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ALGEBRAIC STRUCTURES FOR PAIRWISE COMPARISON MATRICES: CONSISTENCY, SOCIAL CHOICES AND ARROW'S THEOREM

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ABSTRACT. We present the algebraic structures behind the approaches used to work with pairwise comparison matrices and, in general, the representation of preferences. We obtain a general definition of consistency and a universal decomposition in the space of PCMs, which allow us to define a consistency index. Also Arrow's theorem, which is presented in a general form, is relevant.

All the presented results can be seen in the main formulations of PCMs, i.e., multiplicative, additive and fuzzy approach, by the fact that each of them is a particular interpretation of the more general algebraic structure needed to deal with these theories.

1. Introduction and motivation

As shown in [19], Riesz spaces can be used as general framework in the context of pairwise comparison matrices (PCMs, shortly); in fact, it is possible to present at once all approaches and to describe properties in this context. It is undoubtedly the importance of ordered vector spaces in economic analysis, since there is a natural ordering for which "more is better", i.e., preferences are monotonic in the order. Therefore Riesz spaces seem to be the natural framework to deal with multi-criteria methods, too.

Riesz spaces have been studied and widely applied in economics and in several other branches (see, e.g., ([1, 2, 4, 20]).

In this article we investigate the actual mathematical properties behind the most common tools used in the study of pairwise comparison matrices. Pairwise comparison matrices (PCMs) are a way in which one can express preferences: the element $a_{i,j}$ indicates the preference of the element i compared with j (see, e.g., [24]).

They are used in the Analytic Hierarchy Process (AHP) introduced by Saaty in [27], and successfully applied to many Multi-Criteria Decision Making problems.

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Inspired by [19], this work wants to enlighten which kind of algebraic structures are strictly essential to:

- express preferences in the field of PCMs;
- define properties, e.g. consistency, consistency index, weak consistency;
- obtain fundamental theorems, such as Arrow's Theorem.

The paper is structured as follows. In Section 2, we recall the mathematical definitions used in the paper. In Section 3 we focus on consistency in the field of PCMs. By results of Subsection 3, in Section 3.1, we formalize a consistency index. In Section 4 we exhibit Arrow's theorem in the field of PCMs with the minimum amount of properties to require to the algebraic structure which describes preferences. In the conclusions, we recapitulate the obtained results and expose our final considerations.

2. Algebraic structures for preferences

A *partially ordered set* $G = (G, \leq)$ is a set G equipped with a partial order $\leq$, that is a reflexive, antisymmetric and transitive relation.

Definition 1. A *partially ordered vector space* $G = (G, +, \cdot, \leq)$ is a real vector space with an order relation $\leq$ that is compatible with the algebraic structure of G, that is

(1.1) $x \leq y$ implies $x + z \leq y + z$ for each x, y, $z \in G$;

(1.2) $x \leq y$ implies $\alpha x \leq \alpha y$ for every x, $y \in G$ and $\alpha \geq 0$.

In a partially ordered vector space G, the set $\{x \in G : x \geq 0\}$ is a convex cone, called the *positive cone* or the *non-negative cone* of G, denoted by G^+. Any vector of G^+ is said to be *positive*.

For every $x \in G$, the positive part x^+, the negative part x^-, and the absolute value $|x|$ are defined by $x^+ = x \vee 0, x^- = (-x) \vee 0$, and $|x| = x^+ + x^-$, respectively.

Definition 2. A partially ordered vector space $G = (G, +, \cdot, \leq)$ is a *Riesz space* (or *vector lattice*) if the partial order is a lattice order, i.e., every two elements have a unique *supremum* and a unique *infimum*.

Many familiar spaces are Riesz spaces, as the following examples show.

***Examples* 1.** The Euclidean space $\mathbb{R}^n$ is a Riesz space under the usual ordering, where

$$x = (x_1, \ldots, x_n) \leq y = (y_1, \ldots, y_n)$$

whenever $x_i \leq y_i$ for each $i = 1, 2, \ldots, n$.

The supremum and infimum of two vectors x and y are given by

$$x \vee y = (\max\{x_1, y_1\}, \ldots, \max\{x_n, y_n\})$$

and

$$x \wedge y = (\min\{x_1, y_1\}, \ldots, \min\{x_n, y_n\}),$$

respectively.

Alo-groups, presented in [11], are examples of Riesz spaces. An Alo-group is a totally ordered lattice group, and hence also an ℓ-group, and by Freudenthal's theorem (see [20: Theorem 40.2]) every ℓ-group can be embedded into a Riesz space. Let us recall that any archimedean abelian linearly ordered group is isomorphic to a subgroup of $\mathbb{R}$, as Hölder proved. By this, the results contained in this paper generalize the ones contained in [11], i.e., we generalize the additive, the multiplicative and the fuzzy approach (see [5, 23, 26], respectively).

Let G be a Riesz space. For every $n \in \mathbb{N}$, G^n is a Riesz space where the ordering is defined coordinate-wise. In particular, the set of square matrices of order n with entries in a Riesz space is a Riesz space, being isomorphic to G^{n^2}.

Both the vector space $C(X)$ of all continuous real functions and the vector space $C_b(X)$ of all bounded continuous real functions on the topological space X are Riesz spaces, when the ordering is defined pointwise.

The space of piecewise linear functions on an interval of the real line, with the usual pointwise ordering, is a Riesz space.

The vector space $L_p(\mu)\,(0 \leq p \leq \infty)$ is a Riesz space under the almost everywhere pointwise ordering, i.e., $f \leq g$ in $L_p(\mu)$ if $f(x) \leq g(x)$ μ-almost everywhere.

The vector spaces ℓ_p $(0 < p \leq \infty)$ are Riesz spaces under the usual pointwise ordering.

Definition 3. A Riesz space is said to be *order complete* (or *Dedekind complete*) if every nonempty subset that is order bounded from above has a supremum, or equivalently if every nonempty subset that is order bounded from below has an infimum.

Definition 4. A vector space X is the *direct sum* of two subspaces Y and Z if every $x \in X$ has a unique decomposition of the form $x = y + z$, where $y \in Y$ and $z \in Z$.

3. Pairwise comparison matrices and consistency

Let $G = (G, +)$ be an abelian group, and $\mathcal{M}_n$ be the set of all $n \times n$-matrices $A = (a_{i,j})$, whose entries belong to G. Observe that, if, we endow $\mathcal{M}_n$ with an operation $\oplus$, defined by $A \oplus B = (a_{i,j} + b_{i,j})_{i,j}$, where $A = (a_{i,j})_{i,j}$ and $B = (b_{i,j})_{i,j}$, then $(\mathcal{M}_n, \oplus)$ is an abelian group. If G is a vector space over a field $\mathbb{K}$, then, we can define a product by $\alpha \odot A = (\alpha\, a_{i,j})_{i,j}$, $\alpha \in \mathbb{K}$.

An $n \times n$-matrix $A = (a_{i,j})_{i,j}$ is said to be *skew-symmetric* if $a_{j,i} = -a_{i,j}$ for every $i, j = 1, 2, \ldots, n$, or equivalently if $A^T = \ominus A$, where $A^T = (a_{j,i})_{i,j}$ and $\ominus A = (-a_{i,j})_{i,j}$ denote the *transpose* and the *negative* matrix of A, respectively. Note that, in any skew-symmetric matrix, it is

$$a_{ii} = 0 \qquad \text{for every } i \in \{1, 2, \ldots, n\}. \tag{3.1}$$

From now on, when it is not otherwise explicitly specified, $A = (a_{i,j})_{i,j}$ denotes a skew-symmetric matrix and $G = (G, +)$ denotes any abelian group.

We say that A is *consistent* if

$$a_{i,k} = a_{i,j} + a_{j,k} \qquad \text{for all } i, j, k \in \{1, 2, \ldots, n\}. \tag{3.2}$$

We say that A is *totally inconsistent* if $\displaystyle\sum_{j=1}^{n} a_{i,j} = 0$ for each $i \in \{1, 2, \ldots, n\}$.

A vector $\mathbf{v} = (v_1, v_2, \ldots, v_n) \in G^n$ is said to be *coherent* for a matrix A if $v_i - v_j = a_{i,j}$ for every $i, j \in \{1, 2, \ldots, n\}$.

Remark 1. Observe that the sum of any two consistent matrices $A = (a_{i,j})_{i,j}$ and $B = (b_{i,j})_{i,j}$ is still consistent. Indeed, if A and B satisfy condition (3.2), then for every $i, j, k \in \{1, 2, \ldots, n\}$, we have

$$a_{i,k} + b_{i,k} = a_{i,j} + b_{i,j} + a_{j,k} + b_{j,k},$$

getting the consistency of $A \oplus B$. Analogously it is possible to see that, if G is a vector space over a field $\mathbb{K}$, $A = (a_{i,j})_{i,j}$ is consistent and $\alpha \in \mathbb{K}$, then $\alpha \odot A = (\alpha\, a_{i,j})_{i,j}$ is also consistent.

Moreover, if $A = (a_{i,j})_{i,j}$ and $B = (b_{i,j})_{i,j}$ are totally inconsistent, then

$$\sum_{j=1}^{n}(a_{i,j} + b_{i,j}) = \sum_{j=1}^{n} a_{i,j} + \sum_{j=1}^{n} b_{i,j} = 0$$

for each $i \in \{1, 2, \dots, n\}$. Hence, the sum of any two totally inconsistent matrices is still totally inconsistent. Similarly, if G is a vector space over $\mathbb{K}$ and $\alpha \in \mathbb{K}$, then, from the equality

$$\sum_{j=1}^{n}(\alpha\, a_{i,j}) = \alpha \sum_{j=1}^{n} a_{i,j}, \qquad i \in \{1, 2, \dots, n\},$$

we deduce that $\alpha \odot A$ is totally inconsistent whenever A is totally inconsistent and $\alpha \in \mathbb{K}$.

Therefore, the sets of all consistent matrices and of all totally inconsistent matrices are two subgroups of $\mathcal{M}_n$, and two subspaces of $\mathcal{M}_n$ when $\mathcal{M}_n$ is a vector space of $\mathbb{K}$.

Now, we see some examples and fundamental properties of consistent matrices and coherent vectors, extending to our setting [11: Propositions 5.3 and 5.4] and [12: Propositions 13 and 14].

Proposition 3.1. *Let $A = (a_{i,j})_{i,j}$. The following results hold.*

3.1.1) *Any two vectors $\mathbf{v} = (v_1, v_2, \dots, v_n)$, $\mathbf{w} = (w_1, w_2, \dots, w_n)$, coherent for A, differ by a constant $c \in G$, that is $w_i - v_i = c$ for every $i \in \{1, 2, \dots, n\}$.*

3.1.2) *If $\mathbf{v} = (v_1, v_2, \dots, v_n)$ is a coherent vector for A, then A is consistent.*

3.1.3) *If A is consistent, then each column vector $\mathbf{a}^{(h)} = \begin{bmatrix} a_{1,h} \\ a_{2,h} \\ \cdots \\ a_{n,h} \end{bmatrix}$, $h \in \{1,\ 2, \dots, n\}$, is coherent for A.*

3.1.4) *A matrix A is consistent if and only if there is at least a coherent vector for it.*

3.1.5) *A matrix A is consistent if and only if at least one of their column vectors is coherent for it.*

3.1.6) *If G is a real vector space and $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{R}$, $\sum_{r=1}^{n} \alpha_r = 1$, then the vector $\mathbf{v} = (v_1, v_2, \dots, v_n)$ of the* affine combinations $v_i = \sum_{r=1}^{n} \alpha_r\, a_{i,r}$, *$i \in \{1,\ 2, \dots, n\}$, is coherent for A. Moreover, if G is a vector space over the field $\mathbb{Q}$ of the rational numbers, then the vector $\mathbf{w} = (w_1, w_2, \dots, w_n)$ of the* means $w_i = \dfrac{1}{n} \sum_{r=1}^{n} a_{i,r}$, *$i \in \{1,\ 2, \dots, n\}$, is coherent for A.*

Proof. 3.1.1) Let $\mathbf{v} = (v_1, v_2, \dots, v_n)$, $\mathbf{w} = (w_1, w_2, \dots, w_n)$ be such that $v_i - v_j = w_i - w_j = a_{i,j}$ for each $i, j \in \{1, 2, \dots, n\}$. Then

$$w_i - v_i = w_j - v_j \qquad \text{for all } i, j \in \{1, 2, \dots, n\}. \tag{3.3}$$

If, we denote by c the common value in (3.3), then, we get $w_i - v_i = c$ for any $i \in \{1, 2, \dots, n\}$. This proves 3.1.1).

3.1.2) Let $\mathbf{v} = (v_1, v_2, \dots, v_n)$ be as in the hypothesis. For every $i, j, k \in \{1, 2, \dots, n\}$, it is

$$v_i - v_j = -(v_j - v_i), \qquad v_i - v_k = (v_i - v_j) + (v_j - v_k). \tag{3.4}$$

Thus, if $a_{i,j} = v_i - v_j$, $i, j \in \{1, 2, \dots, n\}$, then from (3.4), we deduce that the matrix $A = (a_{i,j})$ is consistent. So, 3.1.2) is proved.

3.1.3) Fix arbitrarily $h \in \{1, 2, \dots, n\}$. Since A is consistent, for every $i, j \in \{1, 2, \dots, n\}$, we get $a_{i,h} = a_{i,j} + a_{j,h}$, and hence $a_{i,h} - a_{j,h} = a_{i,j}$. Thus, 3.1.3) is proved.

3.1.4) and 3.1.5) follow from 3.1.2) and 3.1.3).

3.1.6) Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be as in the hypothesis. As A is consistent, we get

$$v_i - v_j = \sum_{r=1}^{n} \alpha_r (a_{i,r} - a_{j,r}) = \sum_{r=1}^{n} \alpha_r (a_{i,r} + a_{r,j}) = \Big(\sum_{r=1}^{n} \alpha_r\Big) a_{i,j} = a_{i,j}, \tag{3.5}$$

getting the consistency of $\mathbf{v}$.

The proof of the last assertion is analogous to that of the previous one, by replacing α_r with $\frac{1}{n}$ for each $r \in \{1, 2, \dots, n\}$. $\square$

Now, we give an example of a totally inconsistent matrix.

***Example* 1.** Given $A = (a_{i,j})_{i,j}$, for every $i, j, k \in \{1, 2, \dots, n\}$, set

$$e^{(A)}_{i,j,k} = a_{i,j} + a_{j,k} + a_{k,i}, \tag{3.6}$$

and for each $i, j \in \{1, 2, \dots, n\}$ put

$$e^{(A)}_{i,j} = \sum_{k=1}^{n} e^{(A)}_{i,j,k}. \tag{3.7}$$

Let $E^{(A)} = (e^{(A)}_{i,j})_{i,j}$. We prove that $E^{(A)}$ is skew-symmetric.

Since A is skew-symmetric, for any $i, j \in \{1, 2, \dots, n\}$ it is

$$\begin{aligned} e^{(A)}_{i,j} + e^{(A)}_{j,i} &= \sum_{k=1}^{n} (a_{i,j} + a_{j,k} + a_{k,i} + a_{j,i} + a_{i,k} + a_{k,j}) \\ &= n(a_{i,j} + a_{j,i}) + \sum_{k=1}^{n} (a_{j,k} + a_{k,j}) + \sum_{k=1}^{n} (a_{k,i} + a_{i,k}) = 0. \end{aligned} \tag{3.8}$$

Thus, $E^{(A)}$ is skew-symmetric.

Now, we prove that $E^{(A)}$ is totally inconsistent, extending [9: Proposition 11] to the context of arbitrary abelian groups. Choose arbitrarily $i \in \{1, 2, \dots, n\}$. Thanks to the skew-symmetry of A and taking into account (3.1), for each $i \in \{1, 2, \dots, n\}$, we have

$$\begin{aligned} \sum_{j=1}^{n} e^{(A)}_{i,j} &= \sum_{j=1}^{n} \sum_{k=1}^{n} (a_{i,j} + a_{j,k} + a_{k,i}) \\ &= n \sum_{j=1}^{n} a_{i,j} + \sum_{j=1}^{n} \sum_{k=1}^{n} a_{j,k} + n \sum_{k=1}^{n} a_{k,i} \\ &= n \sum_{j=1}^{n} a_{i,j} + \sum_{j=k} a_{j,k} + \sum_{j<k} a_{j,k} + \sum_{j>k} a_{j,k} + n \sum_{k=1}^{n} a_{k,i} \\ &= n \sum_{j=1}^{n} a_{i,j} + \sum_{j=1}^{n} a_{j,j} + \sum_{j<k} a_{j,k} + \sum_{j<k} a_{k,j} + n \sum_{j=1}^{n} a_{j,i} \\ &\qquad \text{(by exchanging } k \text{ with } j\text{)} \\ &= n \sum_{j=1}^{n} (a_{i,j} + a_{j,i}) + \sum_{j<k} (a_{j,k} + a_{k,j}) = 0, \end{aligned} \tag{3.9}$$

getting the total inconsistency of $E^{(A)}$.

The next step is to prove that every skew-symmetric matrix A can be decomposed into the direct sum of a consistent and a totally inconsistent matrix, extending [9: Propositions 12 and 13]. To this aim, we first give some lemmas.

Lemma 3.1. *Let $\widetilde{A} = (n \odot A) \ominus E^{(A)} = (\widetilde{a_{i,j}})_{i,j} = (n\, a_{i,j} - e^{(A)}_{i,j})_{i,j}$, where $E^{(A)}$ is as in* (3.7). *Then $\widetilde{A}$ is consistent.*

Proof. First of all, we claim that $\widetilde{A}$ is skew-symmetric. Indeed, since A and $E^{(A)}$ are skew-symmetric, for every $i, j \in \{1, 2, \dots, n\}$ it is

$$n\, a_{j,i} - e^{(A)}_{j,i} = -n\, a_{i,j} + e^{(A)}_{i,j} = -(n\, a_{i,j} - e^{(A)}_{i,j}).$$

Now, we prove that $\widetilde{A}$ is consistent. Choose arbitrarily $i, j, k \in \{1, 2, \dots, n\}$. Taking into account the skew-symmetry of A, we get

$$\begin{aligned}
\widetilde{a_{i,j}} + \widetilde{a_{j,k}} + \widetilde{a_{k,i}} &= n\, a_{i,j} - e^{(A)}_{i,j} + n\, a_{j,k} - e^{(A)}_{j,k} + n\, a_{k,i} - e^{(A)}_{k,i} \\
&= n\, a_{i,j} - \sum_{h=1}^{n} e^{(A)}_{i,j,h} + n\, a_{j,k} - \sum_{h=1}^{n} e^{(A)}_{j,k,h} + n\, a_{k,i} - \sum_{h=1}^{n} e^{(A)}_{k,i,h} \\
&= n\, a_{i,j} + n\, a_{j,k} + n\, a_{k,i} \\
&\quad - \sum_{h=1}^{n} (a_{i,j} + a_{j,h} + a_{h,i} + a_{j,k} + a_{k,h} + a_{h,j} + a_{k,i} + a_{i,h} + a_{h,k}) \\
&= n\, a_{i,j} + n\, a_{j,k} + n\, a_{k,i} - n\, a_{i,j} - n\, a_{j,k} - n\, a_{k,i} \\
&\quad - \sum_{h=1}^{n} (a_{j,h} + a_{h,j}) - \sum_{h=1}^{n} (a_{h,i} + a_{i,h}) - \sum_{h=1}^{n} (a_{k,h} + a_{h,k}) = 0,
\end{aligned}$$

that is the consistency of $\widetilde{A}$. □

Lemma 3.2. *Let $B = (b_{i,j})_{i,j}$, $C = (c_{i,j})_{i,j}$, $D = (d_{i,j})_{i,j}$, $D = B \oplus C$, where B is totally inconsistent and C is consistent, and let $E^{(D)}$ be as in* (3.7). *Then, $E^{(D)} = n \odot B$.*

Proof. For every i, j, $k \in \{1, 2, \dots, n\}$, we have $d_{i,j} = b_{i,j} + c_{i,j}$, and since C is consistent, we obtain

$$\begin{aligned}
d_{i,j} + d_{j,k} + d_{k,i} &= b_{i,j} + b_{j,k} + b_{k,i} + c_{i,j} + c_{j,k} + c_{k,i} \\
&= b_{i,j} + b_{j,k} + b_{k,i}.
\end{aligned} \tag{3.10}$$

From (3.10), taking into account the skew-symmetry and the total inconsistency of B, we deduce

$$\begin{aligned}
e^{(D)}_{i,j} &= \sum_{k=1}^{n} (d_{i,j} + d_{j,k} + d_{k,i}) = \sum_{k=1}^{n} (b_{i,j} + b_{j,k} + b_{k,i}) \\
&= \sum_{k=1}^{n} b_{i,j} + \sum_{k=1}^{n} b_{j,k} + \sum_{k=1}^{n} b_{k,i} = n\, b_{i,j} + \sum_{k=1}^{n} b_{j,k} - \sum_{k=1}^{n} b_{i,k} = n\, b_{i,j},
\end{aligned}$$

that is the assertion. □

Now, we are ready to prove the result on existence and uniqueness of a decomposition of a skew-symmetric matrix into the direct sum of a consistent and a totally inconsistent matrix.

Theorem 3.2. *Let G be a vector space over the field $\mathbb{Q}$ and A be a skew-symmetric matrix. Then there is a totally inconsistent matrix B_0 and a consistent matrix C_0 such that $A = B_0 \oplus C_0$.*

Moreover, if B_1 is any totally inconsistent matrix and C_1 is any consistent matrix such that $A = B_1 \oplus C_1$, then $B_1 = B_0$ and $C_1 = C_0$.

Proof. Let $E^{(A)}$ be as in (3.7), $B_0 = \frac{1}{n} \odot E^{(A)}$,

$$C_0 = A \ominus B_0 = A \ominus \Big(\frac{1}{n} \odot E^{(A)}\Big) = \frac{1}{n} \odot ((n \odot A) \ominus E^{(A)}). \tag{3.11}$$

It is not difficult to check that B_0 is totally inconsistent, since $E^{(A)}$ is, and that C_0 is consistent, since $(n \odot A) \ominus E^{(A)}$ is.

Moreover, if B_1 and C_1 are as in the hypothesis, then, thanks to Lemma 3.2, we get $E^{(A)} = n \odot B_1$, and so $B_0 = B_1$. From this and (3.11), we deduce that $C_1 = A \ominus B_1 = A \ominus B_0 = C_0$. This ends the proof. □

3.1. Consistency index

Let $A = (a_{i,j})_{i,j}$ be a skew-symmetric matrix, non necessarily consistent. We can estimate the quantity

$$e^{(A)}_{i,j,k} = a_{i,j} + a_{j,k} + a_{k,i} \tag{3.12}$$

as i, j, k vary in $\{1, 2, \ldots, n\}$, taking into account that the expression in (3.12) is equal to 0 for every choice of i, j and k if and only if A is consistent. The *consistency index* of a matrix A will indicate, in a certain sense, "how much A is far from a consistent matrix". In this section, we prove some fundamental properties of the consistency index (see also [7,8,18] for related axiomatic properties and for different kinds of consistency indices existing in the literature).

We begin with proving that $e^{(A)}_{i,j,k}$ is permutation invariant up to the sign, extending [16: Proposition 21] to the setting of arbitrary abelian groups.

Proposition 3.3. *Let $i, j, k \in \{1, \ldots, n\}$, and let $\sigma\colon \{i, j, k\} \to \{i, j, k\}$ be any permutation. Then, either*

$$e^{(A)}_{\sigma(i),\sigma(j),\sigma(k)} = e^{(A)}_{i,j,k} \tag{3.13}$$

or

$$e^{(A)}_{\sigma(i),\sigma(j),\sigma(k)} = -e^{(A)}_{i,j,k}. \tag{3.14}$$

Moreover, if at least two elements among i, j, k are equal, then $e^{(A)}_{i,j,k} = 0$.

Proof. First of all, observe that the equality in (3.13) is obvious when σ is the identity, and is readily seen when $\sigma(i) = j$, $\sigma(j) = k$ and $\sigma(k) = i$ or $\sigma(i) = k$, $\sigma(j) = i$ and $\sigma(k) = j$. When $\sigma(i) = i$, $\sigma(j) = k$ and $\sigma(k) = j$, taking into account the skew-symmetry of A, we get

$$e^{(A)}_{\sigma(i),\sigma(j),\sigma(k)} = e^{(A)}_{i,k,j} = a_{i,k} + a_{k,j} + a_{j,i} = -a_{i,j} - a_{j,k} - a_{k,i} = -e^{(A)}_{i,j,k}. \tag{3.15}$$

From (3.15) it follows that (3.14) holds also when $\sigma(i) = k$, $\sigma(j) = j$ and $\sigma(k) = i$ or $\sigma(i) = j$, $\sigma(j) = i$ and $\sigma(k) = k$.

Now, suppose that the set $\{i, j, k\}$ has at least two equal elements. Without loss of generality, we can assume that $i = j$, since the other cases are analogous. By the skew-symmetry of A, we have

$$e^{(A)}_{i,j,k} = e^{(A)}_{i,i,k} = a_{i,i} + a_{i,k} + a_{k,i} = 0.$$

This completes the proof. □

Now, in order to define the consistency index, we will estimate the "size" of the quantities $e^{(A)}_{i,j,k}$. To this aim, we endow $G = (G, +)$ with an "extended norm".

Definition 5 (see [22: Definition 8.3]). Let $G = (G, +)$ be a vector space over a normed field $(\mathbb{K}, |\cdot|)$, and let $(Y, \leq)$ be a partially ordered vector space. We say that a function $\|\cdot\|\colon G \to Y$ is a *cone norm* over $\mathbb{K}$, on G, with respect to Y, if it satisfies the following properties:

5.1) $\|x\| \geq 0$ for each $x \in G$, and $\|x\| = 0$ if and only if $x = 0$;

5.2) $\|\alpha\, x\| = |\alpha|\, \|x\|$ for every $x \in G$ and $\alpha \in \mathbb{K}$;

5.3) $\|x + y\| \leq \|x\| + \|y\|$ whenever $x,\, y \in G$.

In this case, we say that $G = (G, +, \|\cdot\|)$ is a *cone normed space* over $\mathbb{K}$, with respect to Y.

For example, we observe that any usual norm (with respect to $\mathbb{R}$) on a normed space G is a cone norm on G. Another example of cone norm is the absolute value in any Riesz space G, defined by $|x| = x \vee (-x)$ for each $x \in G$. In this case, we have $G = Y$.

Let G be a Dedekind complete Riesz space, endowed with a *strong order unit* e, that is an element e such that $e \geq 0$, $e \neq 0$ and for every $x \in G$ there is a positive real number β with $|x| \leq \beta\, e$. An example of "real" norm is the Minkowski functional $\|\cdot\|_e$ associated with the interval

$$[-e, e] = \{x \in G : -e \leq x \leq e\},$$

defined by

$$\|x\|_e = \min\{\beta \in \mathbb{R}, \beta \geq 0 : |x| \leq \beta\, e\}. \tag{3.16}$$

The norm in (3.16) has the property that

$$\|x\| \leq \|y\| \text{ whenever } x, y \in G \text{ and } 0 \leq x \leq y \tag{3.17}$$

(see also [6: §4], [21: Proposition 1.2.13]). In this case, $Y = \mathbb{R}$.

From now on, we suppose that $G = (G, +, \|\cdot\|)$ is a cone normed space.

Now, we define the consistency index for matrices.

If $A = (a_{i,j})_{i,j}$ is a 3×3-matrix, then we define the *consistency index* $I_C(A)$ of A by

$$I_C(A) = \|e^{(A)}_{1,2,3}\|. \tag{3.18}$$

Note that, since A is skew-symmetric, $I_C(A)$ indicates how "far" A is from a consistent matrix. Indeed, by Proposition 3.3, we get

$$\{e^{(A)}_{i,j,k} : i, j, k \in \{1, 2, 3\}\} = \{e^{(A)}_{1,2,3}, -e^{(A)}_{1,2,3}, 0\},$$

and hence

$$\{\|e^{(A)}_{i,j,k}\| : i, j, k \in \{1, 2, 3\}\} = \{\|e^{(A)}_{1,2,3}\|, 0\}$$

since, by 5.2), $\|-x\| = \|x\|$ for each $x \in G$.

Now, let $A = (a_{i,j})_{i,j}$ be an $n \times n$-matrix, with $n \geq 4$. Set $T_n = \{(i, j, k) \in \{1, 2, \ldots, n\}^3 : i < j < k\}$, and let $\sharp(T_n)$ denote the cardinality of T_n. Observe that

$$\sharp(T_n) = \frac{n!}{3!(n-3)!} = \frac{n(n-1)(n-2)}{6}.$$

Let us define the *consistency index* $I_C(A)$ of A by

$$I_C(A) = \frac{\sum\limits_{(i,j,k) \in T_n} \left\|e^{(A)}_{i,j,k}\right\|}{\sharp(T_n)}. \tag{3.19}$$

Observe that, when $n = 3$, it is possible to give an analogous definition as in (3.19), which turns out to be equivalent to that given in (3.18), since $\sharp(T_3) = 1$.

Note that, by Proposition 3.3, we get

$$\{\|e^{(A)}_{i,j,k}\| : (i, j, k) \in \{1, 2, \ldots, n\}^3\} = \{\|e^{(A)}_{i,j,k}\| : (i, j, k) \in T_n\} \cup \{0\}.$$

Moreover, from equalities (3.13) and (3.14) of Proposition 3.3, we deduce the following result, which extends [13: Proposition 15] to the cone normed space setting.

Theorem 3.4. *Let $A = (a_{i,j})_{i,j}$, $\sigma\colon \{1,2,\dots,n\} \to \{1,2,\dots,n\}$ be a permutation, and $A^{(\sigma)} = (a_{\sigma(i),\sigma(j)})_{i,j}$.*

Then, $I_C(A^\sigma) = I_C(A)$.

Furthermore, if G is a real vector space, we have the next result, extending [13: Proposition 17] to our context.

Theorem 3.5. *Let $G = (G, +, \|\cdot\|)$ be a cone normed space over $\mathbb{R}$. Let $A = (a_{i,j})_{i,j}$, $\alpha \in \mathbb{R}$, and $\alpha \odot A = (\alpha\, a_{i,j})_{i,j}$.*

Then, $I_C(\alpha \odot A) = |\alpha|\, I_C(A)$.

P r o o f. Choose arbitrarily $\alpha \in \mathbb{R}$. For each $(i,j,k) \in T_n$, we have

$$\begin{aligned} e_{i,j,k}^{(\alpha\odot A)} &= \alpha\, a_{i,j} + \alpha\, a_{j,k} + \alpha\, a_{k,i} \\ &= \alpha\, (a_{i,j} + a_{j,k} + a_{k,i}) = \alpha\, e_{i,j,k}^{(A)}. \end{aligned} \tag{3.20}$$

Taking in (3.20) the norms, and taking into account 5.2), we get

$$\|e_{i,j,k}^{(\alpha\odot A)}\| = |\alpha|\, \|e_{i,j,k}^{(A)}\|.$$

The assertion follows from the arbitrariness of the triple (i,j,k) in T_n and the definition of consistency index. □

Remark 2. Observe that, when the norm $\|\cdot\|$ fulfils (3.17), the consistency index satisfies a "monotonicity-type" property with respect to a fixed single entry, as the involved matrix is farther than a consistent matrix.

To see this, let $A = (a_{i,j})_{i,j}$ be a consistent matrix, and $a_{p,q} \in G$ be a fixed entry of A, such that $p \neq q$. Let $b_{p,q} \neq a_{p,q}$, $b_{q,p} = -b_{p,q}$, and set $B = (b_{i,j})_{i,j}$, where $b_{i,j} = a_{i,j}$ whenever $(i,j) \neq (p,q)$ and $(i,j) \neq (q,p)$. For any $r \in \{1,2,\dots,n\}$ with $r \neq p$ and $r \neq q$, we get $a_{p,r} + a_{r,q} = a_{p,q} \neq b_{p,q}$, so that $a_{p,r} + a_{r,q} + b_{q,p} \neq 0$. This implies, by the definition of the consistency index, that $I_C(B) > 0 = I_C(A)$.

Now, let $(\Lambda, \precsim)$ be a partially ordered set, $(p,q) \in \{1,2,\dots,n\}$ be a fixed pair as above, and suppose that $b_{p,q}^{(\lambda)}$, $b_{q,p}^{(\lambda)}$, $\lambda \in \Lambda$, are two families of elements of G with $0 \le b_{p,q}^{(\lambda_1)} \le b_{p,q}^{(\lambda_2)}$ whenever $\lambda_1 \precsim \lambda_2$, and $b_{q,p}^{(\lambda)} = -b_{p,q}^{(\lambda)}$ for all $\lambda \in \Lambda$. Without loss of generality, we can suppose $p < q$. Set $B^{(\lambda)} = (b_{i,j}^{(\lambda)})_{i,j}$, where

$$b_{i,j}^{(\lambda)} = a_{i,j} \quad \text{whenever} \quad (i,j) \neq (p,q) \text{ and } (i,j) \neq (q,p). \tag{3.21}$$

Let us consider the triples of the type $e_{i,j,k}^{B^{(\lambda)}}$, where $(i,j,k) \in T_n$. If $p \neq i$ or $q \neq k$, then, we get $0 \le e_{i,j,k}^{B^{(\lambda_1)}} \le e_{i,j,k}^{B^{(\lambda_2)}}$ whenever $\lambda_1 \precsim \lambda_2$. If $p = i$ and $q = k$, then $0 \ge e_{i,j,k}^{B^{(\lambda_1)}} \ge e_{i,j,k}^{B^{(\lambda_2)}}$, that is $0 \le -e_{i,j,k}^{B^{(\lambda_1)}} \le -e_{i,j,k}^{B^{(\lambda_2)}}$, whenever $\lambda_1 \precsim \lambda_2$. Thanks to (3.17), in the first case, we obtain

$$\|e_{i,j,k}^{B^{(\lambda_1)}}\| \le \|e_{i,j,k}^{B^{(\lambda_2)}}\| \quad \text{whenever} \quad \lambda_1 \precsim \lambda_2, \tag{3.22}$$

and in the second case, taking into account 5.2) and (3.17), we get

$$\|e_{i,j,k}^{B^{(\lambda_1)}}\| = \| - e_{i,j,k}^{B^{(\lambda_1)}}\| \le \| - e_{i,j,k}^{B^{(\lambda_2)}}\| = \|e_{i,j,k}^{B^{(\lambda_2)}}\| \quad \text{whenever} \quad \lambda_1 \precsim \lambda_2. \tag{3.23}$$

From (3.21), (3.22) and (3.23), we deduce that, if $\lambda_1 \precsim \lambda_2$, then $I_C(B^{(\lambda_1)}) \le I_C(B^{(\lambda_2)})$. Thus, our "monotonicity" property with respect to a fixed single entry is proved. This extends [13: Proposition 19] to our context.

4. Social preferences and Arrow's conditions

Let $G = (G, \leq)$ be any partially ordered set. Given any two elements $a, b \in G$, we say that $b \geq a$ if $a \leq b$, and that $a < b$ or $b > a$ if $a \leq b$ and $a \neq b$. Let q be any positive integer. Given any two elements $\mathbf{a} = (a_1, a_2, \dots, a_q)$, $\mathbf{b} = (b_1, b_2, \dots, b_q) \in G^q$, we say that $\mathbf{a} \leq \mathbf{b}$ or $\mathbf{b} \geq \mathbf{a}$ (resp. $\mathbf{a} < \mathbf{b}$ or $\mathbf{b} > \mathbf{a}$) if $a_i \leq b_i$ (resp. $a_i < b_i$) for every $i \in \{1, 2, \dots, q\}$.

Let $G = (G, +)$ be an abelian group. Given two elements $\mathbf{a} = (a_1, a_2, \dots, a_q)$, $\mathbf{b} = (b_1, b_2, \dots, b_q) \in G^q$, we put

$$\mathbf{a} + b = (a_1 + b_1, a_2 + b_2, \dots, a_q + b_q), \qquad \mathbf{a} - b = (a_1 - b_1, a_2 - b_2, \dots, a_q - b_q).$$

A set $G = (G, +, \leq)$ is called a *partially ordered abelian group* if $(G, +)$ is an abelian group, $(G, \leq)$ is a partially ordered set and $a \leq b$ implies $a + c \leq b + c$ whenever a, b, $c \in G$. Observe that, in this case, we get

$$a + b > 0 \quad \text{whenever} \quad a > 0 \text{ and } b \geq 0. \tag{4.1}$$

Indeed, let $a > 0$ and $b \geq 0$. Since G is a partially ordered abelian group, it is $a + b \geq 0$. Suppose, by contradiction, that $a + b = 0$. Then $a = -b$, and hence $a \leq 0$, because $b \geq 0$. This is impossible, since $a > 0$ by hypothesis.

Let $q \geq 2$ be a positive integer. A function $\phi\colon G^q \to G$ is said to be *increasing* (resp. *strictly increasing*) if $\phi(\mathbf{a}) \leq \phi(\mathbf{b})$ whenever $\mathbf{a} \leq \mathbf{b}$ (resp. $\phi(\mathbf{a}) < \phi(\mathbf{b})$ whenever $\mathbf{a} < \mathbf{b}$). A function $\phi\colon G^q \to G$ is *idempotent* if $\phi(a, a, \dots, a) = a$ for each $a \in G$. A strictly increasing and idempotent function $\phi : G^q \to G$ is called an *averaging functional.* It is not difficult to check that, if G is a real vector space, then every *convex combination*

$$\phi(a_1, a_2, \dots, a_q) = \sum_{i=1}^{q} \alpha_i \, a_i, \tag{4.2}$$

with $\alpha_i \in \mathbb{R}$, $\alpha_i > 0$ for all $i \in \{1, 2, \dots, q\}$ and $\displaystyle\sum_{i=1}^{q} \alpha_i = 1$, is an averaging functional (in particular, note that strict monotonicity follows from (4.1)). As a particular case, if G is a vector space over $\mathbb{Q}$, then the *mean*

$$\phi(a_1, a_2, \dots, a_q) = \frac{1}{q} \sum_{i=1}^{q} a_i$$

is an averaging functional.

In the literature, besides consistency of PCMs, the property of *weak consistency* for skew-symmetric matrices is investigated. Observe that every consistency matrix is also weak consistent, but the converse is not true in general. Moreover, note that weak consistency is sometimes easier to check than consistency (see also [14]). We extend the concepts of ordinal evaluation vector and weak consistency to partially ordered sets.

Definition 6. Let $\mathcal{S}$ be the set of all skew-symmetric $n \times n$-matrices, $A = (a_{i,j})_{i,j} \in \mathcal{S}$, and $\mathbf{v} = (v_1, v_2, \dots, v_n) \in G^n$.

We say that $\mathbf{v}$ is an *ordinal evaluation vector* for A if the following implications hold for every i, $j \in \{1, 2, \dots, n\}$:

6.1) $[a_{i,j} > 0] \Longrightarrow [v_i > v_j]$;

6.2) $[a_{i,j} = 0] \Longrightarrow [v_i = v_j]$.

Remark 3. Observe that condition 6.1) is equivalent to

6.3) $[a_{i,j} < 0] \Longrightarrow [v_i < v_j]$.

Indeed, suppose that $a_{i,j} < 0$. Then, by the skew-symmetry of A, we get $a_{j,i} = -a_{i,j} > 0$. By 6.1), we have $v_j > v_i$, that is $v_i < v_j$. Thus, 6.1) implies 6.3). The proof of the converse implication is analogous.

Definition 7. A matrix $A = (a_{i,j})_{i,j} \in \mathcal{S}$ is said to be *weakly consistent* if for every $i,\ j \in \{1, 2, \dots, n\}$,

$[a_{i,j} > 0] \Longrightarrow [a_{i,k} > a_{j,k}$ for all $k \in \{1,\ 2, \dots, n\}]$, and
$[a_{i,j} = 0] \Longrightarrow [a_{i,k} = a_{j,k}$ for any $k \in \{1,\ 2, \dots, n\}]$.

Now, we see some basic properties of weak consistency, extending [14: Theorems 4.1 and 4.2] to the partially ordered space setting.

Proposition 4.1.

4.1.1) *If A is consistent, then A is weakly consistent.*

4.1.2) *If A is weakly consistent, then every column vector* $\mathbf{a}^{(h)} = \begin{bmatrix} a_{1,h} \\ a_{2,h} \\ \dots \\ a_{n,h} \end{bmatrix}$, $h \in \{1,\ 2, \dots, n\}$, *is an ordinal evaluation vector for A.*

4.1.3) *If $\phi\colon G^n \to G$ is a strictly increasing function, then the vector $\mathbf{w} = (w_1, w_2, \dots, w_n)$ defined by*

$$w_i = \phi(a_{i,1}, a_{i,2}, \dots, a_{i,n}), \qquad i \in \{1, 2, \dots, n\}$$

is an ordinal evaluation vector for A.

Proof. 4.1.1) If A is consistent, then for every $i,\ j,\ k \in \{1, 2, \dots, n\}$ it is $a_{i,j} + a_{j,k} = a_{i,k}$, and hence $a_{i,j} = a_{i,k} - a_{j,k}$. Thus, if $a_{i,j} > 0$ (resp. $a_{i,j} = 0$), then $a_{i,k} > a_{j,k}$ (resp. $a_{i,k} = a_{j,k}$). By the arbitrariness of k, we get that A is weakly consistent.

4.1.2) It is a direct consequence of the definitions of weak consistency and ordinal evaluation vector.

4.1.3) Choose arbitrarily $i,\ j \in \{1,\ 2, \dots, n\}$. By the definition of weak consistency, if $a_{i,j} > 0$, then $a_{i,k} > a_{j,k}$ for each $k \in \{1,\ 2, \dots, n\}$. Since ϕ is strictly increasing, then

$$\phi(a_{i,1}, a_{i,2}, \dots, a_{i,n}) > \phi(a_{j,1}, a_{j,2}, \dots, a_{j,n}).$$

Analogously it is possible to check that, if $a_{i,j} = 0$, then

$$\phi(a_{i,1}, a_{i,2}, \dots, a_{i,n}) = \phi(a_{j,1}, a_{j,2}, \dots, a_{j,n}),$$

getting the assertion. □

Remark 4. Note that, in general, weak consistency does not imply consistency, and the sum of two weakly consistent matrices is not weakly consistent (see, e.g., [14: Example 4.1], [17: Remark 3]).

The next step is to formulate Arrow's conditions in the partially ordered space setting, and extend earlier results of [15] and [17].

Let $\mathcal{S}$ be as in Definition 6, and $\emptyset \neq \mathcal{T} \subset \mathcal{S}^m$. A *profile* is an element of $\mathcal{T}$. A *procedure* on $\mathcal{T}$ for aggregating and/or synthesizing the preferences of a profile in one matrix is any function $\Phi\colon \mathcal{T}_0 \to \mathcal{S}$, where $\emptyset \neq \mathcal{T}_0 \subset \mathcal{T}$.

For every $(A_1, A_2, \dots, A_m) \in \mathcal{S}^m$ and $(i,j) \in \{1, 2, \dots, n\}^2$, set $\mathbf{a_{i,j}} = (a^1_{i,j}, a^2_{i,j}, \dots, a^m_{i,j})$.

Definition 8. We say that a procedure Φ on $\mathcal{T}$ satisfies the *condition of unrestricted domain* (in short, *condition U^**) if $\mathcal{T}_0 = \mathcal{T}$.

A procedure Φ fulfils *pairwise unanimity* (*condition* P^*) if for every profile $(A_1, A_2, \dots, A_m) \in \mathcal{T}_0$, with $A_s = (a^s_{i,j})_{i,j}$, $s \in \{1, 2, \dots, m\}$, we get that, if $a^s_{i,j} > 0$ for each $s \in \{1, 2, \dots, m\}$, then $\widetilde{a}_{i,j} > 0$, where $\widetilde{a}_{i,j} = (\Phi(A_1, A_2, \dots, A_m))_{i,j}$, $(i, j) \in \{1, 2, \dots, n\}^2$.

A procedure Φ satisfies the *condition of independence from irrelevant alternatives* (*condition* I^*) if for each nonempty set $Y \subset \{1, 2, \dots, n\}$ and for any two profiles $(A_1, A_2, \dots, A_m) = ((a^1_{i,j})_{i,j}, (a^2_{i,j})_{i,j}, \dots, (a^m_{i,j})_{i,j})$, $(B_1, B_2, \dots, B_m) = ((b^1_{i,j})_{i,j}, (b^2_{i,j})_{i,j}, \dots, (b^m_{i,j})_{i,j})$, such that

$$A^{(Y)}_s = (a^s_{i,j})_{(i,j)\in Y^2}, \quad B^{(Y)}_s = (b^s_{i,j})_{(i,j)\in Y^2}, \qquad s \in \{1, 2, \dots, m\}, \tag{4.3}$$

it is $(\Phi(A_1, A_2, \dots, A_m))^{(Y)} = (\Phi(B_1, B_2, \dots, B_m))^{(Y)}$.

A procedure Φ satisfies the *condition of nondictatorship* (*condition* D^*) if there is no element $d \in \{1, 2, \dots, m\}$ such that $\Phi(A_1, A_2, \dots, A_m) = A_d$ whenever $A_i \neq A_j$ for at least two different i, $j \in \{1, 2, \dots, n\}$.

We extend to the setting of partially ordered spaces and averaging functionals [17: Proposition 10] and [15: Theorem 1].

Proposition 4.2. *Let $\mathcal{T} = \mathcal{T}_0 = \mathcal{S}^m$, $\varphi\colon G^m \to G$ be an averaging functional and $\Phi\colon \mathcal{S}^m \to \mathcal{S}$ be a procedure defined, for each $(A_1, A_2, \dots, A_m) \in \mathcal{T}$, by*

$$(\Phi(A_1, A_2, \dots, A_m))_{i,j} = \varphi(\mathbf{a_{i,j}}), \qquad (i, j) \in \{1, 2, \dots, n\}^2. \tag{4.4}$$

Then Φ satisfies U^*, P^* and I^* on $\mathcal{T}$. Moreover, if G is a partially ordered real vector space and φ is a convex combination as in (4.2)*, then Φ satisfies also D^* on $\mathcal{T}$.*

P r o o f. U^*) It is readily seen that condition U^* is fulfilled, because Φ is defined on the whole on $\mathcal{T}$.

P^*) Let $(A_1, A_2, \dots, A_m) \in \mathcal{T}$, $A_s = (a^s_{i,j})$, $s \in \{1, 2, \dots, m\}$, be such that

$$a^s_{i,j} > 0 \qquad \text{for each } s \in \{1, \dots, m\}, \quad i, j \in \{1, 2, \dots, n\}. \tag{4.5}$$

Since φ is strictly increasing, from (4.5), we obtain

$$\varphi(a^1_{i,j}, a^2_{i,j}, \dots, a^m_{i,j}) > 0 \tag{4.6}$$

for any $i, j \in \{1, 2, \dots, n\}$. Hence, condition P^* is fulfilled.

I^*) Let $(A_1, A_2, \dots, A_m)$, $(B_1, B_2, \dots, B_m) \in \mathcal{T}$ be as in (4.3), namely such that

$$A^{(Y)}_s = (a^s_{i,j})_{(i,j)\in Y^2} = B^{(Y)}_s = (b^s_{i,j})_{(i,j)\in Y^2}$$

for each $s \in \{1, 2, \dots, m\}$. This means that

$$a^s_{i,j} = b^s_{i,j} \qquad \text{for any } i, j \in Y \text{ and } s \in \{1, 2, \dots, m\}. \tag{4.7}$$

From (4.7) it follows that

$$\varphi(a^s_{i,j}, a^s_{i,j}, \dots, a^s_{i,j}) = \varphi(b^s_{i,j}, b^s_{i,j}, \dots, b^s_{i,j}) \qquad \text{for any } i, j \in Y.$$

Thus, I^* is satisfied. □

The next step is to formulate Arrow's conditions in the context of partially ordered vector spaces and averaging functionals for a procedure, in order to aggregate and/or syntesize the preferences of a profile in a vector, which expresses, in a certain sense, the "order" of preferences, extending [17: Propositions 11–13].

Let $\varphi\colon G^m \to G$ be an averaging functional, and $\Phi\colon \mathcal{S}^m \to \mathcal{S}$ is a procedure defined, for each $(A_1, A_2, \dots, A_m) \in \mathcal{S}^m$, by

$$(\Phi(A_1, A_2, \dots, A_m))_{i,j} = \varphi(\mathbf{a_{i,j}}), \qquad (i, j) \in \{1, 2, \dots, n\}^2. \tag{4.8}$$

We recall that, given an $n\times n$-matrix $A=(a_{i,j})_{i,j}$ and $r\in\{1,2,\dots,n\}$, then $\mathbf{a}_{(r)}=(a_{r,1},a_{r,2},\dots,a_{r,n})$ denotes the r-th row.

Now, let $\varphi\colon G^m\to G$ and $\phi\colon G^n\to G$ be any two fixed averaging functionals, let $\emptyset\neq\mathcal{T}_0\subset\mathcal{T}\subset\mathcal{S}$, and define $\zeta\colon\mathcal{T}_0\to G^n$ by setting, for each $A\in\mathcal{T}_0$ and $r\in\{1,2,\dots,n\}$,

$$\begin{aligned}\zeta(A)&=(\phi(\mathbf{a}_{(1)}),\phi(\mathbf{a}_{(2)}),\dots,\phi(\mathbf{a}_{(n)}))\\&=(\phi(a_{1,1},a_{1,2},\dots,a_{1,n}),\phi(a_{2,1},a_{2,2},\dots,a_{2,n}),\dots,\phi(a_{n,1},a_{n,2},\dots,a_{n,n})).\end{aligned}\tag{4.9}$$

Let $\Psi\colon\mathcal{T}_0\to G^n$ be defined by

$$\Psi(A_1,A_2,\dots,A_m)=\zeta(\Phi(A_1,A_2,\dots,A_m)),\qquad(A_1,A_2,\dots,A_m)\in\mathcal{S}^m,\tag{4.10}$$

where Φ is as in (4.4).

Now, we formulate Arrow's conditions in our context.

Definition 9. A procedure Ψ on $\mathcal{T}$ satisfies the *condition of unrestricted domain* (in short, *condition* U^{**}) if $\mathcal{T}_0=\mathcal{T}$.

A procedure Ψ on $\mathcal{T}$ fulfils *pairwise unanimity* (*condition* P^{**}) if for every profile $(A_1,A_2,\dots,A_m)\in\mathcal{T}_0$, with $A_s=(a^s_{i,j})_{i,j}$, $s\in\{1,2,\dots,m\}$, we get that, if $i,\,j\in\{1,2,\dots,n\}$ are such that $a^s_{i,j}>0$ for every $s\in\{1,2,\dots,m\}$, then $(\Psi(A_1,A_2,\dots,A_m))_i>(\Psi(A_1,A_2,\dots,A_m))_j$.

A procedure Φ satisfies the *condition of independence from irrelevant alternatives* (*condition* I^{**}) if for each nonempty set $Y\subset\{1,2,\dots,n\}$ and for any two profiles $(A_1,A_2,\dots,A_m)=((a^1_{i,j})_{i,j},(a^2_{i,j})_{i,j},\dots,(a^m_{i,j})_{i,j})$, $(B_1,\,B_2,\dots,B_m)=((b^1_{i,j})_{i,j},(b^2_{i,j})_{i,j},\dots,(b^m_{i,j})_{i,j})$, such that

$$A_s^{(Y)}=(a^s_{i,j})_{(i,j)\in Y^2}=B_s^{(Y)}=(b^s_{i,j})_{(i,j)\in Y^2},\qquad,s\in\{1,2,\dots,m\},\tag{4.11}$$

it is

$$\begin{aligned}&((\Psi(A_1,A_2,\dots,A_m))^{(Y)})_i>((\Psi(A_1,A_2,\dots,A_m))^{(Y)})_j\quad\text{if and only if}\\&((\Psi(B_1,B_2,\dots,B_m))^{(Y)})_i>((\Psi(B_1,B_2,\dots,B_m))^{(Y)})_j\end{aligned}$$

for any $i,\,j\in Y$.

A procedure Φ satisfies the *condition of nondictatorship* (*condition* D^{**}) if there is no element $d\in\{1,2,\dots,m\}$ such that

$$\Psi(A_1,A_2,\dots,A_m)=\Psi(A_d,A_d,\dots,A_d)$$

whenever $A_i\neq A_j$ for at least a pair $(i,j)\in\{1,2,\dots,n\}^2$ such that $i\neq j$.

Now, we prove the next result about Arrow's conditions on Ψ in the setting of partially ordered vector spaces and averaging functionals.

Theorem 4.3. *Let $\mathcal{C}$ (resp., $\mathcal{WC}$) $\subset\mathcal{S}$ be the set of all consistent (resp. weakly consistent) $n\times n$-matrices, $\varphi\colon G^m\to G$, $\phi\colon G^n\to G$ be averaging functionals, and Ψ be the preference aggregation procedure in* (4.10). *Then,*

4.3.1) *the function Ψ, on $\mathcal{S}^m$, $(\mathcal{WC})^m$ or $\mathcal{C}^m$, satisfies condition U^{**}, and, when G is a partially ordered real vector space and ϕ, φ are convex combinations, also condition D^{**};*

4.3.2) *the function Ψ, on $(\mathcal{WC})^m$ or $\mathcal{C}^m$, satisfies condition P^{**};*

4.3.3) *the function Ψ, on $\mathcal{C}^m$, satisfies condition I^{**}.*

Proof. 4.3.1) Since Ψ is defined on all elements of $\mathcal{S}^m$ without restrictions, condition U^{**} is fulfilled for any choice of $\mathcal{T}\subset\mathcal{S}$.

Moreover, observe that the convex combinations of vectors defined in (4.2) are not identically equal to anyone of these vectors, and hence they satisfy condition D^{**}.

4.3.2) Pick $\mathcal{T} = (\mathcal{WC})^m$. Let $(A_1, A_2, \ldots, A_m) \in \mathcal{T}$, $A_s = (a^s_{i,j})$, where $s \in \{1, 2, \ldots, m\}$, and $i, j \in \{1, 2, \ldots, n\}$ be such that

$$a^s_{i,j} > 0 \qquad \text{for each } s \in \{1, \ldots, m\}. \tag{4.12}$$

Since, by hypothesis, $A_s = (a^s_{i,j})_{i,j}$ is weakly consistent for all $s \in \{1, 2, \ldots, m\}$, from (4.12) it follows that

$$a^s_{i,h} > a^s_{j,h} \qquad \text{for all } i, j, h \in \{1, 2, \ldots, n\} \text{ and } s \in \{1, 2, \ldots, m\}. \tag{4.13}$$

Now, set $B = (b_{i,j})_{i,j} = (\varphi(a^1_{i,j}, a^2_{i,j}, \ldots, a^m_{i,j}))_{i,j}$. Note that, thanks to (4.4), we get $B = \Phi(A_1, A_2, \ldots, A_m)$. As φ is strictly increasing, from (4.13), we obtain

$$b_{i,h} = \varphi(a^1_{i,h}, a^2_{i,h}, \ldots, a^m_{i,h}) > \varphi(a^1_{j,h}, a^2_{j,h}, \ldots, a^m_{j,h}) = b_{j,h} \tag{4.14}$$

for all $i, j, h \in \{1, 2, \ldots, n\}$. Now, let

$$\zeta(B) = (\phi(b_{1,1}, b_{1,2}, \ldots, b_{1,n}), \phi(b_{2,1}, b_{2,2}, \ldots, b_{2,n}), \ldots, \phi(b_{n,1}, b_{n,2}, \ldots, b_{n,n})). \tag{4.15}$$

Since φ is strictly increasing, from (4.14) and (4.15), we deduce

$$(\zeta(B))_i = \phi(b_{i,1}, b_{i,2}, \ldots, b_{i,n}) > \phi(b_{j,1}, b_{j,2}, \ldots, b_{j,n}) = (\zeta(B))_j.$$

Therefore, condition P^{**} is satisfied.

By arguing analogously as above, it is possible to check that 4.3.2) holds even if one takes $\mathcal{C}^m$ instead of $(\mathcal{WC})^m$.

4.3.3) Pick $\mathcal{T} = \mathcal{C}^m$. For each $Y \subset \{1, 2, \ldots, n\}$ and every matrix $A \in \mathcal{C}$, set $A^{(Y)} = (a_{i,j})_{(i,j) \in Y^2}$. Let $(A_1, A_2, \ldots, A_m)$, $(B_1, B_2, \ldots, B_m) \in \mathcal{T}$. Let $\widetilde{A} = (\widetilde{a}_{i,j})_{(i,j) \in Y^2} = \Phi(A_1, A_2, \ldots, A_m)$, $\widetilde{B} = (\widetilde{b}_{i,j})_{(i,j) \in Y^2} = \Phi(B_1, B_2, \ldots, B_m)$. By hypothesis, we get $\widetilde{A}, \widetilde{B} \in \mathcal{C} \subset \mathcal{WC}$ and hence, by Proposition 4.1, $(\zeta(\widetilde{A}))_i = \phi(\widetilde{a}_{i,1}, \widetilde{a}_{i,2}, \ldots, \widetilde{a}_{i,n})$ and $(\zeta(\widetilde{B}))_i = \phi(\widetilde{b}_{i,1}, \widetilde{b}_{i,2}, \ldots, \widetilde{b}_{i,n})$ are ordinal evaluation vectors for each $i \in Y$. Since $A^{(Y)}_s = B^{(Y)}_s$ for every $s \in \{1, 2, \ldots m\}$, then $(\zeta(\widetilde{A}))_i = (\zeta(\widetilde{B}))_i$ for all $i \in Y$, and hence for every $i, j \in Y$, we get $(\zeta(\widetilde{A}))_i > (\zeta(\widetilde{A}))_j$ if and only if $(\zeta(\widetilde{B}))_i > (\zeta(\widetilde{B}))_j$. Thus, condition I^{**} holds. $\square$

Remark 5. Observe that, in general, condition I^{**} does not hold, when $\mathcal{T} = (\mathcal{WC})^m$ (see e.g. [17: Remark 3]).

5. Conclusions

We propose a generalization of algebraic structures used to work with PCMs. This leads us to a comprehension of which properties, we actually use or need when, we want to represent preferences, social choices and, in this particular case, PCMs. All the presented results can be easily translated in the main formulations of PCMs, i.e., multiplicative, additive and fuzzy approach, by the fact that each of them is a particular interpretation of the more general and essential algebraic structure needed to deal with this theory. We stress also that the generality of the used structures allows us to immediately recognize whether a formulation is enough powerful to express preferences and which kind of properties and theorems can be achieved.

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