcal{G}$ because of Jacobi identity for the double bracket $\{\varsigma(x), \{U(x), V(x)\}'\}'$. Technically the connection is known as Weitzenböck connection [12]. Its non-zero connection coefficients are due to the anholonomy of the basis chosen.

The most general Hamiltonian Θ (2.41) can still be a perfectly consistent object for the NQ-manifold. It is sufficient to impose the master equation $\{\Theta, \Theta\}' = 0$, since solving it puts constraints on the unknown functions ρ and C. Moreover, if a solution was already worked out for the canonical Darboux chart, namely a specific anchor and specific tensors C were already found, transforming them with the diffeomorphism $\mathcal{E}$ of Moser lemma would also return consistent maps with which the Hamiltonian satisfies $\{\Theta, \Theta\} = 0$,

$$\tilde{\rho}^{\alpha i} \mapsto \left(\mathcal{E}^{-1}\right)^{\alpha}{}_{\beta}\tilde{\rho}^{\beta i} =: \underline{\rho}^{\alpha i}, \quad C^{\alpha\beta\gamma} \mapsto \left(\Lambda^3 \mathcal{E}^* C\right)^{\alpha\beta\gamma} =: \underline{C}^{\alpha\beta\gamma}. \tag{3.6}$$

In the context where the map $\mathcal{E}$ is known, as here, we will adopt such viewpoint, which simplifies the search for consistent ρ and C. In other situations we will not be able to resort to it. Suppose hence that in the canonical setting (in particular with $O(d,d)$-invariant pairing η) the map $\tilde{\rho}^{\alpha i} \in \Gamma\left((T^*[1]M \oplus T[1]M)^* \otimes T^*[2]M\right)$ and $C^{\alpha\beta\gamma} \in \bigotimes^3 T^*[1]M \oplus T[1]M$ that solve the master equation with canonical Poisson brackets are given. We are then interested to compute the derived CA bracket, where the total space of the algebroid is $E \cong TM \oplus T^*M$. As discussed in 2.6, this is the derived bracket of $\{,\}'$ with Θ and a pair of generalized vectors $U(x), V(x)$:

$$\begin{aligned} \{\{U(x), \Theta\}', V(x)\}' =& \nabla_{\tilde{\rho}(\mathcal{G}(U))}V(x) - \nabla_{\tilde{\rho}(\mathcal{G}(V))}U(x) + \mathcal{G}\left(\nabla_{\tilde{\rho}(\cdot)}U(x), V(x)\right) \\ &+ \mathcal{G}_{\alpha\delta}\mathcal{G}_{\nu\beta}C^{\alpha\beta\gamma}(x)U^{\delta}(x)V^{\nu}(x) = [U, V]'_{\text{D}}. \end{aligned} \tag{3.7}$$

The standard example of Courant algebroid Dorfman bracket (2.43), in components, is retrieved from (3.7) by plugging in

$$\tilde{\rho}(\chi_k) = p_k, \quad \tilde{\rho}(\theta^k) = 0; \qquad \mathcal{G} = \eta \equiv \langle,\rangle; \qquad C \equiv 0. \tag{3.8}$$

It is now interesting to relax the last option in (3.8). This amounts to twist the bracket by means of the C-tensors. With obvious reference to the stringy fluxes, let us call them

$$H_{ijk}(x)\theta^i\theta^j\theta^k, \quad f_{ij}{}^k(x)\theta^i\theta^j\chi_k, \quad Q_i{}^{jk}(x)\theta^i\chi_j\chi_k, \quad R^{ijk}(x)\chi_i\chi_j\chi_k. \tag{3.9}$$

Then (2.43) is implemented to its twisted version:

$$\begin{aligned}\{\{U(x),\Theta\},V(x)\} =& [U(x),V(x)]_{\mathrm{D}} - X^lY^m\left(H_{lmk}\theta^k + f_{lm}{}^k\chi_k\right) + X^l\sigma_m\left(f_{lk}{}^m\theta^k - Q_l{}^{mk}\chi_k\right)\\ &+ \gamma_l Y^m\left(f_{km}{}^l\theta^k - Q_m{}^{lk}\chi_k\right) + \gamma_l\sigma_m\left(Q_k{}^{lm}\theta^k - R^{lmk}\chi_k\right)\\ =& [U(x),V(x)]_{\mathrm{D}} - H(X,Y,\cdot) - f(X,Y)\frac{\partial}{\partial x} + [f(X,\sigma) - f(Y,\gamma)]\,dx\\ &- [Q(X,\sigma) - Q(Y,\gamma)]\,\frac{\partial}{\partial x} + Q(\gamma,\sigma)dx - R(\gamma,\sigma,\cdot).\end{aligned}$$

In passing from the first to the second equivalence we tacitly made use of the isomorphism $\chi_i \cong \iota_{\partial_i}$ and $\theta^i \cong dx^i$. In other words, f can be interpreted either as a 2-vectors-valued vector field or a vector and a 1-form valued 1-form; on the other hand, Q can be seen as a 2-forms-valued 1-form or a vector and a 1-form valued vector field. Instead $H \in \Omega^3(M)$ and $R \in \mathfrak{X}^3(M)$. The tensors are structure constants of the algebra. Closure of the algebra implies that they fulfill the Bianchi identities. The algebra closes if the Jacobi identity holds for $[,]_{\mathrm{D}}$, but this property is easy to show using graded antisymmetry and the graded Jacobi identity for $\{,\}$. The interested reader can find the derivation of this minor result in the appendix 4.2. However this is tantamount to impose the master equation, as explained in detail in that section. The final step of that computation yields in fact:

$$\begin{aligned}[U,[V,W]] \equiv \{\{U,\Theta\},\{\{V,\Theta\},W\}\} =& \{\{\Theta,\{\{\Theta,U\},V\}\},W\} + \{\{V,\Theta\},\{\{U,\Theta\},W\}\}\\ &- \frac{1}{2}\{\{\{\{\Theta,\Theta\},U\},V\},W\}\\ \equiv& [[U,V],W] + [V,[U,W]] - \frac{1}{2}\{\{\{\{\Theta,\Theta\},U\},V\},W\}.\end{aligned} \tag{3.10}$$

In reference [72], equation (23), the identities for the fluxes were given employing as $\tilde{\rho}$-map some map more articulated than the projector: $\tilde{\rho}(\chi_i) = p_i, \tilde{\rho}(\theta^i) = \beta^{ij}p_j$. Moreover their derivation was performed in the context of Courant algebroids. Our computation here relies instead on the underlying graded symplectic geometry, and stems from the computation of the master equation, carried with the simpler projector as anchor. It gives:

$$\begin{aligned}&\frac{1}{3}\partial_{[i}H_{jkl]} + \frac{1}{2}H_{[ij|m}f_{|kl]}{}^m = 0\\ &\partial_{[i}f_{jk]}{}^l - \frac{1}{2}Q_{[i|}{}^{ml}H_{|jk]m} + f_{[ij|}{}^m f_{m|k]}{}^l = 0\\ &\partial_{[i}Q_{j]}{}^{[kl]} + \frac{1}{2}H_{[ij]m}R^{m[kl]} + \frac{1}{2}f_{[ij]}{}^m Q_m{}^{[kl]} + 2f_{[i|m}{}^{[k|}Q_{j]}{}^{m|l]} = 0\\ &\frac{1}{3}\partial_i R^{[jkl]} + f_{im}{}^{[j|}R^{m|kl]} + Q_i{}^{[j|m}Q_m{}^{|kl]} = 0\\ &R^{[ij|m}Q_m{}^{|kl]} = 0.\end{aligned} \tag{3.11}$$

These agree with the findings of [72] upon imposition of $\beta \equiv 0$ there. A similar derivation in the context of graded NQ-manifolds was also carried in [73].

The novelty of our approach, that will be highlighted in the subsections 3.2 and 3.3, relies heavily on the way we treat the mixed symmetry tensors H, f, Q, R. They are introduced just locally through the gauge potential of the fluxes H and R. In fact, if $dH = 0$ locally the 2-form B rises as the gauge potential for H, $dB = H$; on the other hand, if the Schouten bracket of the differential d_ρ (2.14) with R, seen both as multivectors, is $[d_\rho, R]_{\mathrm{SN}} = 0$, then the bivector Π serves as the gauge potential for R, $R = d_\rho\Pi$. This can take place because the non-canonical choice of coordinates and hence the Poisson brackets too can be made dependent upon the latter tensors. Moreover Π will itself depend on B

through the open-closed metric relation (2.13). These considerations on the fluxes and their locality are dominant in our treatment because at the same time the global fluxes, i.e. the C tensors in the general Hamiltonian, are always set to zero. Note that what is commonly understood under the term *fluxes* are tensors *non-trivial in the cohomology.* We will anyway keep the terminology. When the deformation is implemented by a vielbein $\mathcal{E}$, the proper fluxes C can always be investigated separately thanks to the associativity of the Poisson bracket and also to the fact that the transformation under $\mathcal{E}$ is worked out in (3.6). Therefore the above analysis on H, f, Q, R concludes the excursion on their global properties, in this context.

Let us show how our approach works in detail, for a specific choice of the vielbein. Issues on the globality of the frame are postponed to section 3.3.1, after section 3.3.

3.2 (g, H, ϕ)-Supergravity

This part reviews the publication [3] in detail, moving on from the generalities of the construction for an unknown vielbein in the previous section. The aim was to obtain the (g, H, ϕ)-Supergravity action as a Poisson gauge theory, in the sense that the gauge algebra is the graded Poisson algebra of $T^*[2]T[1]M$ acting on the set of functions. Hence the relevant gauge fields for the theory must be incorporated by means of $\mathcal{E}$. The invertible change of degree-1 coordinates we deployed is:

$$\mathcal{E}(x) = \lambda(x) \begin{pmatrix} \mathbb{1} & 0 \\ g(x) - B(x) & \mathbb{1} \end{pmatrix}, \tag{3.12}$$

where $\lambda(x) := \exp\left[-\frac{\phi(x)}{3}\right]$, $\phi(x)$ scalar field to be identified with the dilaton, and $g \in S^2(T[1]M)$, $B \in \Lambda^2 T[1]M$ appear on the same footing, like in the string sigma model (1.4). The Darboux coordinates χ_i and θ^i are mapped to new coordinates $\underline{\chi}_i$, $\underline{\theta}^i$,

$$\chi_i \mapsto \lambda(x)\chi_i + \lambda(x)\theta^j \left(g(x) - B(x)\right)_{ji} \equiv \underline{\chi}_i, \quad \theta^i \mapsto \lambda(x)\theta^i \equiv \underline{\theta}^i, \tag{3.13}$$

while degree-0 positions x^i and degree-2 momenta p_i are left unchanged. (3.12) could be also thought as the product of two exponentials, $\mathcal{E}(x) = \exp\left[-\frac{\phi(x)}{3}\right] \exp(g(x) - B(x))$, where the second exponential acts on a $2d \times 2d$ nilpotent matrix. This is a generalization of the $O(d, d)$ transformation generated by B, and its effect is to locally reduce the $O(d, d)$ symmetry to the subgroup $(O(d) \times O(d)) \times e^B$. In fact the local curved metric $\mathcal{G}$ (3.1) is thus

$$\mathcal{G} = \lambda^2 \begin{pmatrix} 2g & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}. \tag{3.14}$$

The flat η metric has been conformally rescaled with λ^2 factors and a non-degenerate metric for tangent space has also been brought into existence. Having a non-degenerate entry in the upper left (d-dimensional) block of $\mathcal{G}$ is going to be very convenient for our purposes.

The connection ∇ which arises from this vielbein is

$$\{\upsilon, X + \gamma\}' = \nabla_\upsilon (X + \gamma) = \upsilon.\left(X(x) + \gamma(x)\right) + \upsilon^i X^j \left(\Gamma_{ij}{}^k \chi_k + \Gamma_{ijk}\theta^k\right) + \upsilon^i \gamma^j \Gamma_i{}^j{}_k \theta^k,$$
$$\Gamma_{ij}{}^k = \lambda^{-1}\partial_i \lambda(x) \delta_j{}^k, \quad \Gamma_i{}^j{}_k = \lambda^{-1}\partial_i \lambda(x) \delta^j{}_k, \quad \Gamma_{ijk} = \partial_i \left(g + B\right)_{jk}. \tag{3.15}$$

Before continuing with the discussion on the Hamiltonian and the derived bracket, it turns useful to test the model through a simpler but enlightening example.

Example 3.2.1 (Vielbein with $B \neq 0, g = 0 = \phi$). If in the vielbein we set $\phi(x) = 0$ and $g = 0$, $\mathcal{E}(x)$ implements a symmetry of the η pairing. In fact $\{\xi_\alpha, \xi_\beta\}' = \eta_{\alpha\beta}$, and there is just one non-zero connection symbol, $\Gamma_{ijk} = \partial_i B_{jk}$. For the Hamiltonian simply given by

$$\Theta = \theta^i p_i, \quad C \equiv 0,$$

the non-canonical Poisson brackets corresponding to this instance of $\mathcal{E}(x)$ yields, in the derived bracket with a pair of elements in $\mathcal{O}_1$ (recall that the linear functions in the degree-1 coordinates are isomorphic to sections of $TM \oplus T^*M$), the Dorfman bracket twisted by $dB \in \Omega^3(M, \mathbb{R})$:

$$\left\{\{U^\alpha(x)\xi_\alpha, \Theta\}, V^\beta(x)\xi_\beta\right\} = [U, V]_{\mathrm{D}} + dB(U, V, \cdot). \tag{3.16}$$

This example already highlights that the partial derivatives on B are arranged in such a way that the exterior derivative is built up. The H-class of the Courant algebroid is untouched; such vielbein is hence an automorphism of the CA structure. In the rest of the section we will later see that when the completely symmetric 2-tensor $g(x)$ is turned on in the derived bracket the derivatives will hit it in such a way that they will form the Christoffel connection symbols (of first type, i.e. $\Gamma^{\mathrm{L.C.}} \in \Gamma(\bigotimes^3 T^*M)$).

To be able to show the previous claim we still need to find a Hamiltonian that solves the classical master equation with the non-canonical Poisson brackets. A one-line calculation (omitted here) proves that

$$\Theta = \lambda^{-1}\theta^i p_i \tag{3.17}$$

is a good option. In fact, in virtue of (3.6), this is the vielbein-transformed projector and C is set to 0 again. An alternative way to see this is by direct comparison between $\partial\lambda^{-1}$ from the bracket and the second connection coefficient in (3.15). We can now proceed to compute the derived bracket:

$$\begin{aligned}\{\{U, \Theta\}', V\}' = {} & \lambda X^i \partial_i V(x) - \lambda Y^i \partial_i U(x) + X^i \partial_i \lambda V - Y^i \partial_i \lambda U \\ & + 2\lambda X^{[i} Y^{j]} \partial_i \left(g(x) + B(x)\right)_{jm} \theta^m + \lambda \theta^i X^j Y^k \partial_i \left(g(x) + B(x)\right)_{jk} \\ & + \theta^i \partial_i \lambda \left(2g(X, Y) + X^j \sigma_j + Y^j \gamma_j\right) + \lambda 2g(\theta^i \partial_i X, Y) + \lambda \sigma_j \theta^i \partial_i X^j \\ & + \lambda Y^j \theta^i \partial_i \gamma_j.\end{aligned} \tag{3.18}$$

Thus the deformed Dorfman bracket $[,]'_{\mathrm{D}}$ (3.7) becomes in this case

$$\begin{aligned}[U, V]'_{\mathrm{D}} = {} & \lambda [U, V]_{\mathrm{D}} + \lambda^{-1}\rho(U)\lambda V - \lambda^{-1}\rho(V)\lambda U + \mathcal{G}\left(\lambda^{-1}\rho(\cdot)\lambda U, V\right) \\ & + \lambda \left[\Gamma^{\mathrm{L.C.}}(\cdot, X, Y) + dB(\cdot, X, Y)\right].\end{aligned} \tag{3.19}$$

Here $\mathcal{G}$ is (3.14) and the anchor, given its companion $\tilde{\rho}$ that can be read off from (3.17), corresponds to

$$\rho(U) = \lambda X.$$

The self-explanatory symbol $\Gamma^{\mathrm{L.C.}} \in \Gamma\left(\bigotimes^3 T^*M\right)$ refers to the Christoffel symbol of first kind:

$$\Gamma^{\mathrm{L.C.}}_{kij} = \partial_k g_{ij} + \partial_i g_{jk} - \partial_j g_{ki};$$

to get the Christoffel symbols of second kind dualization occurs in the last entry.

At this point we want to apply proposition 2.3.2 enunciated in the previous section, and get a CA connection. For the CA bracket we can plug in the deformed Dorfman bracket (3.19), while the Lie-like bracket, written with respect to the Darboux chart of reference, is

$$[\![U,V]\!] \overset{(2.15)}{=} \rho(U)V - \rho(V)U = \lambda[U,V]_{\mathrm{D}} + \langle (d_\rho U^\alpha(x))\,\xi_\alpha, V\rangle. \tag{3.20}$$

Therefore the connection $\widetilde{\nabla} : \Gamma(E) \mapsto \Gamma(E^* \otimes E)$ is

$$\begin{aligned}\mathcal{G}\left(\widetilde{\nabla}_W U, V\right) = {} & \lambda^2 \left(2g(X,Y)Z.\lambda + 4g(Z,[Y)\,X].\lambda\right) + \lambda^3 \left[\Gamma^{\mathrm{L.C.}}(Z,X,Y) + dB(Z,X,Y)\right] \\ & + \lambda^2 \left(X^i\sigma_i\,Z.\lambda + Z^i\sigma_i\,X.\lambda\right) + \lambda^2 \left(\gamma_i Y^i\,Z.\lambda - Z^i\gamma_i\,Y.\lambda\right) \\ & + \lambda^2 \left(\kappa_i Y^i\,X.\lambda - \kappa_i Z^i\,Y.\lambda\right) + \mathcal{G}\left(\rho(W)U,V\right).\end{aligned} \tag{3.21}$$

In the first row of this expression we placed only the connection symbols of the first kind which are evaluated against a triplet of vector fields, $\tilde{\Gamma} \in \Gamma\left(\bigotimes^3 T^*M\right)$. In the second addend the square bracket denotes antisymmetrization (with a $\frac{1}{2}$−factor) between Y and X. Out of the whole expression, these connection symbols are the main outcome of our ansatz for the graded Poisson algebra, as it will be soon clear. The derivatives hitting on λ make up the E^*-valued endomorphism of (2.19), with $e = -\frac{1}{3}\begin{pmatrix} 0 \\ d\phi \end{pmatrix}$, where one should pay attention that the pairing here is (3.14); it could be easier to see this in the regular connection (3.25) (second and third term).

The derived bracket approach, with the help of a generalized Lie bracket, allows to start with a curvature-free connection in the graded Poisson algebra and to end up with a torsionful connection whose curvature is non-zero. Let us compute the Gualtieri torsion of the connection with the Dorfman bracket $T_{\mathrm{D}}(U,V,W)$ (2.22) (equivalent to the other torsion definition 2.3.1),

$$T_{\mathrm{D}}(U,V,W) = \langle [V,W]'_{\mathrm{D}}, U\rangle - \langle [U,W]'_{\mathrm{D}}, V\rangle - \langle [\![V,W]\!], U\rangle + \langle [\![U,W]\!], V\rangle - \langle [\![U,V]\!], W\rangle.$$

It is worth pointing out that this is the general torsion in the presence of no fluxes C and untwisted $p-p$ Poisson bracket, therefore it can be applied to more general conditions than the present situation. In the coordinate basis, where the doubled indices α,β,γ respectively refer to d-dimensional indices i,j,k, some trivial algebraic manipulations allow to conclude that the generalized Lie bracket is identically zero, derivatives on λ cancel out with each other, and the rest assembles in this way:

$$T_{\mathrm{D}\,\alpha\beta\gamma} = \Gamma^{\mathrm{L.C.}}_{ijk} - \Gamma^{\mathrm{L.C.}}_{jik} + (dB)_{ijk} - (dB)_{jik} = 2(dB)_{ijk}, \tag{3.22}$$

since in our conventions the Christoffel symbol of the first kind $\Gamma^{\mathrm{L.C.}}$ is symmetric in the first two indices. This is a glaring peculiarity of our generalized connection: it is not torsion-free. We find it interesting because it stands out the literature where the only results known before our analysis regarded the (non-unique!) generalized analogues of the Levi-Civita one, which is metric *and* T_{D}-free. Notice moreover that the torsion 3-form on E is non-trivial just in the component given by a regular 3-form (i.e. $T_{\mathrm{D}} \in \Lambda^3 T^*M$). This confirms that any term of type σ^e as in (2.19) does not contribute to the torsion.

Now, assume that the generalized vectors are constrained to belong to the tangent subspace; to make this possible, a splitting of the exact sequence (2.5) is needed. This splitting $s : \Gamma(TM) \mapsto \Gamma(E)$ shall be isotropic for $\eta \equiv \langle,\rangle$ but non-isotropic for $\mathcal{G}$, so that a non-degenerate metric on tangent vectors could rise. Without it, it would not be possible to dualize the connection coefficients and get the connection symbols of the second kind otherwise. The option we picked up is

$$s(X) = \lambda^{-1} X, \tag{3.23}$$

In fact,

$$\mathcal{G}(s(X), s(Y)) = 2g(X, Y), \quad \langle s(X), s(Y) \rangle = 0.$$

Notice that the induced metric on tangent space is just twice the Riemannian metric g. With such a choice of the splitting the CA connection becomes:

$$\begin{aligned} \mathcal{G}\left(\widetilde{\nabla}_{s(Z)} s(X), s(Y)\right) = 2g(Z.X, Y) + \Gamma^{\mathrm{L.C.}}(Z, X, Y) + dB(Z, X, Y) \\ + 2g(Z, Y)\, \lambda^{-1} X.\lambda - 2g(Z, X)\, \lambda^{-1} Y.\lambda =: 2g(\widetilde{\nabla}_Z X, Y). \end{aligned} \tag{3.24}$$

With an abuse of notation, the connection $\widetilde{\nabla} : \Gamma(TM) \mapsto \Gamma(T^*M \otimes TM)$ is defined. No confusion should arise thanks to the different arguments used. We can therefore present the final expression of this regular connection, which depends upon the Levi-Civita symbols of second kind $\left(\Gamma^{\mathrm{L.C.}}\right)_{ki}{}^{j}$, the 3-form dB and derivatives on the scalar $\phi(x)$:

$$\widetilde{\nabla}_Z X = Z.X + \left[-X^i \partial_i \frac{\phi}{3}\, Z^k \delta_k{}^j + g^{jl} \partial_l \frac{\phi}{3}\, g_{ki} Z^k X^i + Z^k X^i \left(\left(\Gamma^{\mathrm{L.C.}}\right)_{ki}{}^{j} + \frac{1}{2} g^{jl}\, (dB)_{kil} \right) \right] \partial_j. \tag{3.25}$$

Before looking at the curvature, it is instructive to follow how $[\![,]\!]$ (3.20) and $[,]'_{\mathrm{D}}$ (3.19) transform when their arguments are s-vectors, i.e. vectors rescaled with λ^{-1} factors: the former becomes

$$[\![s(X), s(Y)]\!] = X.\left(\lambda^{-1} Y\right) - Y.\left(\lambda^{-1} X\right), \tag{3.26}$$

while the latter is

$$[s(X), s(Y)]'_{\mathrm{D}} = \lambda^{-1} [X, Y]_{\mathrm{Lie}} + \lambda^{-1} \left\{ \Gamma^{\mathrm{L.C.}}(\cdot, X, Y) + dB(\cdot, X, Y) \right\}.$$

Then the torsion $T_{\mathrm{D}}(s(X), s(Y), s(Z)) \in \Lambda^3 T^*M$ on the s-vectors is, in the coordinate basis, the same 3-form (3.22), as the generalized Lie brackets (which is also an endomorphism-valued 1-form) match up together so to cancel, and we are left with $\Gamma^{\mathrm{L.C.}}$ and dB.

Let us now emphasize another peculiar aspect of the connection: the natural Koszul formula. It emerges naturally when constructing the connection, generalizing the Koszul formula for the Levi-Civita connection to the case of a non-symmetric metric. It includes the dilaton too. The Koszul formula is already hidden in (3.24). For a better display of this property it is more instructive to rely on a coordinate-free formulation obtained deploying the homomorphism property of the vielbein $\mathcal{E}(x)$, seen as the map $\mathcal{E} : (E, [,]'_{\mathrm{D}}, \mathcal{G}, \rho') \mapsto (TM \oplus T^*M, [,]_{\mathrm{D}}, \langle,\rangle, \rho)$.

$$\begin{aligned} \mathcal{E}^{-1}[\mathcal{E}s(X), \mathcal{E}s(Y)]_{\mathrm{D}} &= \mathcal{E}^{-1}\big([X, Y] - \iota_Y d\,(g + B)\,(X) + \mathcal{L}_X\,(g + B)\,(Y)\big) \\ &= \lambda^{-1}\big([X, Y] - (g + B)\,([X, Y]) - \iota_Y d\,(g + B)\,(X) + \mathcal{L}_X\,(g + B)\,(Y)\big). \end{aligned}$$

As usual, the first term gets canceled by the Lie-like bracket (3.26) which also introduces the derivatives on λ. Hence we can determine the Koszul formula with the help of the Cartan identities:

$$\begin{aligned} 2g\left(\widetilde{\nabla}_Z X, Y\right) =&\ + X.(g + B)(Y, Z) + Z.(g + B)(X, Y) - Y.(g + B)(X, Z) \\ &- (g + B)([X, Y], Z) + (g + B)(X, [Y, Z]) - (g + B)(Y, [X, Z]) \\ &+ \lambda^{-1}\big(X(\lambda) 2g(Y, Z) - Y(\lambda) 2g(X, Z)\big). \end{aligned} \tag{3.27}$$

It is evident that the above formula yields the Christoffel symbols of the first kind, the exterior derivative of B, and the endomorphism-valued one form $d\lambda$. Contrary to the pure GR case, now it matters if the Lie bracket of vector fields shows up in the first or second argument, and (3.27) is the only consistent combination for the sum of a metric and a 2-form. In the $\lambda = 1$ case, the Koszul formula was already presented in [74].

We can hence compute the Riemann curvature tensor for the regular connection on vector fields (3.25); let us begin with relabeling the derivatives on the dilaton

$$\mu \in \Omega^1(M, \mathrm{End}(TM)), \quad \mu(Y) = Y^j \lambda^{-1} \left(\partial_i \lambda \delta_j^k - g^{kl} \partial_l \lambda g_{ij} \right) dx^i \otimes \partial_k;$$

and denoting the exact 3-form dB with the field strength H. Notice furthermore that

$$\mathbf{Riem}(s(Z), s(Y), s(X)) \equiv \mathbf{Riem}(Z, Y, X) = [\widetilde{\nabla}_Z, \widetilde{\nabla}_Y] X - \widetilde{\nabla}_{[Z,Y]_{\mathrm{Lie}}} X,$$

however such equivalence of the generalized Riemann tensor (2.25) on the s-vector with the standard Riemann tensor is not obvious a priori. It rather follows from a careful check that derivatives on the λ-factors carried by the splitting s cancel between the commutator and the covariant derivative of $[\![,]\!]$. With these premises, $\mathbf{Riem}(Z, Y, X)$ is:

$$\begin{aligned}
\mathbf{Riem}(Z, Y, X) = & \mathrm{Riem}(Z, Y, X) + \frac{1}{2} \nabla_Z^{\mathrm{L.C.}} g^{-1} H(\hat{Y}, \hat{X}, \cdot) - \frac{1}{2} \nabla_Y^{\mathrm{L.C.}} g^{-1} H(\hat{Z}, \hat{X}, \cdot) \\
& - \frac{1}{4} g^{-1} H(Z, g^{-1} H(Y, X, \cdot), \cdot) + \nabla_Z^{\mathrm{L.C.}} \mu(\hat{Y}, \hat{X}) - \nabla_Y^{\mathrm{L.C.}} \mu(\hat{Z}, \hat{X}) \\
& + \frac{1}{2} g^{-1} \big(H(Z, \mu(Y, X), \cdot) - H(Y, \mu(Z, X), \cdot) \big) \\
& + \mu(Z, \mu(Y, X)) - \mu(Y, \mu(Z, X)),
\end{aligned}$$

where the hatted vector fields are not subjected to covariant derivation. Riem is the Riemann tensor for the Levi-Civita connection, and its Ricci curvature will be denoted Ric. The whole Ricci curvature, in components, corresponds to

$$\begin{aligned}
\mathbf{Ric}_{ij} = & \mathrm{Ric}_{ij} + \frac{1}{2} \nabla_l^{\mathrm{L.C.}} H_{ji}{}^l + \frac{1}{4} H_{jm}{}^l H_{li}{}^m - \frac{1}{2} H_{jm}{}^l \left(\nabla^{\mathrm{L.C.}\, m} \phi g_{il} \right) \\
& - \frac{(d-1)}{6} H_{ji}{}^m \nabla_m^{\mathrm{L.C.}} \phi + \frac{(d-2)}{9} \left(\nabla_i^{\mathrm{L.C.}} \phi \nabla_j^{\mathrm{L.C.}} \phi - \left(\nabla^{\mathrm{L.C.}} \phi \right)^2 g_{ij} \right) \\
& + \frac{(d-2)}{3} \nabla_j^{\mathrm{L.C.}} \nabla_i^{\mathrm{L.C.}} \phi + \frac{1}{3} g_{ij} \left(\nabla^{\mathrm{L.C.}} \right)^2 \phi.
\end{aligned} \tag{3.28}$$

The curvature starts to depend on the dimension of the manifold M because of the trace of μ. For $d = 10$ (3.28) is

$$\begin{aligned}
\mathbf{Ric}_{ij} = & \mathrm{Ric}_{ij} + \frac{1}{2} \nabla_l^{\mathrm{L.C.}} H_{ji}{}^l + \frac{1}{4} H_{jm}{}^l H_{li}{}^m - H_{ji}{}^m \nabla_m^{\mathrm{L.C.}} \phi \\
& + \frac{8}{9} \left(\nabla_i^{\mathrm{L.C.}} \phi \nabla_j^{\mathrm{L.C.}} \phi - \left(\nabla^{\mathrm{L.C.}} \phi \right)^2 g_{ij} \right) + \frac{1}{3} \left(8 \nabla_j^{\mathrm{L.C.}} \nabla_i^{\mathrm{L.C.}} \phi + g_{ij} \left(\nabla^{\mathrm{L.C.}} \right)^2 \phi \right).
\end{aligned} \tag{3.29}$$

The bosonic sector of 10-dimensional supergravity, which coincides with the low-energy effective closed string action of type II (NS-NS sector), is retrieved from the Ricci tensor integrated against $(g - B)$ (and other factors). In our present situation the Ricci tensor does not correspond to the sum of the beta functions for g and B, in contrast to what some other authors found, see for example [20]. The beta functions, up to zero order in α', as computed from the open string sigma model [75]

$$\begin{aligned}\beta(g)_{ij} &= \mathrm{Ric}_{ij} - \frac{1}{4} H_{jm}{}^{l} H_{li}{}^{m} + 2\nabla_i^{\mathrm{L.C.}} \nabla^{\mathrm{L.C.}\,j} \phi \\ \beta(B)_{ij} &= \nabla_l^{\mathrm{L.C.}} H_{ji}{}^{l} - 2H_{ji}{}^{m} \nabla_m^{\mathrm{L.C.}} \phi.\end{aligned} \tag{3.30}$$

The function for the dilaton $\beta(\phi)$, apart from a factor dependent on the dimension, is of order 1 in the string coupling constant α' and being its explicit expression irrelevant for the rest of the discussion, it has been omitted. Linear combinations of $\beta(\phi)$ and $\beta(g)_{ij}$ (opportunely multiplied with the metric) give rise to the Einstein equations for g, and to the ϕ-field equation of motion. This led to assign them a further significance: they are also the equations of motion due to the variation of the $S[g, H, \phi]$ action w.r.t. g and ϕ, according to the principle of least action.

The discrepancy between β-functions and Ricci curvature tensor does not cause any harm because eventually the Lagrangian is obtained from the Ricci tensor (3.28) in $d = 10$ dimensions contracted with the antisymmetric combination

$$\exp(-2\phi)\,(g - B)^{ij}\ \sqrt{-\det g},$$

where the metric g is again used to raise indices. New numerical factors which are dimension-dependent arises in this way. Integrating by parts

$$-\int_M \mathrm{Vol}_M\, e^{-2\phi(x)}\, B^{ij} \left(\frac{1}{2} \nabla_l^{\mathrm{L.C.}} H_{ji}{}^{l} - 2H_{ji}{}^{l} \nabla_l^{\mathrm{L.C.}} \phi \right),$$

and applying Stokes' theorem, with $\nabla_l^{\mathrm{L.C.}} B^{ij} = \frac{1}{3} H_l{}^{ij}$, lead to reconstruct the correct $\frac{1}{12}$ factor in front of the H square term. The numerical factor in front of the kinetic term for the dilaton in the action is recovered as well from the trace and integration by parts. Hence for suitable boundary conditions, in plain words that the fields decay sufficiently fast at infinity, we have been able to build the SUGRA action for the NS-NS fields from purely geometric considerations involving a graded Poisson algebra and a generalized tangent bundle endowed with a Courant algebroid.

We find quite surprising that our 10-dimensional Ricci curvature tensor could reproduce the action upon contraction with $g - B$, despite differing from the β-functions (3.30). In fact, up to boundary terms, it is known that [75]

$$e^{-2\phi} \left[\beta(g)_{ij} g^{ij} - \frac{1}{2} \beta(B)_{ij} B^{ij} \right] \equiv \mathcal{L}[g, H, \phi].$$

We could instead derive the same Lagrangian via a tensor ($\mathbf{Ric}_{ij}$ (3.29)) which is definitely not $\beta(g)_{ij} + \frac{1}{2}\beta(B)_{ij}$ (3.30).

3.3 Dual gravity action in the background of Q and R fluxes

In the same conditions than before, when implementing another deformation that deals with both $g + B$ and its inverse (with reverted sign) at the same time, then by construction the derived brackets, in some limit (i.e. through a splitting/projection onto the appropriate subspace), are expected to yield the cohomological terms which are the difference between two tensors in $H_3(M, \mathbb{R})$ but also between two mixed symmetry tensors in $\Gamma\left(\Lambda^2 T^* M \otimes TM\right)$, i.e. $\rho(\xi_\alpha)^i \partial_i \Pi$, for $\Pi \in \mathfrak{X}^2(M)$. In that limit the exterior derivative of $B \in \Lambda^2 T^*M$ is naturally set to zero. Then the field strength of Π can be locally regarded as the stringy Q and R fluxes (although proper fluxes are not globally trivial in the cohomology) and should participate in building up the curvature. Meanwhile, projection to TM with a splitting s portrays again (g, H)-SUGRA as in the previous section. This doubled, mirrored image and the corresponding differential geometry objects are studied in our forthcoming publication [7]. The machinery is pretty much the same one used for the previous deformation.

The vielbein is

$$\mathcal{E}(x) = \begin{pmatrix} \mathbb{1} & (g(x) - B(x))^{-1} \\ -(g(x) + B(x)) & \mathbb{1} \end{pmatrix}. \tag{3.31}$$

It is invertible because the metric $\mathcal{G}$ to which $\mathcal{E}$ gives rise, is Riemannian and thus invertible by definition. Its inverse is:

$$\mathcal{E}^{-1} = \frac{1}{2}\begin{pmatrix} g^{-1}(g-B) & -g^{-1} \\ (g-B)\,g^{-1}(g+B) & (g-B)\,g^{-1} \end{pmatrix}.$$

The Darboux coordinates $\{\chi_i\}$ for $T^*[1]M$ and $\{\theta^i\}$ for $T[1]M$ are mapped to new coordinates $\underline{\chi}$ and $\underline{\theta}$ in the following way:

$$\chi_i \mapsto \chi_i - \theta^j\,(g(x)+B(x))_{ji} =: \underline{\chi}_i, \qquad \theta^i \mapsto \chi_j\,(g(x)-B(x))^{ji} + \theta^i =: \underline{\theta}^i. \tag{3.32}$$

Compared to the vielbein of section 3.2, now the degree 1 coordinates are transformed in a symmetric fashion, one set of coordinates with the map $g+B$, the other with its inverse modulo signs. Moreover, the open-closed string relations (2.13) are nicely implemented in the vielbein itself. Let us clarify once more what is perhaps a point of confusion: the vielbein we consider is purely local, not global.

The non-flat local metric $\mathcal{G}$ associated to this choice for $\mathcal{E}(x)$ is

$$\mathcal{G}(x) = \begin{pmatrix} -2g(x) & 0 \\ 0 & 2\,(g+B)^{-1}\,g\,(g-B)^{-1}\,(x) \end{pmatrix}. \tag{3.33}$$

The symmetry group for the metric is hence $O(d)\times O(d) \subset O(d,d)$. The vielbein $\mathcal{E}$ reduced the orthogonal group of split signature to (the double copy of) the orthogonal group in d-dimensions. When referring to objects in the cotangent space we will also enforce the closed-open string relations

$$(g(x)+B(x))^{-1} = G^{-1}(x) + \Pi(x), \tag{3.34}$$

for $G^{-1}(x) \in S^2(TM)$ and $\Pi(x) \in \mathfrak{X}^2(M)$. In particular,

$$G^{-1} = (g+B)^{-1}\,g\,(g-B)^{-1}\,, \quad \Pi = -(g+B)^{-1}B(g-B)^{-1},$$

and we note the appearance of $2G^{-1}$ in the lower diagonal block of $\mathcal{G}$ (3.33). When we will be restricting our considerations to cotangent space we will express all the tensors w.r.t. G^{-1} and Π. Vice versa, when working in tangent space, our metric and 2-form will be respectively g and B. For the moment being we will express every formula in terms of the latter.

The connection $\nabla = \mathcal{E}^{-1}\partial\mathcal{E}$ corresponds to:

$$\{\upsilon, X+\gamma\}' = \nabla_\upsilon\,(X+\gamma) = \upsilon.\,(X(x)+\gamma(x)) + \upsilon^i X^j\left(\mathit{\Gamma}_{ij}{}^k\chi_k + \mathit{\Gamma}_{ijk}\theta^k\right) + \upsilon^i\gamma_j\left(\mathit{\Gamma}_i{}^j{}_k\theta^k + \mathit{\Gamma}_i{}^{jk}\chi_k\right),$$

$$\mathit{\Gamma}_{ij}{}^k = \frac{1}{2}g^{kl}\partial_i\,(g-B)_{jl}\,, \quad \mathit{\Gamma}_i{}^{jk} = \frac{1}{2}g^{kl}\,(g-B)_{lm}\,\partial_i\left[(g+B)^{-1}\right]^{jm}, \tag{3.35}$$

$$\mathit{\Gamma}_i{}^j{}_k = \frac{1}{2}\left[(g-B)\,g^{-1}\,(g+B)\right]_{km}\partial_i\,(g+B)^{jm}\,, \quad \mathit{\Gamma}_{ijk} = -\frac{1}{2}\,(g+B)_{lk}\,g^{ml}\partial_i\,(g-B)_{jm}\,. \tag{3.36}$$

The non-canonical Poisson brackets and the corresponding symplectic form are preserved by the Poisson action of the following Hamiltonian

$$\Theta = \left(-\chi_i \frac{g^{ik}}{2} + \theta^i \left(g - B\right)_{ij} \frac{g^{jk}}{2}\right) p_k. \tag{3.37}$$

Checking that $\{\Theta, \Theta\}' = 0$ with the above Θ is a lengthy and unwieldy calculation. It is much easier to use a consistent solution in the canonical setting, as the following projector $\tilde{\rho} : \Gamma(E^*) \mapsto \Gamma(TM)$ is:

$$\tilde{\rho} = \begin{pmatrix} 0 & 0 \\ \mathbb{1} & 0 \end{pmatrix},$$

and hence map this to a suitable new $\underline{\tilde{\rho}}$ in virtue of (3.6),

$$\underline{\tilde{\rho}} = \begin{pmatrix} -\frac{g^{-1}}{2} & 0 \\ (g - B)\frac{g^{-1}}{2} & 0 \end{pmatrix}, \qquad \rho(U) = X + (g + B)^{-1}(\gamma) \in \Gamma(TM). \tag{3.38}$$

The derived brackets $[,]'_{\mathrm{D}}$ are thus steadily obtained from the general formula (3.7) through the following replacements: $\tilde{\rho}$ as given in (3.38), $\mathcal{G}$ as given in (3.33), the connection symbols (3.35) and (3.36) and $C \equiv 0$. For the sake of simplicity, it is more convenient to present their expression exploiting the fact that $\mathcal{E}$ is a homomorphism of the CA bracket.

$$\begin{aligned}
[U, V]'_{\mathrm{D}} = \mathcal{E}^{-1}[\mathcal{E}U, \mathcal{E}V]_{\mathrm{D}} = \frac{g^{-1}}{2}(g + B)([\rho(U), \rho(V)]) - \frac{g^{-1}}{2}\mathcal{L}_{\rho(U)}[\sigma - (g - B)(Y)] \\
+ \frac{1}{2}(g - B)\, g^{-1}(g + B)([\rho(U), \rho(V)]) \\
+ \frac{g^{-1}}{2}\iota_{\rho(V)} d[\gamma - (g - B)(X)] \\
+ (g + B)\frac{g^{-1}}{2}\mathcal{L}_{\rho(U)}[\sigma - (g - B)(Y)] \\
- (g + B)\frac{g^{-1}}{2}\iota_{\rho(V)} d[\gamma - (g - B)(X)].
\end{aligned} \tag{3.39}$$

A connection for the CA arises when subtracting the generalized Lie bracket (3.20) of proposition 2.3.2. The connection in its coordinate expression is not particularly meaningful, since the relevant physical fields are put in a dominant place only when we restrict our considerations to the subspaces TM or T^*M. However it can be interesting to take a look at the torsion $T_{\mathrm{D}}(U, V, W)$ of the connection. In coordinates (with indices i, j, k assigned in this order), this is[1]

$$\begin{aligned}
T_{\mathrm{D}}(i, j, k) = 3\,(g + B)^{[j|m}\,\partial_m \Pi^{|ki]} + \partial_{[i} B_{jk]} + 4\partial_j \Pi^{ki} - 4\partial_k \Pi^{ji} + 4\partial_i \Pi^{jk} \\
- 2\,(g - B)_{il}\left(\partial_{[j} + (g + B)^{[j|m}\,\partial_m\right)(g + B)^{|k]l} + \text{cyclic in } i, j, k.
\end{aligned} \tag{3.40}$$

The 3-form component of T_{D} is again just dB; the trivector component $(g + B)^{[j|m}\,\partial_m \Pi^{|ki]}$ will be relevant as well for the rest of the analysis.

The projection onto the physically important subspaces takes place when using a non-isotropic splitting $s : \Gamma(E) \mapsto \Gamma(TM)$ for $\mathcal{G}$ which is actually isotropic for $\langle,\rangle$:

$$s(X) = X; \qquad \mathcal{G}(s(X), s(Y)) = -2g(X, Y), \quad \langle s(X), s(Y)\rangle = 0.$$

[1] Antisymmetrization will occur without factors due to combinatorics unless explicit mentioned.

The induced metric on TM is $-2g(x)$. It is immediate to notice that this option for the splitting covers the previous case of (g, H, ϕ)-Supergravity when $\lambda = 1$ (i.e. $\phi = 0$): in fact the connection becomes

$$\begin{aligned}
\mathcal{G}\left(\widetilde{\nabla}_{s(Z)}s(X), s(Y)\right) =&\ (g-B)\left([X,Y],Z\right) + (g-B)\left(Y,[X,Z]\right) - (g-B)\left(X,[Y,Z]\right) \\
&- X.\,(g-B)\left(Y,Z\right) - Z.\,(g-B)\left(X,Y\right) + Y.\,(g-B)\left(X,Z\right) \\
&- 2g(Z^i\partial_i X, Y) \\
=:&\ -2g\left(\widetilde{\nabla}_Z X, Y\right). \qquad (3.41)
\end{aligned}$$

We have hence obtained a Koszul formula for the non-symmetric metric $(g-B)$ without dilaton, in this case. It is remarkable that the two vielbeins have led to very close results when projecting the CA connection onto TM: this is not trivial even when $\lambda \equiv 1$ since the inverse vielbeins as well as the respective generalized metrics for E differ quite much from each other. Inversion of (3.41) allows to write down the connection $\widetilde{\nabla} : \Gamma(TM) \mapsto \Gamma(T^*M \otimes TM)$:

$$\widetilde{\nabla}_Z X = Z.X + Z^k X^i \left[\left(\Gamma^{\text{L.C.}}\right)_{ki}{}^{j} - \frac{g^{jl}}{2}\,(dB)_{kil}\right]\partial_j.$$

When focusing on the T^*M subspace the physics starts to look quite unwieldy compared to what we described so far. Be reminded that now the open-closed strings relations will be implemented, hence the metric will be G^{-1} while the 2-form B is replaced by the bivector Π. In order to proceed, first we must dualize the short exact sequence for the CA,

$$0 \to TM \xrightarrow{\Delta} E \xrightarrow{\Delta^*} T^*M \to 0,$$

following the prescriptions at the end of subsection 2.2. Δ is an isotropic map for the E-metric $\mathcal{G}$. A possibility is:

$$\Delta(X) = \left(G^{-1} - \Pi\right)\frac{G}{2}(X) + \frac{G}{2}(X). \qquad (3.42)$$

Composition of (3.42) with the anchor (3.38), $\rho\circ\Delta : TM \mapsto TM$, gives the identity of TM. Then having picked up this Δ-map implies that the specific formula for $\Delta^* : E^* \mapsto T^*M$ is

$$\Delta^*(U) = -\left(G^{-1} - \Pi\right)^{-1}(X) + \gamma.$$

Hence a suitable splitting $r : T^*M \mapsto E$, $\Delta^* \circ r = \mathrm{id}_{T^*M}$, non-isotropic for $\mathcal{G}$, is

$$r(\gamma) = \gamma. \qquad (3.43)$$

Clearly, the induced metric on T^*M is $2G^{-1}(x)$. The connection in cotangent space $\widetilde{\nabla} : \Gamma(T^*M) \mapsto \Gamma(TM \otimes T^*M)$,

$$2G^{-1}\left(\widetilde{\nabla}_\zeta \gamma, \sigma\right) \equiv 2\mathcal{G}\left([r(\gamma), r(\sigma)]'_{\text{D}} - [\![r(\gamma), r(\sigma)]\!], r(\zeta)\right),$$

where $[,]'_{\text{D}}$ is given in (3.39), corresponds hence to

$$\begin{aligned}
2G^{-1}\left(\widetilde{\nabla}_\zeta \gamma, \sigma\right) =&\ \iota_{[(G^{-1}+\Pi)(\gamma),(G^{-1}+\Pi)(\sigma)]}\zeta - \iota_{[(G^{-1}+\Pi)(\gamma),(G^{-1}+\Pi)(\zeta)]}\sigma + \iota_{[(G^{-1}+\Pi)(\sigma),(G^{-1}+\Pi)(\zeta)]}\gamma \\
&+ \left(G^{-1}+\Pi\right)(\gamma).\left(G^{-1}+\Pi\right)(\zeta,\sigma) - \left(G^{-1}+\Pi\right)(\sigma).\left(G^{-1}+\Pi\right)(\zeta,\gamma) \\
&+ \left(G^{-1}+\Pi\right)(\zeta).\left(G^{-1}+\Pi\right)(\sigma,\gamma) - 4G^{-1}\left(\left(G^{-1}+\Pi\right)([\gamma).\sigma],\zeta\right). \qquad (3.44)
\end{aligned}$$

The Leibniz rule has been employed to untie the connection into its components. The last term is the contribution of the Lie-like bracket and the square bracket appearing there means antisymmetrization with a $\frac{1}{2}-$factor. The whole expression might look very complicated but it is actually quite easy to simplify it up to (3.45) below, as should be the case since the connection symbols are formally equivalent to $\Gamma^{\text{L.C.}} + g^{-1}dB$ upon replacement of:

$$g \mapsto G^{-1}, \quad B \mapsto \Pi, \quad {\rho_\alpha}^i = {\delta_j}^i \longrightarrow {\rho_\alpha}^i = {\delta_j}^i + \left(G^{-1} - \Pi\right)^{ji}.$$

The final output is:

$$\begin{aligned}\widetilde{\nabla}_\zeta\gamma = &\left(G^{-1}+\Pi\right)(\zeta).\gamma + \frac{1}{2}\zeta_k\gamma_i G_{jl}\left(G^{-1}+\Pi\right)^{km}\partial_m\left(G^{-1}+\Pi\right)^{il}dx^j \\ &+ \frac{1}{2}\zeta_k\gamma_i G_{lj}\left[\left(G^{-1}+\Pi\right)^{im}\partial_m\left(G^{-1}+\Pi\right)^{lk} - \left(G^{-1}+\Pi\right)^{lm}\partial_m\left(G^{-1}+\Pi\right)^{ik}\right]dx^j.\end{aligned} \tag{3.45}$$

For convenience, let us now define:

$$(\Gamma_G)^{ki}{}_j := \frac{1}{2}\left[G^{il}\partial_l G^{mk} + G^{kl}\partial_l G^{im} - G^{ml}\partial_l G^{ik}\right]G_{mj},$$

$$\mathcal{Y}^{ki}{}_j := \frac{1}{2}\left[\Pi^{il}\partial_l G^{mk} + \Pi^{kl}\partial_l G^{im} - \Pi^{ml}\partial_l G^{ik}\right]G_{mj},$$

for the symmetric part of the connection. In fact $(\Gamma_G)^{ki}{}_j = (\Gamma_G)^{ik}{}_j$ and $\mathcal{Y}^{ki}{}_j = \mathcal{Y}^{ik}{}_j$. For the antisymmetric part it is convenient to name the following quantities

$$Q^{ki}{}_j := \partial_j\Pi^{ki}, \qquad R^{kij} := \Pi^{[k|m}\partial_m\Pi^{|ij]}, \tag{3.46}$$

in agreement with the local expressions of the stringy fluxes Q and R when the closed 3-form H is zero, see for example (3.3) in [70], and hence notice the appearance of

$$\mathcal{X}^{ki}{}_l := \frac{1}{2}\left[-Q^{ki}{}_j + 2G_{mj}G^{[i|l}Q^{m|k]}{}_l\right], \quad \text{and} \quad \frac{1}{2}G_{jm}R^{kim}.$$

In the end, the connection symbol $\tilde{\Gamma}$ associated to $\widetilde{\nabla}$ is

$$\tilde{\Gamma}^{ki}{}_j = (\Gamma_G)^{ki}{}_j + \mathcal{Y}^{ki}{}_j + \mathcal{X}^{ki}{}_j + \frac{1}{2}G_{jm}R^{kim}. \tag{3.47}$$

In the fully contracted expression for the torsion $T_{\text{D}}(i,j,k)$ (3.40) just $\mathcal{G}^{-1}\mathcal{X}$ and R contribute and build up the trivector, as we hinted at earlier in this subsection. In fact the same splitting r does not lead to significant changes to the Lie-like bracket, in the coordinate expressions. The connection and its torsion we got here slightly differ from what was studied in [36], where the authors employed a connection on T^*M (and its associated torsion) that depended on a different combination of Q-factors.

Before looking after the consistent curvature tensors, let us combine the developments so far with a new observation, developed further in [61]. It is another interesting viewpoint that can shed light on the matter from another angle.

Redefinition of $\{\theta\}$ only. One more deformation can lead to the Courant algebroid constrained on the r-sections. It is encoded in the vielbein

$$\mathcal{E} = \begin{pmatrix} \mathbb{1} & G^{-1}-\Pi \\ 0 & \mathbb{1}\end{pmatrix}, \tag{3.48}$$

which yields the metric

$$\mathcal{G} = \begin{pmatrix} 0 & \mathbb{1}_d \\ \mathbb{1}_d & 2G^{-1} \end{pmatrix},$$

as well as the metric connection with symbols ${\Gamma_i}^{jk} = \partial_i \left(G^{-1} - \Pi\right)^{jk}$. If the coanchor and thus the Hamiltonian are taken to be:

$$\tilde{\rho}^{\alpha j} = -\left(G^{-1} - \Pi\right)^{ij} + {\delta_i}^j, \quad \Theta = -\chi_i \left(G^{-1} - \Pi\right)^{ij} p_j + \theta^j p_j,$$

then, by construction, the Hamiltonian has null (deformed) Poisson brackets with itself. The derived structure can be recovered. The fully contracted Dorfman bracket is

$$\begin{aligned}\langle [U,V]', W\rangle' = &\left\langle \left(X^k - \left(G^{-1} + \Pi\right)(\gamma)^k\right) \partial_k Y, \kappa \right\rangle + \left\langle \left(X^k - \left(G^{-1} + \Pi\right)(\gamma)^k\right) \partial_k \sigma, Z \right\rangle \\ &+ \sigma_i \kappa_j \left(X^k - \left(G^{-1} + \Pi\right)(\gamma)^k\right) \partial_k \left(G^{-1} + \Pi\right)^{ij} \\ &+ 2G^{-1}\left(\left(X^k - \left(G^{-1} + \Pi\right)(\gamma)^k\right) \partial_k \sigma, \kappa\right) + (W \to U \to V) - (V \leftrightarrow U, W),\end{aligned}$$

$\langle , \rangle$ on the RHS denoting the canonical pairing.

Moreover, since the Lie-like bracket corresponds to

$$\begin{aligned}[\![U,V]\!] &= \rho(U)V - \rho(V)U - U^\alpha V^\beta [\![\xi_\alpha, \xi_\beta]\!] \\ &= \left(X^k - \left(G^{-1} + \Pi\right)(\gamma)^k\right)\partial_k V - \left(Y^k - \left(G^{-1} + \Pi\right)(\sigma)^k\right)\partial_k U\end{aligned}$$

the derived connection is also non-trivial on forms and in T^*M matches (3.45) (the transpose appears when passing from the coordinate description to the index-free notation).

We can now unveil the remaining objects of generalized differential geometry.

A Riemann curvature tensor for a connection on cotangent space can be defined with no ambiguities from **Riem** (2.25), as the 1-forms defining the direction of derivation are always anchored to tangent space with the help of the anchor map, and therefore the connection $\widetilde{\nabla} \in \Gamma(T^*M) \mapsto \Gamma(TM \otimes T^*M)$ can actually be interpreted as the map $\Gamma(T^*M) \mapsto \Gamma(T^*M \otimes T^*M)$. This is glaring in the term of (3.44), involving the generalized Lie bracket, thanks to an easy manipulation:

$$2G^{-1}\left(\dots, j(\zeta)\right) = 2g\left(G^{-1} + \Pi\right)\left(\dots, \rho(j(\zeta))\right),$$

while the other troublesome term, the first addend in (3.44), can be massaged to

$$\iota_{[\rho(j(\gamma)),\rho(j(\sigma))]}\zeta \equiv 2G^{-1}\left(\frac{G}{2}\left([\rho(j(\gamma)), \rho(j(\sigma))]\right), \zeta\right) = G\left([\rho(j(\gamma)), \rho(j(\sigma))], G^{-1}(\zeta)\right).$$

We are hence ready to calculate the Riemann curvature tensor:

$$\mathbf{Riem}(r(\zeta), r(\gamma), r(\sigma)) =: \mathbf{Riem}(\zeta, \gamma, \sigma) = \left[\widetilde{\nabla}_\zeta, \widetilde{\nabla}_\gamma\right]\sigma - \widetilde{\nabla}_{[\![r(\zeta), r(\gamma)]\!]}\sigma.$$

The coordinate-free expression is quite involved and does not add anything interesting to the comprehension of the matter. Because of that we will present just the Ricci tensor in the local coordinate basis. To do so, first we dub $\tilde{D}_G$ the symmetric part of the connection,

$$\tilde{D}_{G\,(\varkappa)}\varsigma := \varkappa_i\bigg(\left(G^{-1} + \Pi\right)^{im} \partial_m \varsigma + \varsigma_k \left({\Gamma_G}^{ik}{}_j + {\mathcal{Y}^{ik}}_j\right) dx^j\bigg). \tag{3.49}$$

Notice that $\tilde{D}_G G^{-1} = 0$. In this notation, the Ricci curvature is

$$\mathbf{Ric}^{jk} = \mathrm{Ric}_G^{jk} + \frac{1}{2}\tilde{D}_G^i R^{jk}{}_i - \frac{1}{4}R^{jm}{}_l R^{lk}{}_m + \tilde{D}_G^i \mathcal{X}^{jk}{}_i - \frac{1}{2}R^{(j|m}{}_i \mathcal{X}^{i|k)}{}_m - \mathcal{X}^{jm}{}_i \mathcal{X}^{ik}{}_m. \quad (3.50)$$

$\mathrm{Ric}_G \in \Gamma(TM \otimes TM)$ is the partial trace of the Riemann tensor of $\tilde{D}_G$. As in the previous section 3.2 we now wish to resort to the natural non-symmetric metric for T^*M, $\left(G^{-1} - \Pi\right)$, with indices raised and lowered with G^{-1}, to build a Lagrangian out of the Ricci tensor (3.50). In this attempt, one should also observe that an exact multivector $d_\rho \alpha \in \mathfrak{X}^\bullet(M)$ with anchor ρ in (3.38) necessarily has the coordinate expression:

$$\partial_i \wedge \left(G^{-1} + \Pi\right)^{ij} \partial_j \alpha^{k_1 k_2 \ldots k_d} \partial_{k_1} \wedge \partial_{k_2} \wedge \cdots \wedge \partial_{k_d}.$$

This has a direct consequence on the half-density $\sqrt{-\det G^{-1}}w$, where w is some scalar:

$$\begin{aligned} d_\rho \left(\sqrt{-\det G^{-1}}w\right) &= \left(G^{-1} + \Pi\right).\left(\sqrt{-\det G^{-1}}w\right) = \frac{1}{2}\sqrt{-\det G^{-1}}\left[\left(G^{-1}+\Pi\right)^{im}\partial_m G_{ij}\right] w^j \\ &= \sqrt{-\det G^{-1}}\left[\left(\Gamma_G\right)^i{}_{ij} + \mathcal{Y}^i{}_{ij}\right] w^j = \sqrt{-\det G^{-1}}\tilde{D}_{G\,j} w^j. \end{aligned}$$

This shows that a divergence $\tilde{\nabla}_j w^j$ accompanied with the half-density $\sqrt{-\det G^{-1}}$ is exact, $d_\rho\left(\sqrt{-\det G^{-1}}w\right)$, similarly to what happens in Riemannian geometry to the volume form for a curved metric g,

$$\sqrt{-\det g}\,\nabla_i w^i = \partial_i \sqrt{-\det g}\, w^i.$$

For suitable boundary conditions, $d_\rho\left(\sqrt{-\det G^{-1}}\, w\right)$ can then be integrated out. Such observation is extremely helpful as it allows to perform integration by parts, which will be applied to the terms $\frac{1}{2}\tilde{D}_G^i R^{jk}{}_i$ and $\tilde{D}_G^i \mathcal{X}^{jk}{}_i$ in (3.50). Notice also that the symmetric covariant derivative $\tilde{D}_G$ acting on Π_{jk} gives:

$$\tilde{D}_G^l \Pi_{jk} \equiv G_{jn} G_{rk} \left(G^{-1} + \Pi\right)^{lm} \partial_m \Pi^{nr} = G_{jn}\left(\frac{1}{3}R^{lnr} + G^{lm} Q^{nr}{}_m\right) G_{rk}.$$

Hence the Lagrangian $\mathcal{L}[G, \Pi]$ describing the dynamics of a metric G^{-1} and of the (local expression of) the non-geometric flux R, together with interactions between the Q-flux and R or Q itself, is:

$$\begin{aligned} \mathcal{L}[G, \Pi] = \sqrt{\det G^{-1}} \Bigg[\mathrm{R}_G - \frac{1}{12}R^2 + \frac{3}{4}\left(Q^{jn}{}_m \left(G^{pm} Q^{lk}{}_p - 2Q^{ml}{}_p G^{pk}\right)\right) G_{jl} G_{nk} \\ - \left(\frac{1}{6}Q^{jn}{}_s + \frac{2}{3}G_{rs} G^{jp} Q^{nr}{}_p\right) R^{slk} G_{jl} G_{nk}\Bigg]. \end{aligned} \quad (3.51)$$

Let us stress once more that the trace procedure, with the non-symmetric metric, turned out to be fundamental in keeping the full **Ric** tensor and in reconstructing the precise numerical factors. The Lagrangian (3.51) can be considered "dual" to $\mathcal{L}[g, B] = \mathrm{R} - \dfrac{1}{12}H^2$ studied previously, given the "duality" transformations

$$(g, B) \mapsto \left(G^{-1}, \Pi\right), \quad \left(\rho(\chi_i) = \partial_i\right) \mapsto \left(\rho(\theta^i) = \left(G^{-1} + \Pi\right)^{ij} \partial_j\right).$$

It is quite natural to conjecture that the minimally coupled Lagrangian (3.51) could be invariant under the gauge symmetries $\Pi \mapsto \Pi + d_\rho \mathcal{P}$, where $\mathcal{P} \in \mathfrak{X}(M)$. We proved already that this must be the case around equation (2.53). The Lie-like bracket, in any reference chart, has also the canonical expression (2.49), with $\check{\mathrm{Q}}$ in (2.48) for the degenerate Poisson structure and $\tilde{\rho}$ in (3.38). The gauge transformations of the $O(d,d)$ generators leave $[\![,]\!]$ unchanged as well. It can be proved that the degree 1 functions in the cohomology of Π with Q are also in the cohomology with differential $\check{\mathrm{Q}}$. We thus are ready to check the expression of the gauge symmetry of Π, as the vielbeins contain the $O(d,d)$-generators, and $[,]$ and $[\![,]\!]$ have a canonical formulation. So how do these shifts of Π by exact bivectors look like in the deformed Poisson algebra? First of all one could think of the gauge symmetries of the $O(d,d)$ generators altogether: from ϱ (of (2.50), with α in (2.51)) of degree 1 and hence linear in χ and θ, one should compute

$$\{a_i\theta^i + \tilde{a}^i\chi_i, \Theta\}'$$

with the rather obvious request that the transformations do depend just on the degree-1 coordinates (as the vielbein does not mix the $\{\xi\}$ and $\{p\}$ sets of coordinates). The general set of equations is the following:

$$\begin{cases} \varrho^\alpha(x)\ \left(\mathcal{E}^T\right)_\alpha{}^j p_j \equiv 0, \quad \forall p_j \\ \left(\mathcal{E}^{-1}\right)^{\alpha j}\ \partial_j\varrho^\beta(x) + \varrho^\gamma(x)\ \left(\mathcal{E}^{-1}\right)^\beta{}_\delta \left(\mathcal{E}^{-1}\right)^{\alpha j}\ \partial_j\mathcal{E}_\gamma{}^\delta = \{\varrho,\Theta\}. \end{cases}$$

In our specialized case, the first condition sets $\tilde{a}^i$ to be $\tilde{a}^i = -a_j\left(G^{-1}+\Pi\right)^{ji}$ (or vice versa $a_i = -\tilde{a}^j(x)\,(g+B)_{ji}$). For the second it should first of all be noticed that since the vielbein shuffles the degree-1 coordinates (3.32), ϱ is going to mix up the gauge transformations of the whole set canonical $O(d,d)$ transformations of the underlying graded Poisson algebra. Hence one should not expect familiar formulas to appear.

Thus the degree-1 function

$$\varrho = a^k(x)\left(\chi_k - \left(G^{-1}+\Pi\right)^{-1}_{km}\theta^m\right), \tag{3.52}$$

yields the following degree-2 function from the Poisson bracket with the Hamiltonian:

$$\begin{aligned}\{\varrho,\Theta\}' = \left(\chi_i\left(G^{-1}+\Pi\right)^{ij}+\theta^j\right)\Big[&2\partial_j a^k(x) g_{kl}\left(\theta^l - \left(G^{-1}-\Pi\right)^{lm}\chi_m\right) \\ &+ a^k(x)\left(2\partial_j g_{ik}\theta^i - 2\left(G^{-1}+\Pi\right)^{ni}\partial_j g_{ik}\chi_n\right)\Big].\end{aligned}$$

The gauge transformations of $\beta^{ij}\chi_i\chi_j$, $\beta\mapsto\beta+\{\varrho,\Theta\}'$ are untied when extracting all the bivectors:

$$-\left.\mathrm{Q}\varrho\right|_{\chi\chi-\text{comp.}} = -2\chi_i\ \left(G^{-1}+\Pi\right)^{ij}\partial_j\left(a^k(x)g_{kl}\right)\left(G^{-1}-\Pi\right)^{ln}\chi_n. \tag{3.53}$$

This holds for any other bivector, such as Π in the derived connection $\widetilde{\nabla}$ (aside from those due to the anchor map, since it descends from the homological vector field itself). Being $\check{\mathrm{Q}} = p_i\dfrac{\partial}{\partial\chi_i} + \left(G^{-1}+\Pi\right)^{ij}p_j\dfrac{\partial}{\partial\theta^i}$, $\check{\mathrm{Q}}$ acting on ϱ (3.52) is identically zero, hence the symmetry is a symmetry for the generalized Lie bracket too. We argue then that the derived objects, i.e. the connection coefficients $(\mathcal{X}+R)^{kij} = \left(G^{-1}+\Pi\right)^{[k}.\Pi^{ij]}$, are hence invariant under $\Pi\mapsto\Pi+d_\rho\varrho$, the transformation being applied just when the bivector does not come from the anchor map.

It should be remarked that with the equivalent deformation (3.48) the computation is more neat. Since in that circumstances old and new χ coordinates coincide, the gauge symmetries of Π do not require much effort to be worked out. Invariance of the Lagrangian

(3.51) under shifts of Π by exact bivectors was inferred in our work [61]. Formula (2.7) there, if $\varrho_l(x) \equiv 2g_{lk}a^k(x)$, is equivalent to eq. (3.53).

A similar analysis in the context of the T^*M Lie algebroid, where the action consisted of a pure Einstein-Hilbert term and the square of the pullback of $H \in \Omega^3(M)$, can be found in section 4.2 of [38].

3.3.1 Generalized parallelizability

Throughout the whole thesis as soon as the graded manifold of reference was designed, a local open set and a chart were always considered. In the course of the exposition we never focused on global aspects of the (ungraded) principal or vector bundles we studied. In this section of the appendix we wish to comment on the global picture perhaps clarifying some less transparent point in the presentation. When in section 3.2 and section 3.3 two peculiar examples of frames were chosen, the connection naturally turned out to be a Weitzenböck connection for $T^*[1]M \oplus T[1]M$. This type of connection is tailored for spaces with a global basis of (generalized) vector fields, since it stems from the condition that the covariant derivative of a global vector along another global vector is zero. However we never faced the global aspects of our graded manifold with Weitzenböck connection: The legitimate question, whether the ungraded principal or vector bundles under investigation were actually trivial, or *generalized parallelizable*, deserves a few more words.

First of all, generalized parallelizability is a weaker condition than usual parallelizability. There are some base manifolds whose tangent bundle is definitely not globally a product manifold, but their generalized tangent bundle is. As Huckleberry and Arial [76] pointed to us, for the sphere embedded in $\mathbb{R}^3$ the trivial bundle as a real vector bundle of rank 4 is the direct sum $TS^2 \oplus T^*S^2$. On the other hand, among the spheres, S^1 and S^3, being the group manifolds of some Lie groups, are known to be parallelizable in the standard sense. S^7 is also parallelizable, although it is not associated to a Lie group, but it is instead the coset space $\mathrm{Spin}(7)/G_2$.

In any case, in the graded symplectic structure the graded 2-forms are always exact (see (2.29)). The construction of non-canonical symplectic forms, via vielbein or through other deformations, is just local. We simply did not look at the coordinate-free description, though there is at least one successful attempt in this sense, by Roytenberg [77]. Anyway this does not mean that an "inverse deformation" that unties the construction brings us back to the original situation we started with (of a flat manifold and a trivial connection). There are other data, such as the Hamiltonian, the splitting, the Lie-like bracket, the induced metric and the tensor with which the trace is taken in the action, that will undergo further changes. B and Π cannot be gauged away already in the local setting.

The issue of gauging B and Π away was first faced in the context of Generalized Geometry with non-geometric fluxes in the paper [30]. To deal with global charges, the authors considered that in the algebra of the CA bracket just structure *constants*, and not functions, appears. Therefore the Courant algebra is a Lie algebra $\mathfrak{h}$, which thanks to the axioms forms actually a subalgebra $\mathfrak{h} \subset \mathfrak{o}(d,d)$, and because of the homomorphism ρ with the Lie bracket of vector fields, $\mathfrak{h}$ can be realized in terms of vector fields, of which some may vanish. The set of generalized vectors that when anchored to TM are null, constitutes a Lie subalgebra $\mathfrak{l}$. The base manifold must hence be the coset space

$$M = H/K, \quad \mathfrak{h} = \mathrm{Lie}(H), \mathfrak{l} = \mathrm{Lie}(K).$$

This confirms the statement that generalized parallelizable manifolds are more general than those parallelizable in the standard sense, shedding a different light on the topic just discussed. Furthermore, when the fluxes are turned on, the authors claimed that it was not possible to arrange them to be all non-zero. In the case of $\mathbb{T}^3$, for example the R flux cannot (it must be zero).

This hence legitimates from another perspective the emergence, in our personal work described in section 3.3, of Q and R combined with a null H (3.46). This is also stressed around (4.1) in the Conclusions.

3.3.2 Deformation with curvature

On the graded symplectic $T^*[2]T[1]M$ there is another class of deformations, more general than the one that yields the non-canonical symplectic form (3.1). Another non-canonical 2-form, invertible and closed, can thus be studied. With respect to the local Darboux chart $\{x, \xi, p\}$, it is given by:

$$\omega' = dx^i \wedge dp_i + d\xi_\alpha G^{\alpha\beta}(x) \wedge d\xi_\beta + dx^i\, \Gamma_i{}^{\alpha\beta}(x)\xi_\beta \wedge d\xi_\alpha + dx^k\, \xi_\alpha \tilde{R}^{\alpha\beta}{}_{kj}(x)\xi_\beta \wedge dx^j.$$

$G^{\alpha\beta}(x) \in \Gamma\left(\bigotimes^2 T^*[1]M \oplus T[1]M\right)$ is symmetric and bilinear, hence it is a metric on the space of ξ-coordinates. It can be appealing to see what closure of ω', $d\omega' = 0$, implies:

$$\begin{aligned}
&\Gamma_i{}^{\alpha\beta}(x) + \Gamma_i{}^{\beta\alpha}(x) = \partial_i G^{\alpha\beta}(x),\\
&\tilde{R}^{\alpha\beta}{}_{ij}(x) = \partial_{[i}\Gamma_{j]}{}^{\alpha\beta}(x),\\
&\partial_{[k|}\tilde{R}^{\alpha\beta}{}_{|ij]}(x) = 0.
\end{aligned} \tag{3.54}$$

The first relation tells us that the metric should be covariantly constant, as in the previous deformation. In fact the Γ's are connection symbols for an affine connection, as Leibniz rule and Jacobi identity for the corresponding Poisson brackets can better determine. The second and the third relation remind of the definition of a curvature tensor and its Bianchi identity, however they do not describe real tensor quantities. The Poisson brackets will shed some more light on (3.54) and accidentally cure the failure of $\tilde{R}$ to be a tensor. The brackets are a mere declination of those in (3.5), where on the left column these are given in their corresponding coordinate notation, while on the right as acting on the functions $\upsilon(x), \varsigma(x), U(x), V(x)$ and $f(x)$ as in (3.5):

$$\begin{aligned}
&\{p_i, x^j\}' = \delta_i{}^j, && \{\upsilon(x), f(x)\}' = \upsilon.f,\\
&\{\xi_\alpha, \xi_\beta\}' = G_{\alpha\beta}(x), && \{U(x), V(x)\}' = G(U, V),\\
&\{p_i, \xi_\alpha\}' = \Gamma_{i\alpha}{}^\beta(x)\xi_\beta, && \{\upsilon(x), U(x)\}' = \nabla_\upsilon U,\\
&\{p_i, p_j\}' = \xi_\alpha R^{\alpha\beta}{}_{ij}(x)\xi_\beta, && \{\upsilon(x), \varsigma(x)\}' = [\upsilon, \varsigma]_{\text{Lie}} + R(\upsilon, \varsigma).
\end{aligned} \tag{3.55}$$

The Greek indices are raised and lowered with G and $R^{\alpha\beta}{}_{ij}(x) \in \Gamma\big(\Lambda^2(T^*[1]M \oplus T[1]M) \bigotimes^2 T^*[2]M\big)$ denotes

$$R^{\alpha\beta}{}_{ij}(x) = \tilde{R}^{\alpha\beta}{}_{ij}(x) - \Gamma_{[i|}{}^{\alpha\gamma} G_{\gamma\delta}(x)\Gamma_{|j]}{}^{\delta\beta} = \partial_{[i}\Gamma_{j]}{}^{\alpha\beta} - \Gamma_{[i|}{}^{\alpha\gamma} G_{\gamma\delta}(x)\Gamma_{|j]}{}^{\delta\beta}.$$

Clearly the true curvature tensor is R. The symplectic model is perfectly consistent with our assumptions. Now a compatible differentiable structure shall be furnished. This is specified as usual by a Hamiltonian function Θ that solve the master equation $\{\Theta, \Theta\}' = 0$ with respect to the non-canonical symplectic structure. The set of conditions is now more challenging to solve than in the previous situation, where the diffeomorphism of Moser lemma was known:

$$\begin{aligned}
&\tilde{\rho}^{\alpha i} G_{\alpha\beta}\tilde{\rho}^{\beta j} = 0,\\
&2\tilde{\rho}^{\gamma i}\nabla_i \tilde{\rho}^{\delta j} + \tilde{\rho}^{\mu i} G_{\mu\nu} C^{\nu\gamma\delta} = 0,\\
&\frac{1}{3}\tilde{\rho}^{\alpha i}\nabla_i C^{\beta\gamma\delta} + \frac{1}{4} C^{\alpha\beta\sigma} G_{\sigma\mu} C^{\mu\gamma\delta} + R^{\gamma\delta}{}_{ij}\tilde{\rho}^{\alpha i}\tilde{\rho}^{\beta j} = 0.
\end{aligned} \tag{3.56}$$

The non-contracted Greek indices are meant to be antisymmetrized, as they belong to either one of the two d-dimensional sets of coordinates which mutually anticommute. The covariant derivative hits just the Greek indices too, e.g. $\nabla_j \tilde{\rho}^{\beta i}(x) = \Gamma_{j\delta}{}^{\beta}\tilde{\rho}^{\delta i}(x) + \partial_j \tilde{\rho}^{\beta i}(x)$. The last condition is certainly the most intriguing, however let us go in order: the first row, if G is locally diagonalized by some vielbeins $\mathcal{F}$, claims that the CA is exact, i.e. that $\mathcal{F}^* \circ \rho^*$ should be in the kernel of $\rho \circ \mathcal{F}$. The second row relates the metric and its connection to C and $\tilde{\rho}$, but above all with the latter one can argue that a covariant field strength for C, taken with $\nabla + C\lrcorner$[2], is $R^{\gamma\delta}{}_{ij}\tilde{\rho}^{\alpha i}\tilde{\rho}^{\beta j}$. It could be as well intended as a generalization of Bianchi identities, where the de Rham differential is replaced by a covariant derivative and the inner contraction with C. An obstruction in the cohomology is brought up, due to the non-zero curvature R of the connection ∇. Thus the metric connection coefficients Γ shall depend on the C tensors (3.9). The interplay between these objects, in the non-canonical coordinate chart, is anyway quite subtle and deserves further attention; our insight on the matter is that deformed Poisson brackets emphasize "exact" 3-generalized tensors, while C is a twist by generalized tensors of the same degree, but not necessarily "exact". When the mutual dependence of Γ and C is settled, exactness, in this case for the short sequence for the CA, still plays a role, in constraining $\tilde{\rho}$ instead.

With this type of deformation it is therefore much more subtle to deal with the fluxes, and construct consistent Hamiltonians for the deformed symplectic structure. This difficulty can also be seen from the fact that the diffeomorphism that produces the deformation is generally not known explicitly. Just the differential equation with the generating vector field can be worked out with a reasonable effort. Due to this, most of the times we will just consider "Hamiltonian-like" functions comprising the first term of (2.41) (which is a sort of Dirac operator), not the C tensors. The choice of the anchor map will be arbitrary and simple anchors are favored. Although the functions will not fulfill the master equation, some appealing results can be nevertheless found. This is the content of the next subsections.

3.3.2.1 Metric connection

In this part we will examine the instance of non-canonical QP $T^*[2]T[1]M$, where the curvature is present (3.55), and study a few concrete examples. Relations with GR and Einstein-Cartan theory will be on the way. Setting the metric on the $(T^*[1]M \oplus T[1]M)$-fiber equal to the $O(d,d)$-invariant pairing η will lead to a symmetric connection.

Let us go in order: consider the following Poisson brackets

$$\begin{aligned}
&\{p_i, x^j\}' = \delta_i{}^j, && \{\upsilon(x), f(x)\}' = \upsilon.f,\\
&\{\chi_i, \theta^j\}' = \delta_i{}^j, && \{U(x), V(x)\}' = \eta(U,V),\\
&\{p_i, \chi_k\}' = \Gamma_{ik}{}^l(x)\chi_l, && \{\upsilon(x), U(x)\}' = \nabla_\upsilon U,\\
&\{p_i, \theta^k\}' = -\eta_{ml}\eta^{kj}\Gamma_{ij}{}^m(x)\theta^l, && \\
&\{p_i, p_j\}' = \chi_l\theta^k R^l{}_{kij}(x), && \{\upsilon(x), \varsigma(x)\}' = [\upsilon,\varsigma]_{\mathrm{Lie}} + R(\upsilon,\varsigma)^l{}_k\chi_l\theta^k.
\end{aligned} \tag{3.57}$$

The minus sign in front of the connection symbol on the θ coordinate ensures that the double bracket between p, χ, θ fulfills Jacobi identity. The connection is metric for η. The same consistency request on the p, p, χ (or θ) triplet tells that the only admissible curvature tensor must contract with a vector field and a 1-form. Other options are available upon raising or lowering the indices via the invariant pairing. Furthermore its well-known structure is confirmed thanks to the Jacobi identity:

$$R^l{}_{kij}(x)\chi_l = \partial_{[i}\Gamma_{j]k}{}^l(x) + \Gamma_{[i|k}{}^m(x)\Gamma_{|j]m}{}^l(x).$$

[2] $\lrcorner$ stands for the inner product with C, taken with G.

Finding the vector field X_t which generates the flow of the diffeomorphism φ_t, $\varphi_t^*\omega_t = \omega_0$, is a particularly simple task. In fact it is almost immediate to get the exact 2-form dA_t, $\omega_t - \omega_0 = dA_t$:

$$\omega_0 = dx^i \wedge dp_i + \frac{1}{2}\delta^i{}_j \, d\chi_i \wedge d\theta^j, \quad \omega_t = \omega_0 + d\left[(\Gamma_t)_{il}{}^k \theta^l \chi_k \, dx^i\right]. \tag{3.58}$$

X_t is presented in (4.5).

In the Hamiltonian function Θ (2.41) the anchor can be given by the projector, and for the moment we set the fluxes C to zero, i.e.

$$\Theta = \theta^i p_i. \tag{3.59}$$

Under these assumptions, the master equation is solved when

$$\Gamma_{ij}{}^k = \Gamma_{ji}{}^k, \quad R^l{}_{[kij]} = 0; \tag{3.60}$$

interestingly, these symmetry conditions define locally nothing but a torsion-free connection, whence by the Levi-Civita theorem $\Gamma_{ij}{}^k$ must be the Christoffel symbols, and the Riemann tensor respects the algebraic Bianchi identity. General Relativity is hence naturally recovered, and the derived bracket with $\mathcal{O}_1(T^*[2]T[1]M)$ functions $U(x), V(x)$ is the covariant Dorfman bracket with Levi-Civita connection $\nabla_U \equiv \nabla_{\rho(U)} = \nabla_X$:

$$[U,V]'_{\mathrm{D}} = \nabla_U V - \nabla_V U + \langle \nabla U, V\rangle, \quad \langle\langle \nabla U, V\rangle, W\rangle = \langle \nabla_W U, V\rangle. \tag{3.61}$$

With this instance of a deformed Dorfman bracket and the standard generalized Lie bracket a secondary connection $\widetilde{\nabla}$ can be derived from proposition 2.3.2. Recall that on the three generalized vector fields U, V, W, $\widetilde{\nabla}$ is given by this formula:

$$\langle \widetilde{\nabla}_W U, V\rangle = \langle [U,V]'_{\mathrm{D}} - [\![U,V]\!], W\rangle.$$

In the Darboux chart which defines with respect to what the Poisson brackets are deformed, the Lie-like bracket is symply

$$[\![U,V]\!] = \rho(U)V - \rho(V)U = [X,Y]_{\mathrm{Lie}} + X.\sigma - Y.\gamma.$$

It does not contribute to further components for the whole $\widetilde{\nabla}$. However the connection $\widetilde{\nabla}$ itself is extremely boring: all its connection coefficients vanish. To prove this statement, begin with the extraction of $\widetilde{\nabla} : \Gamma(E) \mapsto \Gamma(E^* \otimes E)$, which requires inversion with the $O(d,d)$-pairing. Using then the decomposition $U = X + \gamma$, $W = Z + \kappa$ and the fact that the connection ∇ preserves the subbundles TM and T^*M respectively, a useful formula for $\widetilde{\nabla}$ can be untied:

$$\widetilde{\nabla}_W U = \begin{pmatrix} \nabla_Z X + \langle \nabla_X dx^j, Z\rangle \, \partial_j \\ \nabla_Z \gamma - \langle \nabla_{\partial_j}\gamma, Z\rangle \, dx^j \\ \langle \nabla_X \partial_j - \nabla_{\partial_j} X, \kappa\rangle \, dx^j \end{pmatrix} \equiv \begin{pmatrix} Z^k \partial_k X + Z^k X^i \, \tilde{\Gamma}_{ki}{}^j \, \partial_j \\ Z^k \partial_k \gamma + Z^k \gamma_i \, \tilde{\Gamma}_k{}^i{}_j \, dx^j \\ \kappa_k X^i \, \tilde{\Gamma}^k{}_{ij} \, dx^j \end{pmatrix}, \tag{3.62}$$

where the connection symbols $\tilde{\Gamma}$ for the derived connection can be easily worked out from the first equality of (3.62): $\tilde{\Gamma}_{ki}{}^j := \Gamma_{ki}{}^j + \Gamma_i{}^j{}_k$, $\tilde{\Gamma}_k{}^i{}_j := \Gamma_k{}^i{}_j - \Gamma_j{}^i{}_k$ and $\tilde{\Gamma}^k{}_{ij} := \Gamma_{ij}{}^k - \Gamma_{ji}{}^k$. A closer look at them permits to conclude that, when the connection ∇ is Levi-Civita, $\widetilde{\nabla}$ is flat. In all the rows, using the torsionless condition $\Gamma_{ij}{}^k = \Gamma_{ji}{}^k$ and the metricity condition w.r.t. the pairing $\Gamma_i{}^j{}_k = -\Gamma_{ik}{}^j$, the connection coefficients $\tilde{\Gamma}$ turn out to be null. The non-trivial mixing of tangent and cotangent space has led to the trivial connection.

Even if the derived connection is not particularly intriguing, the Poisson structure (3.57) with Levi-Civita connection for some metric $g(x) : TM \vee TM \mapsto C^\infty(M)$ is relevant on its own right. It delivers the graded Poisson theory for the $TM \oplus T^*M$ bundle with connection of [78]. There, the authors deployed as connection the Levi-Civita connection acting on vector fields and forms separately, $\nabla^{\text{L.C.}}_{(X,\gamma)}(Y,\sigma) = \left(\nabla^{\text{L.C.}}_X Y, \nabla^{\text{L.C.}}_X \sigma\right)$. Through the isomorphism between (χ, θ) and the coordinates (∂, dx) for $TM \oplus T^*M$, this is what we are also getting in (3.57). The H-field is then kept by using the general twisted Dorfman bracket; in our framework using the H-twisted Dorfman bracket implies that (2.24) is solved for the Lie-like bracket, and the corresponding generalized torsion $T(U,V) = \nabla^{\text{L.C.}}_X V - \nabla^{\text{L.C.}}_Y U - [\![U,V]\!]$ is non-zero due to H. Hence if as in [78] one wants to get one of the torsion-free generalized connections, compatible with the pairing (and also with the metric g on vector fields), a mixed symmetry tensor $K \in \Omega^1(E) \otimes \Omega^2(E)$ must be worked out. The result is in formula (112) of the cited reference. The geometric action is the effective sting action of type II for the bosonic fields (g, H).

Things get more exciting in the derived brackets if the connection ∇ in the Poisson brackets is allowed to bear torsion, meant in the usual sense, i.e. upon identification of the degree-1 coordinates with the local coordinate basis for tangent space. Hence ${\Gamma_{ij}}^k - {\Gamma_{ji}}^k \neq 0$. The derived Dorfman bracket is formally identical to (3.61), provided that now ∇ is not Levi-Civita, but just a metric connection. Concerning the differentiable structure, clearly now Θ in (3.59) does not make a good Hamiltonian, as it cannot solve the master equation anymore. In fact the first condition in (3.60) is dismissed when the connection has non-zero torsion. There are however two paths worth to be pursued in the present situation: find the appropriate Hamiltonian so that a consistent CA is retrieved, or study the non-topological model described by the differential operator $\mathrm{Q}^2 \neq 0$.

Let us start with the latter: if our inputs fail to satisfy the master equation, then the CA bracket does not fulfill the usual restricted version of Jacobi identity. Nevertheless, proposition (2.3.2) can still be applied, as that property is not required in order to get a consistent affine connection. We are left with

$$\{\Theta, \Theta\}' = -2\theta^k {\Gamma_{km}}^j(x)\theta^m p_j + \theta^k \theta^j {R^l}_{ikj}(x)\chi_l \theta^k.$$

This opens up new possibilities: already the Poisson model is not just topological anymore, but it is possible to give a dynamics to a point particle moving in the graded space, in analogy with relativistic mechanics. The velocity must be a degree 2 object due to the Poisson action of the Hamiltonian. The following $\dfrac{dx^j}{d\tau}$ does the job:

$$\begin{aligned}\dot{\gamma}^j \equiv \frac{dx^j}{d\tau} &= \frac{1}{2}\{\Theta, \{\Theta, x^j\}'\}' = \frac{1}{4}\{\{\Theta, \Theta\}', x^j\}' \\ &= -\frac{1}{2}\theta^k \theta^l {\Gamma_{kl}}^j. \end{aligned} \tag{3.63}$$

Then the acceleration is

$$\begin{aligned}\ddot{\gamma} \equiv \frac{d^2 x^j}{d\tau^2} &= -\frac{1}{2}\dot{\gamma}^i \left(\theta^r \theta^k {\Gamma_{ir}}^l {\Gamma_{[kl]}}^j + \theta^k \theta^l \partial_i {\Gamma_{kl}}^j\right) - \frac{1}{4}{\Gamma_{[cl]}}^j {R^c}_{ikn}\theta^l \theta^i \theta^k \theta^n \\ &= \dot{\gamma}^i \nabla_i \dot{\gamma}^j + \frac{1}{2}\theta^l {\mathrm{T}_{cl}}^j \left(\theta^k \nabla_k \dot{\gamma}^c - \theta^n \nabla_n \dot{\gamma}^c\right), \end{aligned} \tag{3.64}$$

where the torsion, whose definition coincide with the standard differential geometry one, is labeled by $\mathrm{T} \in \Gamma\left(\Lambda^2 T^*M \otimes TM\right)$, ${\mathrm{T}_{cl}}^j := 2{\Gamma_{[cl]}}^j$. We see that this is the geodesic equation. It was obtained from purely algebraic manipulations. This concludes the remarks on the connection ∇ for GR with torsion.

The secondary connection $\widetilde{\nabla}$ for the CA, arising from the difference between the deformed derived bracket $\{\{\mathcal{O}_1, \Theta\}', \mathcal{O}_1\}' = [,]'_{\mathrm{D}}$ (that however does not respects the Jacobi identity), and $[\![,]\!]$, is now non-trivial:

$$\langle \widetilde{\nabla}_W U, V\rangle = \mathrm{T}(X, Y, \kappa) + \mathrm{T}(Y, Z, \gamma) + \mathrm{T}(Z, X, \sigma) + \langle Z.U, V\rangle.$$

As expected, it depends only on the torsion (we previously saw (3.62) that the result from the pure Levi-Civita is trivial). Notice that the connection coefficients $\tilde{\mathit{\Gamma}}_{\alpha\beta}{}^{\gamma}$, for the $2d$-indices α, β, γ being respectively assigned the d-dimensional labels k, i, j, are hence

$$\tilde{\mathit{\Gamma}}_{ki}{}^{j} = \mathrm{T}_{ki}{}^{j}, \quad \tilde{\mathit{\Gamma}}_{k}{}^{i}{}_{j} = \mathrm{T}_{jk}{}^{i}, \quad \tilde{\mathit{\Gamma}}^{k}{}_{ij} = \mathrm{T}_{ij}{}^{k}.$$

There are three corresponding Riemann tensors available: $\mathrm{Riem}_{(1)} : \Gamma(\Lambda^2 T^*M \otimes TM) \mapsto TM$, $\mathrm{Riem}_{(2)} : \Gamma(\Lambda^2 T^*M \otimes T^*M) \mapsto T^*M$, $\mathrm{Riem}_{(3)} : \Gamma(TM \otimes T^*M \otimes TM) \mapsto T^*M$.

$$\begin{aligned}
&\left(\mathrm{Riem}_{(1)}\right)^{l}{}_{mki} = \partial_{[k}\mathrm{T}_{i]m}{}^{l} + \mathrm{T}_{[k|r}{}^{l}\mathrm{T}_{|i]m}{}^{r}\\
&\left(\mathrm{Riem}_{(2)}\right)_{l}{}^{m}{}_{ki} = \partial_{[k}\mathrm{T}_{l|i]}{}^{m} + \mathrm{T}_{l[k|}{}^{r}\mathrm{T}_{r|i]}{}^{m}\\
&\left(\mathrm{Riem}_{(3)}\right)_{lmk}{}^{i} = \partial_k\mathrm{T}_{lm}{}^{i} + \mathrm{T}_{km}{}^{r}\mathrm{T}_{lr}{}^{i} = -\left(\mathrm{Riem}_{(3)}\right)_{lm}{}^{k}{}_{i}
\end{aligned}$$

Now back to the first possibility, namely getting the correct Hamiltonian function when dealing with a torsionful connection. This issue is much more delicate. A more articulated anchor map than the projector has to be sought. Such map needs to satisfy, roughly speaking, the following three conditions: 1) that the coanchor must be in the kernel of the anchor, 2) $\tilde{\rho}^{\alpha i}\left(\Gamma_{i\beta}{}^{\gamma}\tilde{\rho}^{\beta j} + \partial_i\tilde{\rho}^{\gamma j}\right) = 0$, and 3) $\tilde{\rho}^{\alpha i}\tilde{\rho}^{\beta j}R^{l}{}_{kij} = 0$. These are the $C = 0$, $G = \eta$ version of (3.56). Suppose that an anchor map passing these criteria is given; then, in the derived bracket, the Dorfman bracket is again the covariant version (3.61), where the covariant derivative is now just metric, and the anchor definitely does not just simply project to TM. We might as well write the bracket down, in the presence of the C-tensors:

$$\begin{aligned}
\{\{U(x), \Theta\}', V(x)\}' =& \nabla_{\rho(U)}V - \nabla_{\rho(V)}U + \langle\nabla_{\rho(\cdot)}U, V\rangle\\
&+ \frac{1}{3!}U^{\alpha}V^{\varepsilon}\eta_{\alpha\beta}\eta_{\delta\varepsilon}\left(C^{\beta\gamma\delta} - C^{\beta\delta\gamma} - C^{\gamma\beta\delta} + C^{\delta\beta\gamma} + C^{\gamma\delta\beta} - C^{\delta\gamma\beta}\right)\xi_{\gamma}\\
=:& [U, V]'_{\mathrm{D}}
\end{aligned}$$

Although this CA bracket and the derived geometry of $TM \oplus T^*M$ are surely interesting on their own, they do not add anything new to the analysis of gravitational model. This non-canonical $T^*[2]T[1]M$ is related to a vector bundle which is still too close to the standard tangent bundle and Riemannian geometry. What is fascinating, instead, is that already the Poisson brackets (3.57) could undergo some canonical quantization on the lattice, but we shall not focus on this aspect.

Another peculiar possibility concerns the study of metric-affine theories of gravity, within this very same framework. In fact it is sufficient to relax the graded Jacobi identity for the Poisson algebra to produce a non-metric connection. Anyway this will not be investigated here.

In the forthcoming part a deformation of the Poisson algebra ultimately describing the spin connection will be addressed. Since its QP-manifold of reference changes slightly, we first approach such manifold and describe it with more details.

3.4 Non-canonical NQ-$T^*[2]M \oplus T[1]M$

A very elegant, alternative way to formulate classical gravity relies on a Poisson structure on the graded manifold $T^*[2]M \oplus T[1]M$. This situation covers the Palatini formulation of GR, Einstein-Cartan theory and Poincaré gauge theory (the latter just in the instance where some conditions are relaxed). Compared to the other cases investigated so far, here the local description is a bit different. The set of coordinates consists of (x^i, θ^a, p_i), with $|\theta| = 1$ anticommuting coordinates for the fiber of $T[1]M$ and $|p| = 2$ conjugate variables to x. The most general symplectic structure was already illustrated in the proposition 2.5.4 for a generic $E[1]$ bundle with a metric and a compatible connection. In the canonical case (2.32) the metric is flat and the metric connection is trivial too.

3.4.1 Spin connection

Rather than just assuming the existence of a connection on $T[1]M$ as in the realm of proposition 2.5.4 proven in [58], let us find a 2-form, closed and non-degenerate which, when pulled back by means of a diffeomorphism of the base manifold, corresponds to the canonical symplectic form. Some of the facts that will be outlined are present in [63], some other are due to personal research. In the canonical Poisson brackets,

$$\{x^i, p_j\} = \delta^i_j, \quad \{\theta^a, \theta^b\} = \eta^{ab}, \quad \{\theta^a, x^i\} = 0 = \{\theta^a, p_i\} = \{p_i, p_j\}, \tag{3.65}$$

the constant metric on the fibers, η, shall be given Lorentzian signature and shall be linked to a non-flat metric, by interpreting the degree-1 coordinates as vielbeins. Thus the anholonomic $\{\theta^a\}$ are hence a basis of $O(1, n-1)$-frames, while θ^i with spacetime indices $i, j, \dots$ ranging from $1, \dots \dim M$, is related to θ^a thanks to a set of matrices $e^i{}_a(x)$ implementing the Lorentz transformations (rotations and boosts):

$$\theta^i = e^i{}_a(x)\theta^a. \tag{3.66}$$

Then the local relation between the curved spacetime metric $g_{ij}(x)$ and the Lorentzian η is the obvious $g_{ij}(x)\theta^i\theta^j = \eta_{ab}\theta^a\theta^b$. We observe that the spin connection ω, a connection 1-form (endomorphism-valued), is naturally retained in such canonical conditions. Actually just the spin connection with only spacetime indices will pop up; the transformation properties to get the one with Lorentz indices are anyway well-known. Let us show how this happens: recall that the torsion $\mathrm{T}(e)$ is the field strength of the frame:

$$\mathrm{T}(e) = \nabla e = de + \omega \wedge e.$$

The Levi-Civita spin connection $\overset{\circ}{\omega}$ is given by

$$\overset{\circ}{\omega}{}_{i\,b}^{\,a} := \left(e^{-1}\right)_b{}^j \partial_i e^a{}_j - e^a{}_k \left(\Gamma^{\mathrm{L.C.}}\right)_{ij}{}^k \left(e^{-1}\right)^j{}_b.$$

$\mathrm{T}(e) = 0$ for the Levi-Civita spin connection. The curvature tensor of the connection, a tensor-valued 2-form, depends on the vielbein e and the spin connection ω, and whenever the connection is the Levi-Civita one, $\Omega(e, \omega) = d\overset{\circ}{\omega} + \overset{\circ}{\omega} \wedge \overset{\circ}{\omega}$ is equivalent to the Riemann tensor $\mathrm{Riem}\left[g, \nabla^{\mathrm{L.C.}}\right]$. Thus using (3.66), the symplectic form for (3.65) clearly employs the Weitzenböck spin-connection ($\mathrm{T}(e) \neq 0, \Omega(e, \omega) = 0$)

$$\begin{aligned}
\omega &= dp_i \wedge dx^i + \frac{1}{2} d\theta^a \eta_{ab} \wedge d\theta^b \qquad (3.67)\\
&= dp_i \wedge dx^i + \frac{1}{2} dx^k \theta^i \omega_k{}^l{}_i g_{lj}(x) \wedge d\theta^j + \frac{1}{2} d\theta^i g_{ij}(x) \wedge d\theta^j - \frac{1}{2} dx^k \omega_k{}^m_i \theta^i g_{mn}(x) \theta^j \omega_l{}^n_j \wedge dx^l,
\end{aligned}$$

with $\omega = e^{-1}de$. Next we found that the canonical case seems already a good setting for describing gravity in the Palatini formalism. In fact Moser lemma allows to consider, in place of the Poisson brackets (3.65), a graded Poisson algebra where the $p - \theta$ bracket is deformed with the Levi-Civita connection, in spin variables, i.e. $e^a{}_k \left(\Gamma^{\mathrm{L.C.}}\right)^k{}_{ji} e^j{}_b := \Gamma_i{}^a{}_b$:

$$\begin{array}{ll} \{x^i, p_j\}' = \delta^i_j, & \{f(x), \upsilon\}' = \upsilon.f, \\ \{\theta^a, \theta^b\}' = \eta^{ab}(x), & \{\sigma, \gamma\}' = \eta(\sigma, \gamma), \\ \{p_j, \theta^a\}' = \Gamma_i{}^a{}_b(x)\theta^b, & \{\upsilon, \sigma\}'_b = (d\sigma)_b + \Gamma^a_b \sigma_a, \\ \{p_i, p_j\}' = \overset{\circ}{R}_{ijab}(x)\theta^a\theta^b, & \{\upsilon, \varsigma\}' = [\upsilon, \varsigma]_{\mathrm{Lie}} + \overset{\circ}{R}(\upsilon, \varsigma). \end{array} \tag{3.68}$$

Yet one should notice that this is just the simpler instance of a deformation with a generic spin connection that also bears torsion, $\overset{\bullet}{\omega}$. The connection is however still metric. Its Poisson brackets follow:

$$\begin{array}{ll} \{x^i, p_j\}' = \delta^i_j, & \{f(x), \upsilon\}' = \upsilon.f, \\ \{\theta^a, \theta^b\}' = \eta^{ab}(x), & \{\sigma, \gamma\}' = \eta(\sigma, \gamma), \\ \{p_j, \theta^a\}' = \overset{\bullet}{\omega}_j{}^a{}_b(x)\theta^b, & \{\upsilon, \sigma\}'_b = (d\sigma)_b + \overset{\bullet}{\omega}{}^a{}_b \sigma_a, \\ \{p_i, p_j\}' = \overset{\bullet}{R}_{ijab}(x)\theta^a\theta^b, & \{\upsilon, \varsigma\}' = [\upsilon, \varsigma]_{\mathrm{Lie}} + \overset{\bullet}{R}(\upsilon, \varsigma). \end{array} \tag{3.69}$$

Here $\overset{\bullet}{R}$ denotes the curvature built with the more general connection $\overset{\bullet}{\omega}$. Recall that $\overset{\bullet}{\omega} - \overset{\circ}{\omega} \in \Omega^1(\mathcal{M}; \Lambda^2 T^*\mathcal{M})$, i.e. the difference between two metric connections is a 1-form with values in the exterior algebra of $T\mathcal{M}$, this is called *contorsion tensor*. The difference between $\overset{\circ}{R}$ and $\overset{\bullet}{R}$ can be worked out too. The deformed graded Poisson algebra of $T^*[2]M \oplus T[1]M$ can be the setting of Einstein-Cartan theory, while Palatini gravity is modeled by the less general case (3.68). The symplectic form ω' corresponding to the Poisson bivector in (3.69) can be read off the brackets:

$$\omega' = dp_i \wedge dx^i + \frac{1}{2} d\theta^a \eta_{ab} \wedge d\theta^b + \frac{1}{2} dx^i \overset{\bullet}{\omega}_{i\,ab} \theta^a \wedge d\theta^b + \frac{1}{2} dx^i \partial_i \overset{\bullet}{\omega}_{j\,ab} \theta^a \theta^b \wedge dx^j. \tag{3.70}$$

It is not complicated to link the symplectic form for (3.67) to the upper symplectic form. The 1-parameter dependent exact 2-form $dA_t = \omega'_t - \omega$ can be almost directly visualized from the last two terms of (3.70) [63],

$$A_{t=1} = \frac{1}{2} \overset{\bullet}{\omega}_{iab} \theta^a \theta^b \, dx^i. \tag{3.71}$$

To summarize, then, the degree-2 manifold $T^*[2]M \oplus T[1]M$, since $\Gamma(T[1]M) \cong \Omega^1(M)$ gives a prominent position to forms (preferring them over tangent vectors) is more suitable for hosting coframes of the Minkowski metric. This is achieved just by shifting the degree of the tangent bundle. When the symplectic structure of $T^*[2]M \oplus T[1]M$ is taken into account, the local description of gravity as a gauge theory of the Lorentz group is a wonderful outcome of such approach, which comprises either Palatini and Einstein-Cartan theory. Poincaré gauge theory of gravity can only be studied on the premise that the Poisson algebra does not really close, i.e. Jacobi identity is violated. Fermionic matter shall be incorporated pretty straightforwardly.

Concerning the differential structure, we would actually like to consider the function that is Hamiltonian for the symplectic structure in (3.65):

$$\mathbf{H} = \theta^i p_i.$$

This, interpreted as $\mathbf{H} = \theta^a e_a{}^i(x)p_i$, does not have null non-canonical Poisson brackets with itself. Instead it constitutes a Dirac operator. The derived brackets, for a wanna-be algebroid, will hence violate the Jacobi identity. We will not look at the corresponding Courant algebroid structure. It is already intriguing to see how the geodesic equation will arise in the vielbein formulation of gravity. So consider

$$\{\mathbf{H}, \mathbf{H}\}' = p^2 + \theta^a \delta_a{}^i \overset{\circ}{\omega}_i{}^b{}_c \theta^c p_b + \theta^a \theta^b \overset{\circ}{R}_{abcd} \theta^c \theta^d \neq 0. \tag{3.72}$$

In the first addend the square has Lorentzian signature. The dynamics can be retrieved thanks to $\mathbf{H}$. What we personally found for the violated master equation is just a first step, ready to be questioned further.

3.5 Abelian and non-abelian gauge theory

In this part we would like to briefly mention the Poisson brackets for abelian and non-abelian gauge theory. The exposition will not touch new undiscovered topics, but actually this section constitutes the main motivation for the treatment of a gravitational theory as previously done in the text. It should be seen as a reinforcement to the claim that the geometric data of various metric gravitational theories fit perfectly in graded symplectic geometry.

$U(1)$**-gauge theory.** The symplectic manifold of reference is just phase space, T^*M. A $U(1)$-field A (a 1-form with values in $\mathfrak{u}(1)$) can be implemented via Moser lemma with the generating vector field

$$X_t = A_i \frac{\partial}{\partial p_i}.$$

Its action shifts the momenta p to $p + A$, hence the symplectic form becomes $\omega = dx^i \wedge d(p_i + A_i(x))$,

$$\omega = dx^i \wedge dp_i + F_{ij}(x) dx^i \wedge dx^j, \quad F_{ij}(x) = \frac{1}{2}\Big(\partial_i A_j(x) - \partial_j A_i(x)\Big).$$

The gauge transformations $A(x) \mapsto A(x) + d\lambda(x)$ coincide with the canonical transformations of the Poisson structure, which are the transformations that leave the brackets invariant, as we saw for the degree 2 symplectic manifolds. The coordinate transformation associated to the gauge transformation is

$$\begin{cases} x' = x, \\ p' = p + d\lambda. \end{cases}$$

Let us stress once more that the Hamiltonian approach to the $U(1)$-gauge theory is usually disfavored to the Lagrangian approach, typically because until recent years it was not possible to canonically quantize gauge theories. A rare contradiction to the usual approach is [79] which discusses the first quantization of the brackets for Maxwell theory that we just outlined, also in the relativistic setting. Nowadays Kontsevich's deformation quantization is at hand, thus Hamiltonians and symplectic geometry are an available alternative too.

Non-abelian gauge theory. To study the extension to non-abelian groups the graded symplectic manifolds should rather be $T^*[2]M \oplus E[1]$, where E is a Lie group, or $T^*[2]E[1]$. We refer here to [79] and [63]. Then two closed non-degenerate 2-forms on these manifolds shall differ by

$$A_i^\alpha(x) \mathcal{E}_\alpha dx^i, \quad \text{in the circumstances of } T^*[2]M \oplus E[1],$$

and by

$$A_i{}^{\alpha}{}_{\beta}(x)\mathcal{E}_\alpha\tilde{\mathcal{E}}^\beta dx^i, \quad \text{in the circumstances of } T^*[2]E[1].$$

It might still be difficult to impose that the product of these graded symplectic manifolds with the graded manifolds that could be endowed with a $O(1, d-1) \subset GL(d)$ or $O(d, d)$ structure (such as $T^*[2]T[1]M$ and $T^*[2]M \oplus T[1]M$) can be graded symplectic itself. Therefore to the best of our knowledge the theory is still a bit rigid and combining Yang-Mills with Einstein theory is not straightforward.

Chapter 4

Discussion and conclusions

We have concluded our examination of Generalized Geometry and graded Poisson algebra aspects in various gravitational theories. Now a summary of the contents of the previous chapter 3 is due. In the beginning we have been looking at the deformed graded Poisson algebra, with unchanged Hamiltonian, of the shifted cotangent to the shifted tangent space to M, where the canonical Darboux chart of reference was replaced by a chart resulting from a map $\mathcal{E} \in \mathrm{Hom}\,(T^*[1]M \oplus T[1]M)$, which played also the role of a vielbein. After commenting upon the feature of the deformed Poisson brackets yielded by $\mathcal{E}$, in full generality, the focus was switched to two specific instances. First, we characterized $\mathcal{E}$ with the data of a metric g, the Kalb-Ramond field B and exponential factors of the dilaton $\phi(x)$, modifying the anchor map for the Hamiltonian function accordingly. The (free) Hamiltonian however was fixed, only the Poisson brackets were changed. Subsequently the derived bracket of the exact Courant algebroid on $E \cong TM \oplus T^*M$ were computed and proposition 2.3.2 could be applied to that bracket and to a generalized Lie bracket (3.20), in order to extract a CA connection. With the help of a non-isotropic splitting s it was projected to TM, yielding the regular connection $\widetilde{\nabla} : \Gamma(TM) \mapsto \Gamma(T^*M) \otimes \Gamma(TM)$, which was therefore used to build the curvature invariants of the manifold. The NS-NS sector of type II effective string theory arose from the Ricci tensor, contracted with the non-symmetric combination $g - B$, and upon integration by parts.

The second version of $\mathcal{E}$ we studied carried the data of g and B, as well as their companions $G^{-1} \in \vee^2 TM$ and $\Pi \in \mathfrak{X}^2(M)$, specified via the open-closed strings relation for the inverse of $g + B$ (2.13). Following the same directions as in the first case of study, we recovered on tangent space the same action than before, although without dilaton, but this time holding for every arbitrary dimension. Also projection to cotangent space was at hand: it could be performed via a non-isotropic splitting r of the dual sequence (2.20) and thus a genuine dual connection $\widetilde{\nabla} : \Gamma(T^*M) \mapsto \Gamma(TM) \otimes \Gamma(T^*M)$ could be provided. A Ricci tensor $\mathbf{Ric} := \mathrm{Tr}\,\mathbf{Riem}$, $\mathbf{Riem} \in \Gamma(TM \otimes^3 T^*M)$ stemmed from that connection. An invariant could be found via integration against $G^{-1} - \Pi$ but also because with the differential d_ρ, and the anchor map in use $\rho(r(\gamma)) = \left(G^{-1} + \Pi\right)(\gamma)^i \partial_i$, we could spot the presence of exact forms, which were hence integrated by parts. The final outcome was the formulation of an invariant action à la Hilbert-Einstein for the symmetric part of $\widetilde{\nabla}$, $\tilde{D}_G$ (3.49), in the background of the local expression for the Q and R fluxes. These were defined from the bivector Π. The action has the right gauge symmetries (diffeomorphism invariance and the gauge symmetries of Π). On the way, we also noticed that a third vielbein (3.48), the $g - B = 0$ limit of the previous one (3.31) (though G^{-1} and Π are still non-zero), accompanied by the natural generalized Lie bracket for that chart, yielded the same connection $\widetilde{\nabla} : \Gamma(T^*M) \mapsto \Gamma(TM) \otimes \Gamma(T^*M)$ (3.45).

A follow-up to the modification of the graded Poisson algebra by $\mathcal{E}$ was the more general case presented in section 3.3.2. The Poisson bracket between two linear functions in the degree-2 momenta was given by a curvature tensor. Though the new symplectic structure respected the hypotheses of Moser lemma (for graded manifolds), at this point finding the explicit diffeomorphism responsible for the change was a much more difficult task. Hence seeking a consistent Hamiltonian became more difficult, and such task could be accomplished only in very special situations. We gave our viewpoint on the issue below (3.56), the algebraic and differential equations that $\{\Theta, \Theta\} = 0$ produces. Curiously, asking that $\Theta = \theta^i p_i$ could solve the master equation when the Poisson bracket of two $\mathcal{O}_1(\mathcal{M})$ functions was not deformed by means of a symmetric tensor, but was rather constrained to be the η pairing (see (3.57)), forced the connection in the Poisson brackets to be the Levi-Civita one of standard differential geometry. However the derived Courant algebroid bracket and the Lie-like bracket in this case provided nothing but a flat CA connection, which has nor torsion neither curvature. Instead, relaxing the request that the function $\theta^i p_i$ should be Hamiltonian allowed us to consider a non-topological theory and show that the geodesic equation of a test particle moving in a space with covariant derivative given in the Poisson brackets, with torsion and curvature, metric w.r.t. to η, could be reproduced. The proposition 2.3.2 did not necessarily rely on $\{\Theta, \Theta\} = 0$ either, therefore we could notice that it depended just on the (regular) torsion.

Some other interesting consequences where found when a non-canonical symplectic form was given to $T^*[2]M \oplus T[1]M$. Before that, we pointed out that the Weitzenböck spin connection is already naturally contained in the canonical symplectic form. Then the Levi-Civita spin connection and more general types of spin connections could be implemented, as prescribed by Moser lemma. Again, it was stressed that without further assumptions and ansatz a cohomological vector field is not easy to suggest. Few comments on Poincaré gauge theory were made: modeling that theory would require non-associativity of the Poisson brackets.

In the short section 3.5 we mainly highlighted the inclusiveness of graded Poisson algebras by discussing $U(1)$ gauge theory in Hamiltonian formalism, and hinting at the description of non-abelian gauge theories. The exact 1-forms by which one introduces interactions with the non-abelian gauge fields were displayed.

Let us discuss the main developments of our original work. It comprises the new definitions in the context of generalized differential geometry too, section 2.1. Though there are some articles on type II and heterotic Supergravity actions retrieved from Generalized Geometry arguments, first of all the seminal [20] and [21], but also among the others [80], [78], [52], [81], [82], [83], and in M-theory [84], [85], and though NQ-manifolds in this context have attracted some curiosity recently [86], [6], [87], [73], we believe that our work is really original as it interpolates a lot between dg-symplectic structures, Courant algebroids and Supergravity actions.

The new generalized differential geometry objects that we have defined can be handled more easily, as they resembles their counterparts in usual differential geometry. The set comprises d_ρ locally given by (2.14), which is closely related to $\mathcal{D}$ in the CA picture and to the homological vector field Q in the dg-symplectic picture, but only when just the vector field component of Q, $-\tilde{\xi}^\alpha {\rho_\alpha}^i \dfrac{\partial}{\partial x^i}$, is taken into account, and the Poisson structure is actually degenerate (see discussion around (2.46)); The generalized Lie bracket (2.2.2), that could be connected in example 2.2.3 to d_ρ and the Dorfman bracket $[,]_D$ with anchor ρ, and which is quite different from the Courant bracket, although it is an antisymmetric bracket, above all because it respects the Leibniz rule by definition, while the Jacobi identity descends from the homomorphism property of the anchor; The torsion tensor (2.3.1), analogous to the Riemannian geometry counterpart and actually, when the connection comes from proposition 2.3.2, also equivalent to Gualtieri's torsion tensor (and thus completely skewsymmetric); as well the Riemann tensor (2.25) and the Ricci tensor (2.26) had very clean expressions. In the mathematical preliminaries other viable definitions have been presented and compared to ours.

The chapter 2.4 instead just featured a review of the literature on the topic of graded geometry, graded Poisson algebras and differential graded manifolds. The relation with other non-graded types of algebras and algebroids was given particular relevance, as the personal research work was based on it. Small original contributions in the section can be seen in the Q vector fields which were explicitly computed for some cases.

Concerning the body of the manuscript, there are some important novelties introduced in the first section 3.2 compared to the literature: the vielbein (3.12) is not adapted to the generalized metric (2.1.4), therefore the CA connection, as a connection 1-form with values in the Lie algebra of the principal group, does not belong to $\Omega(E, \mathfrak{o}(d) \times \mathfrak{o}(d))$. Its Gualtieri torsion is non-zero, nevertheless via the splitting, which projects the connection to TM, the standard torsion of $\widetilde{\nabla}$ totally coincides with that of a connection (such as the Bismut connection of definition 2.2.4) in the family of torsion-free generalized connections, which are favored in Generalized Geometry. In fact the torsion is given by $H \in H^3(M, \mathbb{R})$. Moreover the curvature tensors did not have to be computed for the whole bundle, but just for tangent space, and integration of the Ricci tensor, which was performed with factors of λ and $g^{-1}(g - B)\, g^{-1}$, showed that our Ricci tensor in (3.29), $\mathbf{Ric}_{ij} \neq \beta(g)_{ij} + \frac{1}{2}\beta(B)_{ij}$, can build up the NS-NS sector of the SUGRA action. Although the β-functions of g and B can be seen as the equations of motion due to variational principle applied to that action, there is hence at least another tensor that upon contraction with the same combination $(g - B)$ gives the Lagrangian of (1.2).

The other soon-to-be-published paper reported in section 3.3 presented some substantial variations with respect to the references we consulted. Apart from the DFT connotation of [36], [37] and [88], where doubled coordinates on the base manifold stand for the usual coordinates and the winding modes, and therefore to constrain the resulting gravity action with non-geometric fluxes Q and R to regular spacetime the section condition of DFT has to be employed, the authors constructed the differential geometry and the covariant derivative for their scopes without referring to NQ-manifolds or Courant algebroids at all. Furthermore, the g, B tensors underwent a formal "T-duality" and were mapped to G^{-1} and Π, which is in fact possible just in the context of DFT. In a follow-up of that investigation, the authors of [38] managed instead to obtain an equivalent action with the T^*M Lie algebroid gathering its symmetries. Our proposal for a gravitational theory with Q and R tensors starts with the graded Poisson algebra of $T^*[2]T[1]M$, and then switches to the corresponding Courant algebroid $E \cong TM \oplus T^*M$, the natural connection for E and the curvature invariants. We do not double the coordinates of M. The non-isotropic splittings of the direct sequence and the dual one allow to deal with the NS-NS SUGRA action in one limit, and with the action for the "T-dual" fields G^{-1}, Π in the other limit. The latter are rather motivated by the closed-open strings relation. Again, the generalized torsion of the connection is non-zero. Another point to compare with the references is the anchor map: we use $\rho(U) = X + \left(G^{-1} + \Pi\right)(\gamma)$, instead [38] uses a map $\rho' : TM \mapsto TM$, given by $\rho' = \mathbb{1} + \Pi G$.

In both the aforementioned articles the SUGRA action and its "dual" had the right symmetries (diffeomorphisms and shifts of B (Π) by an exact 2-form $d\alpha$ (exact bivector $-\mathrm{Q}\varrho|_{\chi\chi-\text{comp}}$, $\varrho = \chi_i a^i(x) - \left(G^{-1} + \Pi\right)^{-1}_{lm} a^l(x)\theta^m$ as in (3.53))): this latter type of symmetry could be checked from the gauge symmetries of the $O(d,d)$ generators (see exposition in section 2.6.1), specialized to the respective cases by careful assignation of the degree-1 ϱ. This was much less straightforward to show in other formulations of the dual gravity action with Q and R. Notice furthermore that in our derivation when these tensors are present, H is not turned on. Hence rather than employing their more general expressions:

$$Q_k{}^{ij} = \partial_k \Pi^{ij} + \Pi^{il}\Pi^{jm}H_{lmk}, \quad R^{ijk} = \Pi^{[i|l}\partial_l \Pi^{|jk]} + \Pi^{il}\Pi^{jm}\Pi^{kn}H_{lmn}, \tag{4.1}$$

we saw only the appearance of the $H = 0$ limit of these expressions in our computations (3.46).

In regards to the results displayed in section 3.3.2 until the end of subsection 4.1 and section 3.4, as well as the completely well-known section 3.5, the clue was to deform the same symplectic manifold giving rise to more intricate deformed Poisson brackets and their Hamiltonian functions, or to deform other kinds of symplectic manifolds such as $T[2]M \oplus T[1]M$, and then study various (natural) possibilities for both the connection in the graded Poisson brackets and the derived CA connection. Weitzenböck, Levi-Civita and a connection with both torsion and curvature, these all appeared under different circumstances. Their spin connection counterparts too. We actually believe that this continuous interplay must really have deep reasons. To dismiss metricity of the connection, instead, the Poisson algebra shall not be associative, i.e. the Jacobi identity shall not hold, but that was not done here.

We believe that the overall strength of our investigation should be seen in the fact that we successfully applied the theory of NQ-manifolds, which usually seeks applications to topological field theories (AKSZ models) and in the quantization techniques of BV and BFV, or even in the α' corrections to string theory [89], to construct "by hand" various gravitational models, not just those motivated by string theory or its DFT formulation [90], [91]. Among other works dealing with more conservative modifications to GR in the setting of dg-manifolds, [92] should be mentioned.

We found also beautiful, from the viewpoint of a mathematical relativist, that various aspects of the theory of gravity as originally formulated (and unsuccessfully extended to account for electromagnetism too) by Einstein shows up and are somehow unified in our approach: non-symmetric metric, frame fields, Weitzenböck connection, Levi-Civita connection, spin connection, Koszul formula, geodesic equation, torsion tensor and Riemann curvature tensor.

4.1 Outlook

The investigation performed here is far from being complete. It leaves some important questions unanswered. Let us collect those which came to our mind.

First, the C tensors in the Hamiltonian function were always left aside. Taking them into account would have not added much more to our examination, when the deformation was carried by a vielbein. Instead when the deformation map was not explicit, having them or not, as well as working out the full transformation of the Hamiltonian, would have made a difference. They must be present in order for Θ to fulfill the classical master equation with non-canonical Poisson brackets, because now the intricate set of algebraic and differential equations interpolates between the connection symbols and the curvature on one side, the anchor map and the C tensors on the other. Only when this matter is settled, the derived CA bracket can be computed in full consistency (i.e. the respect of all the defining axioms is granted). It would be suitable, then, to provide a full classification of general deformed Poisson algebras and the corresponding Hamiltonian functions permitted for the brackets. Such a classification perhaps might work as a generating system for actions with diffeomorphism invariance and other kind of symmetries, and it might as well help in finding concrete examples of T-dual manifolds (T-duality in Generalized Geometry has the clean interpretation of isomorphism classes of Courant algebroids [93]).

Moreover, it is still unclear to us how the R-R fluxes, known to be spinor bundle representations, can be incorporated in the graded Poisson setting. First of all a spin structure exists if the second Stiefel-Whitney class of M, $w_2(M) \in H^2(M, \mathbb{Z}_2)$, is non-zero. Anyway it might not be possible to comprise them, via their gauge fields, in the vielbein (as done for the metrics g or G^{-1} and the Kalb-Ramond field B or the bivector Π).

It would also be good to know how graded symplectic structures can be attached to transitive CAs, i.e. the anchor is just surjective, or equivalently $\mathrm{Im}\rho^*$ is not the whole kernel of ρ. It is clear that a differentiable structure for the graded Poisson manifolds cannot exist if we want to get transitive CAs: $\mathrm{Q}^2 = 0$ was verified provided that $\rho^* \circ g_E \circ \rho = 0$.

We have not been able to settle yet which conditions will be placed on the connection of proposition 2.3.2 if the Jacobi identity for the bracket of the bundle is assumed to hold. Our guess is that its curvature could have some symmetries (e.g. the algebraic Bianchi identity).

Furthermore, we did not establish whether the complex structure naturally existing in Generalized Geometry may play a role here. How these can be included in the graded picture was studied by Grabowski [94]. The construction goes as follow: take a Hamiltonian function H for a graded symplectic manifold of degree 2, and try to find a two-parameter family of solutions to the classical master equation:

$$\{aH + bK, aH + bK\} = 0, \quad \forall\, a, b \in \mathbb{Z}.$$

A class of solutions K is given by canonical transformations of H, that are due to a degree-2 function (actually it is constrained to depend just on the ξ coordinates and not on the momenta p) acting via Poisson bracket on H, $K = \{H, J\}$. To satisfy $\{K, K\} = 0$, however, one should ask for

$$\{\{H, J\}, J\} = cH, \quad c \in \mathbb{Z},$$

as seen from the graded Jacobi identity applied to the master equation. When $c = -1$, the smooth function J yields a generalized complex structure on the CA corresponding to H.[1]

Another extremely useful study that could be carried over concerns the Courant sigma models corresponding to every deformed Poisson algebra for $T^*[2]T[1]M$, implemented by the vielbein. In our first paper [3] the sigma model was just outlined in section 5.2.1: definitely more can be inferred about that, e.g. the quantization. It would be compelling to relate our deformed structures to the gauged sigma models (of Poisson and Dirac type[2]) of [96], [97].

[1] $c = 0$ and $c = 1$ give also some structure of interest, but we will not talk about them.

[2] The Dirac sigma model involves maps from Σ_2 to the Dirac (maximal, isotropic and involutive) subbundle of an exact CA [95].

4.2 Appendix A

This appendix contains the derivations of some minor results that are used along the thesis, and which can already be found in the literature. Afterwards the reader can find a brief discussion on the issue of generalized parallelizability for $TM \oplus T^*M$.

- Proof that the Dorfman algebra $[,]$ closes, from graded symplectic geometry. Start with $[[U,V],W]$ and use graded antisymmetry and graded Jacobi identity for $\{,\}$:

$$\begin{aligned}\{\Theta,\{\{\Theta,U\},V\}\} &= \{\{\Theta,\{\Theta,U\}\},V\} + \{\{\Theta,U\},\{\Theta,V\}\} \\ &= \{\{\{\Theta,\Theta\},U\},V\} - \{\{\Theta,\{\Theta,U\}\},V\} + \{\{\Theta,U\},\{\Theta,V\}\} \\ &= \frac{1}{2}\{\{\{\Theta,\Theta\},U\},V\} + \{\{\Theta,U\},\{\Theta,V\}\}. \end{aligned} \tag{4.2}$$

Notice then that the derived bracket corresponding to $[U,[V,W]]$ satisfies the following relation:

$$\{\{U,\Theta\},\{\{V,\Theta\},W\}\} = \{\{\{U,\Theta\},\{V,\Theta\}\},W\} + \{\{V,\Theta\},\{\{U,\Theta\},W\}\} \tag{4.3}$$

and since $\{U,\Theta\} = \{\Theta,U\}$, one can use the system of the relations (4.2), (4.3) to eliminate $\{\{\{\Theta,U\},\{\Theta,V\}\},W\}$ from them and finally get:

$$\begin{aligned}[U,[V,W]] \equiv \{\{U,\Theta\},\{\{V,\Theta\},W\}\} &= \{\{\Theta,\{\{\Theta,U\},V\}\},W\} + \{\{V,\Theta\},\{\{U,\Theta\},W\}\} \\ &\quad - \frac{1}{2}\{\{\{\{\Theta,\Theta\},U\},V\},W\} \\ &\equiv [[U,V],W] + [V,[U,W]] - \frac{1}{2}\{\{\{\{\Theta,\Theta\},U\},V\},W\}. \end{aligned} \tag{4.4}$$

Therefore for a NQ-manifold $T^*[2]T[1]M$, since by definition the Hamiltonian Θ respects the master equation, the Dorfman algebra closes. One can also ask for a weaker condition, namely to restrict the generalized vectors under consideration to some subspaces for which

$$\frac{1}{2}\{\{\{\{\Theta,\Theta\},U\},V\},W\} = 0, \qquad \{\Theta,\Theta\} \neq 0.$$

This is just enough for the purpose of having a well-defined CA that fulfills all the axioms.

- Proof of (2.45). Let $V = Y + \varsigma$ and $W = Z + \kappa$, where $Y,Z \in \Gamma(TM)$ and $\varsigma,\kappa \in \Gamma(T^*M)$. Then

$$\begin{aligned}[[Y+\varsigma,d],Z+\kappa] =& [[Y,d],Z+\kappa] + [[\varsigma,d],Z+\kappa] \\ =& [[\iota_Y,d],\iota_Z+\kappa] + [\varsigma\wedge d + d\,(\varsigma\wedge)\,,\iota_Z+\kappa] \\ =& [\mathcal{L}_Y,\iota_Z] + [\mathcal{L}_Y,\kappa] + \varsigma\wedge d\iota_Z - \iota_Z\varsigma\wedge d - \varsigma\wedge d\iota_Z - \iota_Z d\,(\varsigma\wedge) \\ & + \varsigma\wedge d(\kappa\wedge) - \kappa\wedge\varsigma\wedge d + (d\varsigma)\wedge\kappa - \varsigma\wedge d(\kappa\wedge) - \kappa\wedge d(\varsigma\wedge) \\ =& \iota_{[Y,Z]} + \mathcal{L}_Y\kappa\wedge - \kappa\wedge\mathcal{L}_Y - \iota_Z(d\varsigma)\wedge \equiv [V,W]_{\mathrm{D}} \end{aligned}$$

- Collection of the vector fields X_t, which generate the flow of the smooth 1-parameter family of diffeomorphisms φ_t such that $\varphi_t^*\omega_t = \omega_0$, by Moser lemma:

(section 3.1): for the symplectic manifold $T^*[2]T[1]M$, with $\varphi_t \equiv \mathcal{E}_t$, $\mathcal{E}_0 = \mathrm{id}$, and $\omega_{t=1} \equiv \omega'$ in (3.2), ω_t belongs to the same cohomology class than the canonical ω_0 if

$$d\dot{A}_t = d\left[\dot{\mathcal{E}}_t\xi\eta^{-1}d\left(\mathcal{E}_t\xi\right)\right].$$

Then from $\iota_{X_t}\omega_t = \dot{A}_t$ the vector field X_t is readily computed:

$$X_t = \left(\dot{\mathcal{E}}_t\right)^{\beta}_{\alpha} \xi_\beta \left(\mathcal{E}_t^{-1}\right)^{\alpha}{}_{\gamma} \frac{\partial}{\partial \xi_\gamma}.$$

(section 3.3.2.1): for the same symplectic manifold, but with undeformed $O(d,d)$ metric and with non-zero curvature, being

$$d\dot{A}_t = d\left[\left(\dot{\Gamma}_t\right)_{ik}{}^{l}\theta^k \chi_l \, dx^i\right],$$

the vector field X_t hence corresponds to

$$X_t = \left(\dot{\Gamma}_t\right)_{ik}{}^{l}\theta^k \chi_l \, \frac{\partial}{\partial p_i}. \tag{4.5}$$

(section 3.4.1): for the symplectic manifold $T^*[2]M \oplus T[1]M$, referring to the non-canonical (3.70) and to the exact 1-form (3.71), the generating vector field is immediate:

$$X_t = \frac{1}{2}\overset{\bullet}{\omega}_{i\,ab}\theta^a\theta^b \frac{\partial}{\partial p_i}.$$

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