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Forecasting banknote circulation during the COVID-19 pandemic using structural time series models

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Non-technical summary

Research Question

Every year the national central banks of the Eurosystem and the European Central Bank jointly define the banknote production requirements. This exercise aims at avoiding shortages or excessive surpluses of banknotes, both at country level and for the euro area as a whole, in order to provide credit institutions and citizens with the quantity demanded, to guarantee the quality of cash in circulation, and to pursue cost efficiency.

Contribution

The forecast models currently in use in the euro area make rather strong assumptions about the properties of the time series in question. In order to overcome some of these weaknesses, we estimate structural time series models for France, Germany, Italy and Spain with the objective of forecasting the net issuance of euro banknotes by denomination for these countries.

Results

According to our study, the structural time series models provide more precise forecasts than the forecast models currently in use. However, this comparison is based on a very short time interval.

Nichttechnische Zusammenfassung

Fragestellung

Jedes Jahr legen die nationalen Zentralbanken des Eurosystems und die Europäische Zentralbank gemeinsam den Bedarf an zu druckenden Banknoten fest. Sowohl auf Länderebene als auch für den Euroraum als Ganzes soll so ein Mangel an oder ein zu großer Überschuss von Banknoten vermieden werden, um die Banknotennachfrage von Kreditinstituten und Privatpersonen bedienen zu können, die Qualität des Banknotenumlaufs sicherzustellen und die Kosteneffizienz bei der Versorgung mit Banknoten zu gewährleisten.

Beitrag

Die derzeit im Euroraum eingesetzten Prognosemodelle machen recht starke Annahmen über die Eigenschaften der betreffenden Zeitreihen. Zur Überwindung einiger dieser Schwächen werden in dieser Arbeit für Frankreich, Deutschland, Italien und Spanien strukturelle Zeitreihenmodelle geschätzt, um die stückelungsweise nationale Nettoemission von Euro-Banknoten durch diese Länder zu prognostizieren.

Ergebnisse

Die strukturellen Zeitreihenmodelle liefern bei unserem Prognosevergleich präzisere Vorhersagen als die derzeit eingesetzten Prognosemodelle. Allerdings basiert dieser Vergleich nur auf einem sehr kurzen Zeitraum.

FORECASTING BANKNOTE CIRCULATION DURING THE COVID-19 PANDEMIC USING STRUCTURAL TIME SERIES MODELS

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Abstract

As part of the Eurosystem's annual banknote production planning, the national central banks draw up forecasts estimating the volumes of national-issued banknotes in circulation for the three years ahead. As at the end of 2021, more than 80 per cent of euro banknotes in circulation (cumulated net issuance) had been issued by the national central banks of France, Germany, Italy and Spain ('4 NCBs'). To date, the 4 NCBs have been using ARIMAX models to forecast the banknotes issued nationally in circulation by denomination ('benchmark models'). This paper presents the structural time series models developed by the 4 NCBs as an additional forecasting tool. The forecast accuracy measures used in this study show that the structural time series models outperform the benchmark models currently in use at each of the 4 NCBs for most of the denominations. However, it should be borne in mind that the statistical informative value of this comparison is limited by the fact the projection period is only twelve months.

JEL Classification: C22, E41, E47, E51

Keywords: euro, demand for banknotes, forecast of banknotes in circulation, structural time series models, ARIMA models, intervention variables

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1. Introduction

Every year the national central banks of the Eurosystem (“NCBs”) and the European Central Bank (ECB) are involved in a joint procedure to define the banknote production requirements. For each denomination, it is necessary to forecast net issuance, defined as the difference between withdrawals from and lodgements to NCBs, and the quantity of fit banknotes retrievable through sorting activity.¹ Avoiding shortages or excessive surpluses of banknotes, both at country level and for the euro area as a whole, is crucial in enabling the Eurosystem to provide credit institutions and citizens with the quantity demanded, to guarantee the quality of cash in circulation, and to pursue cost efficiency.

In the first years after the launch of euro banknotes, net issuance was forecasted at the euro area level by exploiting cointegration relationships with relevant macroeconomic variables, using the so-called “ABCD” model. In the course of time, despite some updates, the “ABCD” model was not able any more to fully reflect developments affecting cash demand. This applied especially to the 2008 financial crisis. In order to tackle this problem a new approach (the “ABCD-2” model), based on a small basket of models, was adopted in 2019 (Bartzsch et al., 2020).² Nevertheless, all of the models that have been proposed since the euro cash changeover fall within the family of traditional time series econometrics techniques such as exponential smoothing, ARIMA, VAR or SUR.

Such approaches make rather strong assumptions about the properties of the time series to be analysed and predicted. Therefore, some components nested within the time series are often filtered out through different pre-treatments: for example, trend or seasonality are, in general, removed by conveniently differentiating the original series.³ Such transformations may have strong implications for the forecasting exercise. Moreover, the traditional framework is not suited to fully accounting for some changes induced by significant shocks that have hit the economy in recent years, such as the global financial crisis (2008) and the European sovereign

¹ Banknote sorting activity, managed at the national level, is characterised by several country-specific features, like sorting capacity, quality policies and pre-existing stocks of banknotes awaiting processing.

² In July 2017, the European System of Central Banks’ Banknote Committee (BANCO) established the Eurosystem Research Network on Cash (EURECA) to focus on research related to cash. The main goal of EURECA is to promote cash-related research by fostering information exchange between NCBs on ongoing works and research methods. ABCD-2 was the first workstream of EURECA.

³ For the ARIMA models used by Banque de France and the ARIMAX models used by the Bundesbank the stationary transformations of the dependent variables are presented in Table 2 in Annex 1 and Table 2 in Annex 2.

debt crisis (2009-2010), or for regulatory interventions, like the discontinuation of the issuance of €500 banknotes announced on 4 May 2016. In addition, the COVID-19 pandemic, occurring in 2020, was associated with circulation dynamics different from those observed after any shock before.⁴

In order to overcome some of these weaknesses, we use Structural Time Series Models (“STSMs”) to forecast the net issuance of euro banknotes for each denomination and country. This class of models, introduced by Harvey (1984), is based on the classical decomposition of a time series into its trend, seasonal, cycle and irregular components and can be augmented with regression variables. It enables all of the dynamics in the time series data to be analysed simultaneously. Moreover, missing data and time-varying regression coefficients are easily handled in the state-space framework.

This paper, which is the product of a collaboration between the NCBs of France, Germany, Italy and Spain (“4 NCBs”), presents forecasts for each denomination of euro banknotes separately for the four countries involved in the project, and a comparison against the results obtained with the benchmark models currently in use.⁵

The remainder of the paper is structured as follows: Section 2 gives a formal description of an STSM; in Section 3 we illustrate developments in cash circulation in the euro area and its four largest economies before the COVID-19 outbreak, and look at the effects exerted on banknote dynamics by the restrictions implemented to combat the spread of the pandemic; in Section 4 we report and comment on the results of the forecasting exercise; Section 5 concludes.

⁴ As shown in Section 4, nationally issued euro banknotes are also impacted by country-specific interventions.

⁵ 4 NCBs’ share in euro banknote circulation (cumulated net issuance) in terms of value amounted to about 82 per cent at the end of 2021.

2. Methodology

Structural time series models (“STSMs”) are a flexible approach to time series forecasting and encompass a broad range of models. STSMs typically consist of interpretable unobserved components such as trend, cycle, seasonal and irregular components (Harvey, 1989).⁶ Each component is separately modelled by an appropriate dynamic stochastic process which usually depends on normally distributed disturbances. Long-term developments in the economy are characterised by the trend component. Mid-term dynamics can be modelled directly by the cycle component. Therefore, the forecasts originate from realistic model representations of the macroeconomic time series rather than black-box methods.

A basic model for representing a time series is the following additive model also known as the Classical Seasonal Decomposition:

$$y_t = \mu_t + \gamma_t + \varepsilon_t, \quad t = 1, \dots, T, \quad [1]$$

where y_t is the observed variable, μ_t denotes a slowly changing component (trend), γ_t is a periodic component (seasonal) and ε_t denotes the irregular component (disturbance). In an STSM or unobserved components model the components in the equation on the right-hand side of [1] are modelled explicitly as stochastic processes. Hereinafter we consider various specifications.

The *local level model* contains only the level component (mean) of the trend and an irregular component.⁷ This simple model assumes that the underlying level of the series varies over time. As the level component is a random walk, the local level model is also referred to as the *random walk plus noise model* (where the noise refers to the irregular component). The specification is therefore:

$$\begin{aligned} y_t &= \mu_t + \varepsilon_t, & \varepsilon_t &\sim \text{NID}(0, \sigma_\varepsilon^2), \\ \mu_{t+1} &= \mu_t + \xi_{t+1}, & \xi_t &\sim \text{NID}(0, \sigma_\xi^2), \quad t = 1, \dots, T. \end{aligned} \quad [2]$$

⁶ The presentation in this section also resorts to Commandeur and Koopman (2007) and Proietti (1991).

⁷ The local level model is equivalent to an ARIMA(0,1,1) model.

μ_t is the unobserved level of the trend and ε_t and ξ_t are normally independent white-noise disturbances with variances σ_ε^2 and σ_ξ^2 . The first equation in [2] is called the *observation* or *measurement equation*, while the second equation is called the *state equation*.

The *local linear trend model* is related to Holt's linear exponential smoothing (Holt, 1957; Harvey, 1984). It extends the previous model by supplementing the trend with a slope:

$$\begin{aligned} y_t &= \mu_t + \varepsilon_t, & \varepsilon_t &\sim \text{NID}(0, \sigma_\varepsilon^2), \\ \mu_{t+1} &= \mu_t + \nu_t + \xi_{t+1}, & \xi_t &\sim \text{NID}(0, \sigma_\xi^2), \\ \nu_{t+1} &= \nu_t + \zeta_{t+1}, & \zeta_t &\sim \text{NID}(0, \sigma_\zeta^2), \quad t = 1, \dots, T, \end{aligned} \quad [3]$$

where ζ_t is a normally independent white-noise disturbance with variance σ_ζ^2 . Moreover, ζ_t and ξ_t are mutually uncorrelated and uncorrelated with ε_t . The trend component μ_{t+1} now consists of a level (its lagged value) and a slope ν_t . In the literature on time series analysis the slope is also referred to as the *drift*.

Finally, if we add a seasonal component, γ_t , the model becomes the *basic structural model* which is similar to the Holt-Winters exponential smoothing scheme. It contains trend, seasonal and irregular components. When the seasonal component is deterministic the seasonal effects $\gamma_j, j = 1, \dots, s$, must sum to zero over the year (*seasonal dummy model*). By adding a white-noise disturbance term, ω_t , the seasonal component can be made stochastic. For quarterly data the number of seasons in the year, s , is equal to four and the basic structural model is specified in the following way:

$$\begin{aligned} y_t &= \mu_t + \gamma_t + \varepsilon_t, & \varepsilon_t &\sim \text{NID}(0, \sigma_\varepsilon^2), \\ \mu_{t+1} &= \mu_t + \nu_t + \xi_{t+1}, & \xi_t &\sim \text{NID}(0, \sigma_\xi^2), \\ \nu_{t+1} &= \nu_t + \zeta_{t+1}, & \zeta_t &\sim \text{NID}(0, \sigma_\zeta^2), \\ \gamma_{1,t+1} &= -\gamma_{1,t} - \gamma_{2,t} - \gamma_{3,t} + \omega_{t+1}, & \omega_t &\sim \text{NID}(0, \sigma_\omega^2), \\ \gamma_{2,t+1} &= \gamma_{1,t}, \\ \gamma_{3,t+1} &= \gamma_{2,t}, \end{aligned} \quad t = 1, \dots, T, \quad [4]$$

where $\gamma_t = \gamma_{1,t}$ denotes the seasonal component and ω_t is a normally independent white-noise disturbance with variance σ_ω^2 .

An alternative way of modelling the seasonal component is through a set of sine and cosine functions (*trigonometric seasonal model*). A fixed seasonal pattern is modelled by the sum of $[s/2]$ cycles defined at the seasonal frequencies $\lambda_j = 2\pi j/s, j = 1, 2, \dots, [s/2]$ where $[s/2] = s/2$ for s even and $(s-1)/2$ if s is odd:

$$\gamma_t = \sum_{j=1}^{[s/2]} \gamma_{j,t}, \quad \gamma_{j,t} = \alpha_j \cos \lambda_j + \alpha_j^* \sin \lambda_j,$$

where $\gamma_{j,t}$ is the effect of season j at time t . The trigonometric seasonal model may be extended stochastically, whereby the seasonal effect at time t then arises from the combination of $[s/2]$ stochastic cycles formulated as in [5], setting $\varrho_\psi = 1$ to allow for a persistent pattern:

$$\begin{aligned} \gamma_t &= \sum_{j=1}^{[s/2]} \gamma_{j,t}, \\ \gamma_{j,t+1} &= \gamma_{j,t} \cos \lambda_j + \gamma_{j,t}^* \sin \lambda_j + \omega_{j,t+1}, \\ \gamma_{j,t+1}^* &= -\gamma_{j,t} \sin \lambda_j + \gamma_{j,t}^* \cos \lambda_j + \omega_{j,t+1}^*, \quad \omega_{j,t}, \omega_{j,t}^* \sim \text{NID}(0, \sigma_\omega^2), \quad t = 1, \dots, T, \end{aligned}$$

where $\lambda_j = 2\pi j/s$ is the frequency, in radians, and the seasonal disturbances $\omega_{j,t}$ and $\omega_{j,t}^*$ are two mutually uncorrelated normally independent disturbances with zero mean and common variance σ_ω^2 . Without the disturbance, the trigonometric specification is identical to the deterministic dummy specification. The component $\gamma_{j,t}^*$ appears as a matter of construction, and its interpretation is not particularly important. When s is even, the last component, defined at $\lambda_{s/2} = \pi$, reduces to $\gamma_{s/2,t+1} = \gamma_{s/2,t} + \omega_{s/2,t}$.

Digression: In addition to trend and season, the unobserved components can also include one or more cycles. In economics, *cycles* represent recurrent, though not exactly periodic, deviations from the long-run path of a series. A model for the cyclical component should be capable of reproducing acknowledged fundamental characteristics, such as the presence of strong autocorrelation, determining the recurrence and alternation of phases, as well as the dampening of fluctuations or rather zero long-run persistence. The statistical specification of a stochastic cycle, ψ , is given by:

$$\begin{bmatrix} \psi_t \\ \psi_t^* \end{bmatrix} = \rho_\psi \begin{bmatrix} \cos \lambda_c & \sin \lambda_c \\ -\sin \lambda_c & \cos \lambda_c \end{bmatrix} \begin{bmatrix} \psi_{t-1} \\ \psi_{t-1}^* \end{bmatrix} + \begin{bmatrix} \kappa_t \\ \kappa_t^* \end{bmatrix}, \quad t = 1, \dots, T, \quad [5]$$

where ρ_ψ , in the range $0 \leq \rho_\psi \leq 1$, is a damping factor, λ_c is the angular frequency measured in radians, in the range $0 \leq \lambda_c \leq \pi$; κ_t and κ_t^* are two mutually uncorrelated normally independent disturbances with zero mean and common variance σ_κ^2 . The *period* of the cycle is equal to $2\pi/\lambda_c$, that is the cycle repeats itself every $2\pi/\lambda_c$ time units.

Additionally, *intervention variables* are added to the basic structural model [4] in order to explain the time series. These variables represent special events which lead to outliers⁸ and trend breaks.⁹ Moreover, calendar effects like the Easter holidays are modelled by means of *regression variables*.¹⁰ To do this, the observation equation in [4] is augmented as follows:

$$y_t = \mu_t + \gamma_t + \sum_{i=1}^k \beta_i x_{it} + \sum_{j=1}^h \lambda_j z_{jt} + \varepsilon_t, \quad \varepsilon_t \sim \text{NID}(0, \sigma_\varepsilon^2), \quad t = 1, \dots, T, \quad [4a]$$

where x_{it} is a regression variable and β_i is an unknown coefficient, for $i = 1, \dots, k$. z_{jt} represents the j th intervention variable. The intervention variables in equation [4a] comprise outliers and trend breaks in the local level. In the former case, these are impulse dummies and, in the latter, step dummies. In order to also incorporate a trend break in the local slope, the second state equation in the basic structural model [4] is augmented accordingly:

$$v_{t+1} = v_t + \bar{\lambda} \bar{z}_{t+1} + \zeta_{t+1}, \quad \zeta_t \sim \text{NID}(0, \sigma_\zeta^2), \quad t = 1, \dots, T, \quad [4b]$$

where \bar{z}_t is an impulse dummy variable. It takes on the value of one in the period in which the underlying event occurs and zero otherwise. The final STSM [6], corresponding to the construction of most of the 4 NCBS' STSMs presented in Section 4, consists of the basic

⁸ An outlier is an unusually high value of the irregular component at a given point in time.

⁹ In a first step, the times at which potential intervention variables occur (e.g. October 2008) are identified using exceptionally high values of the smoothed estimates of the irregular, level and slope disturbances (auxiliary residuals). In a second step, the identified potential intervention variables are interpreted as exceptional events (e.g. financial crisis).

¹⁰ The STSMs presented here are meant for forecasting. We have therefore opted not to incorporate regression variables which stand for demand motives. For example, the transaction motive could be modelled by cash consumption and non-euro area foreign demand by the exchange rate. In Section 4.3 of Deutsche Bundesbank (2019), such econometric models are estimated as ARDL models for the demand for small, medium-sized and high denominations as well as for domestic demand for euro banknotes issued by the Bundesbank.

structural model [4] augmented by the regression variables and intervention variables according to equations [4a] and [4b]:

$$\begin{aligned}
y_t &= \mu_t + \gamma_t + \sum_{i=1}^k \beta_i x_{it} + \sum_{j=1}^h \lambda_j z_{jt} + \varepsilon_t, & \varepsilon_t &\sim \text{NID}(0, \sigma_\varepsilon^2), \\
\mu_{t+1} &= \mu_t + \nu_t + \xi_{t+1}, & \xi_t &\sim \text{NID}(0, \sigma_\xi^2), \\
\nu_{t+1} &= \nu_t + \bar{\lambda} \bar{z}_{t+1} + \zeta_{t+1}, & \zeta_t &\sim \text{NID}(0, \sigma_\zeta^2), \\
\gamma_{1,t+1} &= -\gamma_{1,t} - \gamma_{2,t} - \gamma_{3,t} + \omega_{t+1}, & \omega_t &\sim \text{NID}(0, \sigma_\omega^2), \\
\gamma_{2,t+1} &= \gamma_{1,t}, \\
\gamma_{3,t+1} &= \gamma_{2,t}, & t &= 1, \dots, T.
\end{aligned} \tag{6}$$

As shown below, STSMs fit nicely in the state-space form. The state-space form provides the key to the statistical treatment of STSMs. It enables maximum likelihood estimators (MLE) of the unknown parameters in a Gaussian model to be computed via the Kalman filter and the prediction error decomposition. Once estimates of these parameters have been obtained, it provides algorithms for estimating the unobserved components and predicting future observations. An advantage of STSMs and Kalman filtering techniques is that a variety of explanatory variables, dummies, and missing observations can be included in the model without difficulty.

Particularly, in order to estimate model [6] in the state-space form, the log-likelihood function is maximised regarding the unknown parameters, especially the observation and state disturbance variances σ_ε^2 , σ_ξ^2 , σ_ζ^2 and σ_ω^2 .¹¹ To do this, we apply the Kalman filter to calculate and minimise the prediction errors of the one-step forecasts of the observable time series and their variances. These forecast errors and their variances are also used to calculate the standardised forecast errors, based on which we examine whether the residuals of the state-space model are independent, homoskedastic and normally distributed (diagnostic tests). All significance tests in linear Gaussian models are based on these three assumptions.

All of the STSMs presented above are special cases of *state-space models*. A state space model consists of a measurement equation and a transition equation. The *state-space form* can be written in the following general matrix format (Commandeur and Koopman, 2007, Section 8.1):

¹¹ The observation and state disturbances are also known as *hyperparameters*.

$$y_t = z_t' \alpha_t + \varepsilon_t, \quad \varepsilon_t \sim \text{NID}(0, \sigma_\varepsilon^2), \quad [7.1]$$

$$\alpha_{t+1} = T_t \alpha_t + R_t \eta_t, \quad \eta_t \sim \text{NID}(0, Q_t), \quad t = 1, \dots, T. \quad [7.2]$$

The terms y_t and ε_t are still scalars as before. However, the remaining terms in [7.1] and [7.2] denote vectors or matrices. z_t is an $m \times 1$ *observation vector*, T_t is an $m \times m$ *transition matrix*, α_t is an $m \times 1$ *state vector*, and m therefore denotes the number of elements in the state vector. In many state-space models R_t is simply the identity matrix of order $m \times m$. The $r \times 1$ vector η_t contains the r state disturbances with zero means, and unknown variances collected in an $r \times r$ diagonal matrix Q_t . In this general formulation, equation (7.1) is called the *observation* or *measurement equation*, while (7.2) is called the *transition* or *state equation*. The measurement equation relates the time series y_t to the vector of unobservable components or state vector, α_t . The transition equation is a first order autoregression for the states α_t .

By appropriately defining the vectors z_t , α_t and η_t , and of the matrices T_t , R_t and Q_t , all the STSMs described above can be derived as special cases of [7.1] and [7.2]. For example, the basic structural time series model [4] for quarterly data ($s = 4$) requires a 5×1 state vector, that is one element for the level μ_t , one element for the slope v_t and three elements for the stochastic seasonal dummy effect. By defining:

$$\alpha_t = \begin{pmatrix} \mu_t \\ v_t \\ \gamma_{1,t} \\ \gamma_{2,t} \\ \gamma_{3,t} \end{pmatrix}, \quad \eta_t = \begin{pmatrix} \xi_{t+1} \\ \zeta_{t+1} \\ \omega_{t+1} \end{pmatrix}, \quad T = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad z_t = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

$$Q_t = \begin{bmatrix} \sigma_\xi^2 & 0 & 0 \\ 0 & \sigma_\zeta^2 & 0 \\ 0 & 0 & \sigma_\omega^2 \end{bmatrix}, \quad \text{and} \quad R_t = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

we easily verify that the scalar notation of [7.1] and [7.2] leads to [4].

According to equation [4a] the effects of explanatory variables and intervention variables can be investigated by adding these variables to the observation equation [7.1]. The addition of an explanatory variable to the observation equation [7.1] is achieved by defining:

$$\alpha_t = \begin{pmatrix} \mu_t \\ v_t \\ \beta_t \\ \gamma_{1,t} \\ \gamma_{2,t} \\ \gamma_{3,t} \end{pmatrix}, \quad \eta_t = \begin{pmatrix} \xi_{t+1} \\ \zeta_{t+1} \\ \omega_{t+1} \end{pmatrix}, \quad T = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad z_t = \begin{pmatrix} 1 \\ 0 \\ x_t \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

$$Q_t = \begin{bmatrix} \sigma_\xi^2 & 0 & 0 \\ 0 & \sigma_\zeta^2 & 0 \\ 0 & 0 & \sigma_\omega^2 \end{bmatrix}, \quad \text{and} \quad R_t = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

In scalar notation, we obtain the model:

$$\begin{aligned} y_t &= \mu_t + \gamma_{1,t} + \beta_t x_t + \varepsilon_t, & \varepsilon_t &\sim \text{NID}(0, \sigma_\varepsilon^2), \\ \mu_{t+1} &= \mu_t + v_t + \xi_{t+1}, & \xi_t &\sim \text{NID}(0, \sigma_\xi^2), \\ v_{t+1} &= v_t + \zeta_{t+1}, & \zeta_t &\sim \text{NID}(0, \sigma_\zeta^2), \\ \beta_{t+1} &= \beta_t, \\ \gamma_{1,t+1} &= -\gamma_{1,t} - \gamma_{2,t} - \gamma_{3,t} + \omega_{t+1}, & \omega_t &\sim \text{NID}(0, \sigma_\omega^2), \\ \gamma_{2,t+1} &= \gamma_{1,t}, \\ \gamma_{3,t+1} &= \gamma_{2,t}, & t &= 1, \dots, T, \end{aligned}$$

which is a special case of model [6].

The Kalman filter (KF) plays the same role for time series in state-space form as least squares computations for a regression model. The KF is mainly a set vector and matrix recursions. The KF enables:

- 1) the calculation of one-step ahead predictions of observation and state vectors, and the corresponding mean squared errors;
- 2) diagnostic checking via one-step ahead prediction errors;

- 3) computation of the likelihood function by means of the one-step ahead prediction error decomposition;
- 4) calculation of smoothed estimates of the unobserved components (level, slope and seasonal).¹²

The unobserved state α_t can be estimated from the observations with the KF in the following way.

Writing $Y_t = \{y_1, \dots, y_t\}$, define $a_{t+1} = E(\alpha_{t+1} | Y_t)$ and $P_{t+1} = \text{var}(\alpha_{t+1} | Y_t)$. Then

$$\begin{aligned} v_t &= y_t - z'_t a_t, \\ F_t &= z'_t P_t z_t + \sigma_\varepsilon^2, \\ K_t &= T_t P_t z_t F_t^{-1}, \\ a_{t+1} &= T_t a_t + K_t v_t, \\ P_{t+1} &= T_t P_t T'_t + R_t Q_t R'_t - K_t F_t K'_t, \quad t = 1, \dots, n, \end{aligned}$$

for given values a_1 and P_1 . F_t is a scalar, K_t is an $m \times 1$ vector and P_t is an $m \times m$ matrix. Then, using the smoothing algorithm these estimates are used for predicting the development of the time series. The one-step ahead prediction error is:

$$\begin{aligned} v_t &= y_t - E(y_t | Y_{t-1}) \\ &= y_t - E(z'_t \alpha_t + \varepsilon_t | Y_{t-1}) \\ &= y_t - z'_t E(\alpha_t | Y_{t-1}) \\ &= y_t - z'_t a_t; \end{aligned}$$

the one-step ahead prediction error variance is $F_t = \text{var}(v_t) = z'_t P_t z_t + \sigma_\varepsilon^2$.

To evaluate the performance of the model, a common practice is to start the estimation with the most general specification, run the maximum likelihood estimation and evaluate the significance of the hyperparameters (i.e. disturbance variances). If some of the components are not significant one can re-estimate the model until all the components included are significant. A well specified STSM should achieve strong convergence of the (maximum likelihood) estimation

¹² *Smoothing* or *signal extraction* means to estimate the unobserved components at all points in the sample using all the observations.

of the hyperparameters. Moreover, the model has to pass the diagnostic tests and, if needed, exhibit good forecasting performance.

3. Banknotes in circulation in the euro area

Developments in the cash industry have, in recent years, been marked by a number of fundamental trends. There has been rapid innovation and technological progress where means of payment are concerned. In addition, the industry has also had to respond to changes in payment habits and consumption patterns (in particular the growth of e-commerce).

The dematerialisation of payment instruments has accelerated over the last five years. The ratio between card payments and ATM withdrawals, a good indicator of cash payments (see Figure 1), rose from 1.2 in 2017 to 2.1 in 2021 in the euro area. This strong growth can be mainly attributed to the outbreak of e-commerce – an area still expanding with particular speed – and, in 2020's case, to the sharp decline in ATM withdrawals.¹³ The increase is less marked for card payments at point-of-sale (POS) terminals alone, with the ratio rising from 1.1 to 1.7 over the 2017-2021 period.¹⁴ This increase was boosted by the introduction of contactless card payments.

According to the latest SPACE survey on the use of cash by consumers in the euro area, conducted in 2022 by the ECB, cash was used in 59 per cent of transactions at the point of sale, down from 79 per cent in 2016 and 72 per cent in 2019. In terms of value of payments cash accounted for 42 per cent in 2022, down from 54 per cent in 2016 and 47 per cent in 2019.¹⁵ Moreover, Zamora-Pérez (2021) estimates that transactional use of euro banknotes accounted for around 20 per cent of their circulation in value terms in 2019. Given the estimates of foreign demand in 2019 in Lalouette et al. (2021), the remaining 80 per cent is split roughly evenly between euro area citizens' cash savings (including, at the margin, cash held by credit institutions) and cash held outside the euro area, which has been constantly increasing.

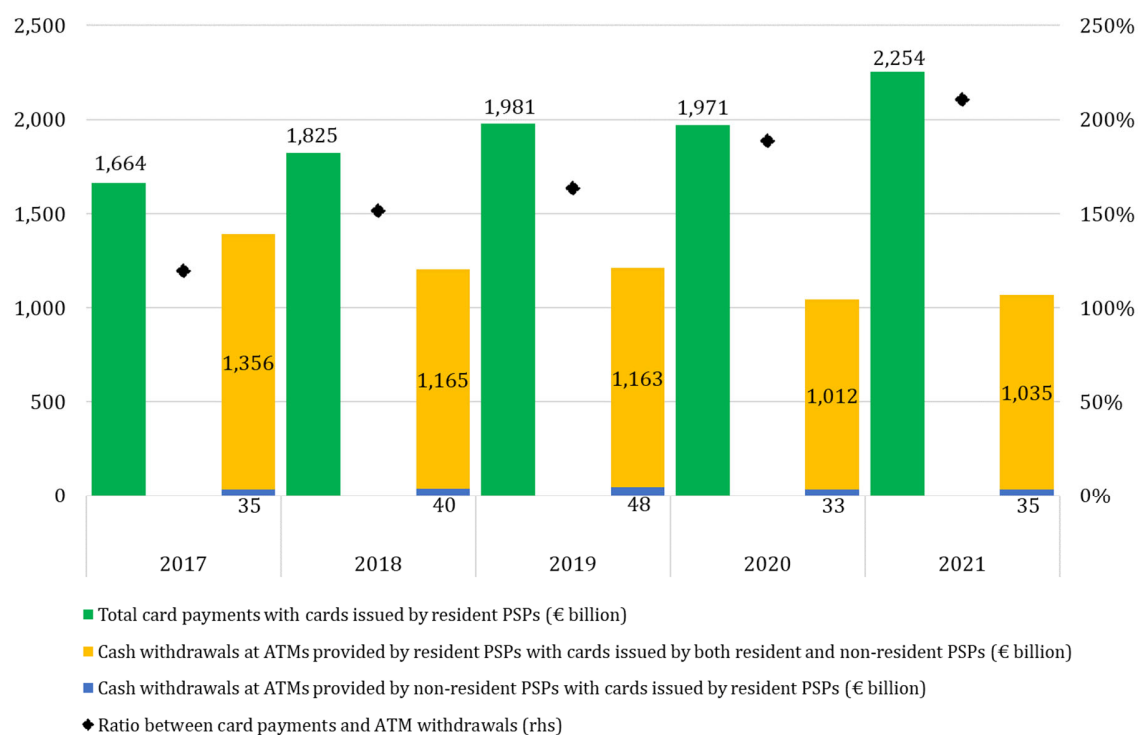
¹³ According to European Central Bank (2022a), the value of cash withdrawals at ATMs in the euro area decreased sharply from €1,210 billion in 2019 to €1,045 billion in 2020. This was primarily due to the COVID-19 restrictions. However, cash withdrawals recovered somewhat afterwards and amounted to €1,070 billion in 2021.

¹⁴ See European Central Bank (2022a, Tables 13.1 and 15.1).

¹⁵ See European Central Bank (2022b).

Figure 1

Card payments and ATM cash withdrawals in the euro area (€ billion)



Source: ECB (2022a).

Note: PSPs stands for payment service providers.

These trends resulted in a decline in the use of cash for transaction purposes, and hence in the volumes processed by the cash industry as a whole. However, the decline in inflows of banknotes to the cash industry did not prevent net issuance from growing at a sustained pace in the euro area.

This phenomenon is what has been referred to in recent years as the “cash paradox”: the use of payment instruments alternative to cash has never been so high but, at the same time, there has never been so much demand for cash among the public. This paradox can be explained by the wide range of reasons for holding cash (Zamora-Pérez, 2021). As well as being used as a means of payment, euro banknotes, and especially those with a high face value, are held by European households for store of value purposes, particularly in times of economic uncertainty and historically low interest rates. For these reasons the demand for banknotes driven by non-transactional needs has never been higher.

In order to better describe the dynamics of euro banknotes in circulation both in the euro area as a whole and in the main countries belonging to the monetary union, we need to take into account two additional factors relating to foreign demand for banknotes. The first one is represented by net shipments. The shipments cover the registered imports and exports of euro banknotes between the Eurosystem and the rest of the world (non-resident credit institutions and national central banks). These shipments are processed by international banknote wholesale banks that are active in the global market for currency dealing. Net shipments represent banknotes exported minus banknotes imported. A second factor to be taken into account are cross-border flows of euro banknotes *within* the euro area. Within a monetary union banknotes issued by the different central banks can flow between member countries. Therefore, countries visited by large numbers of international tourists and with high levels of tourism related activity may benefit from a second source of banknotes, namely those issued by other Eurosystem NCBs that flow into the country primarily through tourism.

3.1 *Banknote circulation before the pandemic*

Euro banknote circulation, defined as cumulated net issuance (“CNI”), has been constantly rising in the euro area since the new single currency was introduced. This is due to the stronger growth of cumulated withdrawals compared to cumulated lodgements of euro banknotes at Eurosystem NCBs.¹⁶

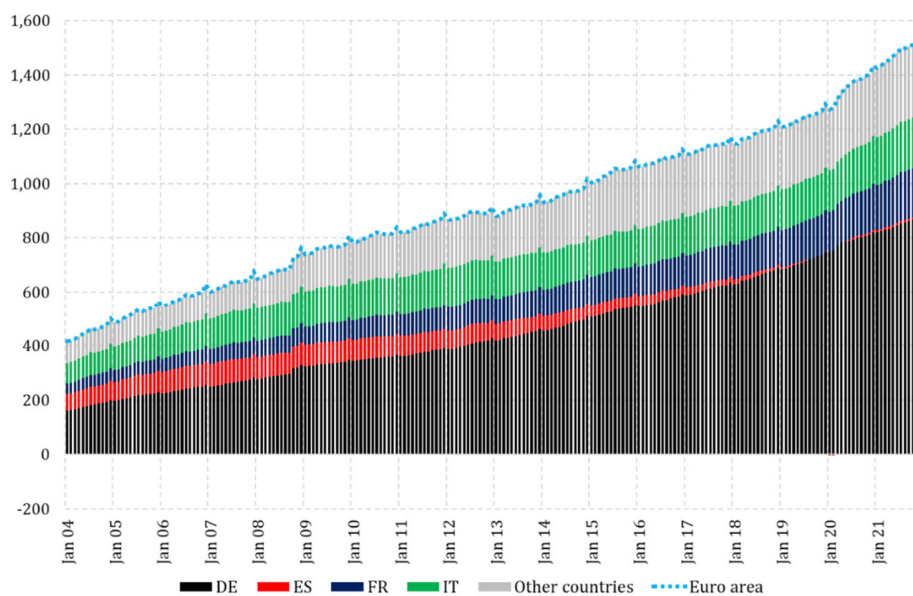
From 2002 to 2019, three main phases can be discerned (see Figures 2 and 3): i) a decade (2002-2012) of high but declining growth rates in the aftermath of the cash changeover, with two temporary recoveries following the financial and sovereign debt crises (2008 and 2011, respectively), when there was exceptional demand for high denomination notes, mostly for precautionary reasons; ii) a phase showing an upward trend in growth rates (2013-2015); iii) a period of moderate and stable positive growth (2016-2019). In terms of nominal GDP,

¹⁶ Cumulated net issuance is computed as the sum of withdrawals at a Eurosystem central bank net of the corresponding sum of lodgements since the euro’s introduction in 2002. Due to migration of notes amongst euro area countries, the aforementioned variable does not necessarily coincide with the actual amount of banknotes in circulation in a member country. Moreover, CNI does not necessarily reflect the use of cash in the economy, not least with regard to cash management activities performed in many countries by what are known as cash handlers (CIT companies and commercial banks). These cash handlers provide their customers with cash, choosing between recirculation and central bank issuance. The factors that feed into their choice include the operational and logistic costs involved (Baldo et al., 2021).

banknotes in circulation have been steadily increasing in the last two decades from slightly lower than 5 per cent to about 11 per cent at the end of 2019.

Figure 2

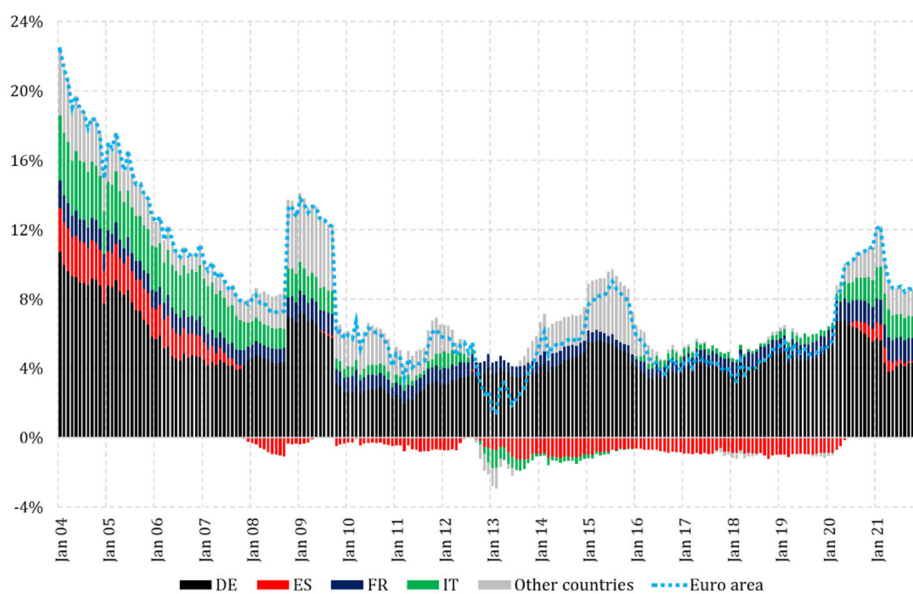
Cumulated net issuance of euro banknotes (€ billion)



Sources: ECB's CIS2 data base and authors' calculations.

Figure 3

Cumulated net issuance of euro banknotes (12-month growth rates and national contributions)



Sources: ECB's CIS2 data base and authors' calculations.

A closer look at the dynamics reveals a degree of heterogeneity between the largest countries belonging to the monetary union: while developments of cash in circulation in Italy, France and Germany broadly followed the pattern traced by the euro area apart from temporary deviations, the evolution observed in Spain is quite different.

Turning first to Italy, we see that, during the sovereign debt crisis, a legislative intervention fixing the limit for cash payments at €1,000 in combination with new anti-money laundering controls took circulation into negative growth rates in the period 2012-2016. At the same time, the cash-to-GDP ratio stopped increasing and remained broadly stable between 8 per cent and 9 per cent from 2012 to 2019.

In France, cash deposits made at Banque de France counters by banks and other cash handlers decreased by 250 million banknotes per year in the years 2014-2016 and by 500 million banknotes per year in the years 2017-2019, reducing the level of inflows from more than 7 billion banknotes in the early 2010s (peak) to about 4.6 billion in 2019. The preference of French citizens for cashless means of payment is relatively high and the digitalisation of the economy has continued, accentuated by the COVID-19 crisis. However, the decline in flows of banknotes did not prevent net issuance from growing at an average sustained pace of 7 per cent per year (in value) over the period 2010-2019. In terms of nominal GDP, CNI increased at a steady pace to around 6 per cent of GDP in 2019 compared with 2 per cent in 2002.

In Spain, after the changeover, banknotes in circulation accounted for 7 per cent of nominal GDP. CNI reached its peak at the end of 2006 and afterwards showed a continuous descent until the outbreak of the COVID-19 pandemic. It even turned negative at the end of the first quarter of 2020 (-0.2 per cent of nominal GDP). At that time, the central bank was issuing less banknotes than were being returned. This particular profile can be explained by the dynamics of foreign net cash inflows (e.g. cash flowing into Spain from abroad mainly due to tourism) which strongly increased in the last decade, providing a substantial supplementary source of banknotes.¹⁷

The development of German-issued euro banknotes in circulation was similar to that of total euro banknotes in circulation, albeit with significantly higher annual growth rates (Deutsche

¹⁷ In 2019, a historical high of 83 million tourists visited the country, tourism reached a weight of 12.4 per cent of GDP and the typical tourism-related branches generated 2.7 million jobs (12.9 per cent of total employment).

Bundesbank, 2022, page 68). In the period from January 2013 to December 2019, the average growth rate of German-issued banknotes in circulation was about 3.2 percentage points above the average annual growth rate of total euro banknotes in circulation. This growth rate differential can be attributed to the traditionally important role of foreign demand for German-issued banknotes. According to an update of the estimate of foreign demand based on net shipments and travel data in Bartzsch et al. (2011), about 60 per cent of German-issued euro banknotes on a value basis were held outside of Germany at the end of 2019 (40 per cent outside the euro area and 20 per cent in the rest of the euro area).¹⁸ However while the share of foreign demand in German-issued banknotes had been rising continuously from about 40 per cent at the end of 2004 to 70 per cent at the end of 2013, since then it has been declining. Correspondingly, the share of German domestic demand had been rising steadily from about 30 per cent at the end of 2013 to about 40 per cent at the end of 2019. This increase in the share of domestic demand is due to a significant increase in banknotes held as a domestic store of value. Their share in German-issued banknotes in circulation increased from 17 per cent to 33 per cent in the period under review whereas the corresponding share of domestic transaction balances fell from 11 per cent to about 8 per cent.¹⁹

3.2 *Banknote circulation during the COVID-19 pandemic*

The outbreak of the COVID-19 pandemic accelerated the downward trend in the use of cash as a means of payment that had been observed for years but, at the same time, total euro banknotes in circulation increased by 9.9 per cent in the second quarter of 2020.²⁰ This was the result of the exceptional decline in banknote lodgements at the national central banks of the euro area (-19.1 per cent, on a yearly basis) and of the smaller reduction in withdrawals (-9.5 per cent). The 12-month growth rate of total euro banknotes in circulation kept increasing strongly up to a high

¹⁸ Correspondingly, German-issued banknotes held outside of the euro area came to €300 billion at the end of 2019. This can be contrasted with the findings of Lalouette et al. (2021) according to which total foreign demand for euro banknotes amounted to between €400 billion (or 30 per cent of total euro banknotes in circulation) and €600 billion (or 50 per cent of total euro banknotes in circulation) at the end of 2019.

¹⁹ According to estimates by Zamora-Pérez (2021), given the foreign demand estimates of Lalouette et al. (2021) of foreign demand, the share of domestic store of value balances in total euro banknotes in circulation amounted to between 28 per cent to 50 per cent in 2019 and the corresponding share of domestic transaction balances added up to between 20 per cent to 22 per cent.

²⁰ The corresponding increase in the average value of banknotes was also reflected in higher average ATM withdrawals during the lockdown.

of 12.2 per cent in February 2021; since then it has been steadily decreasing, mainly due to base effects, though has still remained at levels markedly higher than in the years after the great financial crisis and before the pandemic (8.6 per cent in September 2021).

A number of factors may have played a role in explaining these developments. First, the pandemic induced an increase in the demand for banknotes as a store of value and for precautionary reasons: credit institutions, businesses and citizens increased their cash reserves amid increased uncertainty and a lack of expenditure opportunities. Second, international travel restrictions led to a sharp decline in tourism flows as well as of the high migratory flows of banknotes derived from it. Finally, the subdued economic activity reduced consumption and, in turn, banknote lodgements; in the years prior to 2020, consumption had increased steadily, which contributed positively to banknote lodgements. All these circumstances suggest that technical factors, too – elements not necessarily related to macroeconomic conditions or payment habits – have been at work. This makes it challenging to determine whether the increase in the non-transaction component of cash demand was “forced” (businesses and households were unable to spend on some of their usual activities) or “voluntary” (economic operators refrained from spending, in reaction to the pandemic).

Overall, during the pandemic both components of net issuance were negatively affected by travel and commercial restrictions, but lodgements reported a stronger reduction than withdrawals – automatically leading to an increase in net issuance. These patterns set the recent increase of CNI apart from any other surge observed before, previous instances having always been characterised by an increase in withdrawals, rather than a reduction.

Finally, it is worth mentioning that the COVID-19 crisis, accelerating the downward trend in the use of cash as a means of payment, led to particularly subdued dynamics for withdrawals of low denomination euro banknotes, especially in the early months of the health crisis.

Unlike in the pre-pandemic period, quite similar patterns are in evidence here across the main countries in the euro area.

In Germany, the rise in net issuance seen in 2020 – the highest increase ever except for the cash changeover year 2002 – was due solely to the sharp increase in domestic circulation in connection

with banknotes held as a store of value and in line with the increased household saving ratio.²¹ This significant increase was due to the rise in large denominations in circulation at the beginning of the COVID-19 crisis (March and April 2020), while the growth in small-denomination banknotes was more modest than usual. Large denominations predominantly used for store of value purposes (€100, €200 and €500 banknotes) as a share of annually cumulated net issuance (in value terms) rose from 58 per cent at the end of 2019 to 73 per cent at the end of 2020. The corresponding share attributable to small denominations predominantly used for transactions (€5, €10 and €20 banknotes) fell in the same period from 11 per cent to 3 per cent. The – still positive – contribution from foreign circulation was negligible, while the annually cumulated net shipments in 2020 were even in slightly negative territory due to pandemic-related restrictions on foreign travel, a key motive behind demand for shipments.²²

In Italy, CNI rose by 9.3 per cent on a yearly basis in the second quarter of 2020, as a result of the exceptional decline recorded in lodgements at Banca d'Italia (-26.3 per cent) and the more modest contraction in withdrawals (-7 per cent). Cash circulation in Italy has been particularly affected by travel restrictions. Up to the COVID-19 pandemic net foreign inflows of euro banknotes connected to tourism activity had reduced the net issuance of Banca d'Italia owing to increased lodgements. During the pandemic, however, this channel became less important due to the travel restrictions. The breakdown by denomination shows that the observed dynamic was mostly driven by €50 banknotes.²³

The outbreak of the COVID-19 pandemic broke the downward trend observed in the Spanish CNI since late 2006. In 2020, CNI was almost €10 billion compared to €0.4 billion in 2019. Similarly to Italy, international travel restrictions led to a sharp decline in tourism flows as well as in the high migratory flows of banknotes derived from it. As a consequence, the inflows of banknotes on the central bank's balance sheet fell notably leading to an increase in net issuance. The breakdown by denomination of euro currency issued by Banco de España was similar to that observed in Italy and France: the pandemic crisis led to a large increase in the demand for €50 banknotes, whereas the rate of return for the larger banknotes slowed down.

²¹ At an estimated €70 billion, the total store of value formed in 2020 was twice the size of those formed in each of the previous two years.

²² For a comprehensive description of German-issued banknotes in circulation, see Deutsche Bundesbank (2022).

²³ For an extensive description of Italian-issued banknotes in circulation since the changeover see Baldo et al. (2021).

Against the backdrop of the pandemic, the decline of inflows in France became even more pronounced in 2020 (–1.1 billion banknotes), bringing the level of inflows to an all-time low of 3.5 billion banknotes. Over the same period, French CNI rose by almost 11 per cent or rather €16 billion. When it comes to annually cumulated net issuance (on a 12-months rolling basis), this was even more pronounced with an increase of 30% in 2020 compared to 2019. The high denominations have remained negative (up to €2.4 billion in 2020), but to a lower extent than in the previous year (up to €3.7 billion in 2019). The €50 was the most impacted denomination as it grew by 16 per cent compared with the preceding year. The higher demand for the €50 denomination in recent years can also be illustrated by the outflow statistics, as its corresponding share increased from 20 per cent of the volumes withdrawn in 2016 to 27 per cent in 2020. Between 2019 and 2020, the rise in the share of the €50 denomination at 2.5 percentage points was particularly relevant. By contrast, the share of the €20 banknote remained broadly stable (around 39 per cent) and the share of the €10 banknote decreased (from 34 per cent in 2016 to 30 per cent in 2020).²⁴

4. Forecasting national-issued banknotes in circulation with structural time series models

In this section we present the structural time series forecasting models that were developed by the Banque de France, Deutsche Bundesbank, Banca d'Italia and Banco de España in order to forecast their national banknotes in circulation. The purpose of those exercises is to improve the statistical forecasts that feed into the Eurosystem's yearly banknote production requirement planning. At present, banknote production requirements are calculated using two different approaches so as to gain robust results: (i) bottom-up based on national forecasts provided by the national central banks and (ii) top-down using a centralised euro area forecast made by the ECB (ABCD-2 forecast). National expertise is thus combined with a euro-area-wide perspective. The national forecasts of the bottom-up approach are not harmonised: each national central bank of the Eurosystem can decide for itself which forecasting models to choose and how to evaluate them. This also applies to the evaluation of the forecasts presented in this section. That is why the goodness-of-fit measures as well as the measures for forecast accuracy and the

²⁴ For more details, see Seitz, Devigne and de Pastor (2022) as well as Banque de France (2021).

corresponding benchmark models may differ between the 4 NCBs. The benchmark models are those models that are currently being used in the framework of banknote production requirement planning at the national central banks. In France, Germany and Spain these are ARIMA models.²⁵ Banca d'Italia derives its forecast of banknotes in circulation from combining the predictions of withdrawals and lodgements generated by a basket of models which also includes ARIMA models and exponential smoothing.

This section juxtaposes the main features of the structural times series models (“STSMs”) developed by the 4 NCBs for forecasting their banknotes in circulation based on the methodology described in Section 2.²⁶ We first compare the regression specifications. We then check the forecasting performance of the models against the national benchmark models currently employed by the 4 NCBs.

The regression specifications are summarised in Tables 1 to 3. For ease of analysis the low-denomination banknotes (Table 1) and the large-denomination banknotes²⁷ (Table 3) are not shown separately but as an aggregate. This means that – unless stated otherwise – the entries in these tables represent the sum of the specifications of the underlying denominations. The models of Banca d'Italia and Banco de España employ estimation samples running from January 2004 to September 2020. The observations of the first two years of euro cash have been excluded in order to remove the effects of the euro cash changeover. In contrast, the estimation samples of the STSMs of Germany run from January 2002 to September 2020 and the effects of the cash changeover are accounted for by trend breaks. The STSMs of Banque de France are mainly based on the whole sample, except for the €100 denomination where observations relating to the cash changeover have been removed to ensure convergence. All of the estimation samples end in September 2020 in order to include enough observations to capture the impact of the COVID-19 pandemic (March 2020 to September 2020). Each of the 4 NCBs has chosen the time period from October 2020 to September 2021 as a forecast horizon. In view of the considerable uncertainty about the further effects of the pandemic, we have refrained from extending the length of the forecast horizon. The specifications of the STSMs usually include a

²⁵ The benchmark ARIMA models used by Banque de France and the Bundesbank are described in Annex 1 and Annex 2.

²⁶ The national forecasting models are presented in more detail in Annex 1 to Annex 4.

²⁷ €500 banknotes in circulation are not considered since they are not issued anymore.

trend consisting of a stochastic level and a stochastic slope, a trigonometric seasonal component, an irregular component, intervention variables²⁸ and (usually for small- and medium-denomination banknotes in circulation) dummy variables for Ascension Day and Easter as explanatory variables. The explanatory variables represent calendar effects that lead to an increase in banknote demand for transaction purposes. The intervention variables play a prominent role as they allow for all kinds of events that have had an impact on banknotes in circulation. Most of these events are related to various crises that led to a rise in banknote demand. This is in line with the fact that crises usually coincide with an increased demand for cash. In times of crises people prefer to rely on tried and tested concepts and place their confidence in the default-free central bank money that we know as cash.²⁹

The specifications for Italian-issued banknotes in circulation stand out in three respects. First, they do without regression variables for calendar effects. Second, the Italian model for €10 banknotes is the only STSM to contain a (deterministic) cycle component as an additional unobserved component. Third, they include only a few intervention variables and the effects of these are temporary (outlier interventions). This result probably follows from the prominent role of the trend component in describing the series. The outlier interventions represent the great financial crisis (in the case of medium to large denominations) and measures related to the banknote issuance policy in the case of low-denomination banknotes.³⁰

The model specifications of the STSMs for small-denomination banknotes in circulation are shown in Table 1. Banknotes belonging to this grouping are usually used for transaction purposes. The great financial crisis resulted in a positive level break – that is a permanent increase – in French-issued €20 banknotes. The restrictions on withdrawals from ATMs in Greece in the summer of 2015, when the debt crisis came to a head, had a positive and permanent impact on low-denomination (and €50) German-issued banknotes. At that time, the Greek government was in negotiations with the European Commission, the rest of the euro area, the ECB and the

²⁸ *Outlier interventions* (impulse dummies) represent transitory effects on banknotes in circulation. They are only taken into account here if they are explicable, that is, if they can be related to some event like for example a policy measure. In contrast to outliers, *trend breaks* have a permanent effect on banknotes in circulation. A *level break* is modelled by means of a step dummy and a *slope break* by means of a staircase function.

²⁹ Rösl and Seitz (2021) examine the demand for small-denomination and large-denomination banknotes since the 1990s from an international perspective. They show that demand for cash always increases in times of crisis, independent of the nature of the crisis itself.

³⁰ Specifically, these measures comprise the introduction of the €5 banknote from the Europa series (ES2) and policy measures to improve the quality of banknotes in circulation.

International Monetary Fund about a further bailout programme in return for reform measures. As a result, German tourists took more €10 and €20 banknotes with them on their travels to Greece. This additional demand was transaction-related.

The intervention variables for the COVID-19 pandemic are of particular importance for the purpose of this study. The pandemic stands for major trend breaks in banknotes in circulation at the end of the estimation sample. As a result, forecasts cannot be evaluated in-sample by retaining some observations for pseudo out-of-sample forecasts.³¹ Instead, we had to content ourselves with an out-of-sample evaluation of the forecast accuracy that is based on only 12 observations (October 2020 to September 2021).³² The models in Table 1 contain a positive trend break at the onset of the pandemic which is sometimes accompanied by a negative trend break in the subsequent month. As an explanation for this development in Germany it stands to reason that consumers and professional cash handlers built up considerable cash holdings for precautionary reasons at the start of the first lockdown in March 2020, which is reflected in the positive level break.³³ Concerns regarding potential logistical restrictions on the supply of cash during lockdown were probably behind this precautionary motive. As from April 2020, the month in which the reduction of precautionary holdings began, German net issuance of small-denomination banknotes was well below the previous year's figure.³⁴ This intervention effect is allowed for by the negative slope break in April 2020.

€50 banknotes are in demand for use both in transactions and as a store of value. The model specifications of the STSMs for €50 banknotes in circulation, shown in Table 2, are similar to those for small-denomination banknotes in circulation. However, unlike in the case of low-denomination banknotes, we now have the great financial crisis coming into play in Germany

³¹ For example, without the COVID-19 health crisis the models for small-denomination banknotes in circulation could have been estimated with data running from January 2002 or rather January 2004 to December 2014 (estimation sample). Pseudo out-of-sample forecasts running from January 2015 to September 2020 could then have been calculated and compared with actual banknotes in circulation.

³² In view of the considerable uncertainty relating to the COVID-19 sanitary crisis, we abstained from expanding the forecasting horizon.

³³ The first lockdown in response to the COVID-19 health crisis was between March 2020 and May 2020 and the second lockdown lasted from November 2020 to May 2021. Since September 2021, varying degrees of restrictions on public life linked to vaccination status were in effect in Germany.

³⁴ Looking at 2020 as a whole, net issuance of German-issued low-denomination banknotes was around 64 per cent lower than the level seen in the previous year. This was due to the significant decline in the use of cash at the point of sale as a result of limited opportunities to spend cash – in retail, restaurants and at fairs, for example – owing to the pandemic.

and Italy as well as the end of €500 banknote issuance in France and Germany. These interventions resulted in a sustained and significant increase in French-issued and German-issued €50 banknotes in circulation.³⁵ In the financial crisis liquid, secure assets for use as a store of value were sought after, which caused demand for large-denomination banknotes especially to grow steeply.³⁶ The ECB Governing Council's decision on 4 May 2016 to discontinue the production and issuance of the €500 banknote resulted in declines in these banknotes and shifts to the next smaller denominations all the way to the German-issued €50 banknote. In France, this decision impacted €50 banknotes in circulation only after issuance of the €500 banknote was actually stopped at the end of January 2019. The Spanish-issued €50 banknotes in circulation were impacted by the uncertainty resulting from the financial bailout in May 2012. This entailed an instantaneous and permanent increase in Spanish net issuance of €50 banknotes. The reduction of the precautionary holdings of €50 banknotes that were built up in Germany at the beginning of the COVID-19 pandemic is reflected in the negative slope break in May 2020. This process took longer for the €50 and the high denominations, as it was obscured by the simultaneous build-up of cash reserves as a store of value. This sharp rise in the store of value was mainly due to the restrictions on opportunities for spending on account of the measures adopted to contain the pandemic.

Large-denomination banknotes are typically used for store-of-value purposes. The model specifications of the STSMs for large-denomination banknotes in circulation (Table 3) differ from those for €50 banknotes in circulation in a number of ways. In addition to the positive level break in October 2008 the effect of the financial crisis on large-denomination German-issued banknotes contains a negative level break in December 2008 and a negative slope break in February 2009. Therefore, the rise in large-denomination banknotes in circulation in Germany in October 2008 was (at least partly) reduced in the subsequent months.³⁷ In contrast to the medium-denomination banknotes, the cessation of €500 banknote issuance and the COVID-19 pandemic did not have any effect on French-issued large-denomination banknotes in circulation. Unlike the smaller denominations driven by transactional demand, German-issued large-denomination banknotes were not affected by the limitation of outpayments at ATMs in Greece

³⁵ The financial crisis also affected French-issued €20 banknotes in circulation.

³⁶ See Deutsche Bundesbank (2009).

³⁷ For details, see Table 1a in Annex 2.

in 2015. In order to model the impact of the halt to €500 banknote issuance in Germany, positive slope breaks in April 2016 and in April 2019 were added to the STSM for large-denomination banknotes. The latter intervention variable reflects the additional substitution of German-issued €500 banknotes in circulation after issuance was actually discontinued. The models for Spain contain four additional intervention variables. First, the great financial crisis led to a decrease (that is, a negative slope break) in Spanish-issued €200 banknotes in circulation in September 2008. This can be easily explained by a lower demand for banknotes in Spain owing to the concomitant housing bust. Second, the decision of the ECB Governing Council on 4 May 2016 to stop the issuance of €500 banknotes resulted in an increase of €200 banknotes in circulation.³⁸ Third, in 2012 a series of tax reforms were introduced in order to prevent and fight tax fraud. These reforms limited cash payments to a sum of €2,500 where one of the parties is a professional business person. This led to a decrease in the net issuance of €100 banknotes in November 2012 (negative slope break). Fourth, the political instability in Catalonia entailed an increase in €100 banknotes in circulation in October 2017 (positive level break).

³⁸ See Table 2a in Annex 4.

Table 1

Specifications of structural time series models for small-denomination banknotes

National central bank	Dependent variable y : national-issued small-denomination banknotes in circulation (€5 to €20) ¹⁾			
	France	Germany	Italy	Spain
Estimation sample, number of observations	2002:01 - 2020:09, 225	2002:01 - 2020:09, 225	2004:01 - 2020:09, 201	2004:01 - 2020:09, 201
Forecast horizon, length in months	2020:10 - 2021:09, 12	2020:10 - 2021:09, 12	2020:10 - 2021:09, 12	2020:10 - 2021:09, 12
Model specification ²⁾	$y = \text{trend} + \text{seasonal} + \text{irregular} + \text{explanatory variable} + \text{interventions}$	$y = \text{trend} + \text{seasonal} + \text{irregular} + \text{explanatory variables} + \text{interventions}$	$\log(y) = \text{trend} (+ \text{cycle}) + \text{seasonal} + \text{irregular} + \text{interventions}$	$y = \text{trend} + \text{seasonal} + \text{irregular} + \text{explanatory variable} + \text{intervention}$
Unit of dependent variable y	number in million pieces	number in million pieces	number in million pieces	number in million pieces
Specification of trend	stochastic level and stochastic slope	stochastic level and stochastic slope	stochastic level and stochastic slope	stochastic level and stochastic slope
Specification of cycle	-	-	€10: deterministic cycle	-
Specification of seasonal	fixed trigonometric	fixed trigonometric	fixed trigonometric	stochastic trigonometric
Explanatory variables	dummy variable for Easter (+)	dummy variables for Ascension Day (+) and Easter (+)	-	dummy variable for Easter (+)
Interventions for <i>cash changeover</i>	level breaks in 2002:03 (+), 2002:04 (+), 2002:05 (-) and 2002:12 (+), slope breaks in 2002:05 (+) and 2002:07 (+)	level breaks in 2002:02 (-) and 2002:04 (-)	-	-
Interventions for <i>great financial crisis</i>	€20: level break in 2008:10 (+)	-	-	-
Interventions for <i>Europa series issuance</i>	-	-	€5: outliers in 2013:04 (-) and 2013:05 (+)	-
Interventions for <i>limitation of outpayments at ATMs in Greece in 2015</i>	-	level break in 2015:07 (+)	-	-
Interventions for <i>policy to improve the quality of banknotes in circulation</i>	-	-	€10: outliers in 2019:07 (+) and 2019:12 (+)	-
Interventions for <i>COVID-19 crisis</i>	€5: level break in 2020:04 (-) and slope break in 2020:03 (+), €20: level breaks in 2020:03 (+) and 2020:05 (+)	level break in 2020:03 (+) and slope break in 2020:04 (-)	-	slope break in 2020:04 (+)

Source: Authors' calculations.

Notes:

1) Each of the three denominations were estimated separately. However, for ease of analysis, the specifications shown in this table apply to the total of small-denomination banknotes (unless stated otherwise). That means they are equal to the sum of the specifications for €5, €10 and €20 banknotes in circulation.

2) +/-: sign of the (significant) coefficients of intervention and explanatory variables. Only explicable outlier interventions (impulse dummies) are shown in this table.

Table 2

Specifications of structural time series models for €50 denomination banknotes

	Dependent variable y : national-issued €50 banknotes in circulation			
	France	Germany	Italy	Spain
National central bank				
Estimation sample, number of observations	2002:01 2020:09, 225	2002:01 - 2020:09, 225	2004:01 2020:09, 201	2004:01 2020:09, 201
Forecast horizon, length in months	2020:10 - 2021:09, 12	2020:10 - 2021:09, 12	2020:10 - 2021:09, 12	2020:10 - 2021:09, 12
Model specification ¹⁾	$y = \text{trend} + \text{seasonal} + \text{irregular} + \text{explanatory variable} + \text{interventions}$	$y = \text{trend} + \text{seasonal} + \text{irregular} + \text{explanatory variables} + \text{interventions}$	$\log(y) = \text{trend} + \text{seasonal} + \text{irregular} + \text{interventions}$	$y = \text{trend} + \text{seasonal} + \text{irregular} + \text{explanatory variable} + \text{interventions}$
Unit of dependent variable y	number in million pieces	number in million pieces	number in million pieces	number in million pieces
Specification of trend	stochastic level and stochastic slope	stochastic level and stochastic slope	stochastic level and stochastic slope	stochastic level and stochastic slope
Specification of seasonal	fixed trigonometric	fixed trigonometric	fixed trigonometric	stochastic trigonometric
Explanatory variables	dummy variable for Easter (+)	dummy variable for Easter (+)	-	dummy variable for Easter (+)
Interventions for <i>cash changeover</i>	-	level break in 2002:03 (+)	-	-
Interventions for <i>the great financial crisis</i>	level break in 2008:10 (+)	level break in 2008:10 (+)	outlier in 2008:09 (-)	-
Interventions for <i>financial bailout in Spain</i>	-	-	-	level break in 2012:05 (+)
Interventions for <i>limitation of outpayments at ATMs in Greece in 2015</i>	-	level break in 2015:07 (+)	-	-
Interventions for <i>stop of issuance of €500 banknotes</i> ²⁾	slope break in 2019:03 (+)	level break in 2016:06 (+)	-	-
Interventions for <i>COVID-19 crisis</i>	level break in 2020:05 (+)	level break in 2020:03 (+) and slope break in 2020:05 (-)	-	level break in 2020:04 (+)

Source: Authors' calculations.

Notes:

1) +/-: sign of the (significant) coefficients of intervention and explanatory variables. Only explicable outlier interventions (impulse dummies) are shown in this table.

2) On 4 May 2016 the ECB Governing Council decided to halt the production of €500 banknotes.

Table 3

Specifications of structural time series models for large-denomination banknotes

National central bank	Dependent variable y : national-issued large-denomination banknotes in circulation (€100 and €200) ¹⁾			
	France	Germany	Italy	Spain
Estimation sample, number of observations	2002:01 - 2020:09, 225	2002:01 - 2020:09, 225	2004:01 - 2020:09, 201	2004:01 - 2020:09, 201
Forecast horizon, length in months	2020:10 - 2021:09, 12	2020:10 - 2021:09, 12	2020:10 - 2021:09, 12	2020:10 - 2021:09, 12
Model specification ²⁾	€100: $y = \text{trend} + \text{seasonal} + \text{irregular} + \text{explanatory variables} + \text{interventions}$	€100: $y = \text{trend} + \text{seasonal} + \text{irregular} + \text{explanatory variables} + \text{interventions}$	$\log(y) = \text{trend} + \text{seasonal} + \text{irregular} + \text{interventions}$	$y = \text{trend} + \text{irregular} + \text{interventions}$
	€200: $y = \text{trend} + \text{seasonal} + \text{irregular} + \text{interventions}$	€200: $\log(y) = \text{trend} + \text{seasonal} + \text{irregular} + \text{interventions}$		
Unit of dependent variable y	number in million pieces	number in million pieces	number in million pieces	number in million pieces
Specification of trend	stochastic level and stochastic slope	stochastic level and stochastic slope	stochastic level and stochastic slope	stochastic level and stochastic slope
Specification of seasonal	fixed trigonometric	stochastic trigonometric	fixed trigonometric	-
Explanatory variables	€100: dummy variable for Easter (+)	€100: dummy variable for Easter (+)	-	-
Interventions for cash changeover	€200: level break in 2002:12 (+)	€200: slope break in 2002:03 (-)	-	-
Interventions for the great financial crisis	level break in 2008:10 (+)	level breaks in 2008:10 (+) and 2008:12 (-), slope break in 2009:02 (-)	€100: outlier in 2008:09 (+), €200: outliers in 2008:09 (-) and 2008:10(+)	€200: slope break in 2008:09 (-)
Interventions for introduction of limits for cash payments	-	-	-	€100: slope break in 2012:11 (-)
Interventions for stop of issuance of €500 banknotes ³⁾	-	level break in 2016:06 (+), slope breaks in 2016:04 (+) and 2019:04 (+)	-	€200: level break in 2016:09 (+)
Interventions for political instability in Catalonia	-	-	-	€100: level break in 2017:10 (+)
Interventions for COVID-19 crisis	-	level break in 2020:03 (+) and slope break in 2020:05 (-)	-	€100: slope break in 2020:04 (+), €200: level breaks in 2020:03 (-) and 2020:04 (+)

Source: Authors' calculations.

Notes:

1) Each of the two denominations was estimated separately. However, for ease of analysis, the specifications shown in this table apply to the total of large denomination banknotes (unless stated otherwise). That means they are equal to the sum of the specifications for €100 and €200 banknotes in circulation.

2) +/-: sign of the (significant) coefficients of intervention and explanatory variables. Only explicable outlier interventions (impulse dummies) are shown in this table.

3) On 4 May 2016 the ECB Governing Council decided to halt the production of €500 banknotes. For logistical reasons, issuance was not stopped in Germany until 26 April 2019, the other three NCBs halted issuance on 27 January 2019.

The forecasting performance of the STSMs and the respective benchmark models currently used for national banknote requirement planning are shown in Table 4. According to the forecast accuracy measures employed, the STSMs outperform the benchmark models for each denomination in Spain. In France and Italy, STSMs do a better job at forecasting banknotes in circulation for all but the €50 denomination and in Germany for four out of six denominations (€10, €20, €50 and €200). However, it should be borne in mind that the statistical informative value of this comparison is limited by the short projection period of just 12 months. All things considered, STSMs seem to be a promising extension to time series models currently employed in the context of national banknote requirement planning at the national central banks of France, Germany, Italy and Spain. Nevertheless, in order to assess the robustness of this finding, further exercises of this kind are advisable.

Table 4

Forecast accuracy of structural time series models versus benchmark models

	National-issued banknotes in circulation					
	€5 banknotes	€10 banknotes	€20 banknotes	€50 banknotes	€100 banknotes	€200 banknotes
France:						
Estimation sample, number of observations	2002:01 - 2020:09, 225	2002:01 - 2020:09, 225	2002:01 - 2020:09, 225	2002:01 - 2020:09, 225	2003:01 - 2020:09, 213	2002:01 - 2020:09, 225
Forecast horizon, length in months	2020:10 - 2021:09, 12	2020:10 - 2021:09, 12	2020:10 - 2021:09, 12	2020:10 - 2021:09, 12	2020:10 - 2021:09, 12	2020:10 - 2021:09, 12
RMSE of STSM	4.9	7.3	14.4	28.5	2.7	0.1
RMSE of benchmark ARIMA model	8.4	8.3	21.2	28.1	6.6	0.8
Germany:						
Estimation sample, number of observations	2002:01 - 2020:09, 225	2002:01 - 2020:09, 225	2002:01 - 2020:09, 225	2002:01 - 2020:09, 225	2002:01 - 2020:09, 225	2002:01 - 2020:09, 225
Forecast horizon, length in months	2020:10 - 2021:09, 12	2020:10 - 2021:09, 12	2020:10 - 2021:09, 12	2020:10 - 2021:09, 12	2020:10 - 2021:09, 12	2020:10 - 2021:09, 12
RMSE of STSM	49.1	62.2	44.8	24.4	22.3	4.3
RMSE of benchmark ARIMA model	31.5	79.7	75.5	67.8	7.0	83.6
Italy:						
Estimation sample, number of observations	2004:01 2020:09, 201	2004:01 2020:09, 201	2004:01 2020:09, 201	2004:01 2020:09, 201	2004:01 2020:09, 201	2004:01 2020:09, 201
Forecast horizon, length in months	2020:10 - 2021:09, 12	2020:10 - 2021:09, 12	2020:10 - 2021:09, 12	2020:10 - 2021:09, 12	2020:10 - 2021:09, 12	2020:10 - 2021:09, 12
WMAPE of STSM	1.0%	1.3%	1.8%	0.9%	5.2%	0.1%
WMAPE of benchmark models	2.9%	1.8%	4.3%	0.9%	11.0%	0.2%
Spain:						
Estimation sample, number of observations	2004:01 2020:09, 201	2004:01 2020:09, 201	2004:01 2020:09, 201	2004:01 2020:09, 201	2004:01 2020:09, 201	2004:01 2020:09, 201
Forecast horizon, length in months	2020:10 - 2021:09, 12	2020:10 - 2021:09, 12	2020:10 - 2021:09, 12	2020:10 - 2021:09, 12	2020:10 - 2021:09, 12	2020:10 - 2021:09, 12
MAPE of STSM	4.7%	0.4%	1.5%	2.4%	1.3%	16.1%
MAPE of benchmark ARIMA model	4.9%	1.7%	2.1%	3.8%	2.8%	24.3%

Source: Authors' calculations.

Notes:

RMSE: root mean square error, MAPE: mean absolute percentage error, WMAPE: weighted mean absolute percentage error,

STSM: structural time series model.

5. Summary and conclusions

Avoiding shortages or excessive surpluses of banknotes is crucial for the Eurosystem to provide credit institutions and citizens with the quantity demanded in a cost-effective way. To this end, the Eurosystem determines banknote production requirements once a year. For this process it is necessary to forecast the net issuance of each denomination together with other variables. Previously, banknote circulation was forecasted by exploiting cointegration relationships (the “ABCD model”). Despite some updates, the ABCD model was not able to fully reflect developments affecting cash demand over recent years at the time. This applied especially to the ramifications of the 2008 financial crisis. A new Eurosystem workstream was set up with the aim of improving the previous forecasting model of euro banknote demand (“ABCD-2” workstream). Drawing on the experience from the work of ABCD-2, experts at the NCBs of France, Germany, Italy and Spain believed that it was worthwhile developing structural time series models as an alternative tool for forecasting their national-issued banknotes in circulation by denomination. This paper presents the outcome of this endeavour.

The benchmark models currently used for national banknote requirement planning by the 4 NCBs essentially belong to the ARIMAX family. STSMs represent a very broad class of time series models including ARIMAX models and exponential smoothing models. They break a time series down into its unobservable components such as trend and season. As in regression models, explanatory variables and intervention variables can also be incorporated. Here the explanatory variables represent calendar effects on banknotes in circulation, and intervention variables stand for special events which lead to trend breaks and outliers. In this analysis a variety of crisis-related events turned out to be significant. These include global crises like the great financial crisis and the COVID-19 pandemic as well as local crises like the Catalan political crisis or the financial bailout in Spain. Moreover, intervention variables for the euro cash changeover and the halt to €500 banknote issuance also play a role. The structural time series models developed by Banca d'Italia are quite parsimonious in that they mainly consist of the unobserved components of Italian-issued banknotes in circulation like trend and season, and include dummies to account for the introduction of the second series of €5 banknotes, the implementation of measures to improve the quality of €10 banknotes, and the impacts the great financial crisis exerted on the circulation of higher denominations. This might stem from the prominent role of the trend component in describing Italian-issued banknotes in circulation.

Comparing the models of Banque de France, Deutsche Bundesbank and Banco de España with one another leads us to a number of findings, as detailed below. In France and Germany, the great financial crisis led to a positive level break in €50 and higher denomination banknotes in circulation in October 2008. This was because people sought liquid and secure assets as a store of value in this period. In Spain, meanwhile, the great financial crisis resulted in a decrease in €200 banknotes in circulation. This might be explained by a lower demand for banknotes as a consequence of the corresponding housing bust in Spain. In Germany, the COVID-19 pandemic had an impact on each denomination; in Spain it had an impact on all denominations apart from €5 banknotes. In France it only affected the €5, €20 and €50 denominations. A lowest common denominator can be identified across all the countries under review, with a significant increase in banknotes in circulation occurring more or less at the onset of the crisis in spring 2020. This was probably due to precautionary motives or concerns regarding potential logistical restrictions on the supply of cash. In Germany these precautionary holdings were scaled back again in the subsequent months. On 4 May 2016 the ECB Governing Council announced that the production and issuance of the €500 banknote would be discontinued. This led to declines in these banknotes and shifts to the next smaller denominations. In addition to these “global” intervention variables, country specific interventions play a role for Spanish-issued banknotes in circulation.

According to the forecast accuracy measures employed, the STSMs outperform the benchmark models for each denomination in Spain. In France and Italy, STSMs do a better job at forecasting banknotes in circulation for all but the €50 banknotes and in Germany for four out of six denominations (€10, €20, €50 and €200). However, it should be borne in mind that the statistical informative value of this comparison is limited by the short projection period of just 12 months. All things considered, STSMs seem to be a promising extension to the time series models currently employed in the context of national banknote requirement planning at the 4 NCBs. Nevertheless, in order to assess the robustness of this finding, further exercises of this kind are advisable.

Annex A: Forecasting French-issued banknotes in circulation

Table 1a (at the end of this section) contains the results of the maximum likelihood estimation of structural time series models (STSMs) for the number of French-issued banknotes in circulation. The models were estimated using monthly data from January 2002 to September 2020 except for the €100 denomination. The sample for €100 banknotes ranges from January 2003 to September 2020 to ensure convergence of the model. The forecast horizon runs from October 2020 to September 2021. All of the models contain a trend, a seasonal component, an irregular component and interventions. In most of the models the trend consists of a stochastic level and a stochastic slope and the seasonal component is fixed. While the underlying development of banknotes in circulation is described using trend and season as unobservable components, “intervention variables” are added to explain the time series. Additionally, for all of the denominations except for the high denominations (€100 to €500 banknote) the calendar effect of the Easter holidays is modelled by means of a regression variable. The intervention variables capture special factors resulting in trend breaks.³⁹ These comprise 1) the euro cash changeover at the beginning of 2002, 2) the escalation of the great financial crisis in October 2008, 3) the ECB Governing Council’s decision to discontinue the production and issuance of the €500 banknote around the end of 2018 and 4) the COVID-19 pandemic.

All of the estimated coefficients of the STSMs in Table 1a are significant at the 1 per cent level at least.⁴⁰ All significance tests in linear Gaussian models are based on three assumptions concerning the standardised one-step-ahead prediction errors. These residuals should satisfy the following three properties, which are listed in decreasing order of importance: 1) independence, 2) homoscedasticity and 3) normality.⁴¹ These and other diagnostic statistics and goodness-of-fit measures are shown in Table 1b at the end of this section. Independence was tested by means of the Ljung-Box Q -statistic. Accordingly, most of the models show serial correlation at four lags except for the €100 banknotes. According to the $H(b)$ test statistic the null hypothesis of homoskedastic residuals is rejected in the STSMs for €100 and €200 banknotes and, to a lesser extent, for the €5 denomination. Similarly, according to the Doornik-Hansen statistic, the null

³⁹ The intervention variables also include outliers

⁴⁰ Moreover, when estimating the models, in each case very strong convergence could be achieved.

⁴¹ See Commandeur and Koopman (2007, Section 8.5).

hypothesis of normally distributed residuals is rejected in the STSMs for the €50 to €500 banknotes. Hence, with regard to the three desirable criteria mentioned above, the STSMs for French denominations are acceptable. The parameters are stable in the last two years of the sample at least. In a correctly specified model, the ratio of the prediction error variance and the prediction error mean deviation should be approximately equal to unity. This condition is fulfilled by the models especially for the €10 denomination. The fit of the models in terms of the coefficient of determination based on the difference around the seasonal mean lies in a range of 0.56 to 0.97, which seems to be reasonable.

The out-of-sample forecasts resulting from the STSMs and the corresponding benchmark ARIMA models are compared in Figures A1 to A6. In the bottom part of Table 1b the root mean squared errors of 12-step-ahead forecasts are listed for the STSMs and benchmark ARIMA models. Correspondingly, the structural time series models outperform the ARIMA models for all of the denominations except for the €50 banknotes. However, the statistical informative value of this comparison is limited by the short forecasting horizon of just 12 months.

The benchmark ARIMA models, which are traditionally used in banknote requirement planning at Banque de France, are described in Table 2 at the end of this section. For example, an $ARIMA(4,1,4)(0,1,1)_{12}$ was fitted to €50 banknotes in circulation. Except for €50 banknotes in circulation, the ARIMA models outperform the STSMs in terms of serial correlation at higher lags. The null hypothesis of homoskedastic residuals cannot be rejected for any denomination. The null hypothesis of normally distributed residuals is rejected for all but the €10 denomination.

These findings can be recapitulated as follows. In terms of diagnostic tests, the benchmark ARIMA models perform better than STSMs. When it comes to forecasting, the STSMs generally perform better than the ARIMA models according to the rather limited evidence at hand. Hence it might be promising to use STSMs in addition to the traditional ARIMA forecasting models in the context of banknote requirement planning at Banque de France. Running these models head to head against each other in real time will shed more light on their relative performance.

Table 1a

Structural time series forecasting models: estimation results

	€5 banknotes	€10 banknotes	€20 banknotes	€50 banknotes	€100 banknotes	€200 banknotes
Estimation sample, number of observations	2002:01 - 2020:09, 225	2002:01 - 2020:09, 225	2002:01 - 2020:09, 225	2002:01 - 2020:09, 225	2003:01 - 2020:09, 213	2002:01 - 2020:09, 225
Forecast horizon, length in months	2020:10 - 2021:09, 12	2020:10 - 2021:09, 12	2020:10 - 2021:09, 12	2020:10 - 2021:09, 12	2020:10 - 2021:09, 12	2020:10 - 2021:09, 12
Model specification ^{1),2)} (y : number of French-issued banknotes in circulation in million pieces)	$y = \text{trend} + \text{seasonal} + \text{irregular} + \text{explanatory variable} + \text{interventions}$	$y = \text{trend} + \text{seasonal} + \text{irregular} + \text{explanatory variable} + \text{interventions}$	$y = \text{trend} + \text{seasonal} + \text{irregular} + \text{explanatory variable} + \text{interventions}$	$y = \text{trend} + \text{seasonal} + \text{irregular} + \text{explanatory variable} + \text{interventions}$	$y = \text{trend} + \text{seasonal} + \text{irregular} + \text{interventions}$	$y = \text{trend} + \text{seasonal} + \text{irregular} + \text{interventions}$
Specification of trend	stochastic level and stochastic slope	stochastic level and stochastic slope	fixed level and stochastic slope	stochastic level and stochastic slope	stochastic level and stochastic slope	stochastic level and stochastic slope
Specification of trigonometric seasonal	stochastic	fixed	fixed	fixed	fixed	fixed
Explanatory variable:						
dummy variable for <i>Easter</i>	0.64***	5.93***	9.57***	2.57***		
Interventions for cash changeover :						
level break in 2002:03	14.98***					
level break in 2002:04	29.56***					
level break in 2002:05			-28.32***			
level break in 2002:12	2.70***					0.83***
slope break 2002:05	37.12***					
slope break 2002:07	3.35***	14.49***				
Interventions for the great financial crisis :						
level break in 2008:10			15.02***	12.04***	4.93***	1.04***
Intervention for stop of issuance of €500 banknotes³⁾:						
slope break in 2019:03				6.21***		
Interventions for COVID-19 crisis :						
level break in 2020:03			14.11***			
level break in 2020:04	-4.12***					
level break in 2020:05			15.26***	19.42***		
slope break in 2020:03	2.02***					

Source: Authors' calculations.

Notes: Outlier interventions (impulse dummies) are not shown here.

1) *** significant at the 1 per cent level.

2) All of the models were estimated by maximum likelihood (exact score) using the software package STAMP of Koopman et al. (2007).

The estimation of all of the models showed very strong convergence relative to $1e-007$, comprising the convergence criteria likelihood, gradient and parameter.

3) On 4 May 2016 the ECB Governing Council decided to halt the production and distribution of the €500 banknotes. In France the issuance of €500 banknotes was stopped on 27 January 2019.

Table 1b
Structural time series forecasting models: diagnostics and goodness-of-fit

	€5 banknotes	€10 banknotes	€20 banknotes	€50 banknotes	€100 banknotes	€200 banknotes
Goodness-of-fit statistics:						
Log-Likelihood	48.32	-325.52	-397.89	-296.81	63.65	324.95
Akaike Information Criterion (AIC)	-0.70	2.98	3.64	2.67	-0.87	-3.37
Bayesian Information Criterion (BIC)	-0.36	3.33	3.94	2.99	-0.63	-3.10
Prediction error variance (p. e. v.)	0.41	16.08	31.89	11.94	0.36	0.03
Prediction error mean deviation (m. d.)	0.35	12.83	23.71	9.08	0.28	0.02
Ratio p. e. v./m. d. in squares ¹⁾	0.88	0.99	1.15	1.10	1.08	1.25
Coefficient of determination based on difference around seasonal mean (R_s , 2)	0.97	0.63	0.56	0.81	0.63	0.58
Basic diagnostics of residuals (standardised one-step-ahead prediction errors):						
Test of the null hypothesis of normally distributed residuals by means of the Doornik-Hansen statistic. ²⁾	$3.32 < \chi^2(2; 0.05) = 5.99$	$0.09 < \chi^2(2; 0.05) = 5.99$	$3.69 < \chi^2(2; 0.05) = 5.99$	$19.55 > \chi^2(2; 0.05) = 5.99$	$14.65 > \chi^2(2; 0.05) = 5.99$	$75.86 > \chi^2(2; 0.05) = 5.99$
Test of the null hypothesis of homoskedastic residuals by means of the $H(h)$ test statistic. ³⁾	$1/H(68) = 1/0.54 = 1.65 > F(68, 68; 0.025) = 1.62$	$H(67) = 1.02 < F(67, 67; 0.025) = 1.62$	$H(68) = 1.58 < F(68, 68; 0.025) = 1.62$	$H(68) = 1.44 < F(68, 68; 0.025) = 1.62$	$H(66) = 3.48 > F(66, 66; 0.025) = 1.63$	$1/H(69) = 1/0.45 = 2.23 > F(69, 69; 0.025) = 1.61$
Test of the null hypothesis of serially uncorrelated residuals by means of the Ljung-Box Q -statistic. ⁴⁾	The null hypothesis is rejected for $q = 12$ to 24 at the 1% significance level.	The null hypothesis is rejected for $q = 4$ to 24 at the 1% significance level.	The null hypothesis is rejected for $q = 4$ at the 5% significance level.	The null hypothesis is rejected for $q = 4$ to 24 at the 1% significance level.	The null hypothesis cannot be rejected for any lag.	The null hypothesis is rejected for $q = 4$ to 24 at the 1% significance level.
Period with stable parameters according to recursive residuals	2018:05 - 2020:09	2017:01 - 2020:09	2017:08 - 2020:09	2018:12 - 2020:09	2017:11 - 2020:09	2018:02 - 2020:09
Test of the null hypothesis of no seasonality by means of a chi-square test. ⁵⁾	$994.43 > \chi^2(11; 0.05) = 19.68$	$1404.69 > \chi^2(11; 0.05) = 19.68$	$1917.08 > \chi^2(11; 0.05) = 19.68$	$1399.13 > \chi^2(11; 0.05) = 19.68$	$723.37 > \chi^2(11; 0.05) = 19.68$	$195.59 > \chi^2(11; 0.05) = 19.68$
Forecast horizon, length in months	2020:10 - 2021:09, 12	2020:10 - 2021:09, 12	2020:10 - 2021:09, 12	2020:10 - 2021:09, 12	2020:10 - 2021:09, 12	2020:10 - 2021:09, 12
Root mean squared error of 12-step-ahead forecast	4.9	7.3	15.7	28.5	2.7	0.1
Root mean squared error of 12-step-ahead forecast with benchmark ARIMA model	8.4	8.3	21.2	28.1	6.6	0.8

Source: Authors' calculations.

Notes: 1) In a correctly specified model this ratio should be approximately equal to unity.

2) The Doornik-Hansen statistic is the Bowman-Shenton statistic with the correction of Doornik and Hansen (1994), distributed approximately as χ^2 with 2 degrees of freedom under the null hypothesis. The critical value is $\chi^2(2; 0.05) = 5.99$.

- 3) $H(b)$ is distributed approximately as $F(b, b)$. If $H(b)$ is larger than 1, it is enough to check whether $H(b) < F(b, b; 0.025)$. On the other hand, if $H(b)$ is smaller than 1, we have to use the reciprocal of $H(b)$, and check whether $1/H(b) < F(b, b; 0.025)$.
- 4) The Ljung-Box Q -statistic, $Q(q, q - p)$ is based on the first q residual autocorrelations and distributed approximately as χ^2 with $q - p$ degrees of freedom under the null hypothesis of uncorrelated residuals. The parameter p denotes the number of relative variance parameters. For all of the models presented here p is equal to 3. The maximum number of residuals taken into account is 24, that is $q = 1, \dots, 24$.
- 5) This is a test joint test of significance of the 11 seasonal effects at the end of the sample period. Under the null of no seasonal pattern the test statistic is asymptotically χ^2 with 11 degrees of freedom when the seasonal is deterministic. The critical value is $\chi^2(11; 0.05) = 19.68$. In the case of a stochastic seasonal, the joint seasonal test is also performed although a formal joint test of significance of the seasonal effect is inappropriate. However, when the seasonal pattern is persistent throughout the series and when the seasonal pattern changes relatively slowly, which is usually the case, the test statistic can provide a useful guide to the relative importance of the seasonal.

Table 2
Benchmark ARIMA models

	€5 banknotes	€10 banknotes	€20 banknotes	€50 banknotes	€100 banknotes	€200 banknotes
Model specification ¹⁾	ARIMA(8,1,8)(0,1,1) ₁₂	ARIMA(0,1,1)(0,1,1) ₁₂	ARIMA(0,1,1)(0,1,1) ₁₂	ARIMA(4,1,4)(0,1,1) ₁₂	ARIMA(1,1,1)(0,1,1) ₁₂	ARIMA(1,1,3)(0,1,1) ₁₂
Stationary transformation of dependent variable (number of banknotes in circulation) ²⁾	$\Delta\Delta_{12}nn005$	$\Delta\Delta_{12}nn010$	$\Delta\Delta_{12}nn020$	$\Delta\Delta_{12}nn050$	$\Delta\Delta_{12}nn100$	$\Delta\Delta_{12}nn200$
ARMA terms (ar, ma) and multiplicative seasonal ARMA terms (sar, sma)	ar(8), ma(8), sma(12)	ma(1), sma(12)	ma(1), sma(12)	ar(4), ma(4), sma(12)	ar(1), ma(1), sma(12)	ar(1), ma(3), sma(12)
Estimation sample (adjusted)	2003:10 - 2020:09	2003:02 - 2020:09	2003:02 - 2020:09	2003:06 - 2020:09	2003:03 - 2020:09	2003:03 - 2020:09
Adjusted R-squared	0.30	0.61	0.61	0.21	0.53	0.55
Breusch-Godfrey Serial Correlation LM Test ⁴⁾	The null hypothesis is rejected for lags 12, 13 and 20 at the 1% significance level.	The null hypothesis is rejected for lags 8 and 13 at the 5% significance level.	The null hypothesis is rejected for lag 8, 13 and 23 at the 1% significance level.	The null hypothesis is rejected for lag 1, 7, 13 to 14 and 23 at the 1% significance level and for lags 6, 8 and 12 at the 5% significance level.	The null hypothesis cannot be rejected for any lag.	The null hypothesis cannot be rejected for any lag.
White test of heteroskedasticity ³⁾ : Obs*R ² statistic [p-value]	13.78 [0.47]	9.23 [0.42]	8.76 [0.46]	13.11 [0.52]	5.17 [0.98]	15.15 [0.37]
Jarque-Bera test of normal distribution ³⁾ : JB [p-value]	17.98 [0.00]	1.95 [0.38]	17.86 [0.00]	120.07 [0.00]	579.34 [0.00]	1122.97 [0.00]
Forecast sample	2020:10 - 2021:09	2020:10 - 2021:09	2020:10 - 2021:09	2020:10 - 2021:10	2020:10 - 2021:09	2020:10 - 2021:09
Root mean squared error of 12-step-ahead forecast	8.41	8.33	21.21	28.05	6.64	0.83

Source: Authors' calculations.

Notes: 1) Multiplicative seasonal models are written in the form $ARIMA(p,d,q)(P,D,Q)_s$, where X = exogenous regressors, p and q = the nonseasonal ARMA coefficients, d = number of nonseasonal differences, P = number of multiplicative seasonal autoregressive coefficients, D = number of seasonal differences, Q = number of multiplicative seasonal moving-average coefficients and s = seasonal period. All of the models are estimated by conditional least squares and heteroskedasticity and autocorrelation consistent covariance (HAC) standard errors and covariance.

2) Δ denotes the first difference of a series: $(1 - L)y_t$. Δ_{12} denotes the first seasonal difference of a series: $(1 - L^{12})y_t$.

3) No exogenous regressors were specified.

4) Null hypothesis: no serial correlation of residuals up to the specified number of lags.

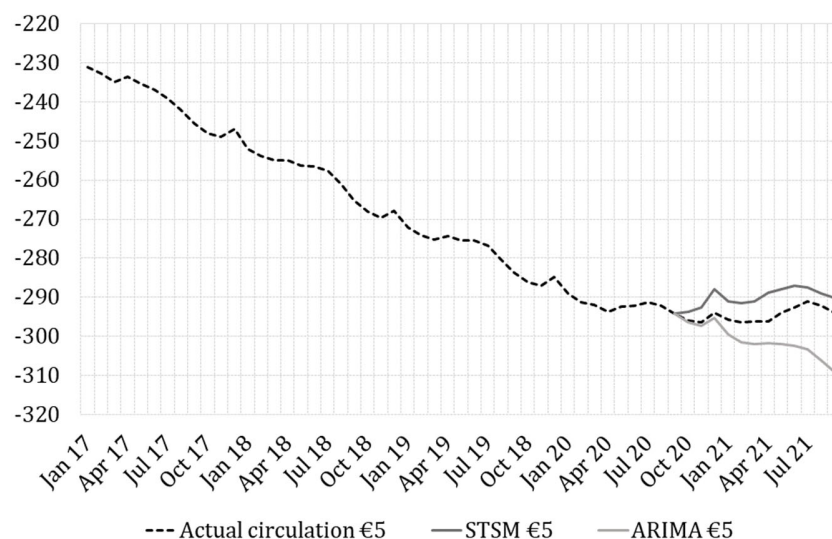
5) Null hypothesis: homoskedasticity of residuals.

6) Null hypothesis: normal distribution of residuals.

7) Coefficients are considered to be stable if they are within a ± 2 coefficient standard deviation corridor in the range of recursive parameter estimation. This range runs from 2015:01 to 2020:09. The corresponding date in the entry indicates the time from which an instable coefficient had been within the ± 2 coefficient standard error corridor. There are no entries for stable coefficients.

Figure A1

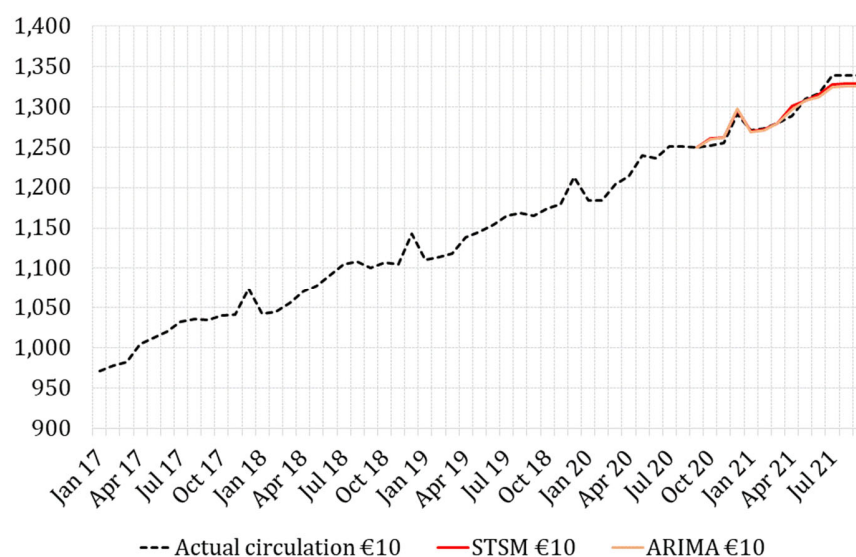
Forecasts of €5 banknotes in circulation obtained by STSM and by benchmark ARIMA model
(million pieces)



Sources: Banque de France and authors' calculations.

Figure A2

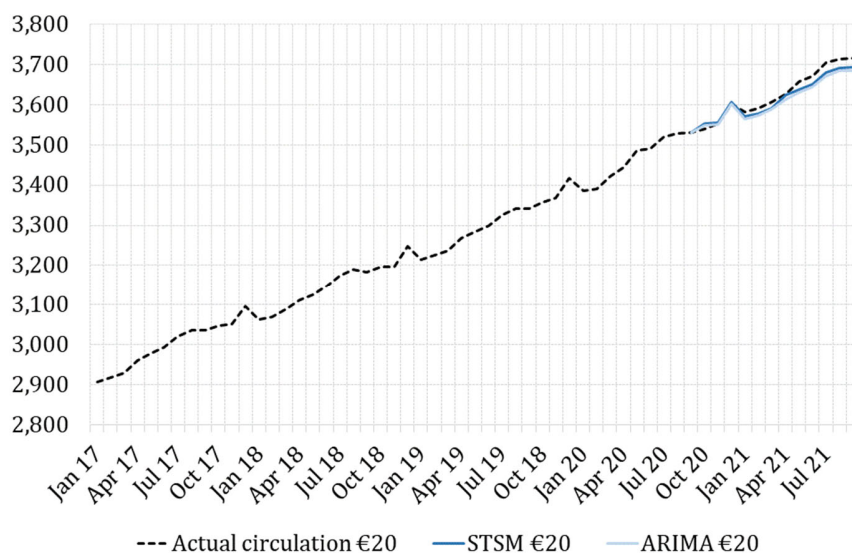
Forecasts of €10 banknotes in circulation obtained by STSM and by benchmark ARIMA
model (million pieces)



Sources: Banque de France and authors' calculations.

Figure A3

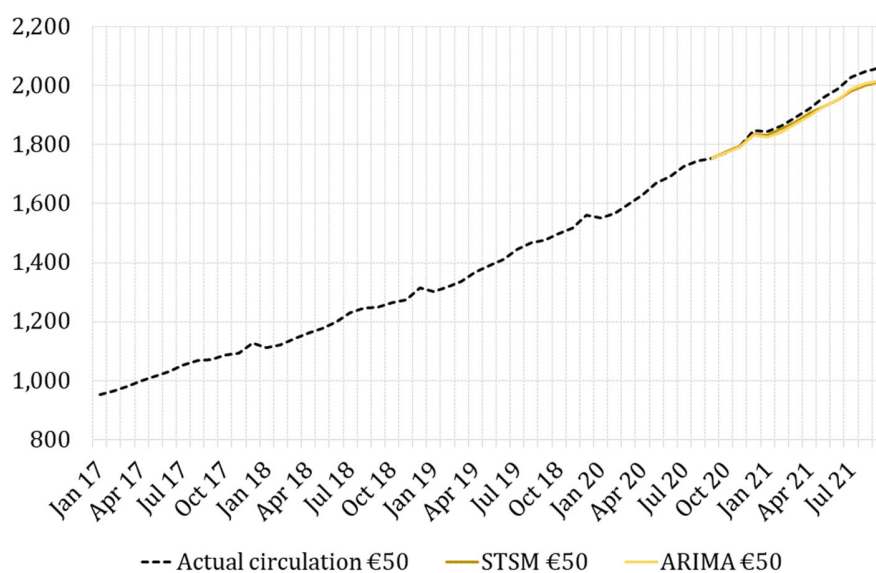
Forecasts of €20 banknotes in circulation obtained by STSM and by benchmark ARIMA model (million pieces)



Sources: Banque de France and authors' calculations.

Figure A4

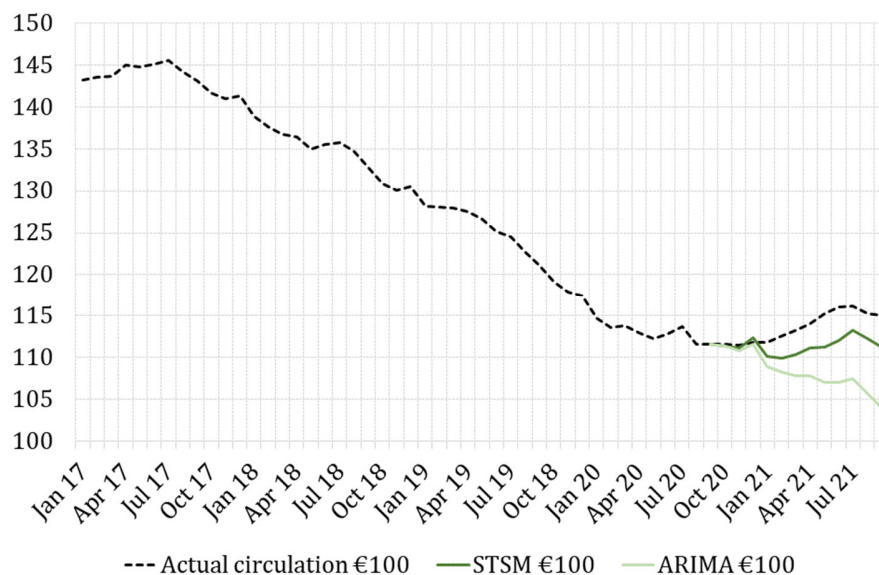
Forecasts of €50 banknotes in circulation obtained by STSM and by benchmark ARIMA model (million pieces)



Sources: Banque de France and authors' calculations.

Figure A5

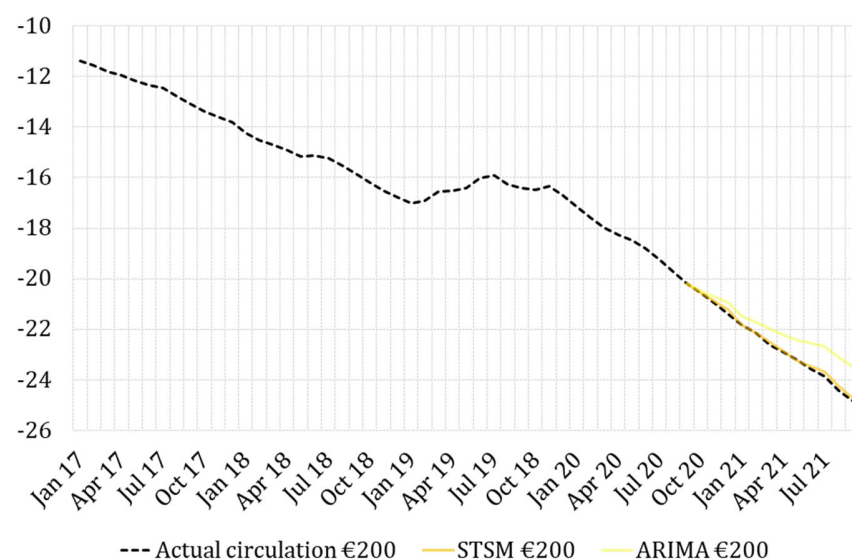
Forecasts of €100 banknotes in circulation obtained by STSM and by benchmark ARIMA model (million pieces)



Sources: Banque de France and authors' calculations.

Figure A6

Forecasts of €200 banknotes in circulation obtained by STSM and by benchmark ARIMA model (million pieces)



Sources: Banque de France and authors' calculations.

Annex B: Forecasting German-issued banknotes in circulation

Table 1a (at the end of this section) contains the results of the maximum likelihood estimation of STSMs for the number of banknotes put into circulation by the Bundesbank. The models were estimated using monthly data from January 2002 to September 2020. The forecast horizon runs from October 2020 to September 2021. All of the models contain a trend, a seasonal component, an irregular component and interventions. In all of the models, the trend consists of a stochastic level and a stochastic slope and the trigonometric seasonal component is usually fixed. While the underlying development of banknotes in circulation is described using trend and season as unobservable components, “intervention variables” are added to explain the time series. Additionally, for all of the denominations except for the €200 banknote and the €500 banknote, the calendar effect of the Easter holidays is modelled by means of a regression variable. The intervention variables capture special factors resulting in trend breaks.⁴² These comprise 1) the euro cash changeover at the beginning of 2002, 2) the escalation of the great financial crisis in October 2008, 3) the limitation of outpayments at ATMs in Greece in 2015 when the debt crisis came to a head, 4) the ECB Governing Council’s decision from 4 May 2016 to discontinue the production and issuance of the €500 banknote around the end of 2018⁴³ and 5) the COVID-19 pandemic. The components of the estimated STSMs are presented in Figures B1 to B6. For a detailed description of the model specification and a comprehensive interpretation of the estimated model using the example of €50 banknotes in circulation, see Deutsche Bundesbank (2022).

All of the estimated coefficients of the STSMs in Table 1a are significant at the 5 per cent level at least.⁴⁴ All significance tests in linear Gaussian models are based on three assumptions concerning the standardised one-step-ahead prediction errors. These residuals should satisfy the following three properties, which are listed in decreasing order of importance: 1) independence, 2) homoscedasticity and 3) normality.⁴⁵ These and other diagnostic statistics and goodness-of-fit measures are shown in Table 1b at the end of this annex. Independence was tested by means of

⁴² The intervention variables also comprise outliers but these are not shown here.

⁴³ The Bundesbank stopped issuing €500 banknotes on 26 April 2019.

⁴⁴ Moreover, when estimating the models, in each case very strong convergence could be achieved.

⁴⁵ See Commandeur and Koopman (2007, Section 8.5).

the Ljung-Box Q -statistic. Accordingly, in each model except for the €200 banknotes, there is serial correlation in the higher lags. According to the $H(b)$ test statistic the null hypothesis of homoskedastic residuals is rejected in STSMs for the high denominations (€100 to €500). Similarly, according to the Doornik-Hansen statistic, the null hypothesis of normally distributed residuals is rejected in the STSMs for the €200 and €500 banknotes. Hence, with regard to the three desirable criteria mentioned above, the STSMs for the large denominations perform relatively poorly. Except for the model for €20 banknotes, the parameters are stable in the last four years of the sample at least. In a correctly specified model, the ratio of the prediction error variance and the prediction error mean deviation should be approximately equal to unity. This condition is fulfilled by the models for the small denominations and the €50 banknotes and to a lesser extent by the models for the high denominations. The fit of the models in terms of the coefficient of determination based on the difference around the seasonal mean lies in a range of 0.62 to 0.88, which seems to be reasonable.

The out-of-sample forecasts resulting from the STSMs and the corresponding benchmark ARIMAX models (i.e. regARIMA models) are compared in Figures B7 to B12. The bottom part of Table 1b lists the corresponding root mean squared errors (RMSEs) of the 12-step-ahead forecasts. Although the statistical informative value of this comparison is limited by the short forecasting horizon of just 12 months, some tentative conclusions may be drawn. The STSMs outperform the ARIMAX models for all of the denominations except for the €5 banknotes and the €100 banknotes.⁴⁶ In terms of the RMSE the STSMs clearly outperform the ARIMAX models for €50 and €200 banknotes in circulation while the ARIMAX model does a much better job at forecasting €100 banknotes in circulation. The ARIMAX models overestimate small banknotes in circulation while the STSMs underestimate them. Because of the discrepancy in the results yielded, it might make sense to combine the two forecasts.

The benchmark ARIMAX models, which are traditionally used in banknote requirement planning at the Bundesbank, are described in Table 2 at the end of this section. For example, an $\text{ARIMAX}(1,1,0)(1,1,0)_{12}$ was fitted to €50 banknotes in circulation. The exogenous regressors X comprise step dummies (intervention variables) for the financial crisis in October 2008 and the discontinuation of €500 banknote issuance in Germany in April 2019. The effect of the COVID-

⁴⁶ The forecast comparison for the €50 banknote is illustrated in more detail in Deutsche Bundesbank (2022, pages 80-81).

19 pandemic is modelled using impulse dummies (intervention variables) for the months spanning the March 2020 to September 2020 period. As is the case with the STSMs, in the ARIMAX models serial correlation of the residuals at the higher lags can be observed for most of the denominations. The null hypothesis of homoskedastic residuals is rejected for €10 banknotes in circulation at the 1 per cent significance level. The null hypothesis of normally distributed residuals is rejected for €5 and €200 banknotes in circulation. Finally, the ARIMAX models for €20 and €200 banknotes show instability of coefficients.

These findings can be recapitulated as follows. In terms of diagnostic tests, the STSMs and the benchmark ARIMAX models perform similarly. When it comes to forecasting, the STSMs generally outperform the ARIMAX models according to the rather limited evidence at hand. Hence it might be promising to use STSMs in addition to the traditional ARIMA forecasting models in the context of banknote requirement planning at the Bundesbank. Running these models head to head against each other in real time will shed more light on their relative performance.

Table 1a
Structural time series forecasting models: estimation results

	€5 banknotes	€10 banknotes	€20 banknotes	€50 banknotes	€100 banknotes	€200 banknotes
Estimation sample, number of observations	2002:01 - 2020:09, 225	2002:01 - 2020:09, 225	2002:01 - 2020:09, 225	2002:01 - 2020:09, 225	2002:01 - 2020:09, 225	2002:01 - 2020:09, 225
Forecast horizon, length in months	2020:10 - 2021:09, 12	2020:10 - 2021:09, 12	2020:10 - 2021:09, 12	2020:10 - 2021:09, 12	2020:10 - 2021:09, 12	2020:10 - 2021:09, 12
Model specification ^{1), 2)} (y : number of Geman-issued banknotes in circulation in million pieces)	$y = \text{trend} + \text{seasonal} + \text{irregular} + \text{explanatory variables} + \text{interventions}$	$y = \text{trend} + \text{seasonal} + \text{irregular} + \text{explanatory variables} + \text{interventions}$	$y = \text{trend} + \text{seasonal} + \text{irregular} + \text{explanatory variables} + \text{interventions}$	$y = \text{trend} + \text{seasonal} + \text{irregular} + \text{explanatory variables} + \text{interventions}$	$y = \text{trend} + \text{seasonal} + \text{irregular} + \text{explanatory variables} + \text{interventions}$	$\log(y) = \text{trend} + \text{seasonal} + \text{irregular} + \text{interventions}$
Specification of <i>trend</i>	stochastic level and stochastic slope	stochastic level and stochastic slope	stochastic level and stochastic slope	stochastic level and stochastic slope	stochastic level and stochastic slope	stochastic level and stochastic slope
Specification of trigonometric seasonal	fixed	fixed	fixed	fixed	stochastic	stochastic
<i>Explanatory variables:</i>						
dummy variable for <i>Ascension Day</i>	5.93**					
dummy variable for <i>Easter</i>	8.14***	11.30***	8.58***	12.54***	1.03**	
<i>Interventions for cash changeover:</i>						
level break in 2002:02	- 133.99***	- 88.96***	- 86.25***			
level break in 2002:03				29.10***		
level break in 2002:04	- 52.32***	- 34.87***	- 24.55***			
slope break 2002:03						- 0.04***
<i>Interventions for great financial crisis:</i>						
level break in 2008:10				45.23***	28.29***	0.06***
level break in 2008:12						- 0.03***
slope break in 2009:02					- 9.36***	
<i>Interventions for limitation of outpayments at ATMs in Greece in 2015:</i>						
level break in 2015:07		18.38**	24.33***	22.43**		
<i>Interventions for stop of issuance of €500 banknotes³⁾:</i>						
level break in 2016:06				32.96***	5.18**	
slope break in 2016:04					9.49***	
slope break in 2019:04					6.81***	0.04***
<i>Interventions for COVID-19 crisis:</i>						
level break in 2020:03	22.84***	29.47***	27.35***	71.25***	37.48***	0.07***
slope break in 2020:04	- 11.87***	- 20.00***	- 18.11***			
slope break in 2020:05				- 12.66***	- 7.26***	- 0.03***

Source: Authors' calculations.

Notes: Outlier interventions (impulse dummies) are not shown here.

1) *** significant at the 1 per cent level, ** significant at the 5 per cent level.

2) All of the models were estimated by maximum likelihood (exact score) using the software package STAMP.⁴⁷ The estimation of all of the models showed very strong convergence relative to 1e-007 comprising the convergence criteria likelihood, gradient and parameter.

3) On 4 May 2016 the ECB Governing Council decided to halt the production and distribution of the €500 banknotes. In Germany the issuance of €500 banknotes was stopped on 26 April 2019.

⁴⁷ See Koopman et al. (2007).

Table 1b

Structural time series forecasting models: diagnostics and goodness-of-fit

	€5 banknotes	€10 banknotes	€20 banknotes	€50 banknotes	€100 banknotes	€200 banknotes
Goodness-of-fit statistics:						
Log-Likelihood	- 337.66	- 467.06	- 445.14	- 473.60	- 218.96	993.26
Akaike Information Criterion (AIC)	3.13	4.36	4.16	4.49	2.19	- 8.62
Bayesian Information Criterion (BIC)	3.51	4.71	4.51	4.96	2.60	- 8.28
Prediction error variance (p. e. v.)	18.40	63.25	51.12	75.88	6.09	4.97e-005
Prediction error mean deviation (m. d.)	14.14	49.39	40.50	58.39	4.52	3.59e-005
Ratio p. e. v./m. d. in squares ¹⁾	1.08	1.04	1.01	1.08	1.16	1.22
Coefficient of determination based on difference around seasonal mean (R_s^2)	0.87	0.62	0.64	0.74	0.88	0.83
Basic diagnostics of residuals (standardised one-step-ahead prediction errors):						
Test of the null hypothesis of normally distributed residuals by means of the Doornik-Hansen statistic. ²⁾	$0.25 < \chi^2(2; 0.05) = 5.99$	$1.11 < \chi^2(2; 0.05) = 5.99$	$0.27 < \chi^2(2; 0.05) = 5.99$	$0.85 < \chi^2(2; 0.05) = 5.99$	$5.51 < \chi^2(2; 0.05) = 5.99$	$25.60 > \chi^2(2; 0.05) = 5.99$
Test of the null hypothesis of homoskedastic residuals by means of the $H(h)$ test statistic. ³⁾	$1/H(67) = 1/0.99 = 1.01 < F(67, 67; 0.025) = 1.62$	$1/H(68) = 1/0.71 = 1.42 < F(68, 68; 0.025) = 1.62$	$H(68) = 1.47 < F(68, 68; 0.025) = 1.62$	$H(65) = 1.37 < F(65, 65; 0.025) = 1.63$	$H(67) = 3.96 > F(67, 67; 0.025) = 1.62$	$1/H(68) = 1/0.50 = 2.01 > F(68, 68; 0.025) = 1.62$
Test of the null hypothesis of serially uncorrelated residuals by means of the Ljung-Box Q -statistic. ⁴⁾	The null hypothesis is rejected for $q = 12$ to 24 at the 1% significance level.	The null hypothesis is rejected for $q = 12$ to 24 at the 1% significance level.	The null hypothesis is rejected for $q = 12$ to 24 at the 1% significance level.	The null hypothesis is rejected for $q = 14$ to 17 at the 5% significance level.	The null hypothesis is rejected for $q = 14$ to 16 at the 5% significance level.	The null hypothesis cannot be rejected for any lag.
Period with stable parameters according to recursive residuals	2013:07 - 2020:09	2014:09 - 2020:09	2017:11 - 2020:09	2016:04 - 2020:09	2016:09 - 2020:09	2016:04 - 2020:09
Test of the null hypothesis of no seasonality by means of a chi-square test. ⁵⁾	$1478.06 > \chi^2(11; 0.05) = 19.68$	$1125.36 > \chi^2(11; 0.05) = 19.68$	$1109.02 > \chi^2(11; 0.05) = 19.68$	$1827.61 > \chi^2(11; 0.05) = 19.68$	$320.30 > \chi^2(11; 0.05) = 19.68$	$13.18 < \chi^2(11; 0.05) = 19.68$
Forecast horizon, length in months	2020:10 - 2021:09, 12	2020:10 - 2021:09, 12	2020:10 - 2021:09, 12	2020:10 - 2021:09, 12	2020:10 - 2021:09, 12	2020:10 - 2021:09, 12
Root mean squared error of 12-step-ahead forecast	49.1	62.2	44.8	24.4	22.3	4.3
Root mean squared error of 12-step-ahead forecast with benchmark ARIMA model	31.5	79.7	75.5	67.8	7.0	83.6

Source: Authors' calculations.

Notes: 1) In a correctly specified model this ratio should be approximately equal to unity.

2) The Doornik-Hansen statistic is the Bowman-Shenton statistic with the correction of Doornik and Hansen (1994), distributed approximately as χ^2 with 2 degrees of freedom under the null hypothesis. The critical value is $\chi^2(2; 0.05) = 5.99$.3) $H(h)$ is distributed approximately as $F(h, h)$. If $H(h)$ is larger than 1, it is enough to check whether $H(h) < F(h, h; 0.025)$. On the other hand, if $H(h)$ is smaller than 1, we have to use the reciprocal of $H(h)$, and check whether $1/H(h) < F(h, h; 0.025)$.

- 4) The Ljung-Box Q -statistic, $Q(q, q - p)$ is based on the first q residual autocorrelations and distributed approximately as χ^2 with $q - p$ degrees of freedom under the null hypothesis of uncorrelated residuals. The parameter p denotes the number of relative variance parameters. For all of the models presented here p is equal to 3. The maximum number of residuals taken into account is 24, that is $q = 1, \dots, 24$.
- 5) This is a joint test of significance of the 11 seasonal effects at the end of the sample period. Under the null of no seasonal pattern the test statistic is asymptotically χ^2 with 11 degrees of freedom when the seasonal is deterministic. The critical value is $\chi^2(11; 0.05) = 19.68$. In the case of a stochastic seasonal, the joint seasonal test is also performed although a formal joint test of significance of the seasonal effect is inappropriate. However, when the seasonal pattern is persistent throughout the series and when the seasonal pattern changes relatively slowly, which is usually the case, the test statistic can provide a useful guide to the relative importance of the seasonal.

Table 2
Benchmark ARIMAX models

	€5 banknotes	€10 banknotes	€20 banknotes	€50 banknotes	€100 banknotes	€200 banknotes
Model specification ¹⁾	ARIMAX(12,1,1)(0,1,0) ₁₂	ARIMAX(1,1,12)(0,1,0) ₁₂	ARIMAX(4,1,12)(0,1,0) ₁₂	ARIMAX(1,1,0)(1,1,0) ₁₂	ARIMAX(4,1,0)(1,0,0) ₁₂	ARIMAX(12,1,5)
Stationary transformation of dependent variable (number of banknotes in circulation) ²⁾	$\Delta\Delta_{12}nn005$	$\Delta\Delta_{12}nn010$	$\Delta\Delta_{12}nn020$	$\Delta\Delta_{12}nn050$	$\Delta\log(nn100)$	$\Delta\log(nn200)$
Exogenous regressors (X) ³⁾	absolute term, $\Delta\Delta_{12}impulse_intv_2003$, $\Delta\Delta_{12}impulse_intv_2004$, $\Delta\Delta_{12}trend_break_2007$	absolute term, $\Delta\Delta_{12}impulse_intv_2003$, $\Delta\Delta_{12}impulse_intv_2004$, $\Delta\Delta_{12}trend_break_2007$	absolute term, $\Delta\Delta_{12}impulse_intv_2003$, $\Delta\Delta_{12}impulse_intv_2004$, $\Delta\Delta_{12}impulse_intv_2005$, $\Delta\Delta_{12}trend_break_2008$	absolute term, $\Delta\Delta_{12}step_intv_0810$, $\Delta\Delta_{12}step_intv_1904$, $\Delta\Delta_{12}impulse_intv_2003$, $\Delta\Delta_{12}impulse_intv_2004$, $\Delta\Delta_{12}impulse_intv_2005$, $\Delta\Delta_{12}impulse_intv_2006$, $\Delta\Delta_{12}impulse_intv_2007$, $\Delta\Delta_{12}impulse_intv_2008$, $\Delta\Delta_{12}impulse_intv_2009$	absolute term, $\Delta seas_{12}$, $\Delta step_intv_0810$, $iv0810_100$, $\Delta step_intv_1605$, $\Delta step_intv_1904$, $\Delta impulse_intv_2003$, $\Delta impulse_intv_2004$, $\Delta impulse_intv_2005$, $\Delta impulse_intv_2006$, $\Delta impulse_intv_2007$, $\Delta impulse_intv_2008$, $\Delta impulse_intv_2009$	absolute term, $\Delta step_intv_0508$, $\Delta step_intv_0810$, $iv0810_200$, $\Delta step_intv_1605$, $\Delta trend_break_1905$, $\Delta step_intv_2003$
ARMA terms (ar, ma) and multiplicative seasonal ARMA terms (sar, sma)	ar(12), ma(1)	ar(1), ma(12)	ar(1), ar(3), ar(4), ma(12)	ar(1), sar(12)	ar(1), ar(4), sar(12)	ar(12), ma(1), ma(2), ma(4), ma(5)
Estimation sample (adjusted)	2004:02 - 2020:09	2003:03 - 2020:09	2003:06 2020:09	2004:03 2020:09	2003:06 2020:09	2003:02 2020:09
Adjusted R-squared	0.39	0.49	0.48	0.35	0.85	0.81
Breusch-Godfrey Serial Correlation LM Test ⁴⁾	The null hypothesis is rejected for lags 13 to 17 and 22 to 23 at the 5% significance level and for lag 24 at the 1% significance level.	The null hypothesis is rejected for lags 14 to 17, 19 to 20 and 22 to 24 at the 5% significance level.	The null hypothesis is rejected for lag 20 at the 5% significance level and for lags 14 to 19 and 21 to 24 at the 1% significance level.	The null hypothesis is rejected for lag 4 at the 5% significance level and for lags 12 to 24 at the 1% significance level.	The null hypothesis is rejected for lags 13 to 21 at the 5% significance level and for lag 12 at the 1% significance level.	The null hypothesis is rejected for lags 3, 10, 11 and 14 at the 5% significance level.
White test of heteroskedasticity ⁵⁾ : Obs* R^2 statistic [p-value]	11.93 [0.75]	67.00 [0.00]	46.76 [0.64]	11.78 [0.99]	62.73 [0.41]	103.31 [0.06]
Jarque-Bera test of normal distribution ⁶⁾ : JB [p-value]	27.71 [0.00]	2.16 [0.34]	2.15 [0.34]	3.71 [0.16]	3.42 [0.18]	17.86 [0.00]
Instable coefficients ⁷⁾	-	-	absolute term 2017:10	-	-	d(step_intv_0810) 2020:05, d(step_intv_1605) 2020:04, d(trend_break_1905) 2020:04
Forecast sample	2020:10 - 2021:09	2020:10 - 2021:09	2020:10 - 2021:09	2020:10 - 2021:09	2020:10 - 2021:09	2020:10 - 2021:09
Root mean squared error of 12-step-ahead forecast	31.5	79.7	75.5	67.8	7.0	83.6

Source: Authors' calculations.

Notes: 1) Multiplicative seasonal models are written in the form $ARIMA(p,d,q)(P,D,Q)_s$, where X = exogenous regressors, p and q = the nonseasonal ARMA coefficients, d = number of nonseasonal differences, P = number of multiplicative seasonal autoregressive coefficients, D = number of seasonal differences, Q = number of multiplicative seasonal moving-average coefficients, s = seasonal period. All of the models are estimated by conditional least squares and heteroskedasticity and autocorrelation consistent covariance (HAC) standard errors and covariance.

2) Δ denotes the first difference of a series: $(1 - L)y_t$. Δ_{12} denotes the first seasonal difference of a series: $(1 - L^{12})y_t$.

3) No exogenous regressors were specified.

4) Null hypothesis: no serial correlation of residuals up to the specified number of lags.

3) *impulse_intv_xxyy*: impulse intervention (outlier) in 20xx:yy, *step_intv_xxyy*: step intervention (level break) in 20xx:yy, *iv0810_xxx*: intervention variable for the impact of the great financial crisis on the circulation of €xxx banknotes in circulation after 2008:10, *trend_break_xxyy*: slope break or rather staircase intervention beginning in 20xx:yy. The trend breaks in the regressions for €5 to €20 and €200 banknotes in circulation are smoothed. *seas_12*: seasonal dummy for December.

4) Null hypothesis: no serial correlation of residuals up to the specified number of lags.

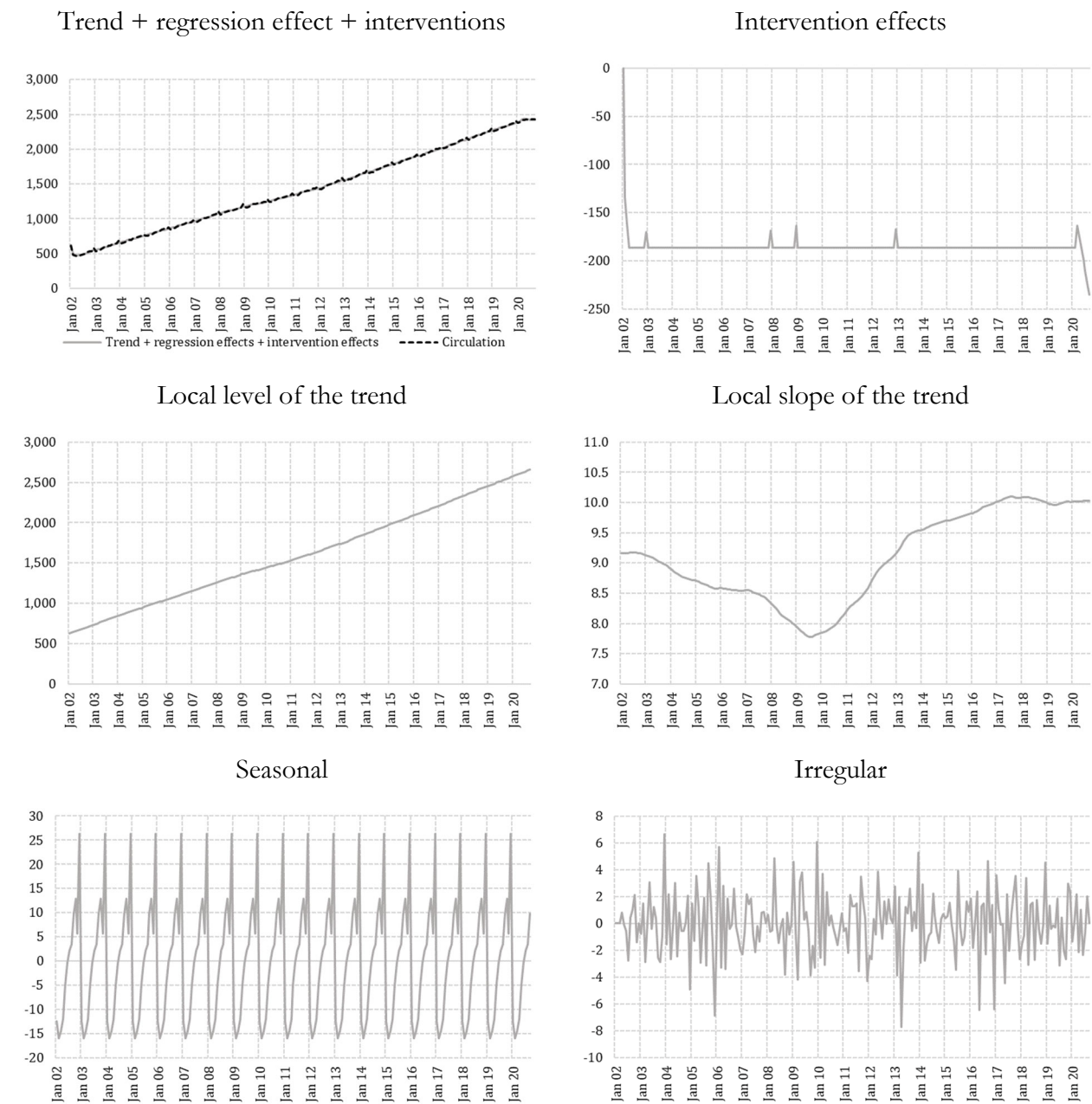
5) Null hypothesis: homoskedasticity of residuals.

6) Null hypothesis: normal distribution of residuals.

7) Coefficients are considered to be stable if they are within a ± 2 coefficient standard deviation corridor in the range of recursive parameter estimation. This range runs from 2015:01 to 2020:09. The corresponding date in the entry indicates the point from which an instable coefficient had been within the ± 2 coefficient standard error corridor. There are no entries for stable coefficients.

Figure B1

STSM for €5 banknotes in circulation (million pieces)

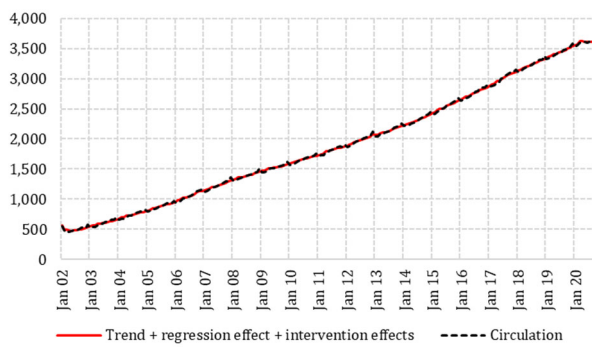


Sources: Deutsche Bundesbank and authors' calculations.

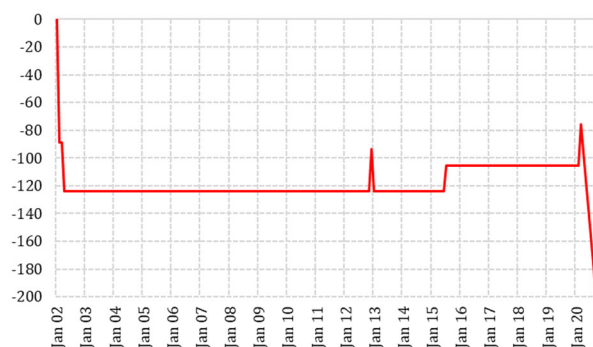
Figure B2

STSM for €10 banknotes in circulation (million pieces)

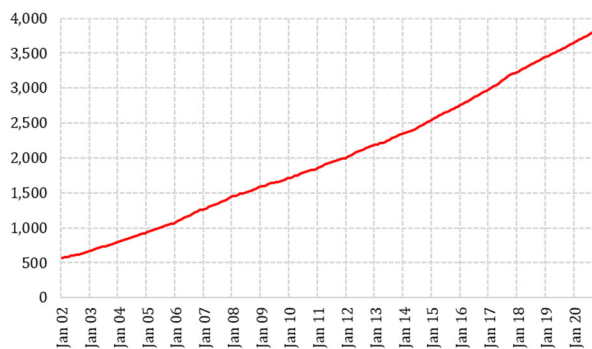
Trend + regression effect + interventions



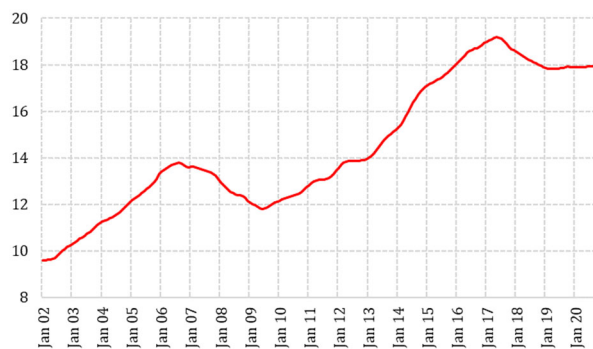
Intervention effects



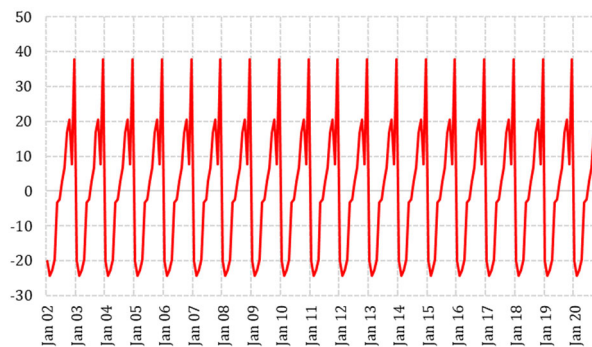
Local level of the trend



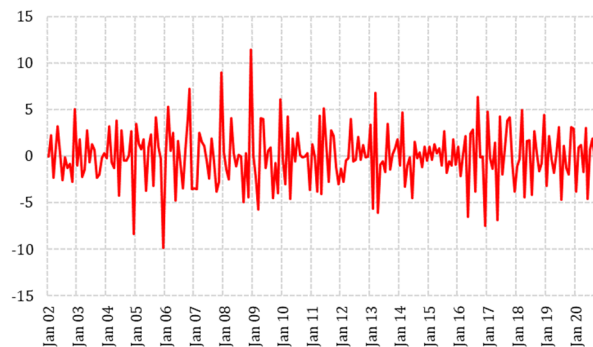
Local slope of the trend



Seasonal



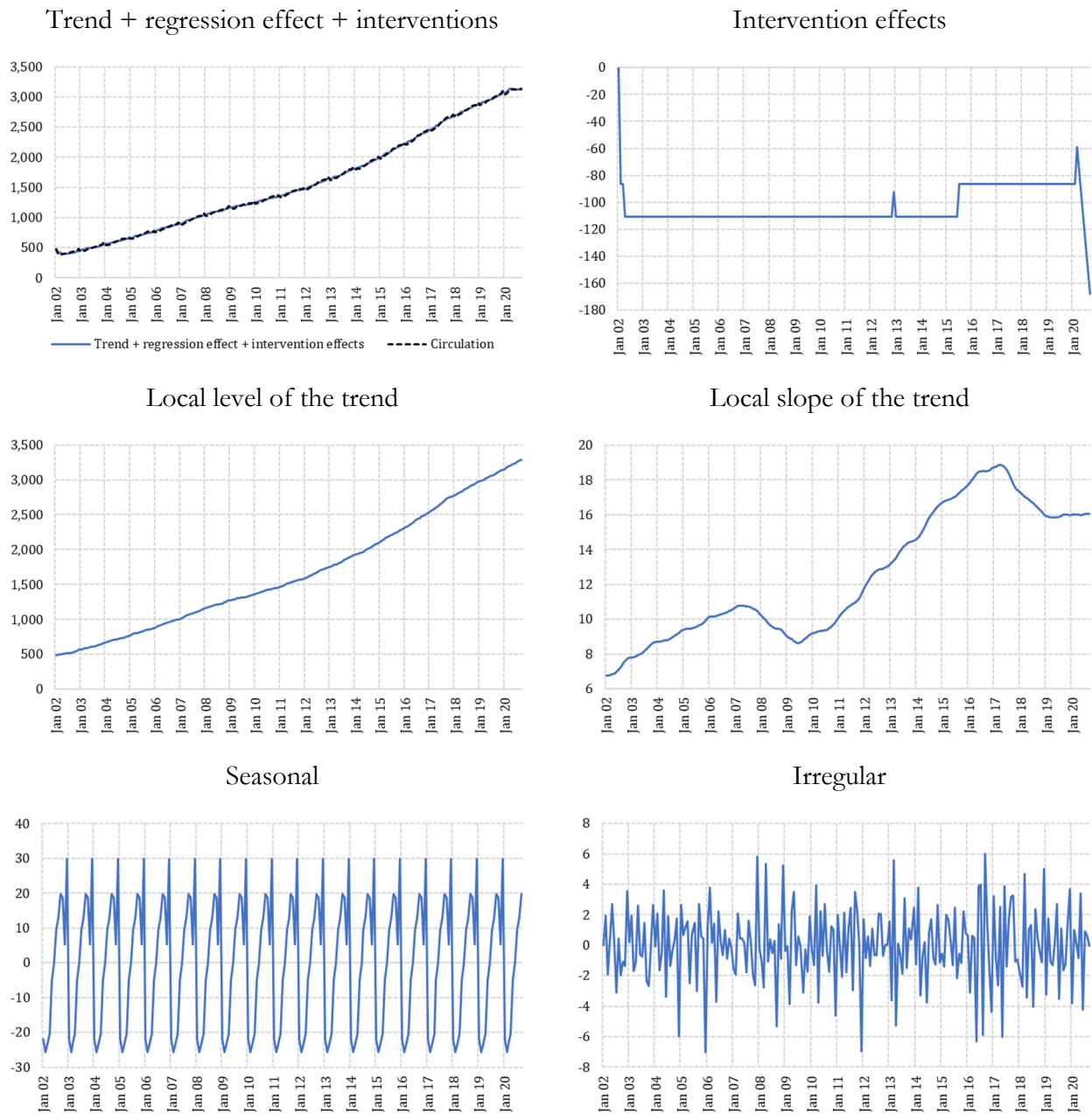
Irregular



Sources: Deutsche Bundesbank and authors' calculations.

Figure B3

STSM for €20 banknotes in circulation (million pieces)

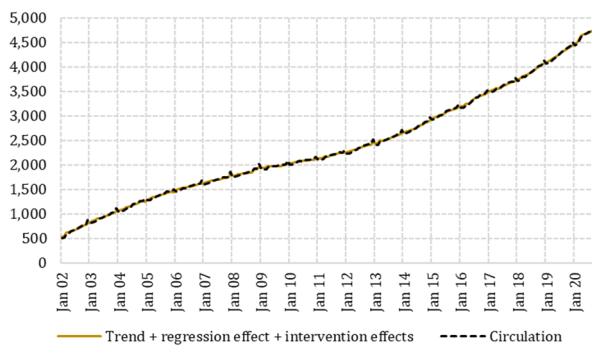


Sources: Deutsche Bundesbank and authors' calculations.

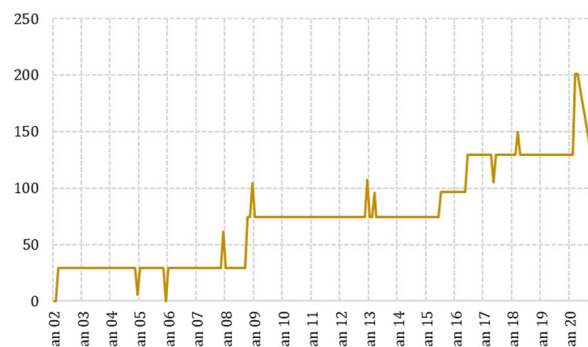
Figure B4

STSM for €50 banknotes in circulation (million pieces)

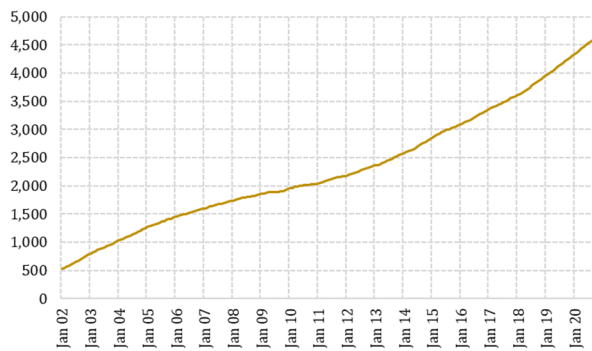
Trend + regression effect + interventions



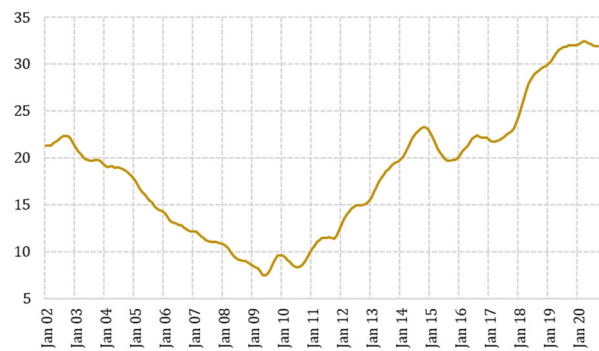
Intervention effects



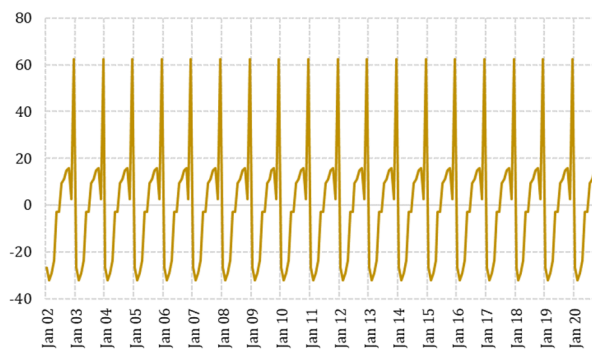
Local level of the trend



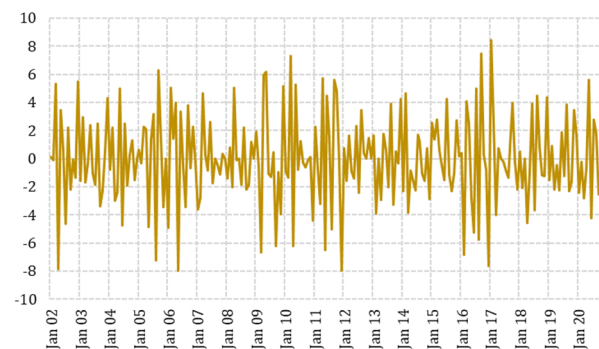
Local slope of the trend



Seasonal



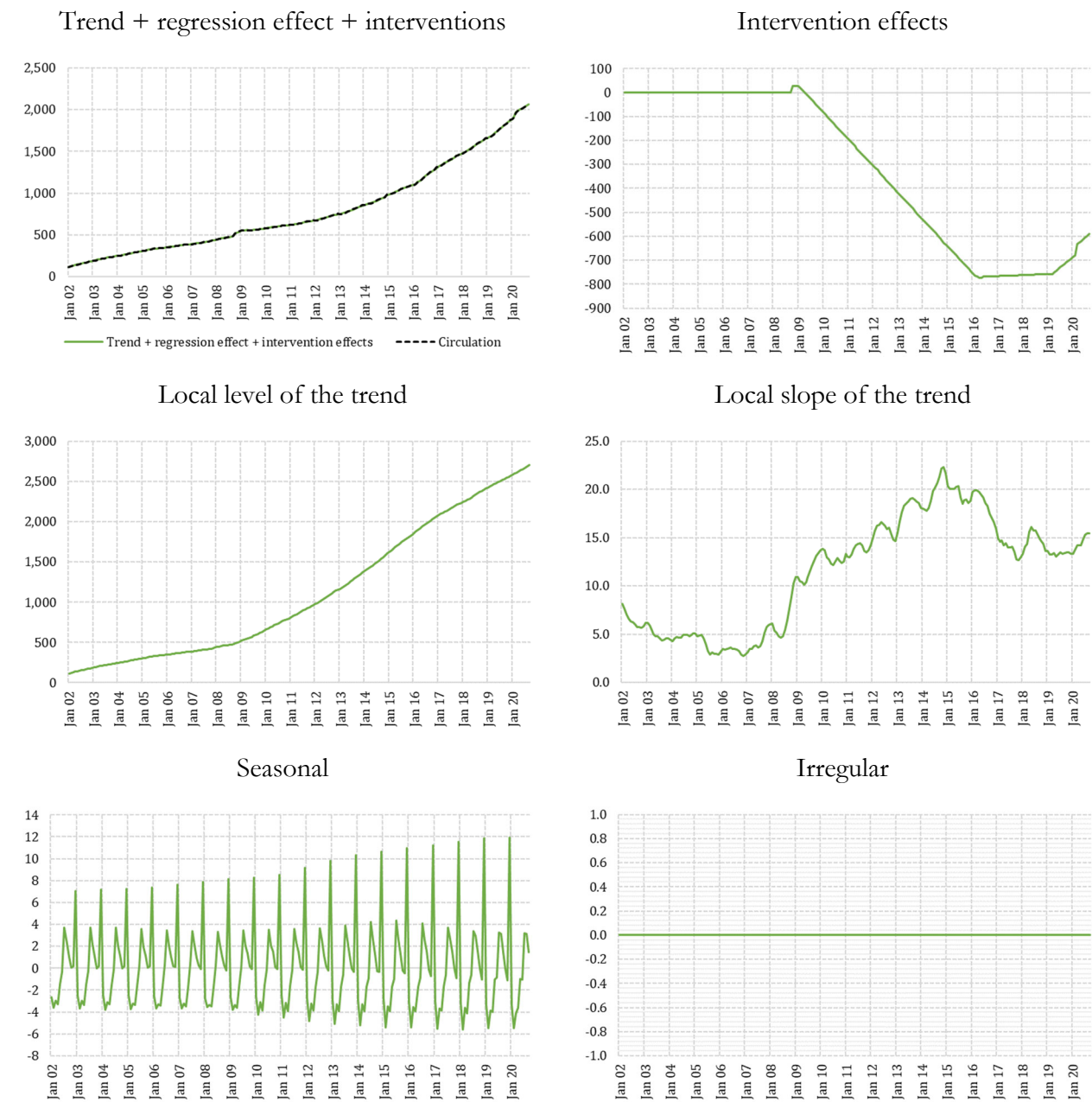
Irregular



Sources: Deutsche Bundesbank and authors' calculations.

Figure B5

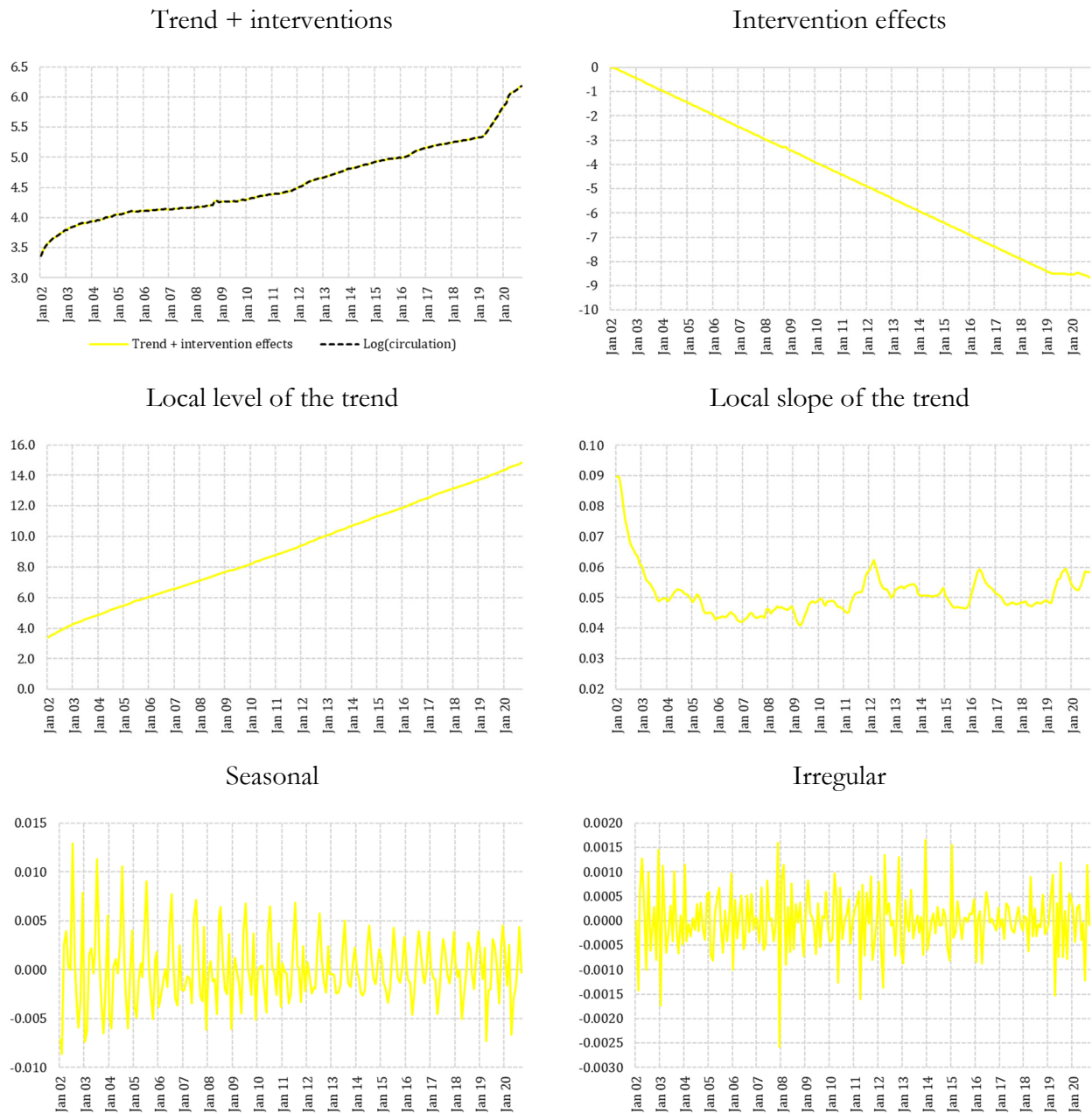
STSM for €100 banknotes in circulation (million pieces)



Sources: Deutsche Bundesbank and authors' calculations.

Figure B6

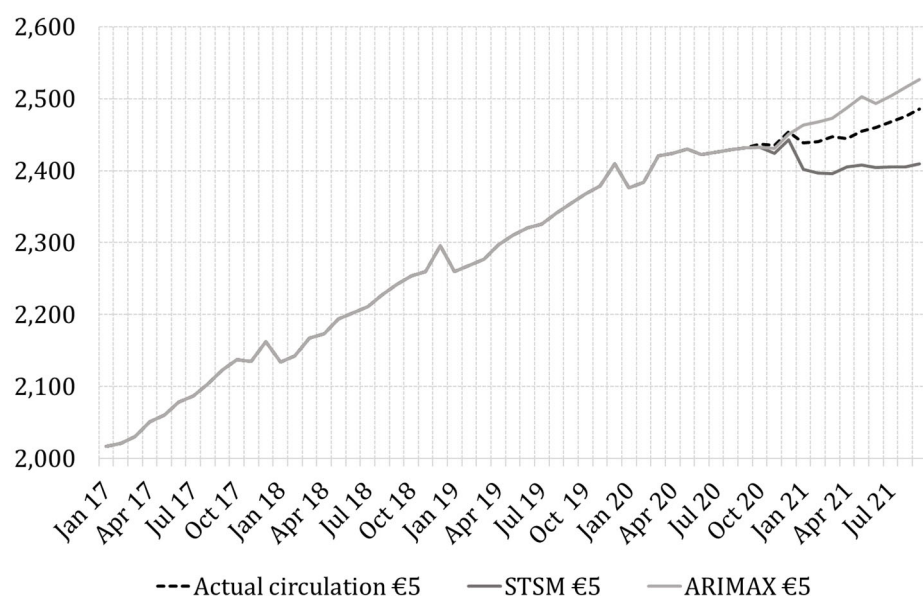
STSM for €200 banknotes in circulation (million pieces)



Sources: Deutsche Bundesbank and authors' calculations.

Figure B7

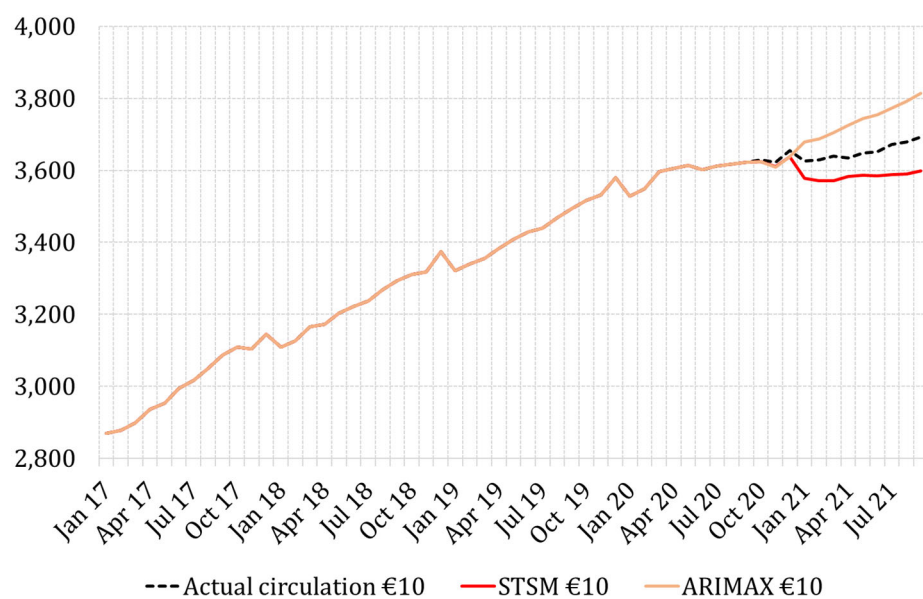
Forecasts of €5 banknotes in circulation obtained by STSM and by benchmark ARIMAX model
(million pieces)



Sources: Deutsche Bundesbank and authors' calculations.

Figure B8

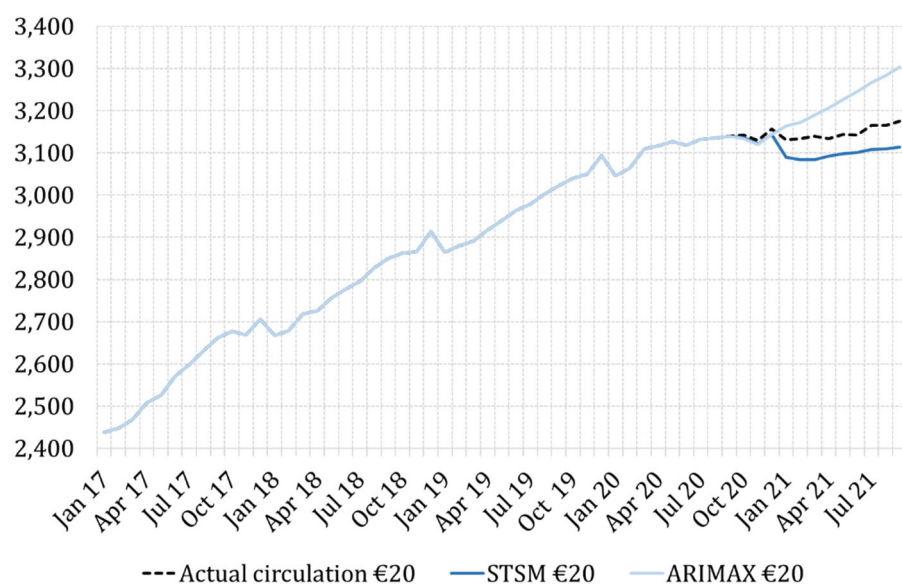
Forecasts of €10 banknotes in circulation obtained by STSM and by benchmark ARIMAX model
(million pieces)



Sources: Deutsche Bundesbank and authors' calculations.

Figure B9

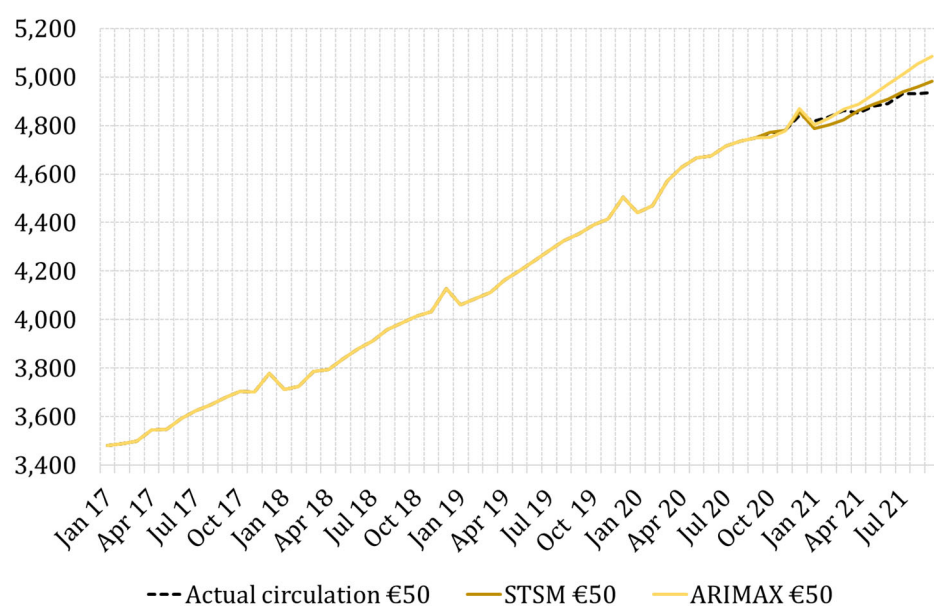
Forecast of €20 banknotes in circulation obtained by STSM and by benchmark ARIMAX model
(million pieces)



Sources: Deutsche Bundesbank and authors' calculations.

Figure B10

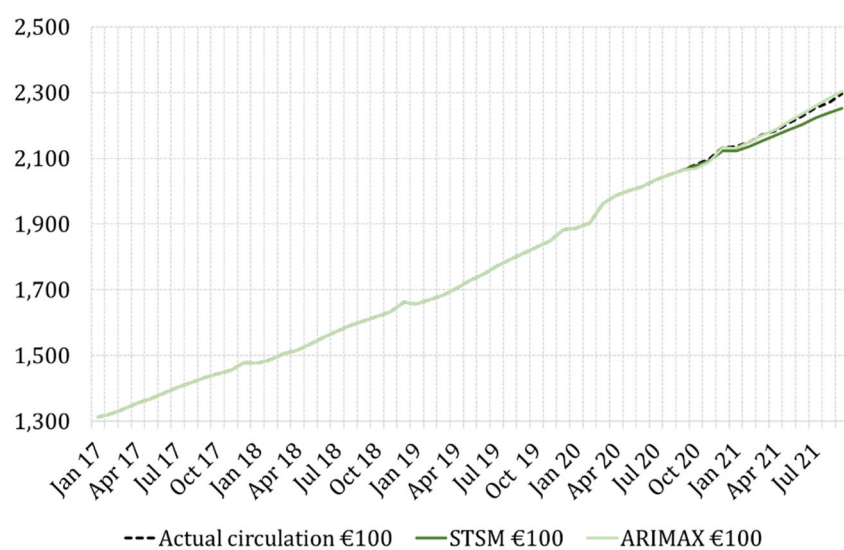
Forecast of €50 banknotes in circulation obtained by STSM and by benchmark ARIMAX model
(million pieces)



Sources: Deutsche Bundesbank and authors' calculations.

Figure B11

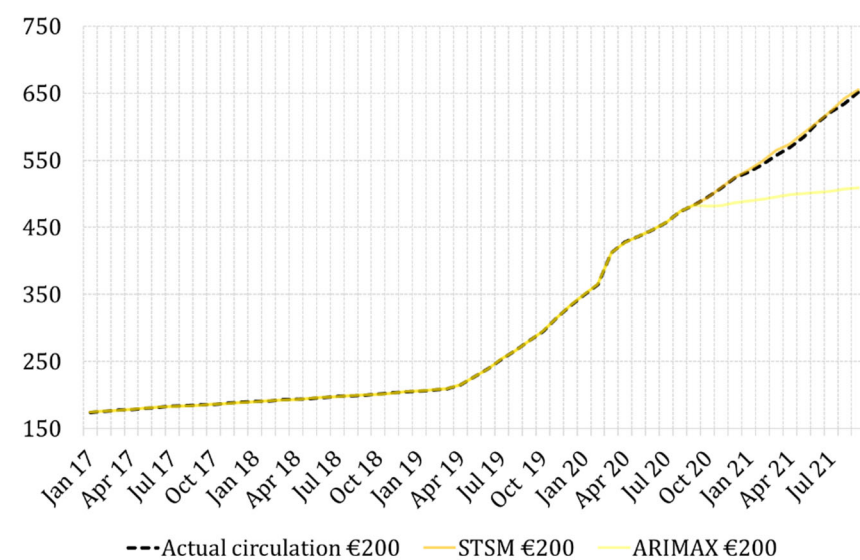
Forecast of €100 banknotes in circulation obtained by STSM and by benchmark ARIMAX model
(million pieces)



Sources: Deutsche Bundesbank and authors' calculations.

Figure B12

Forecast of €200 banknotes in circulation obtained by STSM and by benchmark ARIMAX model
(million pieces)



Sources: Deutsche Bundesbank and authors' calculations.

Annex C: Forecasting Italian-issued banknotes in circulation

The estimation exercise for circulation of Italian-issued banknotes by denomination has been conducted using data from January 2004 to September 2020. By dropping the first two years after the adoption of euro cash we excluded months characterised by unique episodes related to the euro cash changeover and the simultaneous circulation of both the Italian Lira and the new common currency. Moreover, people needed some time to adapt to the new prices and to the different denominations of the newly issued banknotes and coins, and to adjust their habits as regards withdrawing, holding, spending and depositing money.⁴⁸ This choice did not severely affect the length of the time series, since more than two hundred monthly observations were left to conduct the exercise. Excluding this period from the estimation sample, instead of including a set of (step) dummy variables to account for its peculiarity, was more conducive, in our opinion, to obtaining both consistent estimates for the various components in the time span considered and better predictions in the forecast exercise. In the estimation stage, we take the logarithm of the number of banknotes in circulation in order to smooth the series; afterwards, fitted values and forecasts are reconverted into the number of banknotes in order to compute goodness-of-fit statistics and to plot the predicted series. A step-dummy equal to one for March, April and May 2020 is added as an exogenous regressor to take into account the effects exerted by the first wave of the COVID-19 pandemic and the related lockdown. This episode affected the dynamics of circulation differently from any other previous crisis and merited particular attention as illustrated in Section 3.2 in the main text.

Each series was decomposed into trend, cycle, seasonal and irregular components. Not all components are actually informative when it comes to describing circulation dynamics. The trend component (both in its level and its slope) is the one contributing most to the variance explained by all models, reaching, together with the seasonal component, the highest level of significance across all denominations. The cyclic component is found to contribute significantly to describing the dynamic of €10 banknotes in circulation only.⁴⁹ Outliers are automatically detected by the procedure employed for each denomination and the corresponding additive effects are included in the different specifications. The presence of these outliers varies across denominations and may stem from a wide set of events that occurred at the national or European level, such as the introduction of the second series (ES2) of €5 banknotes (outliers in April and May 2013), the implementation of measures aimed at improving the quality of €10 banknotes in circulation (outliers in July and December 2019) and the great financial

⁴⁸ For an extensive description of Italian-issued banknotes in circulation since the changeover, see Baldo et al. (2021).

⁴⁹ With a significance level somewhat higher than 10 per cent, the cyclic component is also significant for €50 banknotes in circulation.

crisis impacting on the circulation of higher denominations (outliers in September and October 2008).⁵⁰ In the case of Italy, the parameters estimated for the COVID-19 step-dummy turn out to be not significant for all denominations. This evidence indicates that the structural time series approach seems able to fit the actual series even in the event of relevant shocks. This result is likely to originate from the above-mentioned predominance of the trend component in describing the series.

To evaluate the model goodness-of-fit, we use the weighted mean absolute percentage error (WMAPE), where weights are represented by actual circulation.⁵¹ This indicator gives a very reliable picture of the fit of the model because, after scaling the estimation error to its actual value, it magnifies percentage errors arising in predicting large amounts, and dampens percentage errors generated for small values. Moreover, from a policy perspective, greater attention should be paid to correctly estimating moments of high circulation – which are usually associated with a rise in banknote demand and necessitate higher production levels – rather than concentrating on periods of low circulation, when lodgements exceed withdrawals.⁵²

The fit of the model is very good for all denominations. The WMAPE, computed using the actual and the estimated number of banknotes in circulation, is very small, ranging from 0.2 per cent for €5 banknotes in circulation to 1.7 per cent for €100 and €200 banknotes in circulation (Table 1a, at the end of this annex). The denominations going into negative circulation in the estimation sample, namely €10 and €200 banknotes, display maximum absolute percentage errors (APEs) which are more than 40 times higher than the respective WMAPE. This happens around the months where the series change their sign and should not cause concern since, in those periods, the values observed are very small and so are absolute errors.⁵³ To support this view, we can compare the mean absolute error (MAE) and the maximum absolute error: for both denominations the maximum absolute error is less than twice the respective MAE, a ratio which is in line with the corresponding ratios for the other denominations, or even lower.

⁵⁰ Other outliers are detected and included as additive effects: April 2019 for €5 banknotes in circulation; July 2020 for €10 banknotes in circulation; March and October 2013 for €20 banknotes in circulation; December 2019 and February 2020 for €100 banknotes in circulation.

⁵¹ The formula for the weighted mean absolute percentage error is $WMAPE = \frac{\sum_{t=1}^T \frac{|y_t - \hat{y}_t|}{|y_t|} |y_t|}{\sum_{t=1}^T |y_t|} = \frac{\sum_{t=1}^T |y_t - \hat{y}_t|}{\sum_{t=1}^T |y_t|}$. The mean absolute percentage error (MAPE), not taking into account the size of each observation, can be misleading.

⁵² As illustrated in Section 3.2 of the main text the circulation growth observed in connection with the COVID-19 pandemic was mainly driven by the strong decrease in lodgements at the central bank. This was unprecedented in Italian circulation dynamics.

⁵³ For these denominations the MAPE is much larger than the WMAPE because estimation errors for actual values near to zero, although very small, contribute heavily to the indicator.

The forecast exercise to predict circulation dynamics from October 2020 to September 2021 performs well for all denominations. In particular, the STSM forecasts largely outperform, in terms of WMAPE (Tables 1b and 1c and Figure C7), the circulation forecasts obtained by combining the predictions of withdrawals and lodgements generated by the models currently in use at Banca d'Italia for €5, €20 and €100 banknotes in circulation, show a slightly higher accuracy for €10 and €200 banknotes in circulation, and display a similar precision for €50 banknotes in circulation. Circulation of €100 banknotes turns negative in the forecast sample, generating very high APEs in the corresponding months. As described for €10 and €200 banknotes in circulation in the estimation sample, this was expected and should not be a concern: we find a maximum APE more than 10 times larger than the WMAPE but this corresponds to a maximum absolute error which is less than three times the MAE. We can conclude that STSMs provide a useful tool for forecasting circulation dynamics in Italy, at least within a forecast horizon of 12 months. This technique may therefore serve as a major aid in checking the consistency of predictions obtained by combining forecasts of withdrawals and lodgements generated by the models currently in use at Banca d'Italia.⁵⁴

Table 1a
STSM estimation statistics

Structural Time Series Model - Estimation Sample (Jan 2004 - Sep 2020)							
	Mean Absolute Percentage Error	Weighted Mean Absolute Percentage Error**	Median Absolute Percentage Error	Maximum Absolute Percentage Error	Mean Absolute Error	Median Absolute Error	Maximum Absolute Error
€ 5	0.2%	0.2%	0.1%	0.4%	223,313	145,970	552,604
€ 10	4.5%	1.6%	1.7%	67.6%*	2,719,400	3,195,528	4,696,471
€ 20	1.3%	1.5%	1.5%	2.0%	2,291,242	1,826,947	8,822,987
€ 50	0.9%	0.8%	0.8%	1.4%	18,354,218	17,537,242	37,906,328
€ 100	2.1%	1.7%	1.6%	13.1%	4,174,658	3,528,076	10,928,322
€ 200	6.3%	1.7%	1.8%	76.1%*	405,138	462,225	761,651

* Excludes the Absolute Percentage Error in the months the circulation changes sign.

** Weighted by actual circulation in corresponding months.

Source: Authors' calculations.

⁵⁴ In the context of the coordinated exercise to predict banknote requirements at the euro area level, national central banks are asked to provide separate forecasts for withdrawals and lodgements. Given that fact, circulation forecasts obtained via STSMs presented in this paper cannot be used directly to complete this particular task. Forecasting withdrawals and lodgements using STSMs is beyond the scope of this paper.

Table 1b
STSM forecast accuracy

Structural Time Series Model - Forecast Sample (Oct 2020 - Sep 2021)							
	Mean Absolute Percentage Error	Weighted Mean Absolute Percentage Error**	Median Absolute Percentage Error	Maximum Absolute Percentage Error	Mean Absolute Error	Median Absolute Error	Maximum Absolute Error
€ 5	1.0%	1.0%	1.0%	2.3%	1,035,367	1,032,847	2,490,224
€ 10	1.3%	1.3%	1.3%	3.6%	3,198,877	3,130,438	8,546,101
€ 20	1.8%	1.8%	1.7%	1.9%	8,589,300	8,565,453	9,410,877
€ 50	0.9%	0.9%	0.9%	0.9%	39,865,853	39,901,092	40,864,795
€ 100	9.2%	5.2%	2.9%	57.8%*	966,920	896,607	2,524,737
€ 200	0.1%	0.1%	0.1%	0.2%	66,425	59,768	149,564

* Excludes the Absolute Percentage Error in the months the circulation changes sign.

** Weighted by actual circulation in corresponding months.

Source: Authors' calculations.

Table 1c
Forecast accuracy of models currently in use at Banca d'Italia

Models currently in use at Bank of Italy - Forecast Sample (Oct 2020 - Sep 2021)							
	Mean Absolute Percentage Error	Weighted Mean Absolute Percentage Error**	Median Absolute Percentage Error	Maximum Absolute Percentage Error	Mean Absolute Error	Median Absolute Error	Maximum Absolute Error
€ 5	2.8%	2.9%	1.7%	7.1%	3,084,176	1,866,942	7,686,504
€ 10	1.8%	1.8%	1.9%	3.9%	4,470,660	4,589,272	9,683,996
€ 20	4.2%	4.3%	4.3%	7.1%	20,870,200	22,293,309	33,717,777
€ 50	0.9%	0.9%	0.9%	2.0%	42,664,059	43,646,476	95,597,569
€ 100	22.0%	11.0%	7.0%	51.7%*	2,049,406	1,640,509	5,130,995
€ 200	0.2%	0.2%	0.2%	0.5%	128,157	130,902	338,548

* Excludes the Absolute Percentage Error in the months the circulation changes sign.

** Weighted by actual circulation in corresponding months.

Source: Authors' calculations.

Table 2a

Structural time series forecasting models: estimation results

	€5 banknotes	€10 banknotes	€20 banknotes	€50 banknotes	€100 banknotes	€200 banknotes
Estimation sample, number of observations	2004:01 - 2020:09, 201	2004:01 - 2020:09, 201	2004:01 - 2020:09, 201	2004:01 - 2020:09, 201	2004:01 - 2020:09, 201	2004:01 - 2020:09, 201
Forecast horizon, length in months	2020:10 - 2021:09, 12	2020:10 - 2021:09, 12	2020:10 - 2021:09, 12	2020:10 - 2021:09, 12	2020:10 - 2021:09, 12	2020:10 - 2021:09, 12
Model specification ¹⁾ (y: number of Italian-issued banknotes in circulation in million pieces)	$\log(y) = \text{trend} + \text{seasonal} + \text{irregular} + \text{interventions}$	$\log(y) = \text{trend} + \text{seasonal} + \text{cycle} + \text{irregular} + \text{interventions}$	$\log(y) = \text{trend} + \text{seasonal} + \text{irregular} + \text{interventions}$	$\log(y) = \text{trend} + \text{seasonal} + \text{irregular} + \text{interventions}$	$\log(y) = \text{trend} + \text{seasonal} + \text{irregular} + \text{interventions}$	$\log(y) = \text{trend} + \text{seasonal} + \text{irregular} + \text{interventions}$
Specification of trend	stochastic level and stochastic slope	stochastic level and stochastic slope	stochastic level and stochastic slope	stochastic level and stochastic slope	stochastic level and stochastic slope	stochastic level and stochastic slope
Specification of cycle	-	deterministic	-	-	-	-
Specification of trigonometric seasonal	fixed	fixed	fixed	fixed	fixed	fixed
<i>Interventions for the great financial crisis:</i>						
outlier in 2008:09				-0.012**	-0.013***	-0.002***
outlier in 2008:10						0.002***
<i>Interventions for Europa series issuance:</i>						
outlier in 2013:04	-0.056***					
outlier in 2013:05	0.041***					
<i>Interventions for policy to improve the quality of banknotes in circulation:</i>						
outlier in 2019:07		0.022***				
outlier in 2019:12		0.031***				
<i>Other interventions:</i>						
outlier in 2013:03			0.166***			
outlier in 2013:10			0.128***			
outlier in 2019:04	0.024***					
outlier in 2019:12					0.024***	
outlier in 2020:02					-0.014***	
outlier in 2020:07		-0.022***				

Source: Authors' calculations.

Note: 1) *** significant at the 1 per cent level, ** significant at the 5 per cent level.

Table 2b

Structural time series forecasting models: diagnostics and goodness of fit

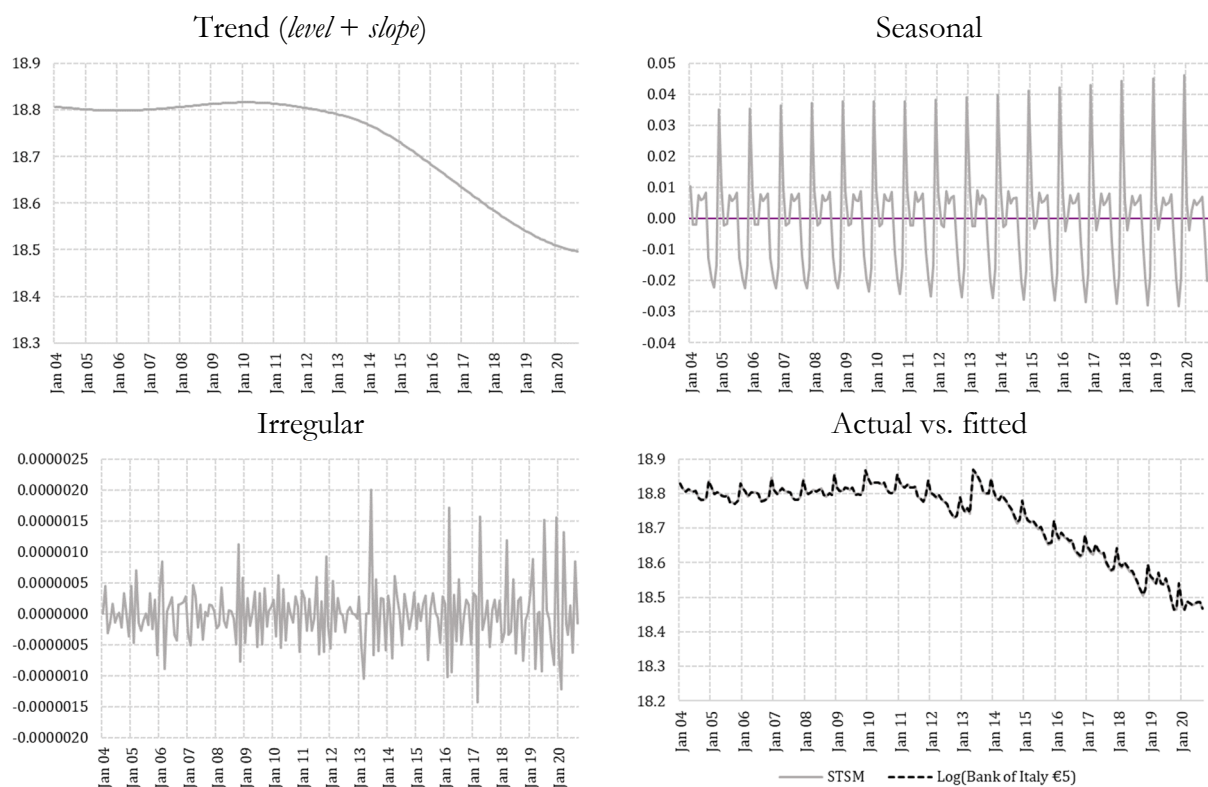
	€5 banknotes	€10 banknotes	€20 banknotes	€50 banknotes	€100 banknotes	€200 banknotes
Goodness-of-fit statistics:						
Log-Likelihood	536.35	662.62	285.01	651.71	591.21	1028.20
Akaike Information Criterion (AIC)	-1057.00	-1039.00	-554.00	-1287.00	-1166.00	-2040.00
Bayesian Information Criterion (BIC)	-1031.00	-1283.00	-528.00	-1262.00	-1141.00	-2015.00
Basic diagnostics of residuals:						
Test of the null hypothesis of no seasonality by means of a chi-square test. ¹⁾	$340.99 > \chi^2(11; 0.05) = 19.68$	$356.09 > \chi^2(11; 0.05) = 19.68$	$89.06 > \chi^2(11; 0.05) = 19.68$	$48.82 > \chi^2(11; 0.05) = 19.68$	$211.07 > \chi^2(11; 0.05) = 19.68$	$59.00 > \chi^2(11; 0.05) = 19.68$
Forecast horizon, length in months	2020:10 - 2021:09, 12	2020:10 - 2021:09, 12	2020:10 - 2021:09, 12	2020:10 - 2021:09, 12	2020:10 - 2021:09, 12	2020:10 - 2021:09, 12
WMAPE of 12-step-ahead forecast	1.1%	1.1%	3.0%	0.3%	4.8%	0.1%

Source: Authors' calculations.

Note: 1) This is a joint test of significance of the 11 seasonal effects at the end of the sample period. Under the null of no seasonal pattern the test statistic is asymptotically χ^2 with 11 degrees of freedom when the seasonal is deterministic. The critical value is $\chi^2(11; 0.05) = 19.68$. In the case of a stochastic seasonal, the joint seasonal test is also performed although a formal joint test of significance of the seasonal effect is inappropriate. However, when the seasonal pattern is persistent throughout the series and when the seasonal pattern changes relatively slowly, which is usually the case, the test statistic can provide a useful guide to the relative importance of the seasonal.

Figure C1

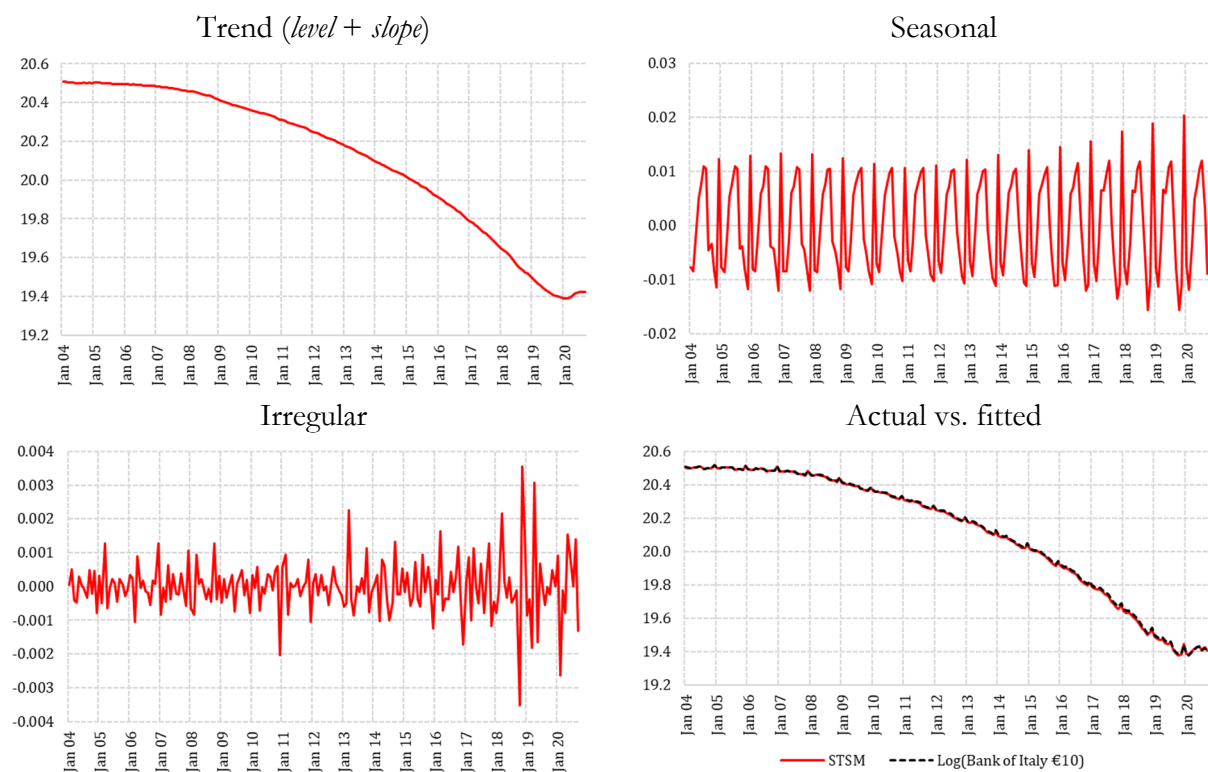
STSM for €5 denomination



Source: Authors' calculations.

Figure C2

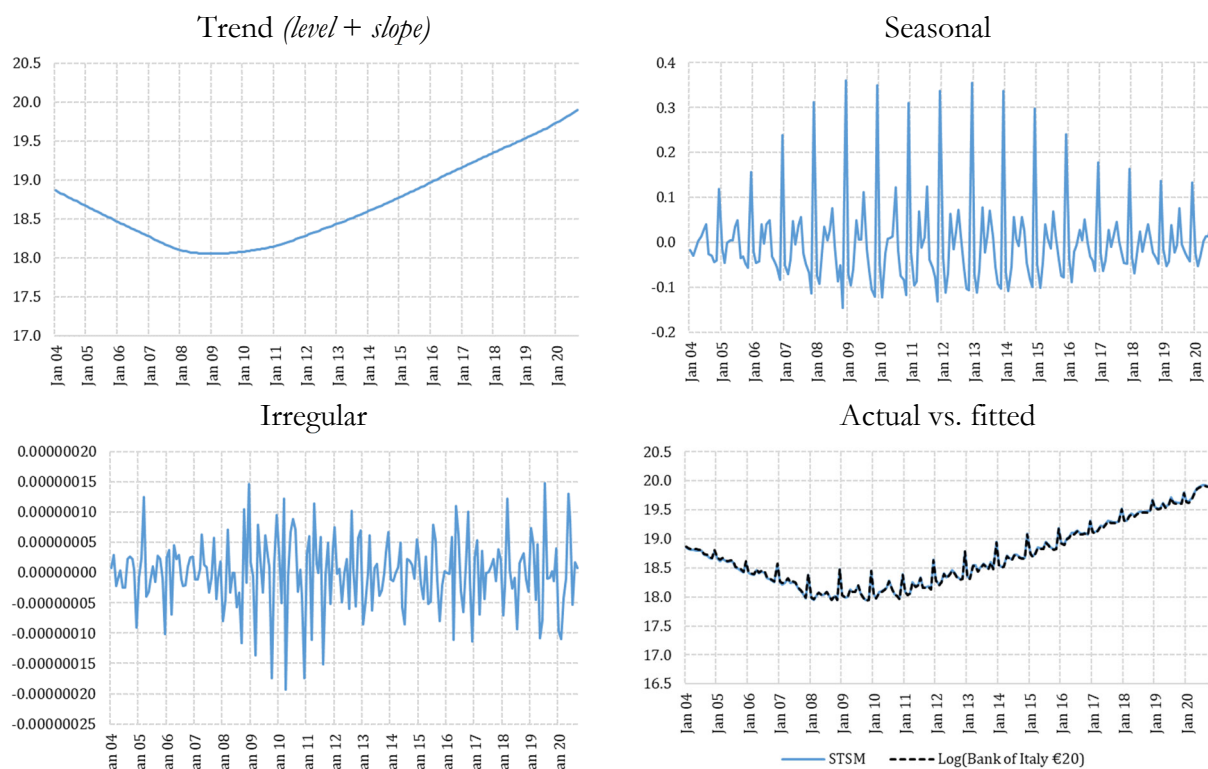
STSM for €10 denomination



Source: Authors' calculations.

Figure C3

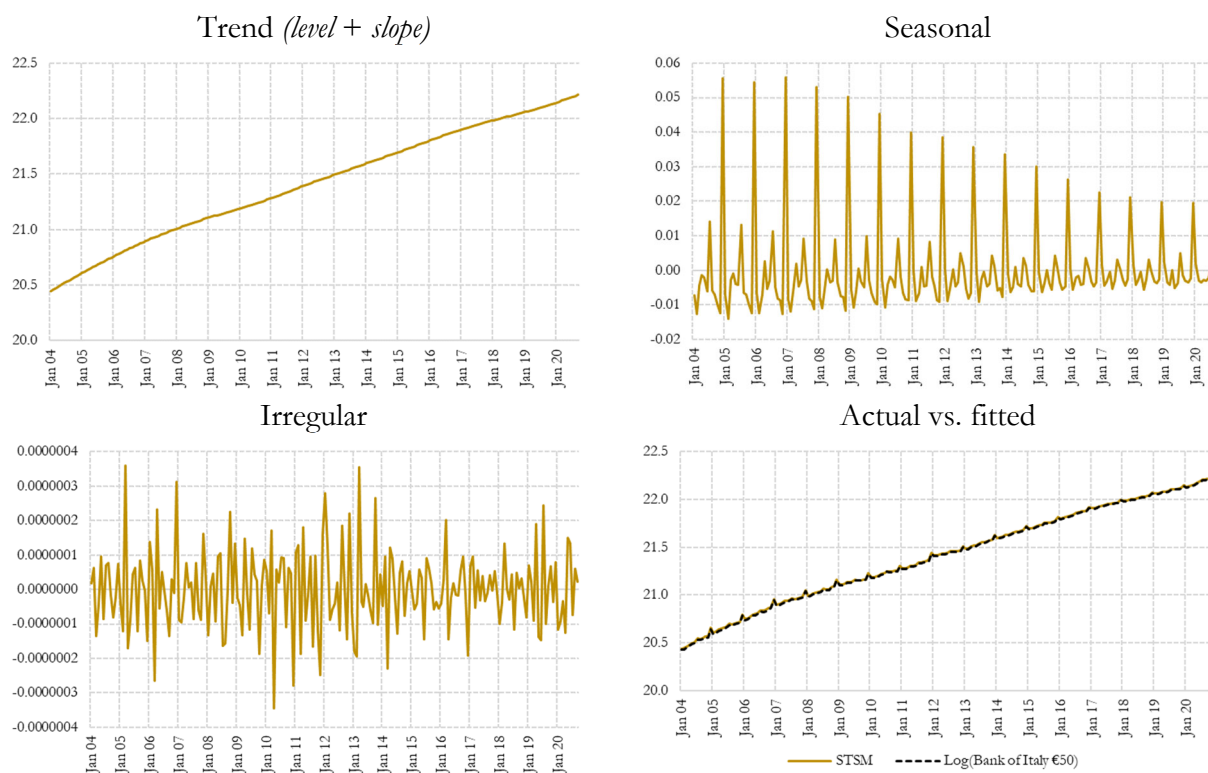
STSM for €20 denomination



Source: Authors' calculations.

Figure C4

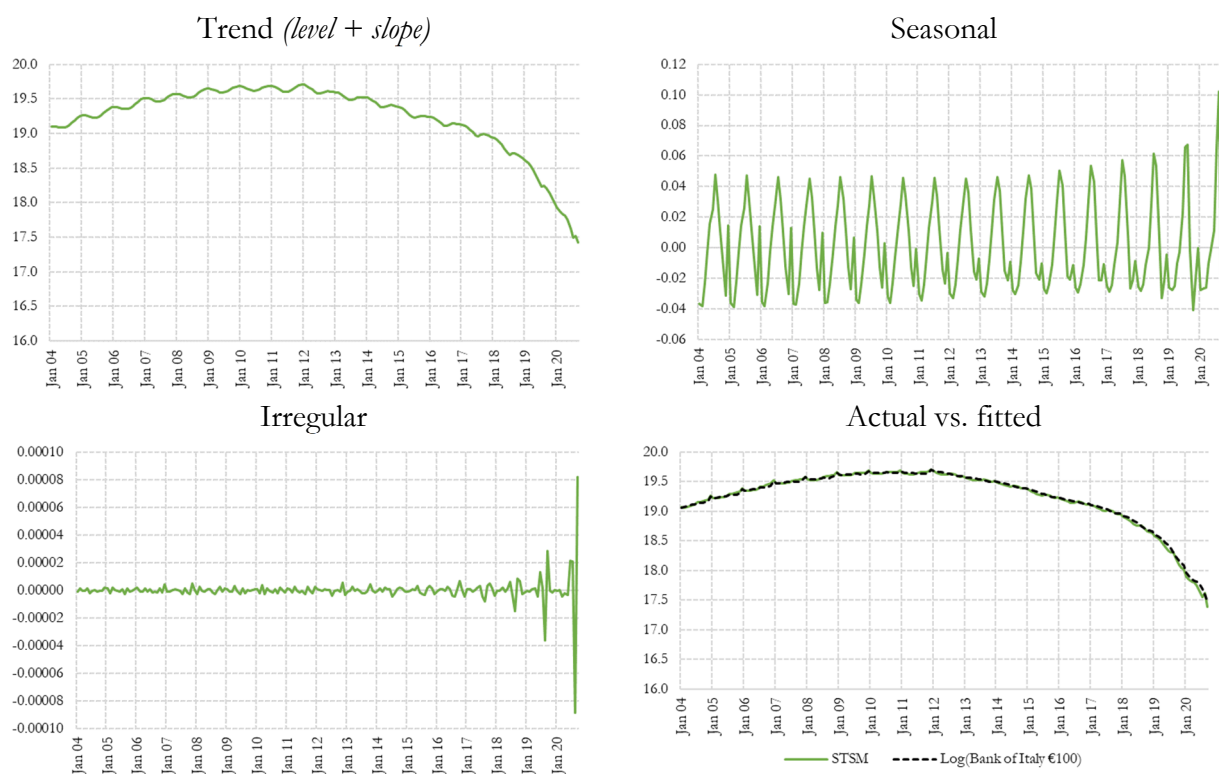
STSM for €50 denomination



Source: Authors' calculations.

Figure C5

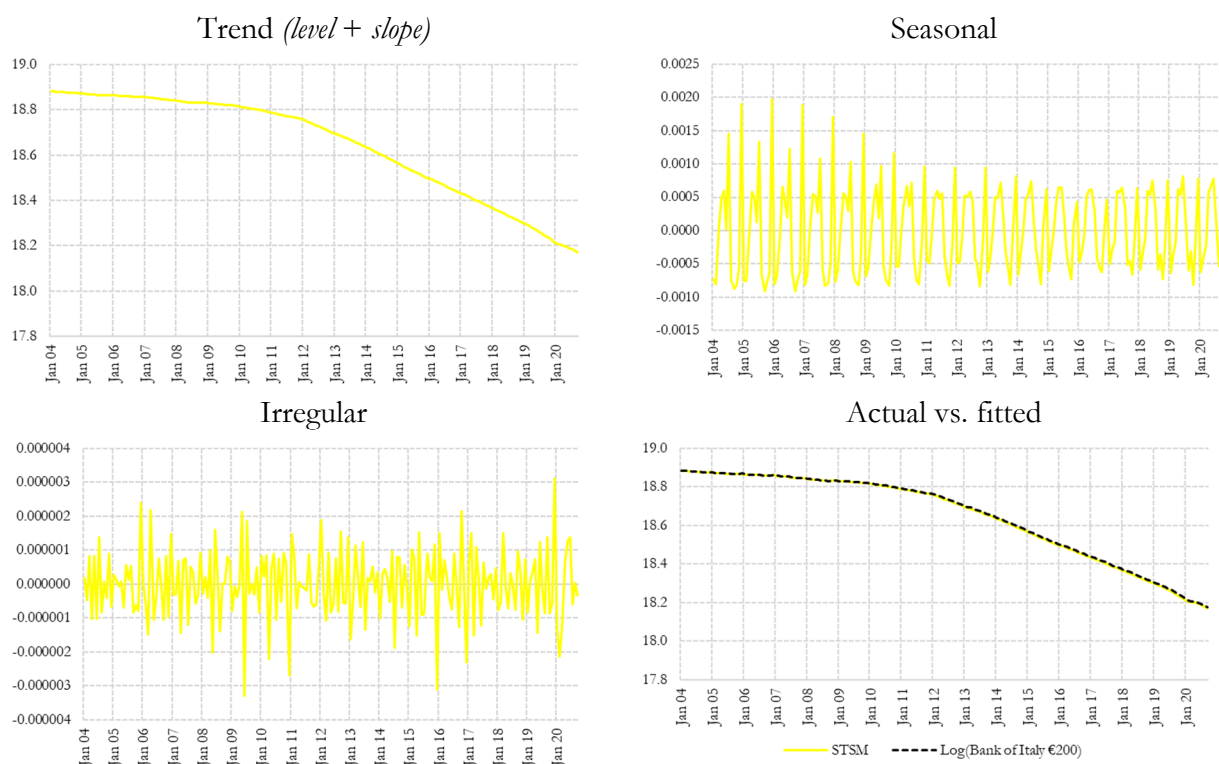
STSM for €100 denomination



Source: Authors' calculations.

Figure C6

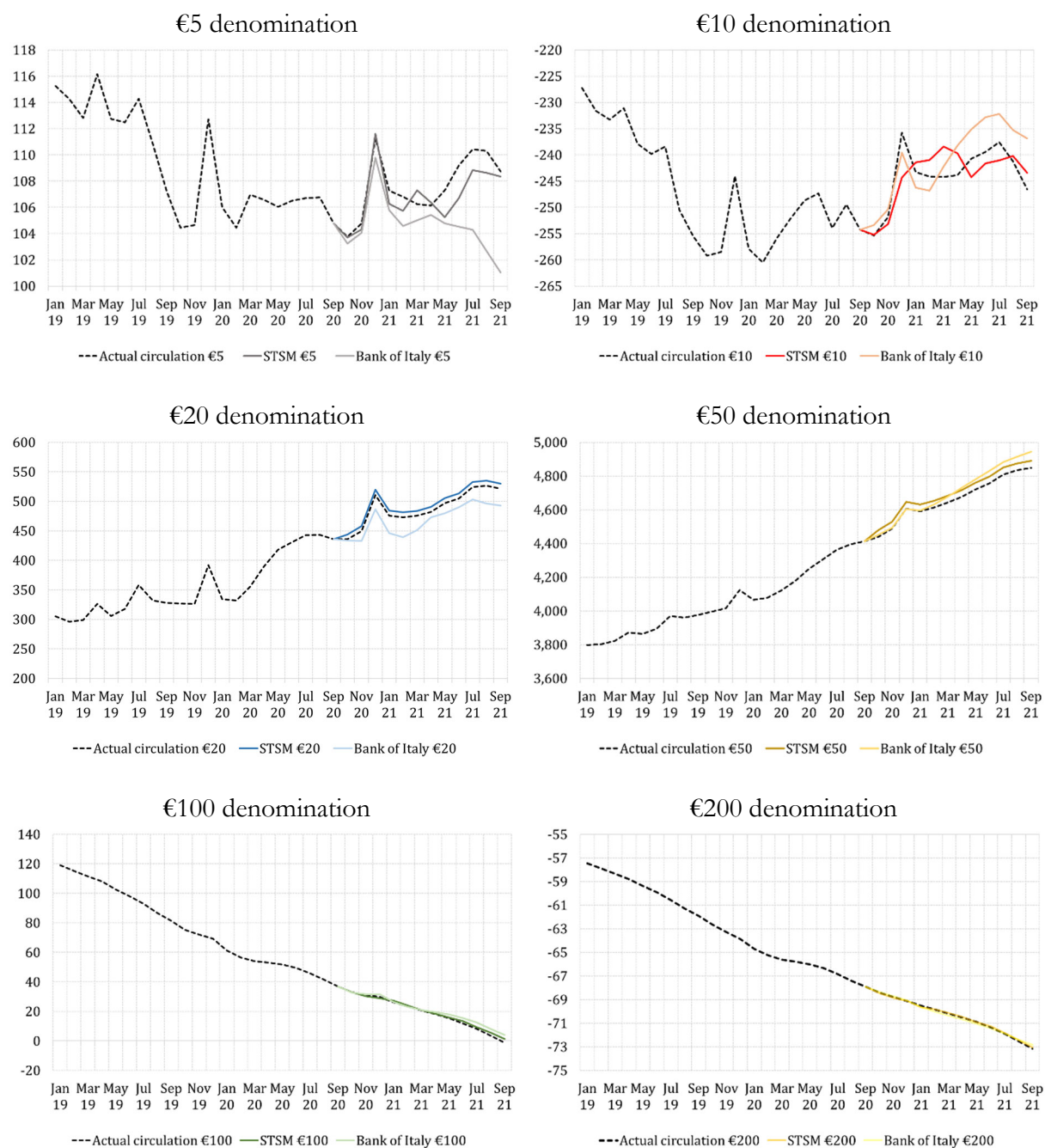
STSM for €200 denomination



Source: Authors' calculations.

Figure C7

Forecasts obtained by STSMs and comparison with models currently in use at Banca d'Italia



Source: Authors' calculations.

Annex D: Forecasting Spanish-issued banknotes in circulation

The main objective of this exercise is to model the monthly series of banknotes issued by Banco de España using STSMs and evaluate their performance against SARIMA-based models.

The results presented in this work refer to the period January 2004 to September 2020 that was used to train the models and the period October 2020 to September 2021 as the test for forecasting performance. The final specification of the STSM for each denomination was done on the basis of the significance of the parameters and the tests on the residuals for normality and serial correlation. The results are reported in Table 2a, Table 2b and Figure D1 to Figure D6 at the end of this annex.

All models include a trend component and an irregular component. Moreover, a seasonal component is included in the models for the low (€5 and €10) and medium (€20 and €50) denominations while in the large denominations (€100 and €200) this component turned out to be non-significant. Similarly, the models for the small and medium denominations contain a regression component to model the calendar effect of the Easter holidays which occur either in March or April each year. Finally, different break points in the trend and outliers were identified and intervention variables were created. Some of these are the result of events of various natures affecting banknote issuance. As an example, the uncertainty stemming from the financial rescue of a Spanish credit institution in the spring of 2012 led to an increase in the net issuance of €50 banknotes. The introduction of a limit to cash payments in late 2012 had a negative effect on the net issuance of €100 banknotes. In October 2017, the uncertainty related to the political situation in Catalonia led to a significant increase in the net issuance of €100 banknotes; and more recently, the outbreak of the COVID-19 pandemic resulted in a rise in net issuance for most denominations.

The Shapiro-Wilk test is applied to test normality. The null hypothesis of normality cannot be rejected for a critical value of 5 per cent for all denominations. The Ljung-Box statistic of serial correlation is computed based on the first p autocorrelations and reported as $Q(p)$ where the values of p are set to 12 and 24. Based on the Ljung-Box statistic we cannot reject the null hypothesis of no serial correlation for a critical value of 1 per cent in all models and frequencies with the exception of the model for €100 banknotes in which we reject the null hypothesis for $Q(24)$.

Table 1
Forecast accuracy

Accuracy of monthly forecasts at different horizons									
denomination	months	STSM				ARIMA			
		MAPE	MAE	MPE	RMSE	MAPE	MAE	MPE	RMSE
5	1	1.756	3.745	-1.756	3.745	1.137	2.426	-1.137	2.426
	3	2.200	4.642	-2.200	4.817	2.171	4.567	-2.171	5.248
	6	3.231	6.862	-3.231	7.306	2.881	6.106	-2.881	6.612
	12	4.738	10.065	-4.738	10.875	4.901	10.402	-4.901	11.626
10	1	0.509	7.812	-0.509	7.812	0.359	5.518	-0.359	5.518
	3	0.470	7.204	-0.470	7.644	0.927	14.192	-0.927	17.057
	6	0.511	7.859	-0.511	8.143	1.173	18.063	-1.173	20.341
	12	0.405	6.262	-0.405	7.021	1.698	26.347	-1.698	29.252
20	1	0.790	16.674	-0.790	16.674	0.388	8.189	-0.388	8.189
	3	0.906	19.109	-0.906	20.008	1.021	21.512	-1.021	25.610
	6	1.254	26.633	-1.254	28.081	1.295	27.466	-1.295	30.563
	12	1.517	32.480	-1.517	33.951	2.078	44.541	-2.078	49.188
50	1	0.883	11.528	0.883	11.528	1.089	14.210	1.089	14.210
	3	0.500	6.627	0.500	7.599	2.564	34.627	2.564	41.244
	6	1.170	16.065	0.865	22.749	3.030	41.428	3.030	45.901
	12	2.408	34.114	2.256	40.863	3.826	53.946	3.826	58.558
100	1	0.369	0.379	-0.369	0.379	0.865	0.887	-0.865	0.887
	3	0.885	0.914	-0.885	1.035	1.663	1.717	-1.663	1.890
	6	1.181	1.238	-1.181	1.347	2.244	2.355	-2.244	2.578
	12	1.317	1.416	-1.186	1.563	2.813	3.056	-2.813	3.296
200	1	0.726	0.038	0.726	0.038	0.334	0.018	-0.334	0.018
	3	1.034	0.053	1.034	0.054	1.823	0.092	-1.823	0.110
	6	2.604	0.112	-0.701	0.171	5.909	0.258	-5.909	0.353
	12	16.054	0.444	-15.103	0.603	24.332	0.697	-24.332	0.887

Source: Authors' calculations.

Forecast accuracy is evaluated using different forecast error measures including the mean absolute percentage error (MAPE), mean absolute error (MAE), the mean percentage error (MPE) and the root mean squared error (RMSE). Table 1 displays the forecast errors across different horizons for both the STSMs and SARIMA models. All measures are consistent in showing that the STSMs generate the most accurate forecast for most denominations with the exception of €5 banknotes where the SARIMA model seems to perform better at certain horizons. However, the differences in forecast accuracy among the models are only marginal in the short term while the STSMs outperform especially at the one year ahead forecast.

All in all, it can be concluded that STSMs perform better for the period evaluated (12 months ahead) in most denominations. It should be taken into account that the issuance of banknotes between October 2020 and September 2021 was still conditioned by the restrictions associated with the public health crisis and it would therefore be prudent to re-run this exercise with the inclusion of a longer horizon.

Table 2a

Structural time series forecasting models: estimation results

	€5 banknotes	€10 banknotes	€20 banknotes	€50 banknotes	€100 banknotes	€200 banknotes
Estimation sample, number of observations	2004:01 - 2020:09, 201	2004:01 - 2020:09, 201	2004:01 - 2020:09, 201	2004:01 - 2020:09, 201	2004:01 - 2020:09, 201	2004:01 - 2020:09, 201
Forecast horizon, length in months	2020:10 - 2021:09, 12	2020:10 - 2021:09, 12	2020:10 - 2021:09, 12	2020:10 - 2021:09, 12	2020:10 - 2021:09, 12	2020:10 - 2021:09, 12
Model specification ¹⁾ (y : number of Spanish-issued banknotes in circulation in million pieces)	$y = \text{trend} + \text{seasonal} + \text{irregular} + \text{explanatory variable}$	$y = \text{trend} + \text{seasonal} + \text{irregular} + \text{explanatory variable} + \text{intervention}$	$y = \text{trend} + \text{seasonal} + \text{irregular} + \text{explanatory variable} + \text{intervention}$	$y = \text{trend} + \text{seasonal} + \text{irregular} + \text{explanatory variable} + \text{interventions}$	$y = \text{trend} + \text{irregular} + \text{interventions}$	$y = \text{trend} + \text{irregular} + \text{interventions}$
Specification of trend	stochastic level and stochastic slope	stochastic level and stochastic slope	stochastic level and stochastic slope	stochastic level and stochastic slope	stochastic level and stochastic slope	stochastic level and stochastic slope
Specification of trigonometric seasonal	stochastic	stochastic	stochastic	stochastic	-	-
Explanatory variable: dummy variable for <i>Easter</i>	0.35**	0.28***	1.10**	2.00**		
Interventions for: <i>Lehman brothers</i> , slope break: 2008:09 <i>financial bailout</i> , level break: 2012:05 <i>cash payments limits</i> , slope break: 2012:11				9.11***	-0.02**	-0.026***
<i>announcement stop issuance €500</i> , level break: 2016:09						0.65***
<i>Catalonia political instability</i> , level break: 2017:10					0.03***	
<i>COVID-19 crisis</i> , level break: 2020:03						-0.53***
<i>COVID-19 crisis</i> , level break: 2020:04				22.21***		0.32***
<i>COVID-19 crisis</i> , slope break: 2020:04		2.14**	13.21***		0.002*	

Source: Authors' calculations.

Note: 1) ***: significant at the 1 per cent level, **: significant at the 5 per cent level, *: significant at the 10 per cent level.

Table 2b

Structural time series forecasting models: diagnostics and goodness-of-fit

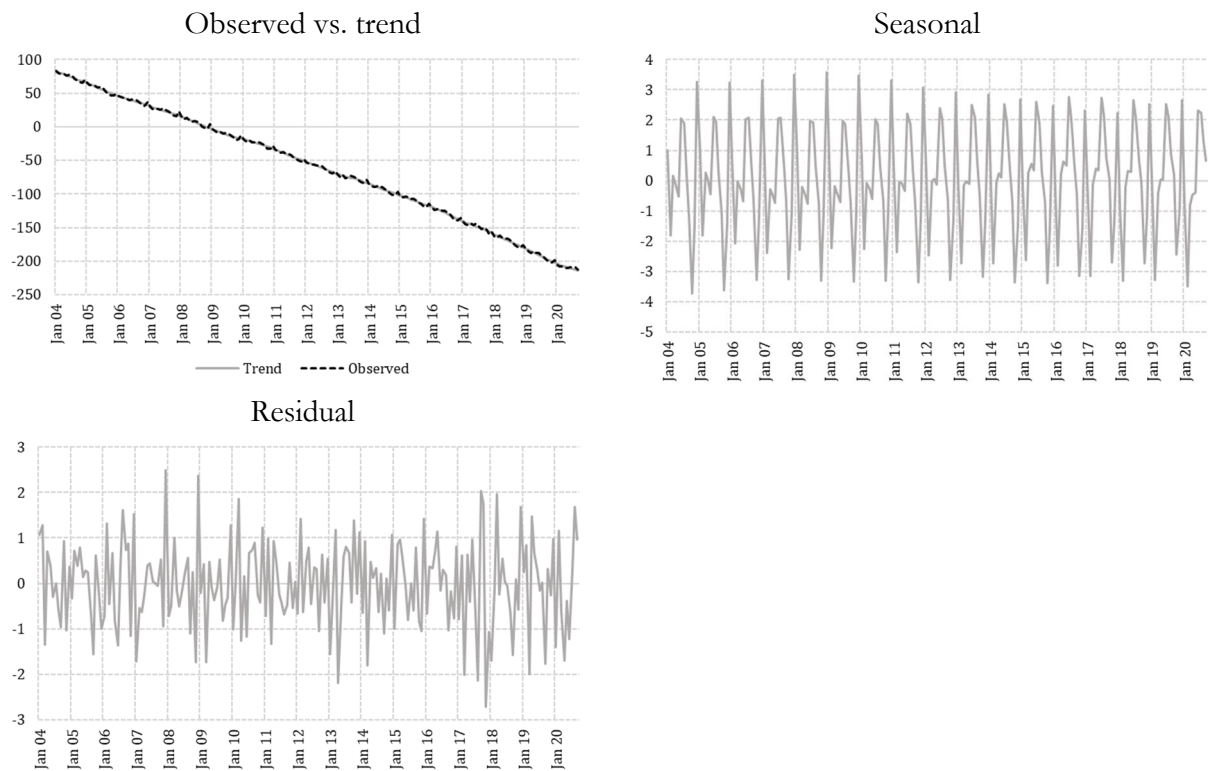
	€5 banknotes	€10 banknotes	€20 banknotes	€50 banknotes	€100 banknotes	€200 banknotes
Goodness-of-fit statistics:						
Log-Likelihood	691.90	834.08	727.54	405.19	629.87	427.19
Akaike Information Criterion (AIC)	-1365.81	-1650.17	-1437.08	-796.37	-1249.75	-848.39
Bayesian Information Criterion (BIC)	-1336.08	-1620.44	-1407.35	-773.25	-1233.23	-838.48
Posterior mean of the residual standard deviation parameter	0.98	3.69	5.34	0.08	0.01	0.08
S.d. of the one-step-ahead prediction errors for the training data	1.40	3.98	6.04	0.19	0.01	0.19
Harvey's goodness of fit statistic	0.75	0.72	0.80	0.81	0.58	0.48
Basic diagnostics of residuals (standardised one-step-ahead prediction errors):						
Test of the null hypothesis of normally distributed residuals by means of the Shapiro-Wilk statistic	$W = 0.9919$ p-value = 0.3311	$W = 0.9616$ p-value = 0.1404	$W = 0.9793$ p-value = 0.2145	$W = 0.9912$ p-value = 0.8633	$W = 0.96$ p-value = 0.1216	$W = 0.9744$ p-value = 0.4136
Test of the null hypothesis of serially uncorrelated residuals by means of the Ljung-Box Q -statistic	$Q(12) = 18.75$ p-value = 0.06582	$Q(12) = 22.153$ p-value = 0.02321	$Q(12) = 15.444$ p-value = 0.1631	$Q(12) = 21.469$ p-value = 0.02883	$Q(12) = 19.586$ p-value = 0.05136	$Q(12) = 17.864$ p-value = 0.08478
	$Q(24) = 38.161$ p-value = 0.02447	$Q(24) = 30.924$ p-value = 0.1247	$Q(24) = 23.505$ p-value = 0.4316	$Q(24) = 32.032$ p-value = 0.09948	$Q(24) = 44.178$ p-value = 0.00505	$Q(24) = 36.223$ p-value = 0.03914
Forecast horizon, length in months	12	12	12	12	12	12
MAPE of 12-step-ahead forecast	4.74	0.40	1.52	2.41	1.32	16.05
MAPE of 12-step-ahead forecast with benchmark ARIMA model	4.90	1.70	2.08	3.83	2.81	24.33

Source: Authors' calculations.

Note: MAPE stands for mean absolute percentage error.

Figure D1

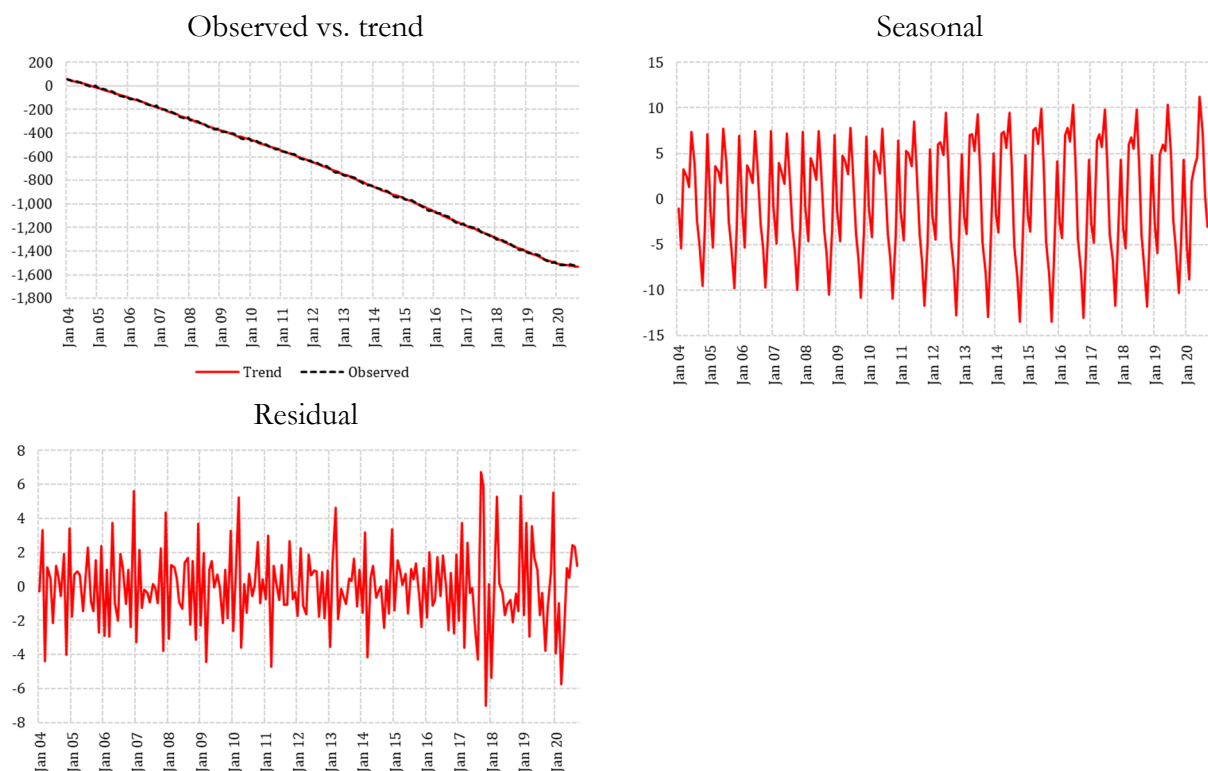
STSM for €5 EUR banknotes in circulation



Source: Authors' calculations.

Figure D2

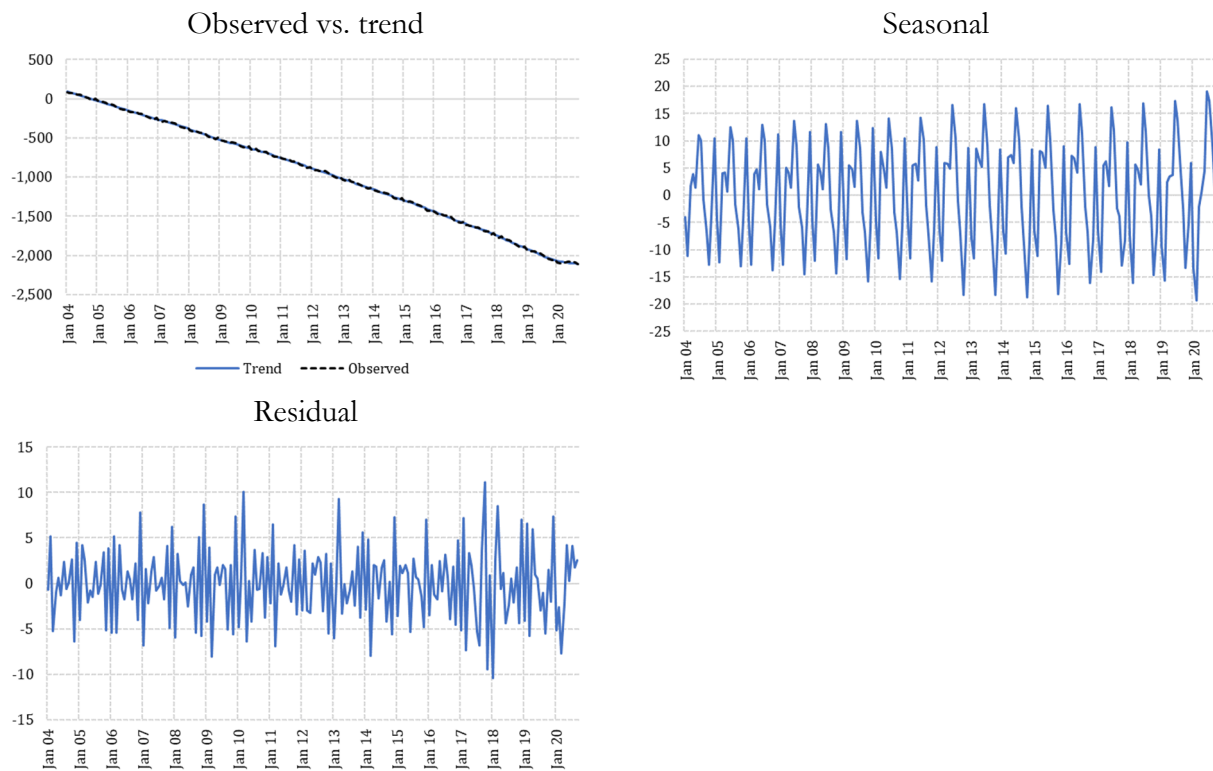
STSM for €10 banknotes in circulation



Source: Authors' calculations.

Figure D3

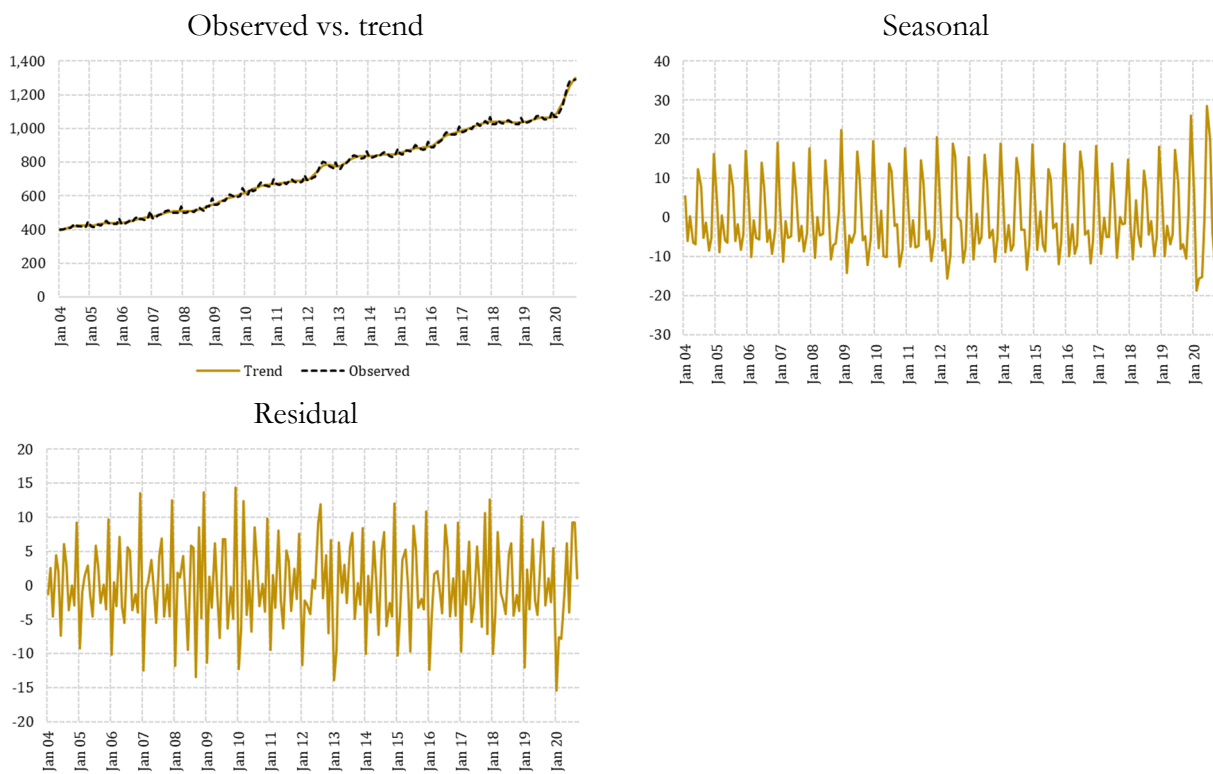
STSM for €20 banknotes in circulation



Source: Authors' calculations.

Figure D4

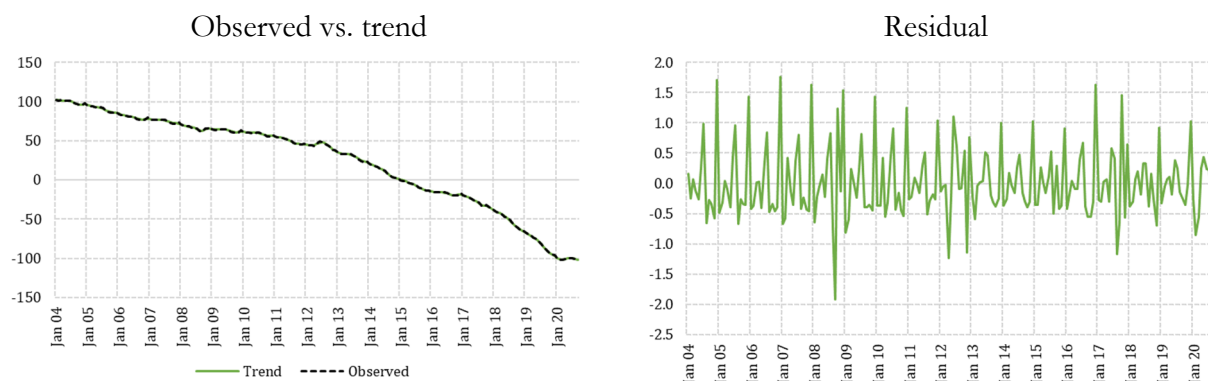
STSM for €50 banknotes in circulation



Source: Authors' calculations.

Figure D5

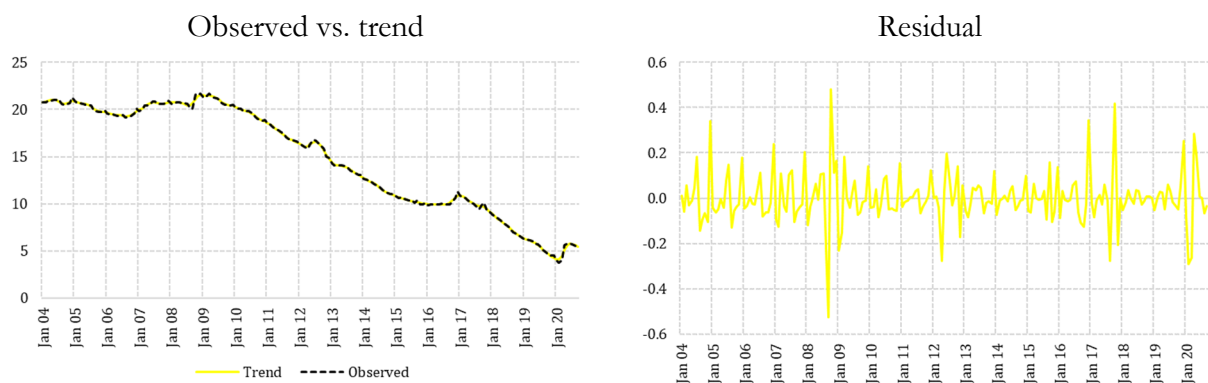
STSM for €100 banknotes in circulation



Source: Authors' calculations.

Figure D6

STSM for €200 banknotes in circulation



Source: Authors' calculations.

References

- Baldo, L., Bonifacio, E., Brandi, M., Lo Russo, M., Maddaloni, G., Nobili, A., Rocco, G., Sene, G. and Valentini, M. (2021), “Inside the black box: tools for understanding cash circulation”, *Markets, Infrastructures, Payment Systems*, Banca d’Italia.
- Banque de France (2021), “The cash industry in the midst of the Covid-19 crisis”, authors: Bayol, L., Laplace, E. and de Pastor, R., *Bulletin*, 235/4, pp. 1-9.
- Bartzsch, N., Gerdesmeier, D., Landau, B., Maddaloni, G., Rocco, G., Roffia, B. and Rua, A. (2020), “Forecasting the demand for banknotes in the euro area”, *International Cash Conference 2019, Conference volume*, Deutsche Bundesbank.
- Bartzsch, N., Rösl, G. and Seitz, F. (2011), “Foreign demand for euro banknotes issued in Germany: estimation using direct approaches”, *Discussion Paper*, Series 1: Economic Studies, No 20, Deutsche Bundesbank.
- Commandeur, J. J. F., Koopman, S. J. (2007), *An introduction to state space time series analysis*, Oxford: Oxford University Press.
- Deutsche Bundesbank (2009), “Demand for banknotes during the financial crisis”, *Monthly Report*, June 2009, pp. 52-53.
- Deutsche Bundesbank (2019), *Cash use in Germany: Macroeconomic estimates of the extent of illicit cash use in Germany*, authors: Bartzsch, N., Schneider, F. and Uhl, M.
- Deutsche Bundesbank (2022), “Demand for euro banknotes issued by the Bundesbank: current developments”, *Monthly Report*, April 2022, pp. 67-83.
- Doornik, J. A. and Hansen, H. (1994), “An omnibus test for univariate and multivariate normality”, *Working Paper*, Nuffield College, University of Oxford.
- European Central Bank (2022a), *ECB Payment Statistics*, July.
- European Central Bank (2022b), *Study on the payment attitudes of consumers in the euro area (SPACE) - 2022*, December.
- Harvey, A. C. (1984), “A unified view of statistical forecasting procedures”, *Journal of Forecasting*, Vol. 3, pp. 245-275.
- Harvey, A. C. (1989), *Forecasting, structural time series models and the Kalman filter*, Cambridge: Cambridge University Press.
- Holt, C. C. (1957), “Forecasting seasonals and trends by exponentially weighted moving averages”, *ONR Memorandum*, Vol. 52, Pittsburgh, PA: Carnegie Institute of Technology.
- Koopman, S. J., Harvey, A. C., Dornik, J. A. and Shephard, N. (2007), *STAMP 8.0: Structural Time Series Analyser, Modeller and Predictor*, London: Timberlake Consultants.
- Lalouette, L., Zamora-Pérez, A., Rusu, C., Bartzsch, N., Politronacci, E., Delmas, M., Rua, A., Brandi, M. and Naksi, M. (2021), “Foreign demand for euro banknotes”, *Occasional Paper Series*, No 253, ECB.
- Proietti, T. (1991), “Forecasting with structural time series models”, in: M. Clements and D. Hendry (eds.), *A companion to economic forecasting*, Oxford: Blackwell Publishers.
- Rösl, G. and Seitz, F. (2021), “Cash and crises: No surprises by the virus”, *Working Paper*, No. 150, Institute for Monetary and Financial Stability at Goethe University Frankfurt.

Seitz, F., Devigne, L. and de Pastor, R. (2022), “Different motives for holding cash in France: an analysis of the net cash issues of the Banque de France”, *Working Paper*, No 888, Banque de France.

Zamora-Pérez, A. (2021), “The paradox of banknotes: understanding the demand for cash beyond transactional use”, *Economic Bulletin*, Issue 2/2021, ECB.