Received: 25 June 2021 | Accepted: 23 September 2021

DOI: 10.1002/pamm.202100246

Finite element analysis of a 2D cantilever on a noisy intermediate-scale quantum computer

André Mielke[1,*] and **Tim Ricken**[1]

[1] Institute of Mechanics, Structural Analysis and Dynamics, Faculty of Aerospace Engineering and Geodesy, University of Stuttgart, 70569 Stuttgart, Germany

Quantum computing is a promising new computing paradigm that may yield both significant speedup over classical algorithms as well as new ways to think about problems and finding novel solution algorithms. While large-scale error-corrected quantum computers are still under development, a selection of noisy intermediate-scale quantum processing units is readily made available by some companies via cloud access. Despite the lack of error correction, these units can be used to test established quantum algorithms on custom problem setups. Here, we solve the finite element problem of a two-dimensional cantilever, completely fixed on one side and loaded on the opposite side, on a 15-qubit QPU from IBM.

1 Problem setup

The size of current Quantum Processing Units (QPUs) constrains the complexity and discretization of problems that can run efficiently on available hardware. As an example for a 2-dimensional mechanical problem, we discretize a 2-dimensional cantilever that is completely fixed on the left and loaded vertically on the right (cf. 1). The discretization is chosen so that the number of degrees of freedom in the model is a power of two. Here, in total 6 nodes are used for two linear finite elements. Since the leftmost two nodes are fixed and the problem is 2-dimensional, each of the four free nodes displays two degrees of freedom, resulting in a total number of 8 degrees of freedom for the whole system.
The material parameters used describe aluminium, with a Young's modulus of $E = 70\,\mathrm{GPa}$ and Poisson's ratio of $\nu = 0.3$. The length of the cantilever is $l = 2\,\mathrm{m}$, its height is $h = 1\,\mathrm{m}$. The load applied on the top right node is $f = -1\,\mathrm{kN}$.

2 Linear equation system and solution algorithm

The linear equation system describing the discretized model is computed by assembling the two local stiffness matrices of the finite elements to get the global stiffness matrix and the global force vector respectively from the local force vectors, then ignoring the rows and columns corresponding to fixed degrees of freedom. The system to solve is $\mathbf{Ku} = \mathbf{f}$, with dimension 8. The analytical solution is computed straightforwardly (cf. Table 1). Naturally, this system will show locking. It is not the goal of this work to develop a perfect FE system of the problem. Instead, we are comparing the analytical solution to the solution found by a QPU. Consequently, we can ignore all the formalities of proper FE setups and concentrate on the solution algorithm.

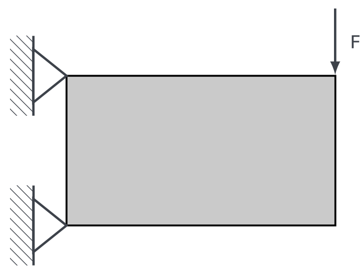

Fig. 1: Problem sketch. The cantilever is fixed on the left and loaded on the top right.

3 Solving the FE problem on a QPU

Instead of working with discrete bits that can represent either 0 or 1, Quantum Computers (QCs) work with *qubits* as elementary units, which can represent a complex superposition of 0 and 1 as $|\Psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle\,, \alpha, \beta \in \mathbb{C}$. The state space of QCs grows exponentially with each qubit added to the system, since the combined basis is formed by the tensor product of the individual qubit bases.

The *Harrow-Hassidim-Lloyd (HHL)* algorithm [1] can be used to solve linear equation systems on QCs. To formulate the problem in quantum terms, the system is expressed in terms of *ket* vectors $\mathbf{K}\,|u\rangle = |f\rangle$ with

$$\mathbf{K} = \mathbf{K}^\dagger, \det(\mathbf{A}) = 1\ , \qquad |b\rangle = \frac{\sum_i b_i\,|i\rangle}{||\sum_i b_i\,|i\rangle\,||} = \sum_i \beta_i\,|i\rangle\ , \qquad |u\rangle = \frac{\sum_i u_i\,|i\rangle}{||\sum_i u_i\,|i\rangle\,||} = \sum_i \alpha_i\,|i\rangle\ ,$$

since quantum states are normalized. The first demand for the matrix to be Hermitian is automatically satisfied, since the stiffness matrix is real and symmetric. Setting the determinant to 1 can be done with an appropriate transformation. Since the HHL algorithm scales strongly with the condition number of the system, a good preconditioner has to be applied before sending the problem to a QPU. We used an incomplete LU decomposition to lower the condition number of $\mathbf{K}$.

* Corresponding author: e-mail andre.mielke@isd.uni-stuttgart.de, phone +49-711-685-69544, fax +49-711-685-63706

Figure 2 shows a sketch of the quantum circuit with the three basic elements comprising the algorithm. These are Quantum Phase Estimation (QPE) [2], a multi-controlled y-rotation R_y, and a conditional measurement performed on the ancilla qubit $|\mathrm{A}\rangle_0$. The algorithm starts by initializing the work register $|\mathrm{W}\rangle$ as the right-hand side vector $|b\rangle$ of the linear equation system. The QPE approximates the eigenvalues of $\mathbf{K}$ by applying a succession of controlled $\mathbf{K}^{2^k}$ rotations, so that the work register contains the eigenvalue approximation. Controlled on this eigenvalue representation, the R_y gate rotates the ancilla qubit into the superposition $|\mathrm{A}\rangle = \sqrt{1 - \frac{c^2}{\tilde{\lambda}_i^2}}\, |0\rangle_\mathrm{A} + \frac{c}{\tilde{\lambda}_i}\, |1\rangle_\mathrm{A}$. The last step is an inverse QPE (uncomputation) that undoes the QPE from the beginning, such that the whole QPU is now in the state

$$|\Psi\rangle = \sum_i \beta_i \left[\sqrt{1 - \frac{c^2}{\tilde{\lambda}_i^2}}\, |0\rangle + \frac{c}{\tilde{\lambda}_i}\, |1\rangle_\mathrm{A} \right] \otimes |v_i\rangle_\mathrm{IO} \ ,$$

where the work register was omitted for brevity since it is not needed anymore, and $|b\rangle = \sum_i \beta_i |v_i\rangle$ is described as a superposition of $\mathbf{K}$'s eigenvectors $|v_i\rangle$. Since a measurement of a qubit projects the system into exactly the measured state, measuring the ancilla qubit brings the system either into the state where $|\mathrm{A}\rangle = |0\rangle_\mathrm{A}$, or into the state where $|\mathrm{A}\rangle = |1\rangle_\mathrm{A}$. The latter option results in the global state

$$|\Psi\rangle = \sum_i \frac{\beta_i}{\tilde{\lambda}_i} |v_i\rangle_\mathrm{IO} = |\tilde{u}\rangle \approx |u\rangle \ ,$$

which provides an approximation $|\tilde{u}\rangle$ of the desired displacement vector $|u\rangle$. Measuring the input-output register puts the bits representing the solution vector into the classical result register.

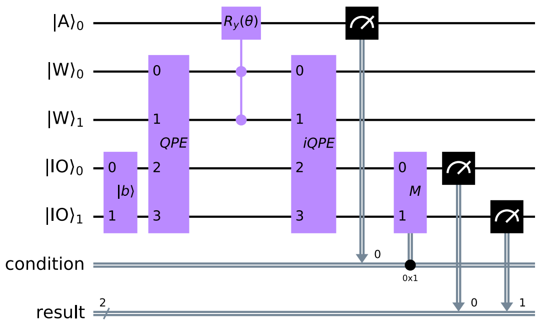

Fig. 2: Sketch of the HHL quantum circuit, produced with Qiskit [3]. An ancilla $|\mathrm{A}\rangle$, a work $|\mathrm{W}\rangle$, and an input-output register $|\mathrm{IO}\rangle$ are necessary to transform the right-hand side $|b\rangle$ of the equation system into an approximation of the solution vector $|u\rangle$.

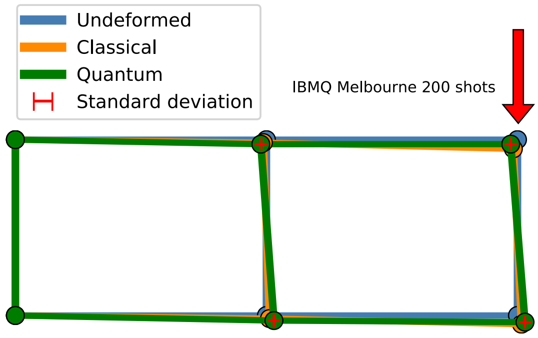

Fig. 3: Undeformed position (blue), analytical solution (orange) and quantum-computed solution (green) for the problem. The red bars show the standard deviation of each nodal position into each direction. The red arrow indicates the load.

4 Results

The code for this work was produced using the qiskit [3] package, which also offers a direct interface to work with the QPUs made available by IBM. The largest available unit is the *ibmq_16_melbourne* device [4], offering 15 qubits connected as a ladder graph. The computation was done 200 times on this QPU. Table 1 compares the analytical solution as a reference to the QPU solution, along with the standard deviation on the 200 shots and the relative error.

	$\mathrm{N3}_x$	$\mathrm{N3}_y$	$\mathrm{N4}_x$	$\mathrm{N4}_y$	$\mathrm{N5}_x$	$\mathrm{N5}_y$	$\mathrm{N6}_x$	$\mathrm{N6}_y$
Reference [mm]	12	-18	-12	-17	16	-49	-16	-52
Mean Quantum [mm]	30	-29	-22	-26	30	-38	-27	-26
Std. Dev. [mm]	3	2	4	3	2	2	2	2
Rel. Err. [%]	-146	-64	-84	-51	-88	24	-63	51

Table 1: Analytial and quantum-computed displacements, along with standard deviation and relative error.

The standard deviation among the 200 shots is in the range of $\approx 10-20\%$, while the relative error with respect to the reference solution is significantly higher. Figure 3 shows a plot of the node positions and the standard deviation for each degree of freedom. Unlike in the case for a 1-dimensional cantilever [5], there is a clear bias in the solution vector. The correct positions are not within the standard deviation of the results. It is unclear where exactly this bias comes from and further work is required to identify the issue.

Acknowledgements Open access funding enabled and organized by Projekt DEAL.

References

[1] A. W. Harrow, A. Hassidim, and S. Lloyd, Physical review letters **113** (15), 150502, (2009)
[2] A. Y. Kitaev, Russian Mathematical Surveys, **52** (6) p. 1191–1249, (1997)
[3] H. Abraham, A. Offei, I. Y. Akhalwaya, et al., Qiskit: An Open-source Framework for Quantum Computing, (2019)
[4] IBM Quantum. https://quantum-computing.ibm.com/, (2021)
[5] A. Mielke and T. Ricken, PAMM **20**(1), (2020)

