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Enriques involutions on pencils of K3 surfaces

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Abstract

The three pencils of K3 surfaces of minimal discriminant whose general element covers at least one Enriques surface are Kondō's pencils I and II, and the Apéry–Fermi pencil. We enumerate and investigate all Enriques surfaces covered by their general elements.

KEYWORDS
elliptic fibration, Enriques surface, K3 surface, transcendental lattice

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1 | INTRODUCTION

Any complex Enriques surface is doubly covered by a K3 surface. On the other hand, a K3 surface X can cover infinitely many Enriques surfaces. The set $\mathrm{Enr}(X)$ of isomorphism classes of Enriques surfaces doubly covered by X, though, is always finite by a result of Ohashi [20]. We call its cardinality $|\mathrm{Enr}(X)|$ the *Enriques number* of the K3 surface X. The Enriques number $|\mathrm{Enr}(X)|$ only depends on the transcendental lattice of X. Shimada and the second named author [23] described a procedure to determine $|\mathrm{Enr}(X)|$ and applied it to K3 surfaces of maximal Picard rank 20.

A K3 surface X of Picard rank 19 can be seen as the generic element of a pencil of K3 surfaces. Its transcendental lattice T_X is an even lattice of signature $(2, 1)$. By a result of Brandhorst, Sonel and the second named author [3], the surface X covers an Enriques surface only if 4 divides $\det(T_X)$, but this condition is not sufficient. In this paper we analyze in detail what happens when $|\det(T_X)|$ is small, more precisely

$$\left|\det(T_X)\right| < 16. \tag{1.1}$$

Henceforth, let X be a K3 surface of Picard rank 19 with transcendental lattice T_X. In the case $T_X \cong \mathbf{U} \oplus [2n]$, $n \geq 1$, it was already noted by Hulek and Schütt [8] that $\mathrm{Enr}(X) \neq \emptyset$ if and only if n is even. Indeed, we prove in Theorem 2.4 under assumption (1.1) that $\mathrm{Enr}(X) \neq \emptyset$ if and only if

$$T_X \cong \mathbf{U} \oplus [4],\ \mathbf{U} \oplus [8] \text{ or } \mathbf{U} \oplus [12].$$

The main reason for bound (1.1) is to keep computations feasible. In particular the enumeration of jacobian elliptic fibrations on K3 surfaces with $T_X \cong \mathbf{U} \oplus [16]$ already becomes quite hard. Moreover, the pencil of K3 surfaces with $T_X \cong \mathbf{U}(2) \oplus [4]$ is not of the form $\mathbf{U} \oplus [2n]$, but it still holds $\mathrm{Enr}(X) \neq \emptyset$, as its generic element is a Kummer surface [11].

Quite interestingly, the first two pencils already feature prominently in Kondō's classification of Enriques surfaces with finite automorphism group [13], which we briefly recall. There are seven families of such Enriques surfaces, numbered I to VII. Families I and II are 1-dimensional, while families III to VII are 0-dimensional. The K3 surfaces covering the generic Enriques surface of type I and II have transcendental lattice $T_X \cong \mathbf{U} \oplus [4]$ and $T_X \cong \mathbf{U} \oplus [8]$, respectively.

The third pencil with $T_X \cong \mathbf{U} \oplus [12]$ has also been extensively studied, because of its arithmetical properties and its appearance in several seemingly unrelated physical contexts (see [6, 21]). Following Bertin and Lecacheux [2], who classified the elliptic fibrations on its generic element (Table 3), we call it the *Apéry–Fermi pencil*.

The aim of this paper is to enumerate and investigate the Enriques surfaces covered by these three pencils. More precisely, for each $m \in \{1, 2, 3\}$ we consider a K3 surface X with $T_X \cong \mathbf{U} \oplus [4m]$ and do the following:

- we compute the Enriques number $|\mathrm{Enr}(X)|$;
- we classify all jacobian elliptic fibrations on X using the extension of the Kneser–Nishiyama method explained in [7];
- we relate the special elliptic pencils on the Enriques quotients to the elliptic fibrations on X.

We summarize here our findings.

Fix $m \in \mathbb{Z}$, $m \geq 1$, and let ω be the number of prime divisors of $2m$ and X a K3 surface with $T_X \cong \mathbf{U} \oplus [4m]$, $m \geq 1$. Among the Enriques quotients of X there are $2^{\omega-1}$ which we call *of Barth–Peters type* (Theorem 2.7). Such quotients admit a cohomologically trivial involution (see [15, 16]) and their presence is explained by the fact that our pencils are subfamilies of the 2-dimensional Barth–Peters family, a fact already noted by Hulek and Schütt [8, 9].

It turns out that if $m = 1$, then X covers only one Enriques surface Y (Theorem 3.1). Therefore, the Enriques surface Y is of Barth–Peters type and, moreover, coincides with Kondō's quotient, so it has finite automorphism group. The list of the 9 elliptic fibrations on X appears in other papers by Scattone [22], Dolgachev [4] and Elkies and Schütt [5], and we confirm it here (Table 1).

If $m = 2$, then X covers two Enriques surfaces Y', Y'', of which only one, say Y', is of Barth–Peters type. We show that the other surface Y'' is Kondō's quotient with finite automorphism group. We include the classification of elliptic fibrations on X up to automorphisms (Table 2). One subtlety arises in this case: two of the 17 elliptic fibrations on X (No. 12 and 13 in Table 2) have the same Mordell–Weil group and two singular fibers of type I_4^*. Nonetheless, the two fibrations are not equivalent under the action of $\mathrm{Aut}(X)$, as they have different frames. We determine which one is the pullback of a special elliptic pencil on Y' and which one is the pullback of a special elliptic pencil on Y'' (Theorem 3.6).

Finally, if $m = 3$, then X covers three Enriques surfaces Y', Y'', Y''', of which two, say Y' and Y'' are of Barth–Peters type (Theorem 3.8). Applying a construction by Hulek and Schütt [8, §3] and using a particular configuration of curves on X found by Peters and Stienstra, we determine a simple description of an explicit Enriques involution for Y'''. In this way we find a configuration of smooth rational curves on Y''' whose dual graph is the union of a tetrahedron and a complete graph of degree 6 (Theorem 3.9).

2 | PRELIMINARY RESULTS

In this section, after explaining our conventions on lattices in §2.1, we collect results regarding K3 surfaces with transcendental lattice $T_X \cong \mathbf{U} \oplus [2m]$, $m \in \mathbb{Z}$, especially regarding their jacobian elliptic fibrations in §2.2. In §2.3 we recall the enumeration formula for Enriques quotients contained in [23] and we prove the lemma that motivates the whole paper. Finally, in §2.4 we introduce the notion of Enriques quotient of Barth–Peters type.

2.1 | Lattices

In this paper, a *lattice of rank r* is a finitely generated free $\mathbb{Z}$-module $L \cong \mathbb{Z}^r$ endowed with an integral symmetric bilinear form $L \times L \to \mathbb{Z}$ denoted $(v, w) \mapsto v \cdot w$. The *signature* of L is the signature of the induced real symmetric form on $L \otimes \mathbb{R}$. We say that L is *even* if $v^2 := v \cdot v \in 2\mathbb{Z}$ for every $v \in L$. The *dual* $L^\vee := \hom(L, \mathbb{Z})$ of L can be identified with $\{ w \in L \otimes \mathbb{Q} \mid w \cdot v \in \mathbb{Z} \text{ for all } v \in L\}$. The *discriminant group* of L is defined as

$$L^\sharp := L^\vee / L,$$

which is a finite abelian group. We denote by $\ell(L^\sharp)$ its *length*, i.e., the minimal number of generators. For a prime number p we denote by $\ell_p(L^\sharp)$ its *p-length*, i.e., the minimal number of generators of its p-part.

If L is an even lattice, then $L^\sharp$ acquires naturally the structure of a finite quadratic form $L^\sharp \to \mathbb{Q}/2\mathbb{Z}$. There is a natural homomorphism $\mathrm{O}(L) \to \mathrm{O}(L^\sharp)$ denoted $\gamma \mapsto \gamma^\sharp$.

We write $\mathbf{U}$ for the indefinite unimodular even lattice of rank 2, and $\mathbf{A}_n, \mathbf{D}_n, \mathbf{E}_n$ for the negative definite ADE lattices. The notation $[m]$, with $m \in \mathbb{Z}$, denotes the lattice of rank 1 generated by a vector of square m. We adopt Miranda–Morrison's notation [14] for the elementary finite quadratic forms $\mathbf{u}_k, \mathbf{v}_k, \mathbf{w}^\varepsilon_{p,k}$. We recall that $\mathbf{u}_k$ (resp. $\mathbf{v}_k$) is generated by two elements of order 2^k, both of square $0 \in \mathbb{Q}/2\mathbb{Z}$ (resp. $1 \in \mathbb{Q}/2\mathbb{Z}$), such that their product is equal to $1/2^k \in \mathbb{Q}/\mathbb{Z}$. The forms $\mathbf{w}^\varepsilon_{2,k}$, with $\varepsilon \in \{1,3,5,7\}$, are generated by one element of order 2^k and square $\varepsilon/2^k \in \mathbb{Q}/2\mathbb{Z}$. For an odd prime p the forms $\mathbf{w}^\varepsilon_{p,k}$, with $\varepsilon \in \{\pm 1\}$, are generated by one element of order p^k and square $a/p^k \in \mathbb{Q}/2\mathbb{Z}$, where a is a square modulo p if and only if $\varepsilon = 1$.

The *genus* of a lattice L is defined as the set of isomorphism classes of lattices M with $\mathrm{sign}(L) = \mathrm{sign}(M)$ and $L^\sharp \cong M^\sharp$. A genus is always a finite set (see [12, Satz 21.3]).

An embedding of lattices $\iota : M \hookrightarrow L$ is called *primitive* if $L/\iota(M)$ is a free group. We denote by $\iota(M)^\perp \subset L$ the *orthogonal complement* of M inside L. We quickly summarize Nikulin's theory of primitive embeddings [17].

By [17, Prop. 1.5.1] a primitive embedding of even lattices $M \hookrightarrow L$ is given by a subgroup $H \subset M^\sharp$ and an isometry

$$\gamma : H \to H' := \gamma(H) \subset \left(\iota(M)^\perp(-1)\right)^\sharp.$$

If Γ denotes the graph of γ in $M^\sharp \oplus \left(\iota(M)^\perp(-1)\right)^\sharp$, the following identification between finite quadratic forms holds (the finite quadratic form on the right side being induced by the one on $M^\sharp \oplus \left(\iota(M)^\perp(-1)\right)^\sharp$):

$$L^\sharp \cong \Gamma^\perp/\Gamma. \tag{2.1}$$

In this paper we call H, γ resp. Γ the *gluing subgroup*, *gluing isometry* resp. *gluing graph* of $M \hookrightarrow L$.

Equivalently by [17, Prop. 1.15.1], assuming that L is unique in its genus, a primitive embedding $M \hookrightarrow L$ is given by a subgroup $K \subset L^\sharp$ and an isometry

$$\xi : K \to K' := \xi(K) \subset M(-1)^\sharp.$$

If Ξ denotes the graph of ξ in $L^\sharp \oplus M(-1)^\sharp$, the following identification between finite quadratic forms holds (the finite quadratic form on the right side being induced by the one on $L^\sharp \oplus M(-1)^\sharp$):

$$\left(\iota(M)^\perp\right)^\sharp \cong \Xi^\perp/\Xi. \tag{2.2}$$

In this paper we call K, ξ resp. Ξ the *embedding subgroup*, *embedding isometry* resp. *embedding graph* of $M \hookrightarrow L$.

2.2 | Elliptic fibrations

Given a K3 surface X, we denote T_X its transcendental lattice, S_X its Néron–Severi lattice, and $\mathcal{J}_X$ the set of jacobian elliptic fibrations on X. The *frame genus* of X is defined as the genus $\mathcal{W}_X$ of negative definite lattices W with $\mathrm{rk}(W) = \mathrm{rk}(S_X) - 2$ and $W^\sharp \cong S_X^\sharp$. The lattices in $\mathcal{W}_X$ are called *frames*. The classes of a fiber and a section of a jacobian elliptic fibration induces a primitive embedding $\iota : \mathbf{U} \hookrightarrow S_X$. As explained in [7], there is a well-defined function

$$\mathrm{fr}_X : \mathcal{J}_X/\mathrm{Aut}(X) \to \mathcal{W}_X$$

which sends each jacobian fibration to the isomorphism class of $\iota(\mathbf{U})^\perp \subset S_X$.

Lemma 2.1. *If X is a K3 surface with transcendental lattice $T_X \cong \mathbf{U} \oplus [2n]$, $n \geq 1$, then on X there are exactly $2^{\omega-1}$ jacobian elliptic fibrations with frame $W := \mathbf{E}_8^2 \oplus [-2n]$ up to automorphisms, where ω is the number of prime divisors of $2n$.*

Proof. Essentially by [7, Thm. 2.8] we want to prove that

$$\left|\mathrm{O}_{\mathrm{h}}^{\sharp}(T_X)\backslash \mathrm{O}\left(T_X^{\sharp}\right)/\mathrm{O}(W)\right| = 2^{\omega-1}.$$

If $n = 1$, then $\mathrm{O}\left(T_X^{\sharp}\right) = \{\,\mathrm{id}\,\}$ and we conclude immediately.

Suppose that $n \geq 2$. Since $\ell\left(T_X^{\sharp}\right) = 1$, the discriminant form $T_X^{\sharp}$ is the direct sum of forms $\mathbf{w}_{p,k}^{\varepsilon}$. It holds

$$\mathrm{O}(q \oplus q') \cong \mathrm{O}(q) \times \mathrm{O}(q')$$

if q and q' are finite quadratic forms with $|q|$ and $|q'|$ coprime, and $\left|\mathrm{O}\left(\mathbf{w}_{p,k}^{\varepsilon}\right)\right| = 2$ if p is odd or $p = 2$ and $k \geq 2$. Hence, $\mathrm{O}\left(T_X^{\sharp}\right)$ is a 2-elementary group of length ω. In particular,

$$\left|\mathrm{O}\left(T_X^{\sharp}\right)\right| = 2^{\omega},$$

As $\mathrm{rk}(T_X)$ is odd, it holds $\mathrm{O}_{\mathrm{h}}^{\sharp}(T_X) = \{\,\pm\mathrm{id}\,\}$ (see for instance [10, Cor. 3.3.5]). Note, moreover, that $\mathrm{id} \neq -\mathrm{id}$ in $T_X^{\sharp}$. The orthogonal group of W is the direct sum of $\mathrm{O}(\mathbf{E}_8^2)$, which has trivial action on the discriminant group because $\mathbf{E}_8$ is unimodular, and $\mathrm{O}([-2n]) = \{\,\pm\mathrm{id}\,\}$. Therefore, it also holds $\mathrm{O}^{\sharp}(W) = \{\,\pm\mathrm{id}\,\}$, so we have

$$\left|\mathrm{O}_{\mathrm{h}}^{\sharp}(T_X)\backslash \mathrm{O}\left(T_X^{\sharp}\right)/\mathrm{O}^{\sharp}(W)\right| = \left|\mathrm{O}\left(T_X^{\sharp}\right)/\{\,\pm\mathrm{id}\,\}\right| = \left|\mathrm{O}\left(T_X^{\sharp}\right)\right|/|\{\,\pm\mathrm{id}\,\}| = 2^{\omega-1}. \qquad \square$$

The *Mordell–Weil group*, i.e., the group of sections of a jacobian elliptic fibration, is naturally endowed with a rational symmetric bilinear form denoted by $(P, Q) \mapsto \langle P, Q\rangle \in \mathbb{Q}$, called the *Mordell–Weil lattice*. The *height* of a section is defined as $\mathrm{ht}(P) := \langle P, P\rangle$. For a clear exposition of this topic we refer to Shioda's original paper [24].

Remark 2.2. Consider one of the elliptic fibrations $\pi : X \to \mathbb{P}^1$ as in Theorem 2.1 and for simplicity assume that $n \geq 2$. Since $W_{\mathrm{root}} \cong \mathbf{E}_8^2$, the fibration π has two singular fibers of Kodaira type II^*. As already remarked by Hulek and Schütt [8, §4.2.2], starting from the fibration π we can construct an involution on X which turns out to be an Enriques involution if n is even. We repeat here their construction directly on the lattice $S_X \cong \mathbf{U} \oplus \mathbf{E}_8^2 \oplus [-2n]$. In the following computations we let $\mathrm{O}(S_X)$ act on S_X *from the right*, so the composition of two isometries in $\mathrm{O}(S_X)$ corresponds to the product of their associated matrices in reversed order.

Let $s_1, \dots, s_{19}$ be a system of generators of S_X such that the corresponding Gram matrix is the standard one. Then, $S_X^{\sharp}$ is generated by $s_{19}/(2n)$. In these coordinates, consider the vectors

$$F := (1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0),$$

$$O := (-1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0),$$

$$P := (n-1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1).$$

Note that $F^2 = 0, O^2 = P^2 = -2, F \cdot O = F \cdot P = 1, P \cdot O = n - 2$.

We can suppose that $s_3, \dots, s_{18}$, generating the two copies of $\mathbf{E}_8$, correspond to the components of the singular fibers which do not intersect O, F to the class of a fiber, O to a section which we take as origin and P to a section of height (cf. [24, Eq. (8.19)])

$$\mathrm{ht}(P) = \langle P, P\rangle = 2\chi(\mathcal{O}_X) + 2P \cdot O = 2 \cdot 2 + 2 \cdot (n-2) = 2n.$$

Let $Q := \boxminus P$ be the inverse section of P. Then $\mathrm{ht}(Q) = \mathrm{ht}(P) = 2n$, hence $Q \cdot O = n - 2$. It follows from $\langle P, Q\rangle = -\langle P, P\rangle$ that (cf. [24, Eq. (8.18)])

$$P \cdot Q = \chi(\mathcal{O}_X) + P \cdot O + Q \cdot O - \langle P, Q\rangle = 2 + (n-2) + (n-2) + 2n = 4n - 2.$$

Recalling moreover that $F \cdot Q = 1$ and $Q \cdot Q = -2$, we see that in our basis we can write

$$Q = (n-1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1).$$

Let $t_P \in \mathrm{O}(S_X)$ be the pullback of the automorphism induced by the translation by P. We have $t_P(Q) = O$, $t_P(O) = P$, $t_P(F) = F$ and t_P acts trivially on the other components of the fibers of type II^*. Therefore, τ_P is given by the following matrix:

$$t_P = \begin{pmatrix} 1 & & & \\ n & 1 & & 1 \\ & & I_{16} & \\ 2n & & & 1 \end{pmatrix}.$$

Let now $\iota \in \mathrm{O}(S_X)$ be the isometry given by

$$\iota = \begin{pmatrix} I_2 & & & \\ & & I_8 & \\ & I_8 & & \\ & & & -1 \end{pmatrix}.$$

Note that ι swaps the two fibers of type II^* and that $\iota(F) = F$, $\iota(P) = Q$ and $\iota(Q) = P$. Constructing an ample divisor as in [7, Prop. 2.7], it is easy to see that ι preserves the ample cone. Moreover, $\iota^\sharp = -\mathrm{id} \in S_X^\sharp$. Therefore, by the Torelli theorem ι is the pullback of a non-symplectic involution (whose quotient is a rational surface).

Consider $\varepsilon := \iota \circ t_P$, whose matrix is then given by

$$\varepsilon = \begin{pmatrix} 1 & & & & \\ n & 1 & & & -1 \\ & & & I_8 & \\ & & I_8 & & \\ 2n & & & & -1 \end{pmatrix}.$$

A computation shows that for even n, the invariant lattice of ε is isometric to $\mathbf{E}_8(2) \oplus \mathbf{U}(2)$, hence it corresponds to an Enriques involution by Nikulin's classification [18, Thm. 4.2.2]. The coinvariant lattice is isometric to $\mathbf{E}_8(2) \oplus [-2n]$.

2.3 | Enriques numbers

Let X be a K3 surface with Néron–Severi lattice S_X and transcendental lattice T_X. We recall briefly the formula for $|\mathrm{Enr}(X)|$ proved in [23]. We define

$$\mathbf{M} := \mathbf{U}(2) \oplus \mathbf{E}_8(2).$$

By Nikulin's classification [18], if $\varepsilon \in \mathrm{O}(S_X)$ is the pullback of an Enriques involution, then the invariant sublattice $S_X^\varepsilon := \{\, x \in S_X \mid \varepsilon(x) = x \,\}$ is isomorphic to $\mathbf{M}$. We denote by $(S_X)_\varepsilon := (S_X^\varepsilon)^\perp$ the coinvariant lattice, whose isometry class depends on the involution ε.

Given a primitive embedding $\iota : \mathbf{M} \hookrightarrow S_X$, we put

$$\mathrm{O}(S_X, \iota) := \{\, \varphi \in \mathrm{O}(S_X) \mid \varphi(\iota(\mathbf{M})) = \iota(\mathbf{M}) \,\}.$$

The Hodge structure on $\mathrm{H}^2(X,\mathbb{Z})$ induces a Hodge structure on T_X. We write $\mathrm{O_h}(T_X)$ for the group of Hodge isometries of T_X. We fix an anti-isometry $T_X^\sharp \cong S_X^\sharp$ (cf. [17, Prop. 1.6.1]), so that we can identify $\mathrm{O}\left(T_X^\sharp\right) \cong \mathrm{O}\left(S_X^\sharp\right)$. We denote the images of $\mathrm{O_h}(T_X)$ and $\mathrm{O}(S_X,\iota)$ under the natural morphisms $\mathrm{O}(T_X) \to \mathrm{O}\left(T_X^\sharp\right)$ and $\mathrm{O}(S_X) \to \mathrm{O}\left(S_X^\sharp\right)$ by $\mathrm{O}_\mathrm{h}^\sharp(T_X)$ and $\mathrm{O}^\sharp(S_X,\iota)$, respectively.

Theorem 2.3 [23, Thm. 3.1.9]. *For any K3 surface X it holds*

$$|\mathrm{Enr}(X)| = \sum \left|\mathrm{O}_\mathrm{h}^\sharp(T_X)\backslash \mathrm{O}\left(T_X^\sharp\right)/\mathrm{O}^\sharp(S_X,\iota)\right|,$$

where the sum runs over all primitive embeddings $\iota : \mathbf{M} \hookrightarrow S_X$ up to the action of $\mathrm{O}(S_X)$ such that there exists no $v \in \iota(\mathbf{M})^\perp$ with $v^2 = -2$.

The main topic of the present paper are K3 surfaces of Picard rank 19 covering an Enriques surface. For computational reasons, we restrict ourselves to K3 surfaces whose transcendental lattice has discriminant $|\det(T_X)| < 16$. By the next lemma, we have three cases to consider.

Lemma 2.4. *Let X be a K3 surface of Picard rank 19 and transcendental lattice T_X and suppose that $|\det(T_X)| < 16$. Then, $\mathrm{Enr}(X) \neq \emptyset$ if and only if $T_X \cong \mathbf{U} \oplus [4m]$ with $m \in \{1,2,3\}$.*

Proof. If $T_X \cong \mathbf{U} \oplus [4m]$, $m \geq 1$, then X covers an Enriques surface by [8, Proposition 4.2].

Conversely, suppose that $\mathrm{Enr}(X) \neq \emptyset$. By [3, Thm. 1.1] the lattice T_X has a Gram matrix of the form

$$\begin{pmatrix} 2a_{11} & a_{12} & a_{13} \\ a_{12} & 4a_{22} & 2a_{23} \\ a_{13} & 2a_{23} & 4a_{33}, \end{pmatrix}, \quad a_{ij} \in \mathbb{Z}.$$

Therefore, $\det(T_X)$ is divisible by 4. Now, the discriminant group $T_X^\sharp$ is a finite quadratic form on an abelian group of order $|\det(T_X)|$ and of signature $2 - 1 = 1$, because T_X has signature $(2,1)$. We classify such finite quadratic forms q using Miranda and Morrison's normal form [14]. For each q in the list we find a lattice T such that $T^\sharp \cong q$, obtaining the following table.

$\det(T)$	-4	-8	-8	-8	-12	-12
$T^\sharp$	$\mathbf{w}^1_{2,2}$	$\mathbf{u}_1 \oplus \mathbf{w}^1_{2,1}$	$\mathbf{w}^1_{2,3}$	$\mathbf{w}^5_{2,3}$	$\mathbf{w}^3_{2,2} \oplus \mathbf{w}^1_{3,1}$	$\mathbf{w}^7_{2,2} \oplus \mathbf{w}^{-1}_{3,1}$
T	$\mathbf{U} \oplus [4]$	$\mathbf{U}(2) \oplus [2]$	$\mathbf{U} \oplus [8]$	$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & -2 \end{pmatrix}$	$\mathbf{U} \oplus [12]$	$\begin{pmatrix} 2 & -1 & 0 \\ -1 & -2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$

In all cases except the second one, T is unique by [17, Thm. 1.14.2]. In the case $T = \mathbf{U}(2) \oplus [2]$, T is unique because $T = T'(2)$, with T' a unimodular indefinite lattice.

If $T_X \cong T$ is such that $T^\sharp \cong \mathbf{w}^5_{2,3}$ or $T^\sharp \cong \mathbf{w}^7_{2,2} \oplus \mathbf{w}^{-1}_{3,1}$, then $\mathrm{Enr}(X) = \emptyset$ because of [3, Prop. 3.9] (in the notation of [3], the two forms do not satisfy condition C(1)). The case $T_X \cong \mathbf{U}(2) \oplus [2]$ is excluded because of [3, Thm. 1.1] (the lattice is an "exceptional lattice").

Therefore, the only cases left are $T_X \cong T \in \{\mathbf{U} \oplus [4], \mathbf{U} \oplus [8], \mathbf{U} \oplus [12]\}$. □

2.4 | Enriques quotients of Barth–Peters type

Let now X be a K3 surface with $T_X \cong \mathbf{U} \oplus [4m]$, $m \geq 1$. A primitive embedding $\iota : \mathbf{M} \hookrightarrow S_X$ depends in general on several data (cf. §2.1 and [17, Prop. 1.15.1]), but in this case one only has to consider the orthogonal complement of the image, thanks to the next lemma.

Lemma 2.5. *Let X be a K3 surface with transcendental lattice $T_X \cong \mathbf{U} \oplus [4m]$, $m \geq 1$. If $\iota : \mathbf{M} \hookrightarrow S_X$ is a primitive embedding, then $\iota(\mathbf{M})^\perp \cong N(2)$, where N is a lattice in the genus of $\mathbf{E}_8 \oplus [-2m]$. Conversely, for each such lattice N there exists exactly one primitive embedding $\iota : \mathbf{M} \hookrightarrow S_X$ with $\iota(\mathbf{M})^\perp \cong N(2)$ up to the action of $\mathrm{O}(S_X)$.*

Proof. The Neron–Séveri lattice S_X is isomorphic to $\mathbf{U} \oplus \mathbf{E}_8^2 \oplus [-4m]$. Consider a primitive embedding $\iota : \mathbf{M} \hookrightarrow S_X$ with embedding subgroup $K \subset S_X^\sharp$ and embedding graph Ξ (see §2.1).

Since $\mathbf{M}^\sharp \cong 5\mathbf{u}_1$ is 2-elementary and $S_X^\sharp$ has length 1, it holds either $|K| = 1$ or $|K| = 2$. The first case, though, is impossible, as otherwise $\left(\iota(\mathbf{M})^\perp\right)^\sharp$ would have length $11 > \mathrm{rk}\left(\iota(\mathbf{M})^\perp\right)$. Therefore, it must be $|K| = 2$, so there is only one choice for the subgroup $K \subset S_X^\sharp$, which is generated by an element of order 2 and square $0 \in \mathbb{Q}/2\mathbb{Z}$.

Moreover, when taking $\Xi^\perp/\Xi$ in the identification (2.2) one copy of $\mathbf{u}_1$ gets killed. Hence, it holds

$$\left(\iota(\mathbf{M})^\perp\right)^\sharp \cong 4\mathbf{u}_1 \oplus [-4m]^\sharp,$$

and in particular $\ell_2\left(\left(\iota(\mathbf{M})^\perp\right)^\sharp\right) = 9 = \mathrm{rk}\left(\iota(\mathbf{M})^\perp\right)$. Therefore (see for instance [3, Lemma 3.10]), it holds $\iota(\mathbf{M})^\perp \cong N(2)$, with N an even lattice. The genus of $N(2)$ determines the genus of N, so we see that N is in the genus of $\mathbf{E}_8 \oplus [-2m]$.

The converse holds by [17, Prop. 1.15.1], because S_X is unique in its genus, K is uniquely determined and $\mathrm{O}(\mathbf{M}) \to \mathrm{O}\left(\mathbf{M}^\sharp\right)$ is surjective (see for instance [1, p. 388]). □

Note that a lattice $N' \cong N(2)$, with N an even lattice, does not contain vectors of square -2. Therefore, Theorem 2.5 essentially says that the terms in the sum of Theorem 2.3 are in one-to-one correspondence with the lattices in the genus of $\mathbf{E}_8 \oplus [-2m]$. In particular, one of them corresponds to $\mathbf{E}_8 \oplus [-2m]$ itself, which we presently consider more in detail.

Barth–Peters introduced a 2-dimensional family of K3 surfaces, whose general element $\mathfrak{X}$ has transcendental lattice $T_{\mathfrak{X}} \cong \mathbf{U} \oplus \mathbf{U}(2)$ and Néron–Severi lattice $S_{\mathfrak{X}} \cong \mathbf{U}(2) \oplus \mathbf{E}_8^2$. Ohashi [20, Remark 4.9(2)] proved that $|\mathrm{Enr}(\mathfrak{X})| = 1$. The coinvariant lattice of an Enriques involution on $\mathfrak{X}$ is isomorphic to $\mathbf{E}_8(2)$.

In the situation of Theorem 2.5, if $\mathbf{E}_8(2)$ embeds into $\left(S_X\right)_\varepsilon \cong \iota(\mathbf{M})^\perp$, then $\iota(\mathbf{M})^\perp \cong \mathbf{E}_8(2) \oplus [-4m]$. This motivates the following definition.

Definition 2.6. We say that an Enriques involution $\varepsilon \in \mathrm{Aut}(X)$ on X is *of Barth–Peters type* if $\left(S_X\right)_\varepsilon \cong \mathbf{E}_8(2) \oplus [-4m]$. The corresponding Enriques quotient is also called *of Barth–Peters type*.

The following lemma provides the number of Enriques quotients of Barth–Peters type up to isomorphisms.

Lemma 2.7. *If X is a K3 surface with transcendental lattice $T_X \cong \mathbf{U} \oplus [4m]$, $m \geq 1$, and $\iota : \mathbf{M} \hookrightarrow S_X$ is a primitive embedding with $\iota(\mathbf{M})^\perp \cong \mathbf{E}_8(2) \oplus [-4m]$, then it holds*

$$\left|\mathrm{O}_{\mathrm{h}}^\sharp(T_X) \backslash \mathrm{O}\left(T_X^\sharp\right) / \mathrm{O}^\sharp(S_X, \iota)\right| = 2^{\omega-1},$$

where ω is the number of prime divisors of $2m$.

Proof. As in the proof of Theorem 2.1, it holds $\left|\mathrm{O}\left(T_X^\sharp\right)\right| = 2^\omega$. As $\mathrm{rk}\left(T_X\right)$ is odd, it holds $\mathrm{O}_{\mathrm{h}}^\sharp\left(T_X\right) = \{\pm \mathrm{id}\}$ (see for instance [10, Cor. 3.3.5]). Note, moreover, that $\mathrm{id} \neq -\mathrm{id}$ in $T_X^\sharp$.

We now want to determine $\mathrm{O}_{\mathrm{h}}^\sharp\left(S_X, \iota\right)$ using the identification (2.1). Let $s \in \mathbf{E}_8(2) \oplus [-4m]$ be the generator of the copy of $[-4m]$, H be the gluing subgroup, $\gamma : H \to H'$ be the gluing isometry, and Γ the gluing graph of ι (see §2.1). By the identification (2.1), it holds $|H| = |H'| = 2^9$. Therefore, $S_X^\sharp \cong \Gamma^\perp/\Gamma$ is generated by an element of the form $(\alpha, s/4m)$, with $\alpha \in \mathbf{M}^\sharp$.

Recall now that $\mathrm{O}(\mathbf{M}) \to \mathrm{O}\left(\mathbf{M}^\sharp\right)$ is surjective (see [1, p. 388]) and that an isometry of a definite lattice preserves its decomposition in irreducible lattices up to order (see for instance [12, Satz 27.2]), so

$$\mathrm{O}\left(\mathbf{E}_8(2) \oplus [-4m]\right) \cong \mathrm{O}\left(\mathbf{E}_8(2)\right) \times \mathrm{O}\left([-4m]\right).$$

These facts imply that the group $\mathrm{O}_{\mathrm{h}}(S_X, \iota)$ can only act as $\pm\mathrm{id}$ on $(\alpha, s/4m)$, i.e., $\mathrm{O}^\sharp_{\mathrm{h}}(S_X, \iota) = \{\pm\mathrm{id}\}$. Therefore, we have

$$\left|\mathrm{O}^\sharp_{\mathrm{h}}(T_X)\backslash \mathrm{O}\left(T_X^\sharp\right)/\mathrm{O}^\sharp(S_X,\iota)\right| = \left|\mathrm{O}\left(T_X^\sharp\right)/\{\pm\mathrm{id}\}\right| = \left|\mathrm{O}\left(T_X^\sharp\right)\right|/|\{\pm\mathrm{id}\}| = 2^{\omega-1}. \qquad \square$$

Remark 2.8. Recall that any elliptic pencil on an Enriques surface has exactly two multiple fibers $2F, 2F'$. The divisors F and F' are called *half-pencils* (necessarily of type I_m for some $m \geq 0$). An elliptic pencil on an Enriques surface is said to be *special* if it has a 2-section which is a smooth rational curve.

As noted by Kondō [13, Lem. 2.6], the pullback of a special elliptic pencil induces a jacobian elliptic fibration π on the K3 surface X. Such pullbacks satisfy the following condition: if the fibration π has exactly n_i fibres type of type J_i (for $i = 1, \dots, r$), where $\mathrm{J}_i, \mathrm{J}_j$ are pairwise distinct Kodaira types if $i \neq j$, then at most two coefficients n_i can be odd; moreover, if n_i is odd, then $\mathrm{J}_i = \mathrm{I}_{2m}$ for some $m \geq 0$.

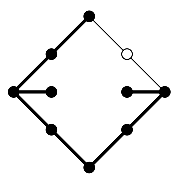
The last sentence comes from the fact that one of the fibers of type J_i is necessarily the pullback of a half-pencil.

Remark 2.9. Let $n = 2m$ be an even integer and consider one of the $2^{\omega-1}$ elliptic fibrations with two fibers of type II^* given in Theorem 2.1. By the construction of Theorem 2.2, we obtain one of the $2^{\omega-1}$ Enriques quotients Y of Barth–Peters type of Theorem 2.7. In the notation of Theorem 2.2, the vector

$$R := (m+1, 2, -4, -5, -7, -10, -8, -6, -4, -2, -2, -3, -4, -6, -5, -4, -3, -2, 1)$$

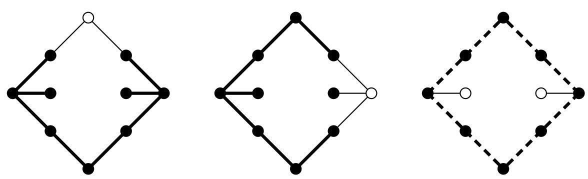
has square -2, satisfies $R \cdot F = 2$ and has intersection number 1 with e_3 and e_{18}. Therefore, it represents a smooth rational curve. Moreover, $R \cdot \varepsilon(R) = 0$.

Thus, the surface Y contains ten smooth rational curves which are the images of R and of the components of the fibers of type II^*. They form the following dual graph, where the white vertex represents the image of R, which is a 4-section of the highlighted elliptic pencil.



This graph appears in [13, Thm. 1.7(i)] and is related to the fact that Y has a cohomologically trivial automorphism (such automorphisms were studied by Mukai and Namikawa [15, 16]).

On the above graph we can recognize three more special elliptic pencils up to symmetries (dashed lines indicates half-pencils):



In our case we retrieve the jacobian elliptic fibrations on X with respectively two fibres of type I_4^*, two fibres of type III^*, and one fibre of type I_{16} (hence the corresponding root lattices W_{root} of the frame contain the sublattices $\mathbf{D}_8^2$, $\mathbf{E}_7^2$ and $\mathbf{A}_{15}$, respectively).

3 | THE THREE PENCILS

This section is divided into three subsections, in which we study K3 surface X with transcendental lattice $T_X \cong \mathbf{U} \oplus [4]$ (§3.1), $T_X \cong \mathbf{U} \oplus [8]$ (§3.2), and $T_X \cong \mathbf{U} \oplus [12]$ (§3.3). In each case we determine $|\mathrm{Enr}(X)|$ and $|\mathcal{J}_X/\mathrm{Aut}(X)|$, then we focus

TABLE 1 Lattices in the frame genus $\mathcal{W}_X$ of a K3 surface X with transcendental lattice $T_X \cong \mathbf{U} \oplus [4]$

W	N_{root}	W_{root}	W/W_{root}	$\lvert\Delta(W)\rvert$	$\lvert \mathrm{O}(W)\rvert$	$\lvert \mathrm{fr}_X^{-1}(W)\rvert$	Rmk.
W_1	$\mathbf{D}_{16}\mathbf{E}_8$	$\mathbf{D}_9\mathbf{E}_8$	0	384	129448569470976000	1	–
W_2	$\mathbf{D}_{24}$	$\mathbf{D}_{17}$	0	544	46620662575398912000	1	–
W_3	$\mathbf{D}_{10}\mathbf{E}_7^2$	$\mathbf{A}_3\mathbf{E}_7^2$	$\mathbb{Z}/2\mathbb{Z}$	264	809053559193600	1	2.9
W_4	$\mathbf{D}_{12}^2$	$\mathbf{D}_5\mathbf{D}_{12}$	$\mathbb{Z}/2\mathbb{Z}$	304	3767021862912000	1	–
W_5	$\mathbf{A}_{11}\mathbf{D}_7\mathbf{E}_6$	$\mathbf{A}_{11}\mathbf{E}_6$	$\mathbb{Z}/3\mathbb{Z}$	204	49662885888000	1	–
W_6	$\mathbf{A}_{15}\mathbf{D}_9$	$\mathbf{A}_1^2\mathbf{A}_{15}$	$\mathbb{Z}/4\mathbb{Z}$	244	334764638208000	1	2.9
W_7	$\mathbf{E}_8^3$	$\mathbf{E}_8^2$	$\mathbb{Z}$	480	1941728542064640000	1	2.9
W_8	$\mathbf{D}_8^3$	$\mathbf{D}_8^2$	$\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})$	224	106542032486400	1	2.9
W_9	$\mathbf{D}_{16}\mathbf{E}_8$	$\mathbf{D}_{16}$	$\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})$	480	1371195958099968000	1	–

on their Enriques quotients, especially those *not* of Barth–Peters type (because those of Barth–Peters type were already considered in §2.4).

Moreover, we show that all jacobian elliptic fibrations satisfying the condition in Theorem 2.8 are indeed pullbacks of elliptic pencils on some Enriques quotient.

3.1 | Kondō's pencil I

Let X be a K3 surface with transcendental lattice

$$T_X \cong \mathbf{U} \oplus [4].$$

Theorem 3.1. *It holds* $|\mathrm{Enr}(X)| = 1$.

Proof. The lattice $\mathbf{E}_8 \oplus [-2]$ is unique in its genus by the mass formula. By Theorem 2.5, the sum in Theorem 2.3 has only one term, which is equal to 1 by Theorem 2.7. □

Therefore, the surface X admits only one Enriques quotient $X \to Y$. Necessarily, the Barth–Peters quotient of Theorem 2.7 coincides with Kondō's quotient [13] (in particular, Y has a finite automorphism group). Indeed, the graph of nodal curves contained in Y, which is pictured in [13, Fig. 1.4], contains the Barth–Peters graph as a subgraph. This Enriques quotient was also studied by Hulek and Schütt [8, §4.6].

For the sake of completeness, we enumerate all jacobian elliptic fibrations on X up to automorphisms (the same list is contained in an unpublished paper by Elkies and Schütt [5]).

Proposition 3.2. *The frame genus $\mathcal{W}_X$ contains exactly 9 isomorphism classes, listed in Table 1, whose Gram matrices are contained in the arXiv ancillary file* `genus_Kondo_I.sage`. *Moreover, it holds*

$$\left|\mathcal{J}_X/\mathrm{Aut}(X)\right| = 9.$$

Proof. It holds $\left|\mathcal{J}_X/\mathrm{Aut}(X)\right| = \left|\mathcal{W}_X\right|$ by [7, Cor. 2.10]. In order to determine $\mathcal{W}_X$, we apply the Kneser–Nishiyama method with $T_0 = \mathbf{D}_7$. The list is complete because the mass formula holds:

$$\sum_{i=1}^{9} \frac{1}{|\mathrm{O}(W_i)|} = \frac{642332179}{18881368343036559360000} = \mathrm{mass}(\mathcal{W}_X). \qquad \square$$

3.2 | Kondō's pencil II

Let X be a K3 surface with transcendental lattice

$$T_X \cong \mathbf{U} \oplus [8].$$

Theorem 3.3. *It holds* $|\mathrm{Enr}(X)| = 2$.

Proof. There is only one more lattice in the genus of $\mathbf{E}_8 \oplus [-4]$, namely $\mathbf{D}_9$, as the mass formula shows. By Theorem 2.5 the sum in Theorem 2.3 has two terms, both equal to 1 as it holds $\mathrm{O}^\sharp_\mathrm{h}(T_X) = \mathrm{O}\left(T^\sharp_X\right)$. □

By Theorem 2.7, one of the two Enriques quotient, say $X \to Y'$, is of Barth–Peters type. The corresponding coinvariant lattice is isometric to $\mathbf{E}_8(2) \oplus [-8]$ and contains 240 vectors of square -4.

By Kondō's classification, the surface X admits an Enriques quotient $X \to Y''$ with finite automorphism group. Kondō's quotient Y'' was also studied by Hulek and Schütt [8, §4.7 and §4.8]. We argue that Y' is not isomorphic to Y''.

Geometrically, this follows from the fact that Y'' contains exactly 12 smooth rational curves whose dual graph is pictured on [13, p. 207, Fig. 2.4]. This dual graph does not contain the graph pictured in Theorem 2.9 as a subgraph.

Algebraically, we can distinguish the two quotients in the following way.

1. The surface X contains 24 smooth rational curves $F^+_1, F^-_1, \dots, F^+_{12}, F^-_{12}$, which intersect as in [13, p. 207, Fig. 2.3] and generate the Néron–Severi lattice S_X.
2. Kondō's Enriques involution exchanges F^+_i with F^-_i, $i = 1, \dots, 12$.
3. Computing explicitly the coinvariant lattice of Kondō's Enriques involution in S_X, we see that it contains 144 vectors of square -4, so it must be isomorphic to $\mathbf{D}_9(2)$. In particular, Kondō's quotient is not of Barth–Peters type.

We now enumerate all jacobian elliptic fibrations on X up to automorphisms.

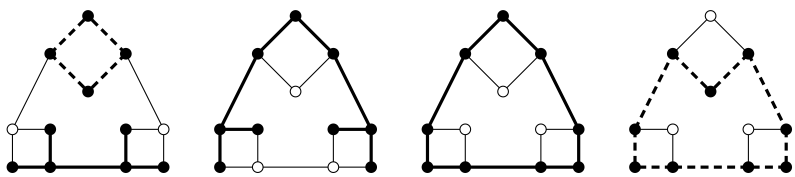
Proposition 3.4. *The frame genus* $\mathcal{W}_X$ *contains exactly* 17 *isomorphism classes, listed in Table 1, whose Gram matrices are contained in the arXiv ancillary file* `genus_Kondo_II.sage`. *Moreover, it holds*

$$\left|\mathcal{J}_X/\mathrm{Aut}(X)\right| = 17.$$

Proof. It holds $\left|\mathcal{J}_X/\mathrm{Aut}(X)\right| = \left|\mathcal{W}_X\right|$ by [7, Cor. 2.10]. In order to determine $\mathcal{W}_X$, we apply the Kneser–Nishiyama method with $T_0 = \mathbf{A}_7$. Note that there are two different primitive embeddings $\mathbf{A}_7 \hookrightarrow \mathbf{D}_8$ (cf. [19, Lem. 4.2]), leading to two distinct frames W with $W_{\mathrm{root}} \hookrightarrow \mathbf{D}_8^3$, namely W_{11} and W_{12} (cf. Theorem 3.6). The list is complete because the mass formula holds:

$$\sum_{i=1}^{17} \frac{1}{|\mathrm{O}(W_i)|} = \frac{642332179}{737553450899865600000} = \mathrm{mass}\left(\mathcal{W}_X\right).$$ □

Remark 3.5. The surface Y'' contains 12 curves on whose dual graph one can recognize the following elliptic pencils (dashed lines indicate half-pencils):



The first three pencils are special pencils and correspond to the elliptic fibrations on X with frames W_1, W_{11} (see Theorem 3.6) and W_{16}, respectively.

The fourth pencil is not special: the highlighted curves on Y'' form a half-pencil and the white vertices represent 4-sections. Indeed, the pullback on X correspond to an elliptic fibration with a fiber of type I_{18}, namely

$$F^+_1 + F^-_4 + F^-_3 + F^-_5 + F^-_6 + F^-_7 + F^-_9 + F^-_{10} + F^-_{11} + F^-_1 + F^+_4 + F^+_3 + F^+_5 + F^+_6 + F^+_7 + F^+_9 + F^+_{10} + F^+_{11}.$$

This fibration is not jacobian, as it does not appear in Table 2.

Remark 3.6. The two frames W_{11} and W_{12} are not isometric, but they can be distinguished neither by the pair $\left(W_{\mathrm{root}}, W/W_{\mathrm{root}}\right)$ nor by their number of automorphisms $|\mathrm{O}(W)|$.

TABLE 2 Lattices in the frame genus $\mathcal{W}_X$ of a K3 surface X with transcendental lattice $T_X \cong \mathbf{U} \oplus [8]$

W	N_{root}	W_{root}	W/W_{root}	$\lvert\Delta(W)\rvert$	$\lvert O(W)\rvert$	$\lvert \mathrm{fr}_X^{-1}(W)\rvert$	Rmk.
W_1	$\mathbf{A}_7^2\mathbf{D}_5^2$	$\mathbf{A}_7\mathbf{D}_5^2$	$\mathbb{Z}/4\mathbb{Z}$	136	594542592000	1	3.5
W_2	$\mathbf{A}_{11}\mathbf{D}_7\mathbf{E}_6$	$\mathbf{A}_3\mathbf{D}_7\mathbf{E}_6$	$\mathbb{Z}$	168	802632499200	1	–
W_3	$\mathbf{A}_{12}^2$	$\mathbf{A}_{12}\mathbf{A}_4$	$\mathbb{Z}$	176	1494484992000	1	–
W_4	$\mathbf{A}_{15}\mathbf{D}_9$	$\mathbf{A}_7\mathbf{D}_9$	$\mathbb{Z}$	200	7491236659200	1	–
W_5	$\mathbf{A}_{17}\mathbf{E}_7$	$\mathbf{A}_9\mathbf{E}_7$	$\mathbb{Z}$	216	21069103104000	1	–
W_6	$\mathbf{A}_{24}$	$\mathbf{A}_{16}$	$\mathbb{Z}$	272	711374856192000	1	–
W_7	$\mathbf{D}_{16}\mathbf{E}_8$	$\mathbf{D}_8\mathbf{E}_8$	$\mathbb{Z}$	342	7191587192832000	1	–
W_8	$\mathbf{D}_{24}$	$\mathbf{D}_{16}$	$\mathbb{Z}$	480	1371195958099968000	1	–
W_9	$\mathbf{E}_8^3$	$\mathbf{E}_8^2$	$\mathbb{Z}$	480	1941728542064640000	1	2.9
W_{10}	$\mathbf{A}_9^2\mathbf{D}_6$	$\mathbf{A}_1\mathbf{A}_9\mathbf{D}_6$	$\mathbb{Z}\oplus(\mathbb{Z}/2\mathbb{Z})$	152	334430208000	1	–
W_{11}	$\mathbf{D}_8^3$	$\mathbf{D}_8^2$	$\mathbb{Z}\oplus(\mathbb{Z}/2\mathbb{Z})$	224	106542032486400	1	3.5, 3.6
W_{12}	$\mathbf{D}_8^3$	$\mathbf{D}_8^2$	$\mathbb{Z}\oplus(\mathbb{Z}/2\mathbb{Z})$	224	106542032486400	1	2.9, 3.6
W_{13}	$\mathbf{D}_{10}\mathbf{E}_7^2$	$\mathbf{A}_1^2\mathbf{E}_7^2$	$\mathbb{Z}\oplus(\mathbb{Z}/2\mathbb{Z})$	256	134842259865600	1	2.9
W_{14}	$\mathbf{D}_{12}^2$	$\mathbf{D}_{12}\mathbf{D}_4$	$\mathbb{Z}\oplus(\mathbb{Z}/2\mathbb{Z})$	288	376702186291200	1	–
W_{15}	$\mathbf{D}_{16}\mathbf{E}_8$	$\mathbf{D}_{16}$	$\mathbb{Z}\oplus(\mathbb{Z}/2\mathbb{Z})$	480	1371195958099968000	1	–
W_{16}	$\mathbf{A}_8^3$	$\mathbf{A}_8^2$	$\mathbb{Z}\oplus(\mathbb{Z}/3\mathbb{Z})$	144	526727577600	1	3.5
W_{17}	$\mathbf{A}_{15}\mathbf{D}_9$	$\mathbf{A}_{15}$	$\mathbb{Z}^2\oplus(\mathbb{Z}/2\mathbb{Z})$	240	83691159552000	1	2.9

Using the command `is_globally_equivalent_to` of the Sage class `QuadraticForm`, we can check that the frame W_{11} corresponds to the fibration with fiber

$$\begin{aligned} F &= F_6^+ + F_8^- + 2F_5^+ + 2F_3^+ + 2F_2^+ + 2F_1^+ + 2F_{11}^+ + F_{10}^+ + F_{12}^+ \\ &= F_6^- + F_8^+ + 2F_5^- + 2F_3^- + 2F_2^- + 2F_1^- + 2F_{11}^- + F_{10}^- + F_{12}^-, \end{aligned}$$

which is the pullback of the second special pencil on Y'' listed in Theorem 3.5.

On the other hand, with the same command we can check that the frame W_{12} corresponds to the fibration with fiber

$$F_2^+ + F_5^+ + 2F_3^+ + 2F_4^+ + 2F_1^- + 2F_2^- + 2F_3^- + F_4^- + F_5^-,$$

which is then the pullback of a special pencil on Y' (cf. Theorem 2.9).

3.3 | Apéry–Fermi pencil

Let X be a K3 surface with transcendental lattice

$$T_X \cong \mathbf{U} \oplus [12].$$

The classification of the jacobian elliptic fibrations on X was carried out by Bertin and Lecacheux [2] and then refined in [7]. For the reader's convenience we reproduce in Table 3 the same table as [7, Table 7].

Theorem 3.7. *It holds* $|\mathrm{Enr}(X)| = 3$.

Proof. The genus of $\mathbf{E}_8 \oplus [-6]$ contains two lattices, namely $\mathbf{A}_2 \oplus \mathbf{E}_7$ and $\mathbf{E}_8 \oplus [-6]$ itself, as the mass formula shows. Thus, by Theorem 2.5, the sum in Theorem 2.3 consists of two terms, one of which is equal to 2 by Theorem 2.7.

Fix a primitive embedding $\iota : \mathbf{M} \hookrightarrow S_X$ with $\iota(\mathbf{M})^\perp \cong \mathbf{A}_2(2) \oplus \mathbf{E}_7(2)$. Note that it holds

$$S_X^\sharp \cong \mathbf{w}_{2,2}^5 \oplus \mathbf{w}_{3,1}^{-1}, \quad \mathbf{A}_2(2)^\sharp \cong \mathbf{v}_1 \oplus \mathbf{w}_{3,1}^{-1}, \quad \mathbf{E}_7(2)^\sharp = 3\mathbf{u}_1 \oplus \mathbf{w}_{2,2}^1.$$

TABLE 3 Lattices in the frame genus $\mathcal{W}_X$ of a K3 surface X with transcendental lattice $T_X \cong \mathbf{U} \oplus [12]$, numbered according to Bertin and Lecacheux (cf. [2, Tables 2 and 3])

W	N_{root}	W_{root}	W/W_{root}	$\lvert\Delta(W)\rvert$	$\lvert O(W)\rvert$	$\lvert \mathrm{fr}_X^{-1}(W)\rvert$	Rmk.
W_3	$\mathbf{D}_{16}\mathbf{E}_8$	$\mathbf{D}_{11}\mathbf{E}_6$	0	292	8475799191552000	1	–
W_1	$\mathbf{E}_8^3$	$\mathbf{A}_3\mathbf{E}_6\mathbf{E}_8$	0	324	3467372396544000	1	–
W_7	$\mathbf{D}_{10}\mathbf{E}_7^2$	$\mathbf{A}_5\mathbf{D}_5\mathbf{E}_7$	$\mathbb{Z}/2\mathbb{Z}$	196	16052649984000	1	–
W_{20}	$\mathbf{A}_{11}\mathbf{D}_7\mathbf{E}_6$	$\mathbf{A}_1^2\mathbf{A}_2^2\mathbf{A}_{11}$	$\mathbb{Z}/6\mathbb{Z}$	148	551809843200	1	3.9
W_{27}	$\mathbf{A}_7^2\mathbf{D}_5^2$	$\mathbf{A}_4\mathbf{A}_7\mathbf{D}_5$	$\mathbb{Z}$	116	18579456000	2	–
W_{21}	$\mathbf{A}_{11}\mathbf{D}_7\mathbf{E}_6$	$\mathbf{A}_1^2\mathbf{A}_8\mathbf{E}_6$	$\mathbb{Z}$	148	300987187200	1	–
W_{18}	$\mathbf{A}_{15}\mathbf{D}_9$	$\mathbf{A}_{12}\mathbf{D}_4$	$\mathbb{Z}$	180	4782351974400	1	–
W_{13}	$\mathbf{D}_{12}^2$	$\mathbf{D}_9\mathbf{D}_7$	$\mathbb{Z}$	228	119859786547200	1	–
W_5	$\mathbf{D}_{16}\mathbf{E}_8$	$\mathbf{A}_3\mathbf{D}_{13}$	$\mathbb{Z}$	324	2448564210892800	1	–
W_6	$\mathbf{D}_{16}\mathbf{E}_8$	$\mathbf{D}_8\mathbf{E}_8$	$\mathbb{Z}$	352	14383174385664000	1	–
W_2	$\mathbf{E}_8^3$	$\mathbf{E}_8^2$	$\mathbb{Z}$	480	1941728542064640000	2	2.9
W_{12}	$\mathbf{D}_{24}$	$\mathbf{D}_{16}$	$\mathbb{Z}$	480	2742391916199936000	1	–
W_{15}	$\mathbf{D}_8^3$	$\mathbf{A}_3\mathbf{D}_5\mathbf{D}_8$	$\mathbb{Z}\oplus(\mathbb{Z}/2\mathbb{Z})$	164	951268147200	1	–
W_8	$\mathbf{D}_{10}\mathbf{E}_7^2$	$\mathbf{A}_1\mathbf{A}_5\mathbf{D}_{10}$	$\mathbb{Z}\oplus(\mathbb{Z}/2\mathbb{Z})$	212	10701766656000	1	–
W_{16}	$\mathbf{D}_8^3$	$\mathbf{D}_8^2$	$\mathbb{Z}\oplus(\mathbb{Z}/2\mathbb{Z})$	224	106542032486400	2	2.9
W_9	$\mathbf{D}_{10}\mathbf{E}_7^2$	$\mathbf{A}_1^2\mathbf{E}_7^2$	$\mathbb{Z}\oplus(\mathbb{Z}/2\mathbb{Z})$	256	269684519731200	1	2.9
W_{14}	$\mathbf{D}_{12}^2$	$\mathbf{D}_4\mathbf{D}_{12}$	$\mathbb{Z}\oplus(\mathbb{Z}/2\mathbb{Z})$	288	753404372582400	1	–
W_4	$\mathbf{D}_{16}\mathbf{E}_8$	$\mathbf{D}_{16}$	$\mathbb{Z}\oplus(\mathbb{Z}/2\mathbb{Z})$	480	1371195958099968000	2	–
W_{19}	$\mathbf{E}_6^4$	$\mathbf{A}_2^2\mathbf{E}_6^2$	$\mathbb{Z}\oplus(\mathbb{Z}/3\mathbb{Z})$	156	773967052800	1	3.9
W_{26}	$\mathbf{A}_7^2\mathbf{D}_5^2$	$\mathbf{A}_1^2\mathbf{A}_7^2$	$\mathbb{Z}\oplus(\mathbb{Z}/4\mathbb{Z})$	116	52022476800	1	3.9
W_{25}	$\mathbf{A}_9^2\mathbf{D}_6$	$\mathbf{A}_6\mathbf{A}_9$	$\mathbb{Z}^2$	132	73156608000	1	–
W_{22}	$\mathbf{A}_{11}\mathbf{D}_7\mathbf{E}_6$	$\mathbf{A}_8\mathbf{D}_7$	$\mathbb{Z}^2$	156	234101145600	2	–
W_{10}	$\mathbf{D}_{10}\mathbf{E}_7^2$	$\mathbf{A}_1\mathbf{D}_7\mathbf{E}_7$	$\mathbb{Z}^2$	212	7491236659200	1	–
W_{11}	$\mathbf{A}_{17}\mathbf{E}_7$	$\mathbf{A}_1\mathbf{A}_{14}$	$\mathbb{Z}^2$	212	10461394944000	1	–
W_{24}	$\mathbf{D}_6^4$	$\mathbf{A}_3\mathbf{D}_6^2$	$\mathbb{Z}^2\oplus(\mathbb{Z}/2\mathbb{Z})$	132	101921587200	1	3.9
W_{23}	$\mathbf{A}_{11}\mathbf{D}_7\mathbf{E}_6$	$\mathbf{A}_{11}\mathbf{D}_4$	$\mathbb{Z}^2\oplus(\mathbb{Z}/2\mathbb{Z})$	156	367873228800	1	–
W_{17}	$\mathbf{A}_{15}\mathbf{D}_9$	$\mathbf{A}_{15}$	$\mathbb{Z}^2\oplus(\mathbb{Z}/2\mathbb{Z})$	240	167382319104000	1	2.9

Let $H \subset \mathbf{M}^\sharp$ be the gluing subgroup (see §2.1). By the identification (2.1) we have $|H| = 2^9$. Thus, the image

$$H' := \gamma(H) \subset \left(\iota(\mathbf{M})^\perp(-1)\right)^\sharp$$

of the gluing isometry is the sum of the copy of $\mathbf{v}_1$ in $\mathbf{A}_2(2)$ and the whole group $\mathbf{E}_7(2)^\sharp$ (with inverted sign).

Consider the isometry $\alpha \in \mathrm{O}\left(\iota(\mathbf{M})^\perp\right)$ defined as $-\mathrm{id}$ on the copy of $\mathbf{A}_2(2)$ and as id on the copy of $\mathbf{E}_7(2)$. Since the natural homomorphism $\mathrm{O}(\mathbf{M}) \to \mathrm{O}\left(\mathbf{M}^\sharp\right)$ is surjective, α extends to an isometry $\tilde{\alpha} \in \mathrm{O}\left(S_X, \iota\right)$ by [17, Cor. 1.5.2].

By construction of α and by the above description of the gluing isometry γ, the element $\tilde{\alpha}^\sharp$ acts as $-\mathrm{id}$ on the 3-part of $S_X^\sharp$ and as id on the 2-part of $S_X^\sharp$. In particular, $\mathrm{O}^\sharp(S_X, \iota)$ contains at least three different elements, namely id, $-\mathrm{id}$ and $\tilde{\alpha}^\sharp$. On the other hand, $\mathrm{O}\left(T_X^\sharp\right)$ contains exactly four elements, as it is generated by multiplication by -1 and by 5. Therefore, we have $\mathrm{O}^\sharp(S_X, \iota) = \mathrm{O}\left(T_X^\sharp\right)$, which implies

$$\left|\mathrm{O}_{\mathrm{h}}^\sharp(T_X) \backslash \mathrm{O}\left(T_X^\sharp\right) / \mathrm{O}^\sharp(S_X, \iota)\right| = 1.$$

In total we get $|\mathrm{Enr}(X)| = 3$. □

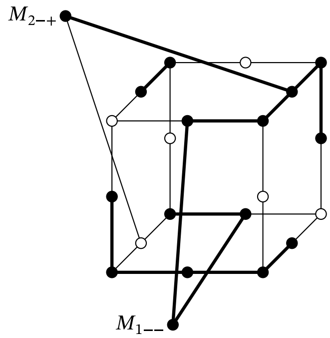
Let Y', Y'', Y''' be the three Enriques quotients of X up to automorphisms. We can suppose that Y', Y'' are of Barth–Peters type (see §2.4). Here we are interested in studying $Y := Y'''$.

Peters and Stienstra [21] showed that X contains $20 + 12$ smooth rational curves, called *L-lines* and *M-lines*, forming a particular configuration, which we call the *Peters–Stientra cube*. The dual graph of the L-lines is pictured in [21, Fig. 1]. We do not reproduce it here, but we follow the same notation. The intersection numbers of the M-lines are described in [21, Lem. 1].

In order to make a connection with the construction of Theorem 2.2, we first look for a fibration with two fibers of type II^* or, equivalently, with frame W_2 in Table 3. We can suppose

$$\begin{aligned} F &= 2L_{++0} + 3M_{2-+} + 4L_{+++} + 6L_{+0+} + 5L_{+-+} + 4L_{0-+} + 3M_{1--} + 2L_{0+-} + L_{-+-} \\ &= 2L_{--0} + 3M_{1++} + 4L_{---} + 6L_{0--} + 5L_{+--} + 4L_{+0-} + 3M_{2+-} + 2L_{-0+} + L_{-++}, \end{aligned}$$

as pictured below in the Peters–Stienstra cube (note that M_{1++} and M_{2+-} and the other M-lines are not displayed):



In the coordinate system of Theorem 2.2, up to substituting P with Q, we can suppose that

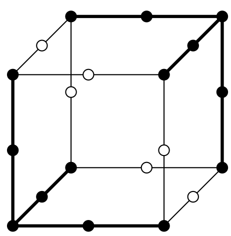
$$\begin{aligned} L_{-+0} &= O, \\ L_{++0} &= (0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0), \\ L_{--0} &= (0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0), \\ L_{++-} &= (4, 4, -6, -8, -11, -16, -13, -10, -7, -4, -6, -9, -12, -18, -15, -12, -8, -4, 1). \end{aligned}$$

The coordinates of all other L-lines and M-lines are determined by these choices.

In order to construct the Enriques involution corresponding to Y, we now consider a fibration with frame W_{19} in Table 3. As $(W_{19})_{\mathrm{root}} \cong \mathbf{A}_2^2\mathbf{E}_6^2$, the fibration has two fibers of type I_3 (or IV) and two fibers of type IV^*.

As pictured below in the Peters–Stienstra cube (omitting the M-lines) we choose

$$\begin{aligned} F_{19} &:= L_{+-+} + L_{++-} + 2L_{+0+} + 2L_{++0} + 3L_{+++} + 2L_{0++} + L_{-++} = M_{3+-} + M_{1+-} + M_{2+-} \\ &= L_{+--} + L_{-+-} + 2L_{0--} + 2L_{-0-} + 3L_{---} + 2L_{--0} + L_{--+} = M_{3--} + M_{1--} + M_{2--}. \end{aligned}$$



Moreover, we choose $O_{19} := L_{+-0}$ as origin. Then, $P_{19} := L_{0+-}$ and $Q_{19} := L_{-0+}$ become the two 3-torsion sections, because it holds $\langle P_{19}, P_{19}\rangle = \langle Q_{19}, Q_{19}\rangle = 0$, whereas $R_{19} := L_{-+0}$ becomes a section of infinite order. From [24, Eq. (8.12) and Table (8.16)] it follows

$$\langle R_{19}, R_{19}\rangle = 2\chi + 2L_{-+0}\cdot L_{+-0} - \sum contr_v(R_{19}) = 2\cdot 2 + 2\cdot 0 - 2\cdot \frac{4}{3} = \frac{4}{3}.$$

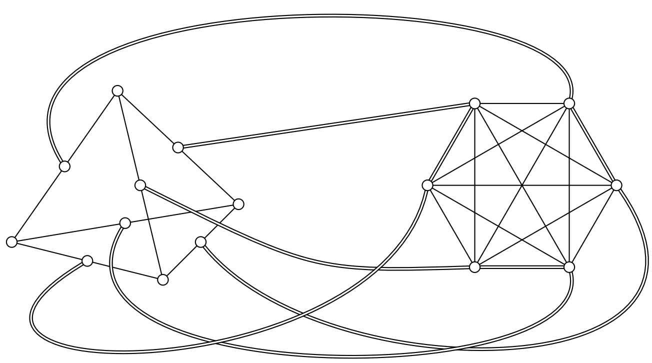
Theorem 3.8. *There is an Enriques involution $\varepsilon \in \mathrm{Aut}(X)$ which acts on the L-lines by exchanging all subscripts "+" with "−" and on the M-lines by exchanging $M_{k+\beta}$ with $M_{k-\beta}$, for all $k \in \{1,2,3\}$, $\beta \in \{+,-\}$. Moreover, ε is not of Barth–Peters type.*

Proof. Let $S_{19} := \boxminus R_{19}$ be the section given by the inverse of R_{19} in the Mordell–Weil group. Then we clearly have $\langle S_{19}, S_{19}\rangle = \langle R_{19}, R_{19}\rangle$ and $\langle S_{19}, R_{19}\rangle = -\langle R_{19}, R_{19}\rangle$. From these equalities, using [24, Theorem 8.6] we obtain $O_{19}\cdot S_{19} = 0$ and $R_{19}\cdot S_{19} = 2$. These intersection numbers explicitly determine S_{19} in the coordinate system of Theorem 2.2:

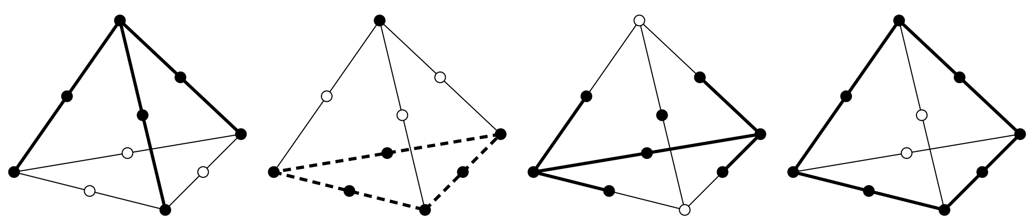
$$S_{19} = (19, 17, -27, -42, -54, -81, -66, -51, -34, -17, -27, -42, -54, -81, -66, -51, -34, -17, 4)\,.$$

We are then able to compute the translation by R_{19}, denoted by t, and involution ι as in Hulek and Schütt's construction [8, §3]. Explicit computations show that the invariant lattice of $\varepsilon := t\circ\iota$ is isomorphic to $\mathbf{M}$, so that ε is the pullback of an Enriques involution. We can verify directly that ε acts on the L-lines and M-lines as described in the statement of the theorem. By Theorem 2.5 and the proof of Theorem 3.7, we know that the coinvariant lattice $(S_X)_\varepsilon$ is isomorphic to either $\mathbf{A}_2(2)\oplus\mathbf{E}_7(2)$ or $\mathbf{E}_8(2)\oplus[-12]$. An explicit computation shows that $(S_X)_\varepsilon$ contains 132 vectors of square -4, so it is necessarily isomorphic to $\mathbf{A}_2(2)\oplus\mathbf{E}_7(2)$, i.e., ε is not of Barth–Peters type. We refer to the ancillary file `calc.Apery_Fermi.sage` for the actual computations in Sage. □

Remark 3.9. Thanks to the description of the Enriques involution in Theorem 3.8, it is immediate to see that the images of the L-lines in Y form a tetrahedron, while the images of the M-lines form a complete graph with six vertices in which three pairs of curves intersect doubly. The tetrahedron and the complete graph are connected in the following way, where double intersections are marked with a double edge.



The following pencils (we omit here the images of the M-lines) are special pencils on Y whose pullbacks correspond to the elliptic fibrations on X with frames W_{19}, W_{20}, W_{24} and W_{26}, respectively.



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