Qualitative properties of solutions to partial differential equations arising in fluid dynamics

vorgelegt von

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Abstract

In this thesis, we consider compressible fluid models that describe both viscous and inviscid fluids. For inviscid fluids, we consider the barotropic Euler system and the complete Euler system, where the term complete indicates that we incorporate the *laws of thermodynamics* including the balance of total energy in the system. In the context of viscous fluids we consider the Navier–Stokes system, where the viscous stress tensor is a linear function of the velocity gradient.

We are interested in the concept of generalized solutions as there are several limitations in the classical existence theory. Various notions of generalized solutions, namely *weak solutions*, *measure-valued solutions*, *dissipative solutions* have been presented in this thesis. To make these generalized solutions compatible with the classical notion we invoke a, *generalized weak-strong uniqueness principle.* The principle asserts that the generalized and strong solutions emanating from same initial data must coincide as long as the strong solution exists.

We study the weak-strong uniqueness problem for the compressible Navier–Stokes system with a general barotropic pressure law. Our results include the case of a hard-sphere pressure law of Van der Waals type with a non-monotone perturbation and a Lipschitz perturbation of a monotone pressure law. Moreover, we consider a renormalized dissipative measure-valued (rDMV) solution of the same system with compactly supported perturbation of monotone pressure law and obtain the generalized weak-strong uniqueness property of this rDMV solution. The relative energy is used as the main tool to prove these results. We emphasize the choice of non-monotone pressure laws, since most previous results consider a monotone pressure law. The viscous term plays an important role in obtaining a weak- strong uniqueness and a generalized weak-strong uniqueness result.

Next, we study the low Mach number limit for a scaled barotropic Euler system and identify its limit as an incompressible Euler system. We also consider the singular limits for a scaled barotropic Euler system modeling a rotating, compressible, and inviscid fluid where the characteristic numbers (the Mach number, the Rossby number and the Froude number) have different scaling with respect to a small parameter ϵ. The fluid is confined to an infinite slab and the limit behavior ($\epsilon \to 0$) is identified as incompressible planar flow, depending on the relation between the characteristic numbers. For *well–prepared* initial data, convergence is shown in the time interval where the strong solution of the target system exists, while for the primitive system a class of generalized dissipative solutions is considered. Since the existence of a

weak solution for an inviscid compressible fluid is not available for a general initial data, it is convenient to consider a generalized solution. The choice of a dissipative solution ensures a certain stability of the target system. In the literature, most of the results are for viscous fluids by considering both strong solutions and weak solutions, although there are several limitations for the strong solutions. Again, we use the relative energy to obtain the desired results.

Finally, we prove that if a weak limit of a consistent approximation scheme of the complete Euler system in the full space $\mathbb{R}^d$, $d = 2, 3$, is a weak solution of the system, then the approximate solutions eventually converge strongly, or at least almost everywhere, under minimal assumptions on the initial data of the approximate solutions. The class of consistent approximate solutions is quite general and includes the vanishing viscosity and heat conductivity limit. In particular, the approximate solutions may not satisfy the minimal principle for entropy. Since both the barotropic Euler system and the complete Euler system are ill-posed in the class of weak solutions, our results ensure that the limit of consistent approximations can be a good selection criterion for a physically relevant solution.

Zusammenfassung

In der vorliegenden Arbeit betrachten wir Modelle für sowohl viskose als auch nichtviskose kompressible Fluide. Für den Fall der nichtviskosen Fluide betrachten wir das System der barotropen Euler-Gleichungen und der vollständigen Euler-Gleichungen, wobei Letzteres bedeutet, dass die *Hauptsätze der Thermodynamik* (inklusive der Erhaltung der Gesamtenergie) im System beinhaltet sind. Im Kontext der viskosen Fluide betrachten wir das System der Navier-Stokes-Gleichungen, in dem der viskose Spannungstensor linear vom Geschwindigkeitsgradienten abhängt.

Da die klassische Existenztheorie gewisse Limitierungen hat, betrachten wir verallgemeinerte Lösungskonzepte, nämlich *schwache*, *maßwertige* und *dissipative Lösungen*. Um sicherzustellen, dass diese verallgemeinerten Lösungen kompatibel mit klassischen Lösungen sind, fordern wir das Prinzip der sogenannten *verallgemeinerten schwach-starken Einzigkeit*. Dieses Prinzip stellt sicher, dass die verallgemeinerte und die starke Lösung zum selben Anfangswert übereinstimmen, sofern Letzere existiert.

Wir untersuchen die schwach-starke Einzigkeit für das System der kompressiblen Navier-Stokes-Gleichungen mit einer allgemeinen barotropen Druck-Dichte-Relation. Unsere Resultate beinhalten den Fall eines Harte-Kugeln-Modells vom Van-der-Waals Typ für die Druck-Dichte-Relation mit einer nichtmonotonen und einer Lipschitz-stetigen Störung einer monotonen Druck-Dichte-Relation. Außerdem betrachten wir eine renormalisierte dissipative maßwertige (rDMV) Lösung desselben Systems mit einer Störung der monotonen Druck-Dichte-Relation mit kompaktem Träger und erhalten die verallgemeinerte schwach-starke Einzigkeit dieser rDMV Lösung. Das Hauptwerkzeug für den Beweis dieser Resultate ist dabei die relative Energie. Wir weisen nochmal darauf hin, dass in dieser Arbeit nichtmonotone Druck-Dichte-Relationen betrachtet werden, da die meisten bisherigen Resultate nur monotone Druck-Dichte-Relationen betrachten. Der viskose Term spielt dabei eine wichtige Rolle, um die schwach-starke und die verallgemeinerte schwach-starke Einzigkeit zu erhalten.

Weiterhin untersuchen wir den Grenzwert für verschwindende Mach-Zahlen eines Systems von skalierten barotropen Euler-Gleichungen und identifizieren den Grenzwert als ein System von inkompressiblen Euler-Gleichungen. Wir betrachten außerdem singuläre Grenzwerte für ein System von skalierten barotropen Euler-Gleichungen, das ein rotierendes, kompressibles und nichtviskoses Fluid modelliert, wobei die Kennzahlen (die Mach-Zahl, die Rossby-Zahl und die Froude-Zahl) unterschiedlich mit einem kleinen Parameter ϵ skalieren. Das Fluid ist begrenzt auf eine unendliche

Platte und das Grenzwertverhalten für $\epsilon \to 0$ wird, abhängig von der Beziehung der Kennzahlen, als inkompressible ebene Strömung identifiziert. Für *wohlgestellte* Anfangsdaten wird die Konvergenz auf dem Existenzintervall der starken Lösung des Zielsystems gezeigt, wobei eine Klasse von verallgemeinerten dissipativen Lösungen für das Ausgangssystem betrachtet wird, da Existenz von schwachen Lösungen für ein nichtviskoses kompressibles Fluid für allgemeine Anfangsdaten nicht bekannt ist. Die Wahl der dissipativen Lösung sichert gewisse Stabilitätseigenschaften des Zielsystems. In der Literatur werden in den meisten Resultaten für viskose Fluide sowohl starke als auch schwache Lösungen betrachtet, obwohl es für starke Lösungen einige Einschränkungen gibt. Wir verwenden wieder die relative Energie als Hauptwerkzeug, um die gewünschten Resultate zu beweisen.

Schließlich zeigen wir, dass, falls der schwache Grenzwert einer konsistenten Approximation des Systems der vollständigen Euler-Gleichungen im ganzen Raum $\mathbb{R}^d$, $d = 2, 3$, eine schwache Lösung dieses Systems ist, die Näherungslösungen sogar stark oder zumindest fast überall konvergieren, wobei nur minimale Annahmen an die Anfangsdaten der Näherungslösungen gestellt werden müssen. Die Klasse der konsistenten Näherungslösungen ist ziemlich allgemein und beinhaltet den Grenzwert für verschwindende Viskosität und Wärmeleitung. Insbesondere kann es sein, dass die Näherungslösungen nicht das Prinzip der minimalen Entropieproduktion erfüllen. Da sowohl das System der barotropen Euler-Gleichungen als auch das System der vollständigen Euler-Gleichungen nicht wohlgestellt in der Klasse der schwachen Lösungen ist, sorgen unsere Ergebnisse dafür, dass der Grenzwert der Näherungslösungen ein gutes Auswahlkriterium für eine physikalisch relevante Lösung sein kann.

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Contents

Notation

The symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ stand for the set of all natural numbers, integers, rational numbers and real numbers, respectively. In this thesis the space dimension is denoted by $d \in \mathbb{N}$ (typically $d = 2$ or $d = 3$ in the case of fluid mechanics applications). The Euclidean norm in

$$\mathbb{R}^d = \left\{ x = (x_1, ..., x_d) = (x_i)_{i=1}^d \,|\, x_i \in \mathbb{R},\ \forall i = 1, \cdots, d,\ d \in \mathbb{N} \right\}$$

is denoted by $x \mapsto |x|$ and the corresponding inner product by $(x, y) \mapsto x \cdot y$.

The space $\mathbb{R}^{d\times d}$ denotes the set of real matrices of order $d \times d$. $\mathbb{I}$ stands for the identity matrix. $\mathbb{R}^{d\times d}_{\text{sym}}$ denotes the set of symmetric matrices, i.e. $\mathbb{A} = \mathbb{A}^T$, where $\mathbb{A} = (a_{ij})_{i,j=1}^d$ and $\mathbb{A}^T = (a_{ji})_{i,j=1}^d$. For $\mathbb{A} \left(= (a_{ij})_{i,j=1}^d \right) \in \mathbb{R}^{d\times d}$ we consider the *symmetric part* and the *traceless part* of $\mathbb{A}$ as

$$\mathbb{D}(\mathbb{A}) = \frac{\mathbb{A} + \mathbb{A}^T}{2}, \quad \text{and } \mathbb{D}_0(\mathbb{A}) = \frac{\mathbb{A} + \mathbb{A}^T}{2} - \frac{1}{d}\mathrm{Tr}(\mathbb{A})\,\mathbb{I},$$

respectively, where $\mathrm{Tr}(\mathbb{A}) = \sum_{i=1}^{d} a_{ii}$.

A space periodic domain $\Omega \subset \mathbb{R}^d$, is usually identified with the flat torus $\mathbb{T}^d$, and is given by

$$\mathbb{T}^d = \left([-1, 1]|_{\{-1,1\}} \right)^d.$$

For the sake of simplicity, we consider the length of the period as 2.

For any multi-index $\alpha = (\alpha_1, ..., \alpha_d) \in \mathbb{N}^d$ we denote its length by $|\alpha| = \alpha_1 + \cdots \alpha_d$. For any function f we define $\partial^\alpha f = \partial^{\alpha_1}_{x_1} \cdots \partial^{\alpha_d}_{x_d} f$ as soon as this partial derivative (in a classical or in a weak sense) exists.

For any open set $\Omega \subset \mathbb{R}^d$, $\overline{\Omega}$ stands for the closure of Ω in $\mathbb{R}^d$ and Γ or $\partial\Omega$ denotes the boundary of Ω. Let $m \in \mathbb{N}$, $C(\overline{\Omega}; \mathbb{R}^m)$ is the space of continuous functions from $\overline{\Omega}$ to $\mathbb{R}^m$. For a bounded set Ω, the space $C(\overline{\Omega}; \mathbb{R}^m)$ is a Banach space with norm

$$\|f\|_{C(\overline{\Omega};\mathbb{R}^m)} = \sup_{x\in\overline{\Omega}} |f(x)|.$$

We denote $C(\Omega; \mathbb{R}^m)$ as the space of continuous functions on Ω. For $m = 1$, we say that they are scalar-valued functions and the space of continuous functions is denoted by $C(\Omega)$ and $C(\overline{\Omega})$, respectively. When $m > 1$, we call them vector-valued functions.

We refer to the space $C_b(\Omega)$ as the space of bounded continuous scalar-valued functions on Ω. This is a Banach space with the supremum norm. If Ω is a bounded subset of $\mathbb{R}^d$ then $C(\overline{\Omega})$ coincides with $C_b(\overline{\Omega})$. They differ if we consider Ω as an unbounded domain of $\mathbb{R}^d$. The space $C_c(\Omega)$ denotes the space of continuous functions on Ω with compact support, where support of a function f is defined as

$$\operatorname{supp}(f) = \text{closure of } \{x \in \Omega \mid f(x) \neq 0\}.$$

The space $C_0(\mathbb{R}^d)$ is the closure under the supremum norm of compactly supported, continuous functions on $\mathbb{R}^d$, i.e., the set of continuous functions on $\mathbb{R}^d$ which vanish at infinity. Similarly, for an unbounded domain Ω we can define the space $C_0(\Omega)$.

$C^k(\Omega;\mathbb{R}^m)$ is the space functions on Ω such that for $f \in C^k(\Omega;\mathbb{R}^m)$ implies $\partial^\alpha f$ exists for $|\alpha| \leq k$. $C^k(\overline{\Omega};\mathbb{R}^m)$ is the space of functions in $C^k(\Omega;\mathbb{R}^m)$ which together with all derivatives possesses continuous extensions to $\overline{\Omega}$.

The symbol $C^{0,\alpha}(\Omega;\mathbb{R}^m)$, with $0 < \alpha \leq 1$, denotes the space of $\alpha-$Hölder continuous functions with the seminorm

$$\|f\|_{C^{0,\alpha}(\Omega;\mathbb{R}^m)} = \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|^\alpha}, \text{ for } f \in C^{0,\alpha}(\Omega;\mathbb{R}^m).$$

In the case $\alpha = 1$, this is the set of Lipschitz continuous functions with seminorm

$$\operatorname{Lip}(f) = \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|}, \text{ for } f \in C^{0,1}(\Omega;\mathbb{R}^m).$$

Similarly, the set $C^{k,\alpha}(\Omega;\mathbb{R}^m)$, $k \in \mathbb{N}$ with $0 < \alpha \leq 1$, of functions in $C^k(\Omega;\mathbb{R}^d)$ and kth order partial derivatives are in $C^{0,\alpha}(\Omega;\mathbb{R}^m)$.

We denote

$$C^\infty(\Omega;\mathbb{R}^m) = \quad \cap_{k=0}^\infty C^k(\Omega;\mathbb{R}^m)$$

and

$$C^\infty(\overline{\Omega};\mathbb{R}^m) = \quad \cap_{k=0}^\infty C^k(\overline{\Omega};\mathbb{R}^m).$$

$\mathcal{D}(\Omega;\mathbb{R}^m)$ is the subspace of $C^\infty(\Omega;\mathbb{R}^m)$ with compact support in Ω. Instead of $\mathcal{D}(\Omega;\mathbb{R}^m)$ we sometimes use the notation $C_c^\infty(\Omega;\mathbb{R}^m)$. For $m = 1$, we denote $C^k(\Omega)$, $C^k(\overline{\Omega})$, $C^\infty(\Omega)$, $C^\infty(\overline{\Omega})$ and $\mathcal{D}(\Omega)$ respectively. Instead of vector-valued functions one can also consider matrix-valued functions.

$\mathcal{D}(\Omega)$ is a topological vector space, the topology is defined as the inductive limit topology of $C_c^k(\Omega)$. $\mathcal{D}'(\Omega)$ is the collection of all continuous linear maps T, $T : \mathcal{D}(\Omega) \to \mathbb{R}$. This is called the space of *distributions*.

Let us introduce some vector and matrix operations. For two vectors $\mathbf{u} = (u_i)_{i=1}^d$ and $\mathbf{v} = (v_i)_{i=1}^d$ scalar product $(\mathbf{u}\cdot\mathbf{v})$, tensor product$(\mathbf{u}\otimes\mathbf{v})$ in $\mathbb{R}^d$ and cross product$(\mathbf{u}\times\mathbf{v})$ in $\mathbb{R}^3$ are defined as

$$\mathbf{u}\cdot\mathbf{v} = \sum_{i=1}^d u_i v_i, \ \mathbf{u}\otimes\mathbf{v} = (u_i v_j)_{i,j=1}^d,$$

$$\mathbf{u}\times\mathbf{v} = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1).$$

For $d = 2$, we have $\mathbf{u} \times \mathbf{v} = (0, 0, u_1 v_2 - u_2 v_1)$.
Scalar product of two matrices $\mathbb{A} \left(= (a_{ij})_{i,j=1}^d\right)$ and $\mathbb{B} \left(= (b_{ij})_{i,j=1}^d\right)$ is given by

$$\mathbb{A} \colon \mathbb{B} = \sum_{i,j=1}^{d} a_{ij} b_{ij}.$$

We say that a matrix is positive semidefinite or nonnegative definite ($\mathbb{A} \geq 0$) if for all $\xi \in \mathbb{R}^d$, $\mathbb{A} \colon (\xi \otimes \xi) \geq 0$ holds. It is positive definite if for all $\xi (\neq 0) \in \mathbb{R}^d$, $\mathbb{A} \colon (\xi \otimes \xi) > 0$ holds .

Here we introduce some standard *differential operators* that we use throughout the thesis.

- The *gradient* of a scalar-valued map $f : \Omega(\subset \mathbb{R}^d) \to \mathbb{R}$ and of a map $\mathbf{u} \left(= (u_i)_{i=1}^d\right) : \Omega(\subset \mathbb{R}^d) \to \mathbb{R}^d$ is defined as

$$\nabla_x f = (\partial_{x_i} f)_{i=1}^d \,, \; \nabla_x \mathbf{u} = \left(\partial_{x_j} u_i\right)_{i,j=1}^d .$$

- The *divergence* of a vector field $\mathbf{u}$ on Ω and a matrix field $\mathbb{A}$ on Ω is defined as

$$\mathrm{div}_x \mathbf{u} = \sum_{i=1}^{d} \partial_{x_i} u_i \text{ and } \mathrm{div}_x \mathbb{A} = ((\mathrm{div}_x \mathbb{A})_i)_{i=1}^d \text{ with } (\mathrm{div}_x \mathbb{A})_i = \sum_{j=1}^{d} \partial_{x_j} a_{ij}.$$

- The *curl* of a vector field $\mathbf{v}$ defined as $\mathbf{Curl}(\mathbf{v}) = \nabla_x \mathbf{v} - \nabla_x^T \mathbf{v}$.
- The *Laplacian* is defined as $\Delta_x \equiv \mathrm{div}_x \nabla_x$.
- For two vectors $\mathbf{u}$ and $\mathbf{v}$ we define a operator as $(\mathbf{v} \cdot \nabla_x)\mathbf{u} = \left(\sum\limits_{j=1}^{d} v_j \partial_{x_j} u_i \right)_{i=1}^{d}$.
  This operator appears in the convective term in fluid models written in the Eulerian setting.

A function $\mathrm{e} : \mathbb{R}^d \to \mathbb{R}$ is convex if

$$\mathrm{e}(\lambda \mathbf{x} + (1-\lambda)\mathbf{y}) \leq \lambda \mathrm{e}(\mathbf{x}) + (1-\lambda)\mathrm{e}(\mathbf{y}),$$

for all $\lambda \in [0,1]$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$. It is also possible to define a convex function ranging in extended real line $\mathbb{R} \cup \{\infty\}$. We say that a vector $\mathbf{z} \in \mathbb{R}^d$ is a *subgradient* of e at $\mathbf{x}$ if it satisfies

$$\mathrm{e}(\mathbf{y}) \geq \mathrm{e}(\mathbf{x}) + \mathbf{z} \cdot (\mathbf{y} - \mathbf{x}).$$

We denote *subdifferential* of e at $\mathbf{x}$ as $\partial \mathrm{e}(\mathbf{x})$ and it is defined as

$$\partial \mathrm{e}(\mathbf{x}) = \{ \mathbf{z} \mid \mathbf{z} \text{ is subgradient of e at } \mathbf{x} \}.$$

Throughout the thesis we use the symbol C for a generic constant, in general it is positive. Furthermore, the symbol $C(\lambda)$ denotes that the constant C is dependent on parameter λ.

In a few cases, we use the notation $\mathbf{u}_\epsilon \to \mathbf{u}$ as $\epsilon \to 0$, that is, we consider $\epsilon \approx \frac{1}{n}$ for $n \in \mathbb{N}$ and the sequence $\mathbf{v}_n (= \mathbf{u}_{\frac{1}{n}}) \to \mathbf{u}$ as $n \to \infty$, in a suitable sense of convergence.

For $\Omega \subset \mathbb{R}^d$, we denote $\mathbf{1}_\Omega$ as

$$\mathbf{1}_\Omega = \begin{cases} 1, & x \in \Omega, \\ 0, & \text{otherwise.} \end{cases}$$

It is called the *characteristics function* of Ω.

Introduction

The aim of this thesis is to study various compressible fluid models from fluid mechanics. These models are given by systems of partial differential equations. The incompressible inviscid fluids were first described by L. Euler in 1755 and the incompressible viscous fluid flows were described by Navier in 1822-1827, followed by Poisson(1831) and Stokes(1845), see [22]. In the last two centuries, a significant development has been observed for these fluid models.

In the context of simplified barotropic fluid models, *the Euler system* describes fluid flow in terms of *density*(ϱ), *velocity*($\mathbf{u}$) and *pressure* (p). These are functions of time t and space x. The relation between pressure and density is given by an *equation of state*. The initial time is fixed at $t = 0$. For $T > 0$ and $\Omega \subset \mathbb{R}^d$ with $d = 2, 3$ the evolution of the variables $(\varrho, \mathbf{u})$ in the time-space cylinder $(0, T) \times \Omega$ is described as:

$$\partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) = 0, \tag{0.0.1}$$

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = 0. \tag{0.0.2}$$

The equation (0.0.1) is the *conservation of mass* and (0.0.2) describes the *conservation of momentum*. The *momentum* is denoted by $\mathbf{m} = \varrho \mathbf{u}$. Since both equations are first order in time, an *initial condition* is given by

$$\varrho(0, x) = \varrho_0(x), \; (\varrho \mathbf{u})(0, x) = (\varrho \mathbf{u})_0(x) \text{ for } x \in \Omega.$$

In general the initial density ϱ_0 is non-negative. Let us make the choice of the space Ω a little more precise. It can be considered as the full domain $\mathbb{R}^d$, a bounded domain, an exterior domain, an infinite slab $\mathbb{R}^{d-1} \times (0, 1)$ or a periodic domain $\mathbb{T}^d$. Depending on the domain, an appropriate *boundary condition* or *far field condition* or both is necessary. If the pressure depends only on the density, it is called *barotropic pressure*.

In a more physically relevant scenario, the pressure depends not only on the density but also on the *temperature* (ϑ). The *first law of thermodynamics* suggests that we consider another equation, namely the *total energy balance*. Here density, momentum/velocity and temperature are considered as independent variables and the system is given by

$$\begin{aligned}
&\partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) = 0, \\
&\partial_t(\varrho \mathbf{u}) + \operatorname{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = 0, \\
&\partial_t \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \right) + \operatorname{div}_x \left(\left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) + p(\varrho, \vartheta) \right) \mathbf{u} \right) = 0,
\end{aligned}$$

where e denotes the *specific internal energy.* Moreover, $p(\varrho, \vartheta)$ and $e(\varrho, \vartheta)$ are interrelated. The *second law of thermodynamics* introduces the *entropy*(s). Then the relation between the specific internal energy, the pressure and the entropy is given by the *Gibbs relation*, i.e.,

$$Ds(\varrho, \vartheta) = De(\varrho, \vartheta) + p(\varrho, \vartheta) D \left(\frac{1}{\varrho} \right),$$

where D stands for the total derivative with respect to ϱ, ϑ.

This helps us to replace the energy balance with the entropy balance. A well-known equation of state is the *Boyle–Mariotte equation of state.* The relation between the internal energy, the temperature and the pressure is given by $e = c_v \vartheta$ and $p = \varrho \vartheta$. Considering $(\varrho, \mathbf{m}, s)$ as state variables, we describe *the complete Euler system* as

$$\begin{aligned}
&\partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) = 0, \\
&\partial_t(\varrho \mathbf{u}) + \operatorname{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, s) = 0, \\
&\partial_t(\varrho s) + \operatorname{div}_x(s \mathbf{m}) = 0.
\end{aligned}$$

In the context of viscous barotropic fluids, the *Navier–Stokes system* in the time space cylinder $(0, T) \times \Omega$ reads

$$\partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) = 0, \tag{0.0.3}$$

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \operatorname{div}_x(\mathbb{S}(\nabla_x \mathbf{u})). \tag{0.0.4}$$

Here $\mathbb{S}$ is called the *viscous stress tensor.* Suitable initial conditions, boundary conditions and far field conditions must also be included for the Navier–Stokes system. Similarly, one can include other laws of thermodynamics, *Fourier's law of heat conduction*, and the Gibbs relation to consider a more general system of equations, namely the *Navier-Stokes-Fourier system.*

Very often the term Euler system or Navier–Stokes system is used to describe an incompressible fluid. Since this thesis is mainly concerned with compressible fluids, we use the term the Euler system or the Navier-Stokes system to refer to the systems for compressible fluids and explicitly state if we refer to incompressible fluids. We will not discuss much about incompressible flows, an interested reader may consult the well-known monographs [116], [38], [83], [95], [112], [14], to name a few.

0.1 An overview of solvability

In the study of systems describing compressible fluids, the first stumbling block is to provide a suitable notion of solution. The classical approach to solving the corresponding initial (boundary) value problems is to find a solution that satisfies the system in pointwise sense. This is called a *classical* or *strong* solution of the system. In order to satisfy a system in a pointwise sense, some differentiability of the state variables is required. The existence of a global in time solution for a nonlinear system is not always possible. There are explicit examples of singularities (shocks) for the Euler system, while the Navier–Stokes admits global in time solutions at least for small initial data. The strong solutions are unique in the time interval of their existence. The absence of a global existence of the strong solution leads to the consideration of a weak solution. In this case, we replace the equations by a set of integral identities. In Chapter 2 we discuss in detail the weak formulation of various systems.

For general initial data, the existence of a global in time weak solution for the compressible Navier–Stokes system has been proved, although there is a certain restriction on the pressure-density relation. Unfortunately, the uniqueness of the solution in this class of solution(s) is still open. The situation is even more delicate in the context of the Euler system. The problem of existence of global in time physically admissible weak solutions for general initial data is still open. Although for some initial data the existence of a weak solution has been proved, there are examples of infinitely many (wild) solutions. Also, in numerical analysis, it is quite difficult to prove the convergence of 'suitable' numerical schemes of these systems to a weak solution.

A new concept of generalized solutions, namely measure-valued or dissipative solutions, is introduced for these systems. There are two properties which justify the concept of generalized solutions:

- **Compatibility:** A sufficiently smooth generalized solution will be a classical solution.

- **Weak-Strong uniqueness:** Given the same initial data, a weak solution will coincide with the strong or classical solution if the latter exists.

The term 'weak-strong uniqueness' can be somewhat ambiguous; in principle, it refers only to weak solutions. However, we use *weak-strong uniqueness* as a general concept that also refers to more general solutions (measure-valued, dissipative) and not only to the weak (distributional) solutions. From now on, we consider *weak-strong uniqueness* to deal with weak solutions, and *generalized weak-strong uniqueness* when we consider measure-valued or dissipative solutions.

There are many results concerning the mathematical theory of the Euler system, as well as the complete Euler system. It is well known that the initial value problem is well posed *locally* in time in the class of smooth solutions, see for example the monograph of Majda [97], Schochet [110] or the recent monograph of Benzoni–Gavage and Serre [12]. Since our interest lies in weak or dissipative solutions of the system, we relax the entropy balance to the inequality that is a physically relevant admissibility criterion for weak solutions. The adaptation of the method of convex integration in the context of incompressible fluids by De Lellis and Székelyhidi [40] leads to the ill-posedness of several problems in fluid mechanics, also in the class of compressible barotropic fluids, see Chiodaroli and Kreml [36], Chiodaroli, De Lellis and Kreml [33] and Chiodaroli et al.[37]. The result of Chiodaroli, Feireisl and Kreml [35] shows that the initial-boundary value problem for the complete Euler system admits infinitely many weak solutions on a given time interval $(0, T)$ for a large class of initial data.

In [61], Feireisl et al. show that the complete Euler system is ill-posed and these solutions satisfy the entropy inequality. Chiodaroli, Feireisl and Flandoli [34] obtain a similar result for the complete Euler system driven by multiplicative white noise. Most of these results, based on the application of the method of convex integration, are non–constructive and exploit the fact that the constraints imposed by the Euler system on the class of weak solutions admit oscillations.

In the articles of Alibert and Bouchitté [4], Gwiazda, Świerczewska-Gwaizda and Wiedemann [86], Březina and Feireisl [20], Březina [24], Basarić [11], Feireisl and Lukáčová-Medvidová [65], we observe the development of the theory on measure-valued solutions for various models describing compressible inviscid fluids, mainly using Young measures. Recently, Feireisl, Lukáčová-Medvidová and Mizerová [66] and Breit, Feireisl and Hofmanová [16] give a new definition for compressible Euler system, termed as *dissipative solution* without involving Young measures.

Addressing the Navier–Stokes system, the existence of a local strong solution was proved in the following articles for different space dimensions: Shelukhin and Khazikov [90], Matsumura and Nishida [100]. For small initial data, global in time existence is also studied by Valli and Zajaczkowski [118], Matsumura and Nishida [100] etc. The existence of a global in time weak solution has been proved, see Antontsev et al. [5], P.-L. Lions [96], Feireisl [50], Feireisl and Novotný [72]. The problem of uniqueness for weak solutions is still open. Conditional uniqueness is provided by Sun, Wang and Zhang [115].

The measure-valued solution of the Navier–Stokes system was introduced by Feireisl et al. [56] using the Young measure. For the Navier–Stokes–Fourier system, the measure-valued solution was defined by Brezina, Feireisl and Novotný [27]. The notion of dissipative solutions is also available for more general viscous stress tensors, see Abbatiello, Feireisl and Novotný [2].

Another important application of measure-valued solutions is their identification as limits of numerical schemes. Together with the existing generalized weak-strong uniqueness principle in the class of measure-valued solutions, one can show that numerical solutions converge strongly to a strong solution of the system as long as the latter exists, see [80], [65], [71]. We use the *relative energy method* to prove *weak-strong uniqueness*, which we will describe in detail in the following chapters.

0.2 Scaled system: Asymptotic analysis

Compressible fluids describe a wide range of possible models in meteorology, geophysics and astrophysics, ranging from sound waves to cyclone waves to models of gaseous stars. Therefore, to gain a deeper understanding of the system, it is important to write it in a dimensionless form. It allows us to compare the relative influence of the different terms that appear in the equations. One can explicitly determine the parameters by scaling the equations, in other words by choosing the system of reference units accordingly. The behavior of the system depends on these parameters, which are called *characteristic numbers*. When these characteristic numbers vanish or become infinite, the study of the system is called *asymptotic analysis* or *singular limit* of the system. Classical textbooks and research monographs are mainly concerned with the way in which the scaling arguments can be used together with other characteristic properties of the data to obtain, usually in a very formal way, a simplified system, see [125]. We refer the reader to the survey by Klein [92] for a thorough discussion of singular limits and the applications of scaling in numerical analysis.

By introducing reference density, velocity, length, time, and other quantities, and by suitably changing the variables, one can describe a set of characteristic numbers which are dimensionless. Including these numbers, the compressible Euler and Navier–Stokes system with source term **f** can be written as

$$\mathrm{Sr}\ \partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{0.2.1}$$

$$\mathrm{Sr}\ \partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\mathrm{Ma}^2} \nabla_x p(\varrho) = \frac{1}{\mathrm{Fr}^2} \varrho \mathbf{f}, \tag{0.2.2}$$

and

$$\mathrm{Sr}\ \partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{0.2.3}$$

$$\mathrm{Sr}\ \partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\mathrm{Ma}^2} \nabla_x p(\varrho) = \frac{1}{\mathrm{Re}} \mathrm{div}_x(\mathbb{S}(\nabla_x \mathbf{u})) + \frac{1}{\mathrm{Fr}^2} \varrho \mathbf{f}, \tag{0.2.4}$$

with the Strouhal number(Sr), Mach number(Ma), Froude number(Fr) and Reynolds number(Re). These characteristic numbers have the following meaning:

- A low Strouhal number corresponds to the longtime behavior of a system. In our application we set it as the unity.
- A low Mach number limit is characteristic for the nearly incompressible regime, the density of the fluid becomes constant and the fluid behaves as an incompressible one.
- A high Reynolds number limit corresponds to a small viscous effect that eventually leads to some turbulent phenomena in the fluid.
- In the consideration of **f** as a gravitational force, the Froude number measures the importance of the stratification of the fluid.

Further if we assume *rotating fluids*, the Rossby number(Ro) will be introduced where a large Rossby number indicates a fast rotation of the fluid. In general, we refer to the system with characteristic numbers as *primitive system* and after performing the limit, we refer to the obtained system as the *target system*.

The classical approach to a singular limit problem is to consider a strong or classical solutions of the primitive system. In this approach there are results by Ebin [45], Kleinermann and Majda [91], Schochet [110], and many others. They consider the low Mach number limit of a compressible fluid. For rotating fluids there are results of Babin, Mahalov and Nicolaenko [6, 7] and Chemin et al. [32]. Here, the main and highly non-trivial issue is to ensure that the lifespan of the strong solutions is bounded below away from zero uniformly with respect to the singular parameter.

Another approach is based on the theory of generalized solution. As mentioned earlier, in Navier–Stokes and Euler systems there is a global time generalized solution. If the initial data are chosen correctly, convergence can be shown, provided that the *target system* admits a smooth solution. In the case of second approach, most of the results dealing with weak solutions have been studied for the compressible Navier–Stokes system with additional consideration of a high Reynolds number limit. The singular limit results for the Navier-Stokes-Fourier are available in Feireisl and Novotný [72]. We also refer the reader to the survey by Masmoudi [99] for a comparative study of the two approaches. We deal with the generalized solution approach in Chapter 4.

0.3 Relative energy or entropy

The concept of the *relative energy* is based on the following mathematical observation of a convex function:
Suppose $\mathrm{e}: \mathbb{R}^d \to [0, \infty)$ *is a strictly convex function. We observe that a function*

$$E(\mathbf{u}\,|\,\mathbf{v}) = \mathrm{e}(\mathbf{u}) - \mathrm{e}(\mathbf{v}) - \mathbf{z} \cdot (\mathbf{u} - \mathbf{v})$$

is always non-negative for any $\mathbf{z} \in \partial \mathrm{e}(\mathbf{v})$ *with the distance property i.e.,* $E(\mathbf{u}\,|\,\mathbf{v}) = 0$ *if and only if* $\mathbf{u} = \mathbf{v}$. This introduces the *Brégman divergence*, see [15], [113].

The concept of relative energy or relative entropy was introduced by Dafermos [39] in the context of hyperbolic conservation laws to study the weak-strong uniqueness property. Later it was used for various systems in fluid dynamics.

If we assume $\mathbf{u}$ is a weak or generalized solution and $\mathbf{v}$ is a strong solution, then $E(\mathbf{u}\,|\,\mathbf{v})$ stands for the distance. It is enough to prove $E = 0$ to ensure that the weak and strong solutions are the same, i.e., the weak-strong uniqueness property holds.

If a system has a convex energy or entropy, then one can define an appropriate relative entropy. Since our main focus is on evolution problems, it is important to study the time evolution of $\frac{\mathrm{d}}{\mathrm{d}t} E(\mathbf{u}\,|\,\mathbf{v})$. Formally we have

$$\frac{\mathrm{d}}{\mathrm{d}t} E(\mathbf{u}\,|\,\mathbf{v}) = D_{\mathbf{u}} E(\mathbf{u}\,|\,\mathbf{v}) \cdot \partial_t \mathbf{u}.$$

This shows that we have to use the term $D_{\mathbf{u}} E(\mathbf{u}|\mathbf{v})$ as a test function for the evolution equation of $\mathbf{u}$. Unfortunately, this is not always available for weak solutions, as it requires a high regularity of the weak solution $\mathbf{u}$. This somehow motivates to consider a system with dissipative energy estimate for weak solutions, i.e.,

$$\frac{d\mathrm{e}(\mathbf{u})}{dt} \leq -f(\mathbf{u}),$$

for some real valued non-negative function f.

Our goal is to obtain an estimate of the type

$$\frac{\mathrm{d}}{\mathrm{d}t} E(\mathbf{u} \,|\, \mathbf{v}) \leq C E(\mathbf{u} \,|\, \mathbf{v}),$$

for any t in the time interval $(0, T)$ and some positive constant C. Using a Grönwall type argument, we conclude that the relative energy vanishes provided $E(\mathbf{u}|\mathbf{v})(0) = 0$.

The relative energy also has an application in singular limit problems. We consider a scaled problem with a characteristic number of order $\epsilon > 0$ and for each $\epsilon > 0$ denote a solution of the primitive system by $\mathbf{u}_\epsilon$. Let $\mathbf{v}$ be the solution of the target system. The relative energy $E_\epsilon(\mathbf{u}_\epsilon \,|\, \mathbf{v})$ measures the distance between them. Furthermore, we establish convergence when this error $E_\epsilon(\mathbf{u}_\epsilon \,|\, \mathbf{v})$ goes to zero for $\epsilon \to 0$. Clearly a lot of hypotheses are required to obtain these kind of results, which we describe in Chapter 4.

Other than these applications, the relative energy plays important role in stability analysis and characterization of steady solutions.

0.4 Structure of the thesis

In Chapter 1 we discuss some preliminaries that we will use in the next chapters. The basic time dependent function spaces are discussed in this chapter along with weak and weak-(*) convergence in these spaces. The concept of Young measures also plays a crucial role in our discussions and main results. Therefore, some important results on Young measures are given in this chapter.

Chapter 2 is dedicated to the derivation and weak formulation of the system. We collect the available definitions of various systems describing a compressible fluid. We also present the similarities and differences of considering the problem in different domains. Furthermore, we try to explain the importance of generalized (measure-valued and dissipative) solutions and consider them as the weak limit of the weak solutions in certain cases.

Chapter 3 is devoted exclusively to the compressible Navier–Stokes system with general barotropic pressure laws. We consider a general non-monotone pressure-density relation. We also consider a singular non-monotone pressure. We prove the generalized weak-strong uniqueness property. The relative energy or entropy is the main tool used here.

In Chapter 4, we consider a scaled Euler system. We consider a general scaling and observe that the target system describes an incompressible flow in the regime of low Mach numbers. The effect of different characteristic numbers is explained in this case. We use relative energy as a main tool. Here we use generalized solutions of the primitive system to identify the limit. This reflects the stability of the target system.

Finally, in Chapter 5 we discuss the convergence of approximation schemes of the complete Euler system in the domain $\mathbb{R}^d$. We define an approximation scheme which we call *consistent approximation* scheme. We will prove that these approximate solutions either converge strongly (at least almost everywhere) to a weak solution of the complete Euler system, or the limit is not a weak solution of the system at all.

This reflects a way to consider 'good' weak solutions, from the vanishing viscosity limit of a viscous system (e.g. Navier–Stokes–Fourier system).

The thesis is based on the following articles and preprints:

- N. Chaudhuri, On weak-strong uniqueness for compressible Navier–Stokes system with general pressure laws, [29]
- N. Chaudhuri, On weak (measure-valued)-strong uniqueness for compressible Navier–Stokes system with non-monotone pressure law, [31].
- N. Chaudhuri, Multiple scales and singular limits of perfect fluids, [28].
- N. Chaudhuri, Limit of a consistent approximation to the complete compressible Euler System, [30].

Chapter 1

Mathematical preliminaries

1.1 Function spaces

In the section on notation we introduce the spaces of continuous and differentiable functions. Here we are mainly concerned with measurable and integrable functions, in general they are described as Lebesgue spaces. The reader is advised to consult basic books on measure theory for a detailed discussion, Rudin [108], Folland [82], Evans and Gariepy [48] to name a few.

For any subset Ω of $\mathbb{R}^d$ we consider the Lebesgue measure space $(\Omega, \mathfrak{M}, \mathcal{L})$. For any integrable function f over this measure space we use the simple notation $\int_\Omega f \mathrm{d}x$, instead of the appropriate notation $\int_\Omega f \mathrm{d}\mathcal{L}$.

L^p space:

Let $1 \leq p < \infty$ and $\Omega \subset \mathbb{R}^d$, we define the space $L^p(\Omega)$ as

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R} \,|\, \text{f is Lebesgue measurable and } \int_\Omega |f(x)|^p dx < \infty \right\}.$$

We define the space $L^\infty(\Omega)$ as

$$L^\infty(\Omega) = \{ f : \Omega \to \mathbb{R} \,|\, \text{f is Lebesgue measurable and ess sup}_{x \in \Omega} |f(x)| < \infty \}.$$

For $1 \leq p < \infty$, $L^p(\Omega)$ with norm $\|f\|_{L^p} \left(= (\int_\Omega |f(x)|^p dx)^{\frac{1}{p}} \right)$ is a Banach space. Similarly, $L^\infty(\Omega)$ with norm $\|f\|_{L^\infty}(= \text{ess sup}_{x\in\Omega}(|f(x)|))$ is also a Banach space.

For $1 \leq p < \infty$, the dual of the space $L^p(\Omega)$ is the space $L^q(\Omega)$, where $\frac{1}{p} + \frac{1}{q} = 1$, with the duality pairing

$$\langle f, g \rangle_{L^p, L^q} : = \int_\Omega fg \, \mathrm{d}x, \text{ for } f \in L^p(\Omega) \text{ and } q \in L^q(\Omega).$$

This is not true for $p = \infty$, we only have $L^1(\Omega)$ is a subset of the dual of $L^\infty(\Omega)$.

Analogously, for vector-valued and matrix-valued functions, we denote them as $L^p(\Omega, \mathbb{R}^m)$ and $L^p(\Omega; \mathbb{R}^{m\times m})$, respectively, for $m > 1$.

We note that the space $L^p + L^r(\Omega)$ is a Banach space with norm

$$\|f\| = \inf\left\{\|g\|_{L^p} + \|h\|_{L^r} \mid f = g + h \in L^p + L^r(\Omega)\right\},$$

for $1 \le p < r \le \infty$.

Sobolev spaces:

In 1930's Sobolev introduces these spaces using the concept of weak derivatives. In modern analysis, *Sobolev spaces* are considered as one of the important tools. Here we give some important properties of these spaces, a detailed discussion and application can be found in Brezis [21], Adams [3], Evans [47].

Let $k > 0$ be an integer and let $1 \le p \le \infty$. The *Sobolev space* $W^{k,p}(\Omega)$ is defined as

$$\begin{aligned} W^{k,p}(\Omega) = \{u \in L^p(\Omega) \,|\, &\text{for all multi-index } \alpha \text{ with } |\alpha| \le k, \\ &\text{weak derivative } \partial^\alpha u \text{ exists and } \partial^\alpha u \in L^p(\Omega)\}, \end{aligned}$$

endowed with the norm

$$\|u\|_{W^{k,p}} = \left(\sum_{|\alpha|\le k} \int_\Omega |\partial^\alpha u|^p\right)^{\frac{1}{p}} = \left(\sum_{|\alpha|\le k} \|\partial^\alpha u\|_{L^p}^p\right)^{\frac{1}{p}}, \text{ for } 1 \le p < \infty$$

and

$$\|u\|_{W^{k,\infty}} = \sum_{|\alpha|\le k} \|\partial^\alpha u\|_{L^\infty}, \text{ for } p = \infty.$$

We further denote the space $W_0^{1,p}(\Omega)$ as the closure of $C_c^\infty(\Omega)$ with respect to $W^{1,p}$-norm. In Sobolev spaces, the boundary of a domain plays a crucial role in some consideration.

Domain of class C^m: We first consider the following subspaces of $\mathbb{R}^d$:

$$\begin{aligned} &\mathbb{R}^d_+ = \left\{x = (x', x_d) |\, x' \in \mathbb{R}^{d-1}, x_d > 0\right\}, \ \mathcal{H} = \left\{(x', x_d) |\, |x'| < 1, |x_d| < 1\right\}, \\ &\mathcal{H}_+ = \mathcal{H} \cap \mathbb{R}^d_+ \text{ and } \mathcal{H}_0 = \{(x', 0) |\, |x'| < 1\}. \end{aligned}$$

We say that an *open set* Ω *is of class* C^m, $m \ge 1$ being an integer, if for every $x \in \partial\Omega$ there exists a neighborhood U of x in $\mathbb{R}^d$ and a bijection $H : \mathcal{H} \to U$ such that

$$H \in C^m(\overline{\mathcal{H}}), \quad H^{-1} \in C^m(\overline{U}), \quad H(\mathcal{H}_+) = U \cap \Omega, \quad H(\mathcal{H}_0) = U \cap \partial\Omega \tag{1.1.1}$$

We also use the term Ω is with C^m boundary. It is of C^∞ if it is of class C^m for all m. Moreover, instead of C^m, if the function H is Lipschitz, i.e., $C^{0,1}$, then we say Ω with Lipschitz boundary $\partial\Omega$.

Next we state the Sobolev embedding theorem from Adams [3, Theorem 5.4].

Theorem 1.1.1. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a Lipschitz boundary. Consider $W^{k,p}(\Omega)$ with $1 \leq p < \infty$, $k \geq 0$ and $u \in W^{k,p}(\Omega)$. Then the following holds:*

- *If $d > kp$,*

$$W^{k,p}(\Omega) \subset L^q(\Omega)$$

for all $q \in [1, p^*]$, where $\frac{1}{p^*} = \frac{1}{p} - \frac{k}{d}$ this embedding is continuous and thus we have

$$\|u\|_{L^q(\Omega)} \leq C(k,p,d)\|u\|_{W^{k,p}(\Omega)}.$$

- *If $d = kp$,*

$$W^{k,p}(\Omega) \subset L^q(\Omega)$$

for all $q \in [p, \infty)$, this embedding is continuous and thus we have

$$\|u\|_{L^q(\Omega)} \leq C(k,p,d)\|u\|_{W^{k,p}(\Omega)}.$$

- *If $d < kp$,*

$$W^{k,p}(\Omega) \subset C^{m,\sigma}(\overline{\Omega})$$

$m = [k - \frac{d}{p}], \sigma = \{k - \frac{d}{p}\}$, this embedding is continuous and thus we have

$$\|u\|_{C^{k,\sigma}} \leq C(k,p,d)\|u\|_{H^s}.$$

Remark 1.1.2. For any $x \in \mathbb{R}$, $[x]$ and $\{x\}$ denote the integral part and fractional part of a real number. The same theorem holds for domain $\Omega = \mathbb{T}^d$. If $\Omega = \mathbb{R}^d$, we refer a similar result in Brezis [21, Corollary 9.13].

Remark 1.1.3. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain. For $1 < p < \infty$, We denote $W^{-1,q}(\Omega)$ is the dual of $W_0^{1,p}(\Omega)$, where $\frac{1}{p} + \frac{1}{q} = 1$.

Homogeneous Sobolev space:

Let $\Omega(\subset \mathbb{R}^d)$ be an unbounded domain. It is easy to verify that the sets $C_c^\infty(\Omega)$ and $C_c^\infty(\overline{\Omega})$ endowed with the norm

$$|u|_{1,q} := \|\nabla_x u\|_{L^q(\Omega)}$$

are normed linear spaces.

Definition 1.1.4. We define the *homogeneous Sobolev space* as

$$\begin{aligned} D_0^{1,q} &= \overline{C_c^\infty(\Omega)}^{|\cdot|_{1,q}}, \\ D^{1,q} &= \overline{C_c^\infty(\overline{\Omega})}^{|\cdot|_{1,q}}, \end{aligned} \tag{1.1.2}$$

where the sign "overline with norm" denotes the completion with respect to the norm.

Remark 1.1.5. If Ω is a bounded domain, $D_0^{1,q}(\Omega)$ coincides with $W_0^{1,q}(\Omega)$. The definition of $D^{1,q}(\Omega)$ as a Banach makes no sense for a bounded domain Ω. The spaces $D^{1,q}(\mathbb{R}^d)$ and $D_0^{1,q}(\mathbb{R}^d)$ are the same.

1.1.1 Important inequalities of function spaces

Here we give some important inequalities related to function spaces.

Young's inequality:

This inequality is used quite frequently in the functional analysis. A standard version is available in [14, Proposition II.2.16]. Here we give a generalized form of it.

Proposition 1.1.6. *Let $a_i \geq 0$ for $i = 1, \ldots, m$ and $p_i \geq 1$ for $i = 1, \cdots, m$ such that $\sum\limits_{i=1}^{m} \frac{1}{p_i} = 1$. Then for $\epsilon_i > 0$ for $i = 1, \cdots, (m-1)$ there exists $c(\epsilon_1, \cdots, \epsilon_{m-1}) > 0$ such that*

$$\prod_{i=1}^{m} a_i \leq \epsilon_1 a_1^{p_1} + \cdots + \epsilon_{m-1} a_{m-1}^{p_{m-1}} + c(\epsilon_1, \cdots, \epsilon_{m-1}) a_m^{p_m}.$$

Grönwall's inequality:

The following lemma is a very useful ingredient for the study of time-dependent partial differential equations, in particular to obtain a priori estimates, in our case the estimation of the relative energy.

Lemma 1.1.7. *Let $T > 0$ and $y \in L^\infty(0, T)$, a non negative function $g \in L^1(0, T)$ and $y_0 \in \mathbb{R}$ such that*

$$y(\tau) \leq y_0 + \int_0^\tau y(t) g(t), \text{ for a.e. } \tau \in (0, T).$$

Then we have

$$y(\tau) \leq y_0 \exp\left(\int_0^\tau g(t)\ dt \right) \text{ for a.e. } \tau \in (0, T).$$

See [14, Lemma II.4.10] for a complete proof.

Poincaré type inequality:

Poincaré type inequalities provide an estimate of the L^p-norm of a Sobolev function by the L^p-norms of its derivative.

Theorem 1.1.8 (Poincaré's inequality, [21, Corollary 9.19])**.** *Let $\Omega (\subset \mathbb{R}^d)$ be a bounded Lipschitz domain and $1 \leq p < \infty$. Then there exists $C(p, d, \Omega)$ such that*

$$\|u\|_{L^p(\Omega)} \leq \|\nabla u\|_{L^p(\Omega)}, \quad \forall u \in W_0^{1,p}(\Omega). \tag{1.1.3}$$

Remark 1.1.9. Poincaré's inequality remains true if Ω has finite measure and also if Ω has a bounded projection on an axis. We observe that the above inequality is not true in the bounded domain Ω for functions in $W^{1,p}(\Omega)$. A simple counterexample can be established by considering $u = 1$ in Ω.

Therefore, we give a general version of the Poincaré inequality.

Theorem 1.1.10 (Poincaré–Wirtinger's inequality, [21, Chapter 9])**.** *Let Ω be a bounded domain with Lipschitz boundary and $1 \leq p < \infty$. Then there exists C such that*

$$\|u - \bar{u}\|_{L^p(\Omega)} \leq \|\nabla u\|_{L^p(\Omega)}, \quad \forall u \in W^{1,p}(\Omega), \quad \textit{where } \bar{u} = \frac{1}{\mathcal{L}(\Omega)} \int_\Omega u \; dx. \tag{1.1.4}$$

Korn type inequalities:

Korn's inequality plays an important role in the theory of linear elasticity. It is also of great importance in the analysis of viscous fluids. The standard formulation of Korn's inequality provides a bound on the L^p-norm of the gradient of a vector field by the L^p-norm of its symmetric part. Following [72, Section 11.10] we state the theorems.

Theorem 1.1.11. *Let $1 < p < \infty$.*

1. *There exists a positive constant $C = c(p, d)$ such that*

$$\|\nabla_x \mathbf{v}\|_{L^p(\mathbb{R}^d;\mathbb{R}^{d\times d})} \leq C \left(\|\nabla_x \mathbf{v} + \nabla_x^T \mathbf{v}\|_{L^p(\mathbb{R}^d;\mathbb{R}^{d\times d})} \right)$$

   for any $\mathbf{v} \in W^{1,p}(\mathbb{R}^d;\mathbb{R}^d)$.

2. *Let $\Omega \subset \mathbb{R}^d$ be bounded Lipschitz domain. There exists a positive constant $c = c(p, d)$ such that*

$$\|\nabla_x \mathbf{v}\|_{L^p(\Omega;\mathbb{R}^{d\times d})} \leq C \left(\|\nabla_x \mathbf{v} + \nabla_x^T \mathbf{v}\|_{L^p(\Omega;\mathbb{R}^{d\times d})} \right)$$

   for any $\mathbf{v} \in W_0^{1,p}(\Omega;\mathbb{R}^d)$.

3. *Let $\Omega \subset \mathbb{R}^d$ be bounded Lipschitz domain. There exists a positive constant $c = c(p, d)$ such that*

$$\|\mathbf{v}\|_{W^{1,p}(\Omega;\mathbb{R}^{d\times d})} \leq C \left(\|\nabla_x \mathbf{v} + \nabla_x^T \mathbf{v}\|_{L^p(\Omega;\mathbb{R}^{d\times d})} + \int_\Omega |\mathbf{v}| \; dx \right)$$

   for any $\mathbf{v} \in W^{1,p}(\Omega;\mathbb{R}^d)$.

Our main goal is to apply these inequalities to compressible viscous fluids. For this application it is useful to replace the symmetric gradient in the previous theorem by its *traceless* part. The result is given in the following theorem.

Theorem 1.1.12. *Let $1 < p < \infty$ and $d \geq 2$.*

1. *There exists a positive constant* $C = C(p, d)$ *such that*

$$\|\nabla_x \mathbf{v}\|_{L^p(\mathbb{R}^d;\mathbb{R}^{d\times d})} \leq C \left(\left\| \nabla_x \mathbf{v} + \nabla_x^T \mathbf{v} - \frac{2}{d} div_x \mathbf{v} \mathbb{I} \right\|_{L^p(\mathbb{R}^d;\mathbb{R}^{d\times d})} \right)$$

for any $\mathbf{v} \in W^{1,p}(\mathbb{R}^d; \mathbb{R}^d)$.

2. *Let* $\Omega \subset \mathbb{R}^d$ *be bounded Lipschitz domain. There exists a positive constant* $c = c(p, d)$ *such that*

$$\|\nabla_x \mathbf{v}\|_{L^p(\Omega;\mathbb{R}^{d\times d})} \leq C \left(\left\| \nabla_x \mathbf{v} + \nabla_x^T \mathbf{v} - \frac{2}{d} div_x \mathbf{v} \mathbb{I} \right\|_{L^p(\mathbb{R}^d;\mathbb{R}^{d\times d})} \right)$$

for any $\mathbf{v} \in W_0^{1,p}(\Omega; \mathbb{R}^d)$.

3. *Let* $\Omega \subset \mathbb{R}^d$ *be bounded Lipschitz. There exists a positive constant* $c = c(p, d)$ *such that*

$$\|\mathbf{v}\|_{W^{1,p}(\Omega;\mathbb{R}^{d\times d})} \leq C \left(\left\| \nabla_x \mathbf{v} + \nabla_x^T \mathbf{v} - \frac{2}{d} div_x \mathbf{v} \mathbb{I} \right\|_{L^p(\mathbb{R}^d;\mathbb{R}^{d\times d})} + \int_\Omega |\mathbf{v}| \ dx \right)$$

for any $\mathbf{v} \in W^{1,p}(\Omega; \mathbb{R}^d)$.

Generalized Korn-Poincaré inequality

We conclude this part with a further generalization and combination of the Poincaré and Korn inequality, see [72, Section 11.10].

Theorem 1.1.13. *Let* $1 < p < \infty$, $M_0 > 0$, $K > 0$, $\gamma > 1$ *and* $d > 2$. *Let* $\Omega \subset \mathbb{R}^d$ *be a bounded Lipschitz domain. There exists a positive constant* $C = C(p, d, M_0, K, \gamma)$ *such that*

$$\|\mathbf{v}\|_{W^{1,p}(\Omega;\mathbb{R}^{d\times d})} \leq C \left(\left\| \nabla_x \mathbf{v} + \nabla_x^T \mathbf{v} - \frac{2}{d} div_x \mathbf{v} \mathbb{I} \right\|_{L^p(\mathbb{R}^d;\mathbb{R}^{d\times d})} + \int_\Omega r|\mathbf{v}| \ dx \right)$$

for any $\mathbf{v} \in W^{1,p}(\Omega; \mathbb{R}^d)$ *and any non negative scalar valued function* r *such that*

$$0 < M_0 \leq \int_\Omega r \ dx, \ \int_\Omega r^\gamma \ dx \leq K.$$

1.1.2 Weak and Weak-(*) convergence

In a general Banach or Hilbert space, the notion of weak and weak-(*) convergence has been developed. We mainly follow Brezis [21] to state the following compactness result:

Theorem 1.1.14 (Banach-Alaoglu-Bourbaki, [21, Theorem 3.16])**.** *Let X be a Banach space, $B \subset X$ be the unit ball in X and $B^* \subset X^*$ also be a unit ball in X^*. Then B^* is always weak-(*)ly compact. If X is reflexive, B is weakly compact.*

Let $\Omega \subset \mathbb{R}^d$ be a domain. For $1 < p < \infty$, in $L^p(\Omega)$ every uniformly bounded sequence $\{f_n\}_{n\in\mathbb{N}}$ has a weakly convergent subsequence. In the case of $L^\infty(\Omega)$ any uniformly bounded sequence poses a weak-(*)ly convergent subsequence. By saying uniformly bounded we mean

$$\|f_n\|_{L^p(\Omega)} \leq C,$$

where C is independent of n and $1 < p \leq \infty$. In the context of $L^1(\Omega)$, the condition is a bit delicate. Here we state the following theorem:

Theorem 1.1.15. *Let $\{f_n\}_{n\in\mathbb{N}}$ be uniformly bounded in $L^1(\Omega)$. Then the following statements are equivalent:*

1. *The sequence $\{f_n\}_{n\in\mathbb{N}}$ contains a subsequence that converges weakly in $L^1(\Omega)$.*
2. *For all $\epsilon > 0$, there exists $\delta > 0$ such that*

$$\int_M |f_n| \; dx < \epsilon$$

whenever $\mathcal{L}(M) < \delta$, for $M \subset \Omega$. Here $\mathcal{L}$ is the Lebesgue measure of $\mathbb{R}^d$.

The condition (2.) is called equi-integrability criterion for weak convergence in L^1.

Remark 1.1.16. There are several other equivalent statements of the theorem 1.1.15, see [72, Theorem 10].

1.2 Spaces involving time

We are interested in time-dependent problems. Therefore, we introduce the spaces of time-dependent functions ranging in a Banach space. There are several ways to define the integrabality in these spaces namely *Bochner integral*, *Petis inegral*, *Dunford integral* etc. Here we stay with the Bochner integral. We follow the textbook Yoshida [122].

Let X be a Banach space. For $T > 0$, we consider the map $f : [0, T] \to X$.

- A function $s : [0, T] \to X$ is called *simple* if it has the form

$$s(t) = \Sigma_{i=1}^m \chi_{E_i}(t) u_i, \; t \in [0, T],$$

  where for each $i = 1, \cdots, m$, E_i is Lebesgue measurable subset of $[0, T]$ and $u_i \in X$.

- A function $f : [0, T] \to X$ is *strongly measurable* if there exists a sequence of simple functions $\{s_k\}_{k\in\mathbb{N}}$ with $s_k : [0, T] \to X$ such that $s_k(t) \to f(t),$ for a.e. $t \in [0, T]$ as $k \to \infty$.

- A function $f : [0, T] \to X$ is *weakly measurable* if for each $u^* \in X^*$, the mapping $t \mapsto \langle f(t), u^* \rangle_{X,X^*}$ is Lebesgue measurable.
- A function $f : [0, T] \to X$ is *separably valued* if there exists $N \subset [0, T]$ with $\mu(N) = 0$, such that the set $\{f(t) \,|\, t \in [0, T] - N\}$ is separable.

Remark 1.2.1. Let X^* be the dual space of X. A function $f : [0, T] \to X^*$ is *weak-(*)ly measurable* if for each $u \in X$, the mapping $t \mapsto \langle u, f(t) \rangle_{X,X^*}$ is Lebesgue measurable.

We consider the following Banach spaces:

- $$L^p(0, T; X) = \Big\{ u : [0, T] \to X \mid u \text{ is a strongly measurable function and } \int_0^T \|u(t)\|_X^p dt < \infty \Big\},$$
  equipped with norm $\|u\|_{L^p(0,T;X)} = (\int_0^T \|u(t)\|_X^p)^{\frac{1}{p}}$, for $1 \le p < \infty$.
- $$L^\infty(0, T; X) = \Big\{ u : [0, T] \to X \mid u \text{ is a strongly measurable function and } \operatorname{ess\,sup}_{t\in[0,T]} \|u(t)\|_X < \infty \Big\},$$
  equipped with norm $\|u\|_{L^\infty(0,T;X)} = \operatorname{ess\,sup}_{t\in[0,T]} \|u(t)\|_X$.
- $$C([0, T]; X) = \Big\{ u : [0, T] \to X \text{ continuous function} \mid \max_{t\in[0,T]} \|u(t)\|_X < \infty \Big\},$$
  equipped with norm $\|u\|_{C([0,T];X)} = \max_{t\in[0,T]} \|u(t)\|_X$.

Analogously, we define the following spaces:

- For $1 \le p < \infty$,
  $$L^p_{\text{weak}}(0, T; X) = \Big\{ u : [0, T] \to X \mid u \text{ is weakly measurable function, } t \mapsto \|u(t)\|_X \text{ is measurable and } \int_0^T \|u(t)\|_X^p dt < \infty \Big\},$$
  equipped with norm $\|u\|_{L^p(0,T;X)} = \Big(\int_0^T \|u(t)\|_X^p\Big)^{\frac{1}{p}}$.

  In the case $p = \infty$, we replace the condition $\int_0^T \|u(t)\|_X^p dt < \infty$ by
  $$\operatorname*{ess\,sup}_{t\in[0,T]} \|u(t)\|_X < \infty,$$
  and $\|u\|_{L^\infty(0,T;X)} = \operatorname{ess\,sup}_{t\in[0,T]} \|u(t)\|_X$.

- For $1 \le p < \infty$,

$$L^p_{\text{weak-}(*)}(0,T;X^*) = \Big\{ u : [0,T] \to X^* |\; u \text{ is weak-(*)ly measurable function}, \\ t \mapsto \|u(t)\|_{X^*} \text{ is measurable and } \int_0^T \|u(t)\|^p_{X^*} dt < \infty \Big\},$$

equipped with norm $\|u\|_{L^p(0,T;X^*)} = \left(\int_0^T \|u(t)\|^p_{X^*}\right)^{\frac{1}{p}}$. Similarly for $p = \infty$, we replace the condition $\int_0^T \|u(t)\|^p_{X^*} dt < \infty$ by

$$\operatorname*{ess\ sup}_{t\in[0,T]} \|u(t)\|_{X^*} < \infty,$$

and $\|u\|_{L^\infty(0,T;X^*)} = \text{ess sup}_{t\in[0,T]}\|u(t)\|_{X^*}$.

We state the following theorem from Pedregal [107, Theorem 6.14]:

Theorem 1.2.2. *Let X be a separable Banach space and $1 \le p < \infty$. Then*

$$[L^p(0,T;X)]^* = L^q_{\textit{weak-(*)}}(0,T;X^*) \ \textit{with}\ \frac{1}{p} + \frac{1}{q} = 1$$

with the duality

$$\langle f, g\rangle = \int_0^T \langle f(t), g(t)\rangle \ dt\ .$$

We also define *weak* and *weak-(*)* continuous time-dependent function spaces in the following way:

- We say a function $f : (0,T) \to X$ is *weak continuous* if for all $\psi \in X^*$, the map $t \mapsto \langle f(t), \psi\rangle_{X,X^*}$ is continuous and the space of all such functions is denoted by $C_{\text{weak}}([0,T];X)$.
- Similarly, we say that a function $f : (0,T) \to X^*$ is *weak-(*) continuous* if for all $\phi \in X$, the map $t \mapsto \langle \phi, f(t)\rangle_{X,X^*}$ is continuous and the space of all such functions is denoted by $C_{\text{weak-}(*)}([0,T];X^*)$.

We end this section with this lemma.

Lemma 1.2.3. *Let $f, g \in L^\infty(0,T;X^*)$ such that*

$$\int_0^T \langle \phi, f(t)\rangle_{X,X^*}\eta'(t)\ dt \ = \int_0^T \langle \phi, g(t)\rangle_{X,X^*}\eta(t)\ dt\ ,\ \forall \phi \in X \ \textit{and}\ \eta \in C_c^\infty(0,T).$$

Then, $f \in C_{\textit{weak-(*)}}([0,T];X^*)$.

Proof. It is enough to prove that the map $t \mapsto \langle \phi, f(t)\rangle_{X,X^*}$ is absolutely continuous in $[0,T]$ for every $\phi \in X$. □

There is a similar lemma for weak continuous functions.

1.3 Young measure and related results

For any space X, a standard measure space is denoted by the triplet $(X, \mu, \mathcal{X})$. In the case of $X = \Omega \subset \mathbb{R}^d$ we consider $\mathcal{L}$ as the standard Lebesgue measure on Ω.

1.3.1 Borel and Radon measure

Let $\mathcal{Q}$ be a locally compact Hausdorff metric space.

- The symbol $\mathcal{B}(\mathcal{Q})$ stands for the space of signed Borel measures on $\mathcal{Q}$. The symbol $\mathcal{B}^+(\mathcal{Q})$ denotes the cone of non-negative Borel measures on $\mathcal{Q}$.
- We say a Borel measure is finite if $\mu \in \mathcal{B}(\mathcal{Q})$ such that $\mu(\mathcal{Q}) < \infty$. Moreover a Borel measure is *locally finite* if $\mu \in \mathcal{B}(\mathcal{Q})$ such that $\mu(K) < \infty$ for all compact subset K of $\mathcal{Q}$
- The symbol $\mathcal{B}(\mathcal{Q}; \mathbb{R}^d)$ means for $\zeta = \{\zeta_i\}_{i=1}^d \in \mathcal{B}(\mathcal{Q}; \mathbb{R}^d)$, $\zeta_i \in \mathcal{B}(\mathcal{Q})$, $\forall i = 1, 2, \cdots, d$, and the notation $\mathcal{B}(\mathcal{Q}; \mathbb{R}^{d\times d})$ stands for $\zeta = \{\zeta_{i,j}\}_{i,j=1}^d \in \mathcal{B}(\mathcal{Q}; \mathbb{R}^{d\times d})$, $\zeta_{i,j} \in \mathcal{B}(\mathcal{Q})$, $\forall i, j = 1, 2, \cdots, d$.

Now we state an important theorem that characterizes a positive linear function on $C_c(\mathcal{Q})$. Since we are interested in $\mathbb{R}^d$, we consider $\mathcal{Q} = \Omega \subset \mathbb{R}^d$, a Borel set.

Theorem 1.3.1 (Riesz Representation,[108, Theorem 2.14]). *Let f be a non-negative linear functional defined on the space $C_c(\Omega)$. Then there exists a σ-algebra of measurable sets containing all Borel sets and a unique non-negative measure on $\mu_f \in \mathcal{B}^+(\Omega)$ such that*

$$\langle f, \phi \rangle = \int_\Omega \phi d\mu_f, \quad \textit{for any } \phi \in C_c(\Omega).$$

Moreover, the measure μ_f has the following properties,

1. *$\mu_f(K) < \infty$ for any compact subset $K \subset \Omega$.*
2. *For any open set $E \subset \Omega$ it holds*
$$\mu_f(E) = \sup\{\mu_f(K) | K \subset E\}.$$
3. *For any Borel set V it satisfies*
$$\mu_f(V) = \inf\{\mu_f(E) | V \subset E,\ E \textit{ open }\}.$$
4. *If E is μ_f measurable, $\mu_f(E) = 0$, $A \subset E$, then A is μ_f measurable.*

We say a Borel measure is a Radon measure if it satisfies properties the first three properties in (1.3.1). Therefore, we can say that μ_f is a positive Radon measure.

- We denote the symbol $\mathscr{M}(\Omega)$ stands for the space of signed Radon measures on Ω. The symbol $\mathcal{M}(\Omega)$ denotes the set of finite Radon measures on Ω, i.e.,

$$\mu \in \mathcal{M}(\Omega) \text{ implies } \mu \in \mathscr{M}(\Omega) \text{ with } \mu(\Omega) < \infty.$$

  The notation $\mathcal{M}_{\text{loc}}(\Omega)$ stands for the set of locally finite Radon measure, i.e.,

$$\mu \in \mathcal{M}(\Omega) \text{ implies } \mu \in \mathscr{M}(\Omega) \text{ and } \mu \text{ is locally finite.}$$

- Moreover, $\mathcal{M}^+(\Omega)$ is the cone of positive finite Radon measures.
- We also define $\mathcal{M}(\Omega; \mathbb{R}^d)$ and $\mathcal{M}(\Omega; \mathbb{R}^{d\times d})$ for vector-valued and matrix-valued finite Radon measures.
- The space $\mathcal{M}^+(\Omega; \mathbb{R}^{d\times d})$ denotes the set of *positive semi-definite* matrix-valued finite Radon measures, i.e., $\mathfrak{D} \in \mathcal{M}^+(\Omega; \mathbb{R}^{d\times d})$ implies for all $\xi \in \mathbb{R}^d$, $\mathfrak{D} \colon (\xi \otimes \xi) \in \mathcal{M}^+(\Omega)$.
- The symbol $\mathcal{P}(\Omega)$ indicates the space of probability measures, i.e., for $\nu \in \mathcal{P}(\Omega) \subset \mathcal{M}^+(\Omega)$ we have $\nu(\Omega) = 1$.

There is another way to define Radon measures, using the duality of continuous function spaces, see [98, Chapter 1]. The two definitions are equivalent, which helps us to make the following observations:

- For $\Omega \subset \mathbb{R}^d$, a Borel set, we have $[C_0(\Omega)]^* = \mathcal{M}(\Omega)$ with the duality pairing

$$\langle \mu, f \rangle_{\mathcal{M}(\Omega), C_0(\Omega)} = \int_\Omega f \mathrm{d}\mu.$$

  In particular, we obtain $[C_0(\mathbb{R}^d)]^* = \mathcal{M}(\mathbb{R}^d)$.
- Further if we consider a bounded Borel set Ω, then we observe $[C(\overline{\Omega})]^* = \mathcal{M}(\overline{\Omega})$ with duality pairing

$$\langle \mu, f \rangle_{\mathcal{M}(\overline{\Omega}), C(\overline{\Omega})} = \int_{\overline{\Omega}} f \mathrm{d}\mu.$$

- If $\mu \in [C_c(\Omega)]^*$ then $\mu \in \mathscr{M}(\Omega)$ and μ is locally finite.
- If $\mu \in M^+(\Omega)$, i.e., $\mu \geq 0$, then we have $\langle \mu, f \rangle_{\mathcal{M}(\Omega), C_0(\Omega)} \geq 0$ for any $f \geq 0$.

Remark 1.3.2. From the above discussion and the Theorem 1.2.2, we have

$$\left[L^1(0,T; C_0(\Omega))\right]^* = L^\infty_{\text{weak-}(*)}(0,T; \mathcal{M}(\Omega)).$$

In particular, we obtain

$$\left[L^1(0,T; C_0(\mathbb{R}^d))\right]^* = L^\infty_{\text{weak-}(*)}(0,T; \mathcal{M}(\mathbb{R}^d)).$$

Also, for a bounded Borel set $\Omega \subset \mathbb{R}^d$, we get

$$\left[L^1(0,T; C(\overline{\Omega}))\right]^* = L^\infty_{\text{weak-}(*)}(0,T; \mathcal{M}(\overline{\Omega})).$$

Jensen's inequality

The standard form of the inequality is given by.

Lemma 1.3.3. *If μ is a probability measure on Ω and g is a $\mu-$measurable map. Then*

$$E\left(\int_\Omega g\,d\mu\right) \leq \int_\Omega E(g)\,d\mu,$$

for any real valued convex function E.

1.3.2 Definition of Young measure

The theory of Young measure has several applications in various fields of mathematical analysis. It was introduced by L. C. Young, see [123]. The main development of this theory is related to the calculus of variations. Later many applications were found in various contexts of differential equations. There are several articles and monographs on the study of the Young measure and its application, to name a few, see Ball [9], Ball and Murat [10], Pedregal [107], Málek et al. [98].

Let $k, m \in \mathbb{N}$ and $Q \subset \mathbb{R}^m$ be an open set. Consider a sequence $\{\mathbf{U}_n\}_{n\in\mathbb{N}}$ of measurable functions with $\mathbf{U}_n\colon Q \to \mathbb{R}^k$. We identify $\mathbf{U}_n(y) \approx \delta_{\mathbf{U}_n(y)}$, where $\delta_{\mathbf{U}} \in \mathcal{P}(\mathbb{R}^k)$ denotes the Dirac measure in $\mathbb{R}^k$ supported at $\mathbf{U}$. It helps us to interpret $\mathbf{U}_n$ as

$$\mathbf{U}_n\colon Q \mapsto \delta_{\mathbf{U}_n(y)} \in \mathcal{P}(\mathbb{R}^k).$$

It is easy to verify that

$$\mathbf{U}_n \in L^\infty_{\text{weak-(*)}}\big(Q; \mathcal{P}(\mathbb{R}^k)\big).$$

We have $L^\infty_{\text{weak-(*)}}\big(Q; \mathcal{M}(\mathbb{R}^k)\big) = [L^1\big(Q; C_0(\mathbb{R}^k)\big)]^*$ and $L^1\big(Q; C_0(\mathbb{R}^k)\big)$ is separable. This implies that there exists a subsequence (not relabeled) such that

$$\mathbf{U}_n \to \mathcal{V} \text{ weak-(*)ly in } L^\infty_{\text{weak-(*)}}\big(Q; \mathcal{M}(\mathbb{R}^k)\big).$$

This limit quantity $\mathcal{V}$ is called a *Young measure* associated with or generated by the sequence $\mathbf{U}_n$. It is interpreted as a parameterized family of Borel measures $\{\mathcal{V}_y\}_{y\in Q}$. The definition of weak-(*) convergence implies

$$\int_Q \phi(y) b(\mathbf{U}_n(y))\,\mathrm{d}y \to \int_Q \phi(y) \left\langle \mathcal{V}_y; b(\tilde{\mathbf{U}})\right\rangle \,\mathrm{d}y \text{ for any } \phi \in L^1(Q) \text{ and } b \in C_b(\mathbb{R}^k),$$

where

$$\left\langle \mathcal{V}_y; b(\tilde{\mathbf{U}})\right\rangle = \int_{\mathbb{R}^k} b(\tilde{\mathbf{U}})\mathrm{d}\mathcal{V}_y(\tilde{\mathbf{U}}).$$

As a trivial consequence, we get

$$\mathcal{V}_y \in \mathcal{M}^+(\mathbb{R}^k) \text{ for a.e. } y \in Q \text{ with } \|\mathcal{V}_y\|_{\mathcal{M}(\mathbb{R}^k)} \leq 1.$$

1.3.3 Fundamental theorems of Young measure

Next we state the fundamental theorem for Young measure from Feireisl et al. [67, Proposition 5.1].

Theorem 1.3.4 (Fundamental theorem)**.** *Let $Q(\subset \mathbb{R}^m)$ be a domain for $m \geq 1$ and $\{\mathbf{U}_n\}_{n\in\mathbb{N}}$ be a sequence such that $\mathbf{U}_n \colon Q \to \mathbb{R}^k$ with*

$$\mathbf{U}_n \in L^1(Q;\mathbb{R}^k),\ \|\mathbf{U}_n\|_{L^1(Q)} \leq C \text{ uniformly for } n \to \infty. \tag{1.3.1}$$

Then there exists a subsequence $\{\mathbf{U}_{n_k}\}_{k\in\mathbb{N}}$ and a parameterized family of probability measures $\{\mathcal{Y}_y\}_{y\in Q}$ with

$$\mathcal{V}_y \in \mathcal{P}(\mathbb{R}^k) \text{ for a.e. } y \in Q, \text{ and } \mathcal{V} \in L^\infty_{weak\text{-}(*)}\big(Q;\mathcal{M}(\mathbb{R}^k)\big)$$

such that

$$\int_Q \phi(y) b(\mathbf{U}_{n_k}(y))\ dy \to \int_Q \phi(y) \left\langle \mathcal{V}_y; b(\tilde{\mathbf{U}}) \right\rangle\ dy$$

for any $\phi \in L^1(Q)$ and $b \in C_b(\mathbb{R}^k)$ for a.e. $y \in Q$. The symbol $\left\langle \mathcal{V}_y; b(\tilde{\mathbf{U}}) \right\rangle$ is given by

$$\left\langle \mathcal{V}_y; b(\tilde{\mathbf{U}}) \right\rangle := \int_{\mathbb{R}^k} b(\lambda)\, d\mathcal{V}_y(\lambda).$$

Remark 1.3.5. It is convenient to introduce the following notation:

$$\left\langle \mathcal{V}; b(\tilde{\mathbf{U}}) \right\rangle := \left\{ y \mapsto \left\langle \mathcal{V}_y; b(\tilde{\mathbf{U}}) \right\rangle \right\}.$$

Also, we denote the barycenter of the Young measure by $\left\langle \mathcal{V}; \tilde{\mathbf{U}} \right\rangle$, it is given as

$$\left\langle \mathcal{V}; \tilde{\mathbf{U}} \right\rangle := \left\{ y \mapsto \left\langle \mathcal{V}_y; \tilde{\mathbf{U}} \right\rangle \right\}$$

Remark 1.3.6. Note that the condition (1.3.1) can be replaced by a weaker assumption

$$\mathbf{U}_n \text{ is measurable, } \int_Q h(\mathbf{U}_n)\ \mathrm{d}y \leq C \text{ uniformly for } n \to \infty,$$

where $h(\xi) \to \infty$ for $\xi \to \infty$.

Remark 1.3.7. Instead of L^1, the Theorem 1.3.4 holds if

$$\mathbf{U}_n \in L^p(Q;\mathbb{R}^k),\ \|\mathbf{U}_n\|_{L^p(Q)} \leq C \text{ uniformly for } n \to \infty, \tag{1.3.2}$$

for any $1 \leq p \leq \infty$.

We give another form of the fundamental theorem due to Pedregal [107, Theorem 6.2]. Let $Q \subset \mathbb{R}^d$, a function $\psi(x,\lambda) : Q \times \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ is called Carathéodory function if it measurable in x and continuous in λ. The theorem states as follows:

Theorem 1.3.8. *Let $Q \subset \mathbb{R}^d$ be a measurable set and let $\mathbf{U}_n : Q \to \mathbb{R}^k$ be measurable functions such that*

$$\sup_n \int_Q g(\mathbf{U}_n)\ dx < \infty,$$

where $g : [0,\infty) \to [0,\infty]$ is a continuous, non-decreasing function such that $\lim_{t\to\infty} g(t) = \infty$. There exists a subsequence, not relabeled, and a family of probability measures, $\nu = \{\nu_x\}_{x\in Q}$ depends measurably on x, with the property whenever the sequence $\{\psi(x, \mathbf{U}_n(x))\}_{n\in\mathbb{N}}$ is weakly convergent to $\bar{\psi}$ in $L^1(Q)$ for any Carathéodary function ψ, the weak limit is the (measurable) function

$$\bar{\psi}(x) = \int_{\mathbb{R}^k} \psi(x,\lambda)\, d\nu_x(\lambda).$$

Remark 1.3.9. From the above discussion, it is clear that the Young measure helps us to give a representation of the weak or weak-(*) limit of a sequence with a nonlinear composition.

We note that the Theorem 1.3.4 does not identify the limit for $b \in C(\mathbb{R}^k)$ unless there is a further information about the convergence of the sequence $b(\mathbf{U}_n)$. Here we give the next proposition that states the properties of the Young measure of composition.

Proposition 1.3.10. *Let $Q \subset \mathbb{R}^m$ be a domain and let $\{\mathbf{U}_n\}_{n\in\mathbb{N}}$ be a sequence such that $\mathbf{U}_n\colon Q \to \mathbb{R}^k$ and let $b \in C(\mathbb{R}^k)$ such that*

$$\mathbf{U}_n \in L^1(Q;\mathbb{R}^k),\ \|\mathbf{U}_n\|_{L^1(Q;\mathbb{R}^k)} \le C,\ \|b(\mathbf{U}_n)\|_{L^1(Q;\mathbb{R}^k)} \le C \text{ uniformly for } n \to \infty.$$

Additionally, we assume that $\{\mathbf{U}_n\}_{n\in\mathbb{N}}$ generates a Young measure $\{\mathcal{V}_y\}_{y\in Q}$. Then

$$\left\langle \mathcal{V}_y; b(\tilde{\mathbf{U}}) \right\rangle \text{ is finite for a.e. } y \in Q, \text{ and } y \in Q \mapsto \left\langle \mathcal{V}_y; b(\tilde{\mathbf{U}}) \right\rangle \in L^1(Q).$$

We recall a standard notation for the approximation of a function by a sequence of non-decreasing functions. Let $b : \mathbb{R}^k \to [0,\infty)$ be a function. We use the notation $b_j \nearrow b$ to describe that there exists a sequence of functions $\{b_j\}_{j\in\mathbb{N}}$ such that $0 \le b_j \le b_{j+1} \le b$ for all $j \in \mathbb{N}$ and $b_j(x) \to b(x)$ for a.e. $x \in \mathbb{R}^k$ as $j \to \infty$.

Proof. Let $b \in C(\mathbb{R}^k)$. It suffices to prove for $b \ge 0$ since we can write $b = b^+ - b^-$ with $b^+, b^- \ge 0$. We consider a sequence $b_j \in C_b(\mathbb{R}^k)$ such that $0 \le b_j \nearrow b$.
Using the monotone convergence theorem[108, Chapter 3], we obtain

$$\left\langle \mathcal{V}_y; b_j(\tilde{\mathbf{U}}) \right\rangle \nearrow \left\langle \mathcal{V}_y; b(\tilde{\mathbf{U}}) \right\rangle \in [0,\infty] \text{ for a.e. } y \in Q.$$

On the other hand, the Theorem 1.3.4 helps us to get

$$\int_B b_j(\mathbf{U}_n)\ \mathrm{d}y \to \int_B \left\langle \mathcal{V}_y; b_j(\tilde{\mathbf{U}}) \right\rangle\ \mathrm{d}y \text{ as } n \to \infty,$$

for a bounded Borel subset $B \subset Q$. Thus we have

$$\sup_n \|b(\mathbf{U}_n)\|_{L^1(Q;\mathbb{R}^k)} \geq \lim_j \int_B b_j(\mathbf{U}_n)\, \mathrm{d}y = \int_B \left\langle \mathcal{V}_y; b_j(\tilde{\mathbf{U}}) \right\rangle \, \mathrm{d}y.$$

From this we conclude

$$\int_B \left\langle \mathcal{V}_y; b(\tilde{\mathbf{U}}) \right\rangle dy \leq \sup_n \|b(\mathbf{U}_n)\|_{L^1(Q;\mathbb{R}^k)} \leq C, \tag{1.3.3}$$

for any bounded Borel set B. It ends the proof of the proposition. □

Corollary 1.3.11. *We know $L^1(\Omega)$ is continuously embedded in $\mathcal{M}(\Omega)$. Thus we have*

$$b(\mathbf{U}_n) \to \overline{b(\mathbf{U})} \text{ weak-(*)ly in } \mathcal{M}(\Omega).$$

If $b \geq 0$ and all the hypothesis of the Proposition 1.3.10 holds true, then from (1.3.3) *we have*

$$\overline{b(\mathbf{U})} \geq \left\langle \mathcal{V}; b(\tilde{\mathbf{U}}) \right\rangle,$$

in the sense of measure.

In later chapters we will note that we need to consider nonlinearities b that are not continuous. To do so, we give the definition of lower semicontinuous functions:

Definition 1.3.12. A function $\phi : \Omega(\subset \mathbb{R}^d) \to \mathbb{R} \cup \{\infty\}$ is said to be lower semicontinuous(l.s.c) if for every $\lambda \in \mathbb{R}$ the set $\{x \in \Omega \mid \phi(x) \leq \lambda\}$ is closed.

The proposition (1.3.10) can be extended for a bounded below l.s.c function $b : \mathbb{R}^k \to \mathbb{R} \cup \{\infty\}$. The key observation is the availability of a suitable approximation of bounded below lower semicontinuous function by bounded continuous increasing functions, i.e., there exists $b_j \nearrow b$ and $b_j \in C_b(\mathbb{R}^k)$ (Baire's theorem, see [8]). An elegant construction of such a bounded continuous sequence of functions b_j can be found in [108, Chapter 2, Exercise 22].

Corollary 1.3.13. *Let $Q \subset \mathbb{R}^m$ be a domain,$\{\mathbf{U}_n\}_{n\in\mathbb{N}}$ be a sequence such that $\mathbf{U}_n : Q \to \mathbb{R}^k$. We have the following assumption*

$$\mathbf{U}_n \in L^1(Q;\mathbb{R}^k),\ \|\mathbf{U}_n\|_{L^1(Q;\mathbb{R}^k)} \leq C,\ \|b(\mathbf{U}_n)\|_{L^1(Q;\mathbb{R}^k)} \leq C \text{ uniformly for } n \to \infty,$$

for a l.s.c function $b : \mathbb{R}^k \to [0,\infty]$. Additionally, we assume that $\{\mathbf{U}_n\}_{n\in\mathbb{N}}$ generates a Young measure $\{\mathcal{V}_y\}_{y\in Q}$. Then

$$\begin{aligned} &\left\langle \mathcal{V}_y; b(\tilde{\mathbf{U}}) \right\rangle \text{ is finite for a.e. } y \in Q, \text{ and } y \in Q \mapsto \left\langle \mathcal{V}_y; b(\tilde{\mathbf{U}}) \right\rangle \in L^1(Q) \\ &\text{and } \overline{b(\mathbf{U})} \geq \left\langle \mathcal{V}; b(\tilde{\mathbf{U}}) \right\rangle. \end{aligned}$$

Here we give a result that allows us to compare the oscillation defect measure for two different nonlinearities. This result is a generalization of the result obtained by Feireisl et al. in [56, Lemma 2.1].

Lemma 1.3.14. *Let* $\mathbf{U}_n : Q(\subset \mathbb{R}^d) \to \mathbb{R}^k$, $E : R^k \to [0, \infty]$ *be a lower semi-continuous function and* $\mathbf{G} : R^k \to R^n$ *be a continuous function with the following properties:*

1. $\{\mathbf{U}_n\}_{n\in\mathbb{N}}$ *is a family of measurable functions, such that*

$$\|\mathbf{U}_n\|_{L^1(Q;\mathbb{R}^k)} \leq C, \text{ and } \int_Q E(\mathbf{U}_n)\ dy \leq C \text{ uniformly for } n. \tag{1.3.4}$$

2. *The functions* E *and* G *satisfy*

$$\limsup_{|\mathbf{U}|\to\infty} |\mathbf{G}(\mathbf{U})| < \liminf_{|\mathbf{U}|\to\infty} E(\mathbf{U}). \tag{1.3.5}$$

Then

$$\overline{E(\mathbf{U})} - \left\langle \nu_y; E(\widetilde{\mathbf{U}}) \right\rangle \geq \left| \overline{\mathbf{G}(\mathbf{U})} - \left\langle \nu_y; \mathbf{G}(\widetilde{\mathbf{U}}) \right\rangle \right|. \tag{1.3.6}$$

Remark 1.3.15. Here $\overline{E(\mathbf{U})} \in \mathcal{M}^+(Q)$ and $\overline{\mathbf{G}(\mathbf{U})} \in \mathcal{M}(Q; R^n)$ are the corresponding weak-(*) limits and ν denotes the Young measure generated by $\{\mathbf{U}_n\}$. The inequality (1.3.6) should be understood as

$$\overline{E(\mathbf{U})} - \left\langle \mathcal{V}; E(\widetilde{\mathbf{U}}) \right\rangle - \left(\overline{\mathbf{G}(\mathbf{U})} - \left\langle \mathcal{V}; \mathbf{G}(\widetilde{\mathbf{U}}) \right\rangle \right) \cdot \xi \geq 0$$

for any $\xi \in R^n$, $|\xi| = 1$.

Proof. The result was proved for continuous functions E, $\mathbf{G}$, see [56, Lemma 2.1]. To extend it to the class of lower semi-continuous functions like E, we first observe that there is a sequence of continuous functions $F_j \in C(R^k)$ such that

$$0 \leq F_j \leq E,\ F_j \nearrow E.$$

In view of (1.3.5), there exists $R > 0$ such that

$$|\mathbf{G}(\mathbf{U})| < E(\mathbf{U}) \text{ whenever } |\mathbf{U}| > R.$$

Consider a function

$$T : C^\infty(R^m),\ 0 \leq T \leq 1,\ T(\mathbf{U}) = 0 \text{ for } |\mathbf{U}| \leq R,\ T(\mathbf{U}) = 1 \text{ for } |\mathbf{U}| \geq R + 1.$$

Finally, we construct a sequence

$$E_j(\mathbf{U}) = T(\mathbf{U}) \max\{|\mathbf{G}(\mathbf{U})|; F_j(\mathbf{U})\}.$$

We have

$$0 \leq E_j(\mathbf{U}) \leq E(\mathbf{U}),\ E_j(\mathbf{U}) \geq |\mathbf{G}(\mathbf{U})| \text{ for all } |\mathbf{U}| \geq R + 1.$$

Applying the lemma [56, Lemma 2.1] we get

$$\overline{E_j(\mathbf{U})} - \left\langle \nu_y; E_j(\widetilde{\mathbf{U}}) \right\rangle \geq \left| \overline{\mathbf{G}(\mathbf{U})} - \left\langle \nu_y; \mathbf{G}(\widetilde{\mathbf{U}}) \right\rangle \right|$$

for any j. Thus the proof reduces to showing

$$\overline{E_j(\mathbf{U})} - \left\langle \nu_y; E_j(\widetilde{\mathbf{U}}) \right\rangle \leq \overline{E(\mathbf{U})} - \left\langle \nu_y; E(\widetilde{\mathbf{U}}) \right\rangle,$$

or, in other words, to showing

$$\overline{H(\mathbf{U})} - \left\langle \nu_y; H(\widetilde{\mathbf{U}}) \right\rangle \geq 0 \text{ whenever } H : R^m \to [0, \infty] \text{ is an l.s.c function.}$$

Repeating the above arguments, we construct a sequence

$$0 \leq H_j \leq H \text{ of bounded continuous functions, } \; H_j \nearrow H.$$

Consequently,

$$0 \leq \overline{H(\mathbf{U})} - \overline{H_j(\mathbf{U})} = \overline{H(\mathbf{U})} - \left\langle \nu_y; H_j(\widetilde{\mathbf{U}}) \right\rangle \to \overline{H(\mathbf{U})} - \left\langle \nu_y; H(\widetilde{\mathbf{U}}) \right\rangle$$

as $j \to \infty$.

□

Remark 1.3.16. The condition (1.3.4) can be replaced with the following assumption,

$$E(\mathbf{U}) \geq |\mathbf{U}| \text{ as } |\mathbf{U}| \to \infty \text{ and } \int_Q E(\mathbf{U}_n)\, \mathrm{d}y \leq C \text{ uniformly for } n. \tag{1.3.7}$$

Our next goal is to state a similar result for time-dependent functions.

Proposition 1.3.17. *Let $T > 0$ and $\mathbf{U}_n \in L^\infty(0, T; L^1(Q))$ be a sequence such that $\mathbf{U}_n \colon (0, T) \times Q \to \mathbb{R}^k$ and let $b \in C(\mathbb{R}^k)$ with*

$$\|\mathbf{U}_n\|_{L^\infty(0,T;L^1(Q))} \leq C, \; \|b(\mathbf{U}_n)\|_{L^\infty(0,T;L^1(Q))} \leq C \textit{ uniformly for } n \to \infty.$$

Additionally, we assume that $\{\mathbf{U}_n\}_{n\in\mathbb{N}}$ generates a Young measure $\{\mathcal{V}_{t,x}\}_{(t,x)\in(0,T)\times Q}$ and

$$b(\mathbf{U}_n) \to \overline{b(\mathbf{U})} \textit{ weak-(*)ly in } L^\infty(0, T; \mathcal{M}(\Omega)).$$

Then

$$\left\langle \mathcal{V}_{t,x}; b(\tilde{\mathbf{U}}) \right\rangle \textit{ is finite for a.e. } (t, x) \in (0, T) \times Q,$$
$$\textit{and } (t, x) \in (0, T) \times Q \mapsto \left\langle \mathcal{V}_{t,x}; b(\tilde{\mathbf{U}}) \right\rangle \in L^\infty(0, T; L^1(Q)).$$

Proof. Without loss of generality we assume $b \geq 0$ and the existence of b_j as in proof of the Theorem 1.3.10. We observe $L^\infty(0,T;L^1(Q)) \subset L^1((0,T)\times Q)$. As a consequence of the Theorem 1.3.10 we obtain $\left\langle \mathcal{V}; b(\tilde{\mathbf{U}})\right\rangle \in L^1((0,T)\times Q)$. Thus we observe

$$t \mapsto \int_Q \left\langle \mathcal{V}_{t,x}; b(\tilde{\mathbf{U}})\right\rangle \, \mathrm{d}x \in L^1(0,T).$$

Using monotone convergence theorem we conclude for a.e. $t \in (0,T)$

$$\int_B \left\langle \mathcal{V}_{t,x}; b(\tilde{\mathbf{U}})\right\rangle \, \mathrm{d}x = \lim_{j\to\infty} \int_B \left\langle \mathcal{V}_{t,x}; b_j(\tilde{\mathbf{U}})\right\rangle \, \mathrm{d}x, \tag{1.3.8}$$

for a bounded Borel set $B \subset Q$. We also have

$$\int_B b_j(\mathbf{U}_n)(t) \, \mathrm{d}x \xrightarrow{n\to\infty} \int_B \left\langle \mathcal{V}_{t,x}; b_j(\tilde{\mathbf{U}})\right\rangle \, \mathrm{d}x, \tag{1.3.9}$$

for a.e. $t \in (0,T)$. From (1.3.8) and (1.3.9), we conclude that

$$\operatorname{ess}\sup_{(0,T)} \int_B \left\langle \mathcal{V}_{t,x}; b(\tilde{\mathbf{U}})\right\rangle \, \mathrm{d}x \leq \operatorname{ess}\sup_{(0,T)} \int_B b(\mathbf{U}_n) \, \mathrm{d}x \leq C, \tag{1.3.10}$$

for any bounded Borel subset B. It is easy to prove that

$$t \mapsto \int_Q \left\langle \mathcal{V}_{t,x}; b(\tilde{\mathbf{U}})\right\rangle \eta(x) \, \mathrm{d}x \text{ is measurable,} \tag{1.3.11}$$

for $\eta \in L^\infty(Q)$. Therefore, we have $t \mapsto \left\langle \mathcal{V}_{t,x}; b(\tilde{\mathbf{U}})\right\rangle$ is weakly measurable. Since $L^1(Q)$ is separable thus the map is strongly measurable. This concludes our desired result. □

Remark 1.3.18. In the Proposition 1.3.10 and 1.3.17, we can replace the space $L^1(Q)$ by $L^1_{\mathrm{loc}}(Q)$.

Remark 1.3.19. The proposition 1.3.17 can be extended for a bounded below l.s.c function $b : \mathbb{R}^k \to \mathbb{R} \cup \{\infty\}$.

1.3.4 Defect measure and its properties

From our earlier observation for a sequence $\mathbf{U}_n$ with the property

$$\|\mathbf{U}_n\|_{L^1(Q;\mathbb{R}^k)} \leq C, \|b(\mathbf{U}_n)\|_{L^1(Q;\mathbb{R}^k)} \leq C \text{ uniformly for } n \to \infty, \tag{1.3.12}$$

for $b \in C(\mathbb{R}^k)$. Let $\{\mathcal{V}_y\}_{y\in Q}$ be the associated Young measure. Then we have

- $b(\mathbf{U}_n) \to \overline{b(\mathbf{U})}$ weak-(*)ly in $\mathcal{M}(Q)$.
- $\left\langle \mathcal{V}_y; b(\tilde{\mathbf{U}})\right\rangle$ is finite for a.e. $y \in Q$, and $y \in Q \mapsto \left\langle \mathcal{V}_y; b(\tilde{\mathbf{U}})\right\rangle \in L^1(Q)$.

- $y \in Q \mapsto \left\langle \mathcal{V}_y; \tilde{\mathbf{U}} \right\rangle \in L^1(Q)$ and $b \in C(\mathbb{R}^k)$ imply $y \in Q \mapsto b(\left\langle \mathcal{V}_y; \tilde{\mathbf{U}} \right\rangle)$ is measurable.

Therefore, we introduce the defect measures:

- **Concentration Defect:**
$$\mathfrak{R}^{\mathrm{cd}} = \overline{b(\mathbf{U})} - \left\langle \mathcal{V}_y; b(\tilde{\mathbf{U}}) \right\rangle \in \mathcal{M}(Q).$$
- **Oscillation defect:**
$$\mathfrak{R}^{\mathrm{od}} = \left\langle \mathcal{V}_y; b(\tilde{\mathbf{U}}) \right\rangle - b\left(\left\langle \mathcal{V}_y; \tilde{\mathbf{U}} \right\rangle\right) \text{ is a measurable map on } Q.$$
- **Total defect:**
$$\mathfrak{R} = \mathfrak{R}^{\mathrm{cd}} + \mathfrak{R}^{\mathrm{od}}.$$

Let us consider $b : \mathbb{R}^k \to [0, \infty]$, a convex l.s.c function and $\{\mathbf{U}_n\}$ satisfies (1.3.12). Then we have
$$b\left(\left\langle \mathcal{V}_y; \tilde{\mathbf{U}} \right\rangle\right) \leq \left\langle \mathcal{V}_y; b(\tilde{\mathbf{U}}) \right\rangle$$
as a direct consequence of Jensen's inequality and $b \not\equiv \infty$. This leads us to conclude

- $\mathfrak{R}^{\mathrm{cd}} \in \mathcal{M}^+(Q)$.
- $\mathfrak{R}^{\mathrm{od}} \in L^1(Q)$ and $\mathfrak{R}^{\mathrm{od}} \geq 0$ for a.e. $y \in Q$.

Now we focus on the time-dependent functions. Let us assume $\|\mathbf{U}_n\|_{L^\infty(0,T;L^1(Q))}$ with $\|b(\mathbf{U}_n)\|_{L^\infty(0,T;L^1(Q))} \leq C$ uniformly for $n \to \infty$ and $b \in C(\mathbb{R}^k)$.

We recall that $L^\infty_{\text{weak-(*)}}(0, T; \mathcal{M}(\mathbb{R}^d))$ is the dual of $L^1(0, T; C_0(\mathbb{R}^d))$. Then, the first observation is
$$b(\mathbf{U}_n) \to \overline{b(\mathbf{U})} \text{ weak-(*)ly in } L^\infty_{\text{weak-(*)}}(0, T; \mathcal{M}(\mathbb{R}^d)).$$
Proposition 1.3.17 gives us $(t, x) \in Q \mapsto \left\langle \mathcal{V}_{t,x}; b(\tilde{\mathbf{U}}) \right\rangle \in L^\infty(0, T; L^1(Q))$. This implies the following results:

- **Concentration defect:**
$$\mathfrak{R}^{\mathrm{cd}} = \overline{b(\mathbf{U})} - \left\langle \mathcal{V}_{t,x}; b(\tilde{\mathbf{U}}) \right\rangle \in L^\infty_{\text{weak-(*)}}(0, T; \mathcal{M}(\mathbb{R}^d)).$$
- **Oscillation defect:**
$$\mathfrak{R}^{\mathrm{od}} = \left\langle \mathcal{V}_{t,x}; b(\tilde{\mathbf{U}}) \right\rangle - b\left(\left\langle \mathcal{V}_{t,x}; \tilde{\mathbf{U}} \right\rangle\right) \text{ is a measurable map on } (0, T) \times Q.$$

If we consider $b : \mathbb{R}^k \to [0, \infty]$ a convex l.s.c then we conclude

- $\mathfrak{R}^{\mathrm{cd}} \in L^\infty_{\text{weak-(*)}}(0, T; \mathcal{M}(\mathbb{R}^d))$,
- $\mathfrak{R}^{\mathrm{od}} \in L^\infty(0, T; L^1(\mathbb{R}^d))$ and $\mathfrak{R}^{\mathrm{od}} \geq 0$ for a.e. $(t, x) \in (0, T) \times Q$.

Consequence of vanishing concentration defect

We state a result related to a defect measure $\mathbb{D} \in \mathcal{M}^+(\mathbb{R}^d; \mathbb{R}^{d\times d}_{\text{sym}})$. For $\mathbb{D} \in \mathcal{M}^+(\mathbb{R}^d; \mathbb{R}^{d\times d}_{\text{sym}})$ means $\mathbb{D} \in \mathcal{M}^+(\mathbb{R}^d; \mathbb{R}^{d\times d})$ and $\mathbb{D}(x)$ is symmetric for each $x \in \mathbb{R}^d$. Feireisl and Hofmanová proved the following proposition in [58, Section 4.2].

Proposition 1.3.20. *Let $\mathbb{D} \in \mathcal{M}^+(\mathbb{R}^d; \mathbb{R}^{d\times d}_{sym})$ satisfy*

$$\int_{\mathbb{R}^d} \nabla_x \boldsymbol{\varphi} : d\mathbb{D} = 0 \textit{ for any } \boldsymbol{\varphi} \in C^1_c(\mathbb{R}^d; \mathbb{R}^d),$$

Then $\mathbb{D} = 0$.

Remark 1.3.21. The result is not true if $\mathbb{D} \in \mathcal{M}^+(\Omega; \mathbb{R}^{d\times d}_{\text{sym}})$ if Ω is a bounded domain. A modified version in a bounded domain is available in [57].

The key ingredient of the proof of Proposition 1.3.20 is the consideration of the sequence of cut off function $\{\chi_n\}_{n\in\mathbb{N}}$ such that

$$\begin{aligned} &\chi_n \in C^\infty_c(\mathbb{R}^d),\ 0 \le \chi_n \le 1,\ \chi_n(x) = 1 \text{ for } |x| \le n,\ \chi_n(x) = 0 \text{ for } |x| \ge 2n, \\ &|\nabla_x \chi_n| \le \frac{2}{n} \text{ uniformly as } n \to \infty. \end{aligned} \tag{1.3.13}$$

This helps us to conclude that

$$\int_{\mathbb{R}^d} \nabla_x \boldsymbol{\varphi} : \mathrm{d}\mathbb{D} = 0 \text{ for any } \boldsymbol{\varphi} \in C^1(\mathbb{R}^d; \mathbb{R}^d) \text{ with } \nabla_x \boldsymbol{\varphi} \in L^\infty(\mathbb{R}^d; \mathbb{R}^{d\times d}).$$

Here we state a corollary of the above proposition and the proof of the corollary lies in the similar lines of the above proposition.

Corollary 1.3.22. *Let $\mathbb{D} = \{\mathbb{D}_{ij}\}_{i,j=1}^d \in L^\infty_{weak\text{-}(*)}(0,T; \mathcal{M}(\mathbb{R}^d; \mathbb{R}^{d\times d}))$ be such that*

$$\int_0^T \int_{\mathbb{R}^d} \nabla_x \phi :\ d\mathbb{D}\, dt = 0$$

for any $\phi \in \mathcal{D}((0,T) \times \mathbb{R}^d; \mathbb{R}^d)$.
Then, for any $\psi \in C^\infty_c(0,T; C^1(\mathbb{R}^d; \mathbb{R}^d))$, $\nabla_x \psi \in C^\infty_c(0,T; L^\infty(\mathbb{R}^d; \mathbb{R}^{d\times d}))$, we have

$$\int_0^T \int_{\mathbb{R}^d} \nabla_x \psi :\ d\mathbb{D}\, dt = 0.$$

1.3.5 Convergence results

It is well known fact that the Young measure captures oscillation. We state the lemma from [98, Chapter 3].

Lemma 1.3.23. *Let $1 < p \le \infty$ and $\{\mathbf{v}_n\}_{n\in\mathbb{N}}$ with $\mathbf{v}_n : \mathbb{R}^d \to \mathbb{R}^m$. Further $\{\mathbf{v}_n\}_{n\in\mathbb{N}}$ is bounded uniformly in $L^p_{\text{loc}}(R^d, ; R^m)$ and generates a Young measure ν.*

- *If* $\mathbf{v}_n \to \mathbf{v}$ *weakly in* $L^p(\mathbb{R}^d;\mathbb{R}^m)$ *for* $1 < p < \infty$ *or weak-(*)ly in* $L^\infty(\mathbb{R}^d;\mathbb{R}^m)$. *Then we have* $\mathbf{v} = \langle \nu_y; \tilde{\mathbf{v}} \rangle$ *for a.e.* y.

- *Further it holds that* $\mathbf{v}_n \to \mathbf{v}$ *strongly in* $L^p(\mathbb{R}^d;\mathbb{R}^m)$ *if and only if* ν *reduces to a Dirac measure i.e.*
$$\nu_y = \delta_{\mathbf{v}(y)} \text{ for a.e. } y \in \mathbb{R}^d.$$

The next result is for the critical case of L^1. We recall the definition of *convergence in measure.*

Definition 1.3.24. Let $\mathbf{u}_n : \Omega(\subset \mathbb{R}^d) \to \mathbb{R}^k$ be a sequence of measurable functions. We say $\mathbf{u}_n$ *converges in measure* to $\mathbf{u}$ if for every $\epsilon > 0$,
$$\lim_{n\to\infty} \mathcal{L}\left(\{x \in \Omega \mid |\mathbf{u}_n(x) - \mathbf{u}(x)| \geq \epsilon\}\right) = 0.$$

Lemma 1.3.25. *We consider a sequence* $\{\mathbf{v}_n\}_{n\in\mathbb{N}}$, $\mathbf{v}_n : \mathbb{R}^d \to \mathbb{R}^m$ *such that* $\{\mathbf{v}_n\}_{n\in\mathbb{N}}$ *is bounded in* $L^1_{\text{loc}}(R^d;R^m)$ *and generates a Young measure* ν. *Suppose* $\mathbf{v}(y) = \langle \nu_y; \tilde{\mathbf{v}} \rangle$ *is the barycenter of the Young measure and* $\nu_y = \delta_{\mathbf{v}(y)}$ *for a.e.* $y \in R^d$, *then* $\mathbf{v}_n \to \mathbf{v}$ *in measure.*

Proof. Consider $0 < \epsilon < 1$ and $M_\epsilon = \{y \in \mathbb{R}^d | |\mathbf{v}_n(y) - \mathbf{v}(y)| \geq \epsilon\}$. We obtain
$$\mathcal{L}(M_\epsilon) \leq \frac{1}{\epsilon}\int_{\mathbb{R}^d} \min\{1, |\mathbf{v}_n(y) - \mathbf{v}(y)|\} \text{ d}y.$$
Let us choose a bounded Carathéodory function $\phi(\lambda, y) = \min\{1, |\lambda - \mathbf{v}(y)|\}$. From the fundamental theorem of Young measure 1.3.8 we get
$$\int_{\mathbb{R}^d} \phi(\mathbf{u}_n(y), y) \text{ d}y \to \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^m} \phi(\lambda, y)\text{d}\nu_y(\lambda) \right) \text{ d}y \text{ as } n \to \infty.$$
Using the fact that the Young measure is a Dirac mass, it holds
$$\lim_{n\to\infty} \mathcal{L}(M\epsilon) \leq \lim_{n\to\infty} \frac{1}{\epsilon} \int_{\mathbb{R}^d} \phi(x, \mathbf{v}(y)) \text{ d}y = 0.$$
This concludes that $\mathbf{v}_n \to \mathbf{v}$ in measure. □

Lemma 1.3.26. *Let* $Q \subset \mathbb{R}^d$ *be a bounded domain, and let* $\{\mathbf{v}_n\}_{n=1}^\infty$ *be sequence of vector–valued functions,*
$$\mathbf{v}_n : Q \to \mathbb{R}^k, \ \int_Q |\mathbf{v}_n| \leq C \text{ uniformly for } n \to \infty,$$
generating a Young measure $\nu_y \in \mathcal{P}[\mathbb{R}^k]$, $y \in Q$. *Suppose that*
$$E(\mathbf{v}_n) \to \langle \nu_y; E(\widetilde{\mathbf{v}}) \rangle \text{ weak-(*)ly in } \mathcal{M}(\overline{Q}), \ \langle \nu_y; E(\widetilde{\mathbf{v}}) \rangle \in L^1(Q),$$
where $E : R^d \to [0, \infty]$ *is an l.s.c. function.*

Then
$$E(\mathbf{v}_n) \to \langle \nu_y; E(\widetilde{\mathbf{v}}) \rangle \text{ weakly in } L^1(Q).$$

Proof. Enough to prove the equi-integrability of $\{E(\mathbf{v}_n)\}_{n\in\mathbb{N}}$. A detailed proof is in [67, Lemma 5.1]. □

1.3.6 Inequalities involving Young measure

Generalized Korn-Poincaré inequality for Young measure

Theorem 1.3.27. *Let $\{\mathbf{v}_n\}_{n\in\mathbb{N}}$ be a uniformly bounded in $L^2(0,T;W_0^{1,2}(\Omega))$. Let $\mathcal{V}$ be an Young measure generated by $\{\mathbf{v}_n, \mathbb{D}(\nabla_x \mathbf{v}_n)\}$. For $\tilde{\mathbf{u}} \in L^2(0,T;H_0^1(\Omega;\mathbb{R}^d))$, the following inequality is true:*

$$\int_0^\tau \int_\Omega \left\langle \mathcal{V}_{t,x}; |\tilde{\mathbf{v}} - \tilde{\mathbf{u}}|^2 \right\rangle \, dx \, dt \leq C \int_0^\tau \int_\Omega \left\langle \mathcal{V}_{t,x}; |\mathbb{D}_0(\widetilde{\mathbb{D}_{\mathbf{v}}}) - \mathbb{D}_0(\nabla_x \tilde{\mathbf{u}})|^2 \right\rangle \, dx \, dt \, . \tag{1.3.14}$$

Remark 1.3.28. Here we consider $\mathbf{v}_n$ and $\nabla_x \mathbf{v}_n$ as two different variable and $\mathcal{V}$ is a Young measure generated by $\{\mathbf{v}_n, \mathbb{D}(\nabla_x \mathbf{v}_n)\}_{n\in\mathbb{N}}$. For a continuous bounded map $f : \mathbb{R}^d \times \mathbb{R}^{d\times d}_{\mathrm{sym}} \to \mathbb{R}$, we denote the weak-(*) limit of $f(\mathbf{v}_n, \mathbb{D}(\nabla_x \mathbf{v}_n))$ by $\bar{f}$ and

$$\bar{f} = \left\langle \mathcal{V}_{t,x}; f(\tilde{\mathbf{v}}, \widetilde{\mathbb{D}_{\mathbf{v}}}). \right\rangle$$

The proof is available in Březina, Feireisl and Novotný [27, Lemma 2.2].

Theorem 1.3.29. *Let $\{\mathbf{v}_n\}_{n\in\mathbb{N}}$ be a uniformly bounded in $L^2(0,T;W^{1,2}(\Omega))$ and $\{\varrho_n\}_{n\in\mathbb{N}}$ be uniformly bounded in $L^\infty(0,T;L^\gamma(\Omega))$, for $\gamma > 1$. Moreover, we assume that for each $n \in \mathbb{N}$, ϱ_n satisfies $\int_\Omega \varrho_n = M_{0,n} > 0$ for a.e. $t \in (0,T)$.*

Let $\mathcal{V}$ be a Young measure generated by $\{\varrho_n, \mathbf{v}_n, \mathbb{D}(\nabla_x \mathbf{v}_n)\}_{n\in\mathbb{N}}$. Then for any $\tilde{\mathbf{u}} \in L^2(0,T;W_0^{1,2}(\Omega;\mathbb{R}^d))$, the following inequality holds:

$$\begin{aligned} \int_0^\tau \int_\Omega \left\langle \mathcal{V}_{t,x}; |\tilde{\mathbf{v}} - \tilde{\mathbf{u}}|^2 \right\rangle \, dx \, dt \leq C \bigg(& \int_0^\tau \int_\Omega \left\langle \mathcal{V}_{t,x}; |\mathbb{D}_0(\widetilde{\mathbb{D}_{\mathbf{v}}}) - \mathbb{D}_0(\nabla_x \tilde{\mathbf{u}})|^2 \right\rangle dx \, dt \\ & + \int_0^\tau \int_\Omega \left\langle \mathcal{V}_{t,x}; \tilde{\varrho} |\tilde{\mathbf{v}} - \tilde{\mathbf{u}}|^2 \right\rangle dx \, dt \bigg). \end{aligned} \tag{1.3.15}$$

Proof. The first observation is $\mathbf{v}_n - \tilde{\mathbf{u}} \in L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^d))$. Thus, using the generalized Korn-Poincaré inequality (1.1.13), we obtain

$$\begin{aligned} & \int_0^\tau \int_\Omega |\mathbf{v}_n - \tilde{\mathbf{u}}|^2 \, \mathrm{d}x \, \mathrm{d}t \\ & \leq C \left(\int_0^\tau \int_\Omega \mathbb{D}_0(\nabla_x(\mathbf{v}_n - \tilde{\mathbf{u}})) : \mathbb{D}_0(\nabla_x(\mathbf{v}_n - \tilde{\mathbf{u}})) \, \mathrm{d}x \, \mathrm{d}t + \int_0^\tau \int_\Omega \varrho_n |\mathbf{v}_n - \tilde{\mathbf{u}}|^2 \, \mathrm{d}x \, \mathrm{d}t \right). \end{aligned}$$

Since $\mathcal{V}$ is the Young measure generated by $\{\varrho_n, \mathbf{u}_n, \mathbb{D}(\nabla_x \mathbf{u}_n)\}$, we obtain our desired inequality.

□

Sharp form of Jensen's inequality

We already have Jensen's inequality in Lemma 1.3.3, a sharp form of it is also available. We state it in the following lemma.

Lemma 1.3.30. *Suppose that $E : R^d \to [0, \infty]$ is an l.s.c. convex function satisfying:*

- *E is strictly convex on its domain of positivity, meaning for any $y_1, y_2 \in R^m$ such that $0 < E(y_1) < \infty$, $E(y_2) < \infty$, $y_1 \neq y_2$, we have*

$$E\left(\frac{y_1 + y_2}{2}\right) < \frac{1}{2}E(y_1) + \frac{1}{2}E(y_2).$$

- *If $y \in \partial\mathrm{Dom}[E]$, then either $E(y) = \infty$ or $E(y) = 0$, in other words,*

$$E(y) = 0 \text{ whenever } y \in \mathrm{Dom}[E] \cap \partial\mathrm{Dom}[E].$$

Let $\nu \in \mathcal{P}[R^d]$ be a (Borel) probability measure with finite first moment satisfying

$$E(\langle \nu; \widetilde{y} \rangle) = \langle \nu; E(\widetilde{y}) \rangle < \infty.$$

Then **(i)** *either*

$$\nu = \delta_Y,\ Y = \langle \nu; \widetilde{y} \rangle \in \mathrm{Dom}[E],\ E(Y) > 0,$$

(ii) *or*

$$\mathrm{supp}[\nu] \subset \left\{ y \in R^m \;\middle|\; E(y) = 0 \right\}.$$

The proof of the above lemma is available at [57, Lemma 3.1].

Chapter 2

Fluid models and generalized solutions

2.1 Continuum fluid models

A fluid consists of a large number of molecules in motion without having a precise shape at rest (unlike a solid). One way to describe it is to write down equations of motion for each particle by considering their interactions (collisions, characterized by the mean free path). This approach leads to the study of the kinetic theory of fluids and statistical mechanics in general.

If the mean density of the fluid is not too small (i.e., if the characteristic lengths of the problem are large compared to the mean free path of the particles), then the fluid can be considered as a continuous medium. In this thesis, our goal is to discuss a *phenomenological theory* of fluid dynamics based on observable macroscopic quantities such as density, velocity, internal energy, and so on. These depend on the spatial reference coordinate $x \in \mathbb{R}^d$, and, since we are interested in fluids in motion, on time $t \in \mathbb{R}$. We say that the description of the continuous medium is valid, if there exists a non-negative function $(t, x) \mapsto \varrho(t, x)$ such that the mass contained in any finite volume $\omega \subset \mathbb{R}^d$ at any time t can be expressed as follows

$$\text{Mass in } \omega \text{ at time } t = \int_\omega \varrho(t, x) \, \mathrm{d}x.$$

This function ϱ is called the *density* of the fluid.

The motion of fluid particles is determined by the velocity field $\mathbf{u}(t, x) \in \mathbb{R}^d$. Let ω be the volume space occupied by the fluid at the initial time. The trajectory of the set ω is given by

$$\mathbf{X}(t, \omega), \ t \geq 0, \ \text{where } \partial_t \mathbf{X}(t, x) = \mathbf{u}(t, \mathbf{X}(t, x)), \ \mathbf{X}(0, x) = x, \ x \in \omega.$$

The individual trajectories $t \mapsto \mathbf{X}(t, x)$ are called *streamlines.* We note that the streamlines are well defined and bijective if the velocity $\mathbf{u}$ enjoys certain regularity, i.e.,

$$\text{for any interval } I \subset (0,\infty),\ \nabla_x \mathbf{u} \in L^\infty(I; L^\infty_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^{d\times d})).$$

Eulerian and Lagrangian description

Let ω be the volume of space occupied by the fluid at the initial time. Given a velocity field $\mathbf{u}$ that generates a family of streamlines $\mathbf{X} = \mathbf{X}(t,x)$ in the physical space $\Omega \subset \mathbb{R}^d$, a quantity Q can be expressed in terms of the *Eulerian variables* as

$$Q = Q_E(t,x),\ t \in I,\ x \in \mathbf{X}(t,\omega),$$

or, in terms of the *Lagrangian variables* as

$$Q = Q_L(t,Y),\ t \in I,\ Y \in \omega,$$

where $Q_L(t,Y) = Q_E(t,\mathbf{X}(t,Y))$ or $Q_L(t,\mathbf{X}^{-1}(t,x)) = Q_E(t,x)$ provided the streamlines are bijective. In particular, the time derivative in the Lagrangian setting is equivalent to the material derivative in the Eulerian description:

$$\partial_t Q_L(t,Y) = \partial_t Q_E(t,\mathbf{X}(t,x)) + \mathbf{u}(t,\mathbf{X}(t,x)) \cdot \nabla_x Q_E(t,\mathbf{X}(t,x)).$$

Although it seems that the Lagrangian description is simpler, the invertibility of the mapping $\mathbf{X}$ is the main problem, since the regularity of $\nabla_x \mathbf{v}$ is not always available. Therefore, in this thesis we stick to the Eulerian description of the fluid.

2.1.1 Balance laws

In the introduction we mentioned some systems of partial differential equations describing fluid flows. The formulation of these equations follows from the fundamental laws of physics in the form of balance laws. The time evolution of any macroscopic quantity (D) is determined by its volume density $d = d(t,x)$, flux $\mathbf{F} = \mathbf{F}(t,x)$ and source term $\sigma = \sigma(t,x)$. Given an arbitrary time interval $[t_1,t_2]$ and an arbitrary volume element $\omega \subset \Omega$, the corresponding *balance law* can be written as:

$$\left[\int_\omega d(t,x)\ \mathrm{d}x\right]_{t=t_1}^{t=t_2} = -\int_{t_1}^{t_2}\int_{\partial\omega} \mathbf{F}(t,x)\cdot \mathbf{n}\ \mathrm{dS}_x\ \mathrm{d}t + \int_{t_1}^{t_2}\int_\omega \sigma(t,x)\ \mathrm{d}x\ \mathrm{d}t\,, \tag{2.1.1}$$

where $\mathbf{n}$ is the outer normal vector to $\partial\omega$ and S_x is the surface measure on $\partial\omega$. The balance law (2.1.1) states that the total amount of quantity D in volume ω is proportional to the amount acting through the boundary $\partial\omega$ and the amount contributed by the source term. In the absence of the source term, the relation (2.1.1) represents a *conservation law*. We note that the balance law written in this form (2.1.1) requires very little regularity of variables:

1. the (local) integrability of the variables $d, \mathbf{F}$ and σ,

2. the existence of the normal trace of the field $[d, \mathbf{f}]$ in the time-space cylinder $(t_1, t_2) \times \omega$.

Note that for simplicity we assume that d and σ are scalar-valued and the corresponding flux is vector-valued. We can consider a more generalized form by considering a vector-valued quantity $\mathbf{d}$, a vector-valued source term $\boldsymbol{\sigma}$ and a matrix-valued flux $\mathbb{F}$. Then the balance law is

$$\left[\int_\omega \mathbf{d}(t,x)\ \mathrm{d}x\right]_{t=t_1}^{t=t_2} = -\int_{t_1}^{t_2}\int_{\partial\omega} \mathbb{F}(t,x)\cdot\mathbf{n}\ \mathrm{dS}_x\ \mathrm{d}t + \int_{t_1}^{t_2}\int_\omega \boldsymbol{\sigma}(t,x)\ \mathrm{d}x\ \mathrm{d}t. \tag{2.1.2}$$

Assuming that all quantities written in equation (2.1.1) are smooth enough, we divide the equation by $(t_2 - t_1)$, and consider the limit $t_2 \to t_1$ to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\omega d(t,x)\ \mathrm{d}x + \int_{\partial\omega} \mathbf{F}(t,x)\cdot\mathbf{n}\ \mathrm{dS}_x = \int_\omega \sigma(t,x)\ \mathrm{d}x,$$

for any $t \in [t_1, t_2]$. Then the Gauss-Green theorem yields

$$\int_\omega \left[\partial_t d(t,x) + \mathrm{div}_x \mathbf{F}(t,x)\right]\ \mathrm{d}x = \int_\omega \sigma(t,x)\ \mathrm{d}x.$$

Since $\omega \subset \Omega$ is arbitrary, we conclude that for any $(t,x) \in (0,T)\times\Omega$ the following holds

$$\partial_t d + \mathrm{div}_x \mathbf{F} = \sigma. \tag{2.1.3}$$

We call (2.1.3) as a *strong* or *differential* form of the balance law (2.1.1). Similarly, the vector-valued version of a balance law is

$$\partial_t \mathbf{d} + \mathrm{div}_x \mathbb{F} = \boldsymbol{\sigma}.$$

Assuming that d and $\mathbf{F}$ are weakly differentiable, we derive the following from (2.1.1):

$$\begin{aligned}\int_{t_1}^{t_2}\int_\Omega d(t,x)\partial_t\phi(t,x)\ \mathrm{d}x\ \mathrm{d}t + \int_{t_1}^{t_2}\int_\Omega \mathbf{F}\cdot\nabla_x\phi(t,x)\ \mathrm{d}x\ \mathrm{d}t&\\ = -\int_{t_1}^{t_2}\int_\Omega \sigma(t,x)\phi(t,x)\ \mathrm{d}x\ \mathrm{d}t&,\end{aligned} \tag{2.1.4}$$

where $\phi \in C_c^1((t_1,t_2)\times\Omega)$ and ϕ is called a *test function*. For a simpler consideration, assume that Ω has a Lipschitz boundary. This version of the balance law is called as the *weak form of balance law*. Similarly, we have a corresponding vector-valued version. It reads as follows

$$\begin{aligned}\int_{t_1}^{t_2}\int_\Omega \mathbf{d}(t,x)\cdot\partial_t\boldsymbol{\phi}(t,x)\ \mathrm{d}x\ \mathrm{d}t + \int_{t_1}^{t_2}\int_\Omega \mathbb{F} : \nabla_x\boldsymbol{\phi}(t,x)\ \mathrm{d}x\ \mathrm{d}t&\\ = -\int_{t_1}^{t_2}\int_\Omega \boldsymbol{\sigma}(t,x)\cdot\boldsymbol{\phi}(t,x)\ \mathrm{d}x\ \mathrm{d}t&,\end{aligned} \tag{2.1.5}$$

where $\boldsymbol{\phi} \in C_c^1((t_1,t_2)\times\Omega;\mathbb{R}^d)$.

Initial and boundary condition

As mentioned in the introduction, the physical problems are supplemented with initial states, for $d \in C([0,T) \times \overline{\Omega})$ we write

$$d(0,\cdot) = d_0 \tag{2.1.6}$$

where d_0 is given. Unfortunately, when considering the weak form of the balance law, the pointwise values are not clear. Therefore, we need to find a suitable explanation for them. In this case, the following proposition helps us.

Proposition 2.1.1. *Let $\Omega \subset \mathbb{R}^d$ be a domain. We assume that $d \in L^\infty(0,T; L^1(\Omega))$, $\mathbf{F} \in L^1((0,T) \times \Omega; \mathbb{R}^d)$ and $s \in L^1((0,T) \times \Omega)$ solves the weak form of balance law* (2.1.4). *Then*

$$d \in C_{\textit{weak-(*)}}([0,T]; \mathcal{M}(\Omega))$$

and

$$\left\{ t \mapsto \int_\Omega d(t,x)\psi(t,x)\ dx \right\} \in C[0,T] \text{ for any } \psi \in C_c(\Omega).$$

Remark 2.1.2. Thus for any $d_0 \in L^1(\Omega)$, the initial data is satisfied in the following sense

$$\lim_{t\to 0} \int_\Omega d(t,x)\psi(x)\ \mathrm{d}x = \int_\Omega d_0(x)\psi(x)\ \mathrm{d}x.$$

For $d_0 \in \mathcal{M}(\Omega)$ we can define in a similar way by substituting the duality bracket $\big(\langle \cdot\,,\,\cdot \rangle_{\mathcal{M}(\Omega),C_c(\Omega)}\big)$ for the integral. Proposition 2.1.1 is related to the Lemma 1.2.3.

Suppose a *boundary condition* is given by the normal component of the flux $\mathbf{F}$ on $\partial\Omega$ as

$$\mathbf{F}\cdot\mathbf{n} = \mathbf{F}_{bd},$$

where $\mathbf{n}$ is the outer normal vector to $\partial\Omega$. It is well defined for a smooth function $\mathbf{F}$. In weak formulation it can be included in the equation (2.1.4) as

$$\begin{aligned}
&\int_{t_1}^{t_2}\int_\Omega d(t,x)\partial_t\phi(t,x)\ \mathrm{d}x\ \mathrm{d}t + \int_{t_1}^{t_2}\int_\Omega \mathbf{F}\cdot\nabla_x\phi(t,x)\ \mathrm{d}x\ \mathrm{d}t \\
&= \int_{t_1}^{t_2}\int_{\partial\Omega} \mathbf{F}_{bd}\phi(t,x)\mathrm{dS}_x\ \mathrm{d}t + \int_{t_1}^{t_2}\int_\Omega \sigma(t,x)\phi(t,x)\ \mathrm{d}x\ \mathrm{d}t,
\end{aligned} \tag{2.1.7}$$

where $\phi \in C_c^1((t_1,t_2)\times\overline{\Omega})$.

Finally, we summarize the above discussion and give the weak form of balance law as

$$\begin{aligned}
\left[\int_\Omega d(t,x)\phi(t,x)\ \mathrm{d}x\right]_{t=0}^{t=\tau} &= \int_0^\tau\int_\Omega d(t,x)\partial_t\phi(t,x)\ \mathrm{d}x\ \mathrm{d}t \\
&+ \int_0^\tau\int_\Omega \mathbf{F}\cdot\nabla_x\phi(t,x)\ \mathrm{d}x\ \mathrm{d}t - \int_0^\tau\int_{\partial\Omega}\mathbf{F}_{bd}\phi(t,x)\mathrm{dS}_x\ \mathrm{d}t \\
&+ \int_0^\tau\int_\Omega \sigma(t,x)\phi(t,x)\ \mathrm{d}x\ \mathrm{d}t,
\end{aligned}$$

where $\phi \in C_c^1([0,T) \times \overline{\Omega})$.

2.1.2 Balance laws in fluid dynamics

To obtain the basic equations in fluid dynamics, we first choose the basic principles such as *conservation of mass*, *conservation of momentum*, and *conservation of energy*. Then we find the interrelations between the unknown variables such as density, velocity, momentum, temperature, internal energy, entropy, etc. Finally, depending on the domain, the initial, boundary and far field conditions are included to complete the formulation.

Conservation of mass

The distribution of mass is given by the mass density$\varrho(t,x)$. The *conservation of mass* is expressed as

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{2.1.8}$$

where $\mathbf{u}(t,x)$ is a vector field representing the velocity of the fluid. The equation is also called *equation of continuity* or *continuity equation*. For a bounded domain, if we include the *impermeability boundary condition* on $\partial\Omega$, i.e.

$$\mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega, \tag{2.1.9}$$

the suitable weak form of the balance law is as follows:

$$\int_0^\tau \int_\Omega \left[\varrho \partial_t \phi + \varrho \mathbf{u} \cdot \nabla_x \phi\right] \,\mathrm{d}x \,\mathrm{d}t = \left[\int_\Omega \varrho(t,x)\phi(t,\cdot) \,\mathrm{d}x,\right]_{t=0}^{t=\tau} \tag{2.1.10}$$

for any $\tau \in (0,T)$ and $\phi \in C_c^1([0,T) \times \overline{\Omega})$. The equation remains the same if we consider Ω as a flat torus $\mathbb{T}^d$ instead of a bounded domain. We note that (2.1.10) also holds for $\phi \in C^1([0,T) \times \overline{\Omega})$ if we assume (2.1.9). Thus, if we substitute $\phi \equiv 1$ in a bounded domain with impermeability condition or in a flat torus, we obtain the *total mass conservation*

$$\left[\int_\Omega \varrho(t,x) \,\mathrm{d}x\right]_{t=0}^{t=\tau} = 0, \tag{2.1.11}$$

for any $\tau \in (0,T)$. The situation is delicate for an unbounded domain or the full domain $\mathbb{R}^d$. We have to impose certain limits of density and velocity as $|x| \to \infty$. One can consider a condition like

$$\begin{gathered} \varrho \to \bar{\varrho} \text{ and } \mathbf{u} \to 0 \text{ as } |x| \to \infty, \\ \text{or, } |\varrho - \bar{\varrho}| \to 0 \text{ and } \mathbf{u} \to 0 \text{ as } |x| \to \infty, \end{gathered} \tag{2.1.12}$$

where $\bar{\varrho} > 0$. The above condition is termed as a *far field* condition.

In a physically relevant scenario, density should be considered as a positive or non-negative function. It is clear from Proposition 2.1.1 that we need only $\varrho \in L^\infty(0,T;L^1(\Omega))$ and $\varrho\mathbf{u} \in L^1((0,T)\times\Omega)$ to make sense of (2.1.10). Unfortunately, this is not enough to conclude the strict positivity of ϱ, even if we assume $\varrho_0 > 0$. For $\varrho_0 > 0$, DiPerna and Lions [42] show that the density is strictly positive if the velocity $\mathbf{u}$ satisfies the following condition

$$\|\mathrm{div}_x\mathbf{u}\|_{L^1(0,T;L^\infty(\Omega))} < \infty.$$

We rewrite the equation(2.1.8) in terms of density ϱ and momentum $\mathbf{m}(=\varrho\mathbf{u})$ as

$$\partial_t\varrho + \mathrm{div}_x\mathbf{m} = 0. \tag{2.1.13}$$

The weak form of it can be considered accordingly.

Renormalized continuity equation

The hyperbolic system of conservation laws is not well posed in the class of weak solutions, since it develops discontinuities(Shock). To identify physically admissible solutions in addition to weak solutions, a set of inequalities (entropy inequality) must be added. These solutions are called *entropy solution*, see [98, Chapter 2].

For a transport equation such as the continuity equation, DiPerna and Lions [42] adapt the concept of entropy solution, in the form of *renormalized equation of continuity.* The renormalized equation of continuity is obtained by multiplying the equation of continuity by a smooth function $b'(\varrho)$ with $b \in C^1(0,\infty)$, b is bounded in $[0,\infty)$ and $b(0) = 0$, i.e.

$$\partial_t(b(\varrho)) + \mathrm{div}_x(b(\varrho)\mathbf{u}) = (b(\varrho) - \varrho b'(\varrho))\mathrm{div}_x\mathbf{u}. \tag{2.1.14}$$

A similar weak formulation is also possible by multiplication with appropriate test functions. Moreover, if ϱ and $\mathbf{u}$ solves the equation of continuity in distributional sense and, $\varrho \in L^2(0,T;L^2(\Omega))$ and $\mathbf{u} \in L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^d))$, then then satisfy the renormalized equation of continuity in the sense of distribution, see Feireisl [50, Section 4.1] for further generalizations.

Conservation of momentum

We include Newton's second law to write the *conservation of momentum* as

$$\partial_t(\varrho\mathbf{u}) + \mathrm{div}_x(\varrho\mathbf{u}\otimes\mathbf{u}) = \mathrm{div}_x\mathbb{T} + \varrho\mathbf{f}, \tag{2.1.15}$$

where $\mathbb{T}$ is the Cauchy stress tensor and $\mathbf{f}$ is an external force.

We rewrite the above equation by considering the momentum $\mathbf{m}$ as

$$\partial_t\mathbf{m} + \mathrm{div}_x\left(\frac{\mathbf{m}\otimes\mathbf{m}}{\varrho}\right) = \mathrm{div}_x\mathbb{T} + \varrho\mathbf{f}. \tag{2.1.16}$$

We note that it is reasonable to consider $\mathbf{1}_{\varrho>0}\frac{\mathbf{m}\otimes\mathbf{m}}{\varrho}$ instead of $\frac{\mathbf{m}\otimes\mathbf{m}}{\varrho}$ as the term is not well defined for $\varrho = 0$. This term $\mathrm{div}_x\left(\mathbf{1}_{\varrho>0}\frac{\mathbf{m}\otimes\mathbf{m}}{\varrho}\right)$ or $\mathrm{div}_x\,(\varrho\mathbf{u}\otimes\mathbf{u})$ is called *the convective term.*

By *Stokes law*, we consider the Cauchy stress tensor as

$$\mathbb{T} = \mathbb{S} - p\mathbb{I}, \tag{2.1.17}$$

where $\mathbb{S}$ is the *viscous stress tensor* and p is a scalar field called *pressure.*

Remark 2.1.3. Note that in the case of a perfect(inviscid) fluid, we consider $\mathbb{S} \equiv 0$.

We rewrite the equation (2.1.15) as

$$\partial_t(\varrho\mathbf{u}) + \mathrm{div}_x(\varrho\mathbf{u}\otimes\mathbf{u}) + \nabla_x p = \mathrm{div}_x\mathbb{S} + \varrho\mathbf{f}.$$

A weak formulation of this equation is as follows

$$\begin{aligned}&\left[\int_\Omega \varrho\mathbf{u}(t,\cdot)\cdot\boldsymbol{\varphi}(t,\cdot)\ \mathrm{d}x\right]_{t=0}^{t=\tau}\\ &= \int_0^\tau\int_\Omega \left(\varrho\mathbf{u}\cdot\partial_t\boldsymbol{\varphi} + \varrho\mathbf{u}\otimes\mathbf{u} : \nabla_x\boldsymbol{\varphi} + p\mathrm{div}_x\boldsymbol{\varphi} - \mathbb{S} : \nabla_x\boldsymbol{\varphi} - \varrho\mathbf{f}\cdot\boldsymbol{\varphi}\right)\ \mathrm{d}x\ \mathrm{d}t,\end{aligned} \tag{2.1.18}$$

for any $\tau \in (0,T)$ and any $\boldsymbol{\varphi} \in C_c^1([0,T]\times\Omega;\mathbb{R}^d)$.

In the context of a perfect fluid ($\mathbb{S} = 0$) that satisfies the *impermeability boundary condition* (2.1.9), the set of admissible test functions extends to

$$C^1([0,T]\times\overline{\Omega};\mathbb{R}^d) \text{ with } \boldsymbol{\varphi}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega. \tag{2.1.19}$$

For a viscous fluid ($\mathbb{S} \neq 0$) the weak formulation (2.1.18) is compatible with the *complete slip* or *Navier Slip* boundary condition, i.e.

$$\mathbf{u}\cdot\mathbf{n} = 0 \text{ and } [\mathbb{S}\cdot\mathbf{n}]_{\tan} = 0 \text{ on } \partial\Omega, \tag{2.1.20}$$

where $[\mathbb{S}\cdot\mathbf{n}]_{\tan} = 0$ on $\partial\Omega$ means that the tangential component of the normal viscous stress vanishes at the boundary.

Further we observe that the following weak formulation:

$$\begin{aligned}&\left[\int_\Omega \varrho\mathbf{u}(t,\cdot)\cdot\boldsymbol{\varphi}(t,\cdot)\ \mathrm{d}x\right]_{t=0}^{t=\tau}\\ &= \int_0^\tau\int_\Omega \left(\varrho\mathbf{u}\cdot\partial_t\boldsymbol{\varphi} + \varrho\mathbf{u}\otimes\mathbf{u} : \nabla_x\boldsymbol{\varphi} + p\mathrm{div}_x\boldsymbol{\varphi} - \mathbb{S} : \nabla_x\boldsymbol{\varphi} - \varrho\mathbf{f}\cdot\boldsymbol{\varphi}\right)\ \mathrm{d}x\ \mathrm{d}t,\end{aligned} \tag{2.1.21}$$

for any $\tau \in (0,T)$ and any $\boldsymbol{\varphi} \in C_c^1([0,T]\times\Omega;\mathbb{R}^d)$ corresponds to the *no-slip* boundary condition, i.e.

$$\mathbf{u} = 0 \text{ on } \partial\Omega. \tag{2.1.22}$$

Conservation of energy

The *first law of thermodynamics* requires that total energy be conserved. The total energy is a scalar function $\mathrm{e} = \mathrm{e}(t, x)$. Multiplying the momentum equation by $\mathbf{u}$, gives the expression for the kinetic energy $\mathrm{e}_{\mathrm{kin}}$. Since $\mathrm{e}_{\mathrm{kin}} = \frac{1}{2}\varrho|\mathbf{u}|^2$, the time evolution is obtained as

$$\partial_t \mathrm{e}_{\mathrm{kin}} + \mathrm{div}_x(\mathrm{e}_{\mathrm{kin}}\mathbf{u}) - \mathrm{div}_x(\mathbb{S}\mathbf{u}) = -\mathbb{S} \colon \nabla_x \mathbf{u} + p\mathrm{div}_x\mathbf{u} + \varrho\mathbf{f} \cdot \mathbf{u}.$$

Thus, the internal energy $\mathrm{e}_{\mathrm{int}}(= \varrho e)$ should satisfy

$$\partial_t \mathrm{e}_{\mathrm{int}} + \mathrm{div}_x(\mathrm{e}_{\mathrm{int}}\mathbf{u}) + \mathrm{div}_x\mathbf{q} = \mathbb{S} \colon \nabla_x \mathbf{u} - p\mathrm{div}_x\mathbf{u},$$

where $\mathbf{q}$ is the *diffusive internal energy flux* and e is *specific internal energy.* In a bounded domain, we assume that the system is thermally insulated. i.e.

$$\mathbf{q} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega.$$

Together, we have the energy balance for the total energy $\mathrm{e} = \mathrm{e}_{\mathrm{kin}} + \mathrm{e}_{\mathrm{int}}$ as

$$\partial_t \mathrm{e} + \mathrm{div}_x((\mathrm{e} + p)\mathbf{u}) + \mathrm{div}_x(\mathbf{q} - \mathbb{S}\mathbf{u}) = \varrho\mathbf{f} \cdot \mathbf{u}. \tag{2.1.23}$$

Similarly, we obtain the weak form of the energy balance by multiplying the above equation by test functions.

If $\mathbf{f} = 0$, then in a bounded domain with suitable boundary conditions, we observe the energy balance:

$$\left[\int_\Omega \mathrm{e}(t, x)\ \mathrm{d}x\right]_{t=0}^{t=\tau} = 0, \tag{2.1.24}$$

for any $\tau \in (0, T)$.

Constitutive relation

We can clearly see that so far we have more unknowns compared to the number of equations, so the system is not closed. The thermodynamic variables are interrelated by various *constitutive equations.* Here we consider a few of them.

- **Perfect gas equation of state:** We introduce the absolute temperature ϑ. The equation of state is given by the Boyle-Mariotte law, i.e.

  $$e = c_v\vartheta,\ c_v = \frac{1}{\gamma - 1},\ \text{where } \gamma > 1 \text{ is the adiabatic constant.} \tag{2.1.25}$$

  The relation between pressure p and absolute temperature ϑ is

  $$p = \varrho\vartheta.$$

  As a simple consequence of it, we have

  $$(\gamma - 1)\varrho e = p.$$

- **Fourier's law of heat conduction:** Similarly, a simplest possible choice for heat conduction is given by
$$\mathbf{q} = -\kappa \nabla_x \vartheta, \tag{2.1.26}$$
where κ is the heat conductivity coefficient.

- **Viscous stress:** In general, the viscous stress is a function of the velocity gradient $\nabla_x \mathbf{u}$ and the temperature ϑ. It is given by the Newton's rheological law as
$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu(\vartheta)\left(\frac{\nabla_x \mathbf{u} + \nabla_x^T \mathbf{u}}{2} - \frac{1}{d}(\mathrm{div}_x \mathbf{u})\mathbb{I}\right) + \lambda(\vartheta)(\mathrm{div}_x \mathbf{u})\mathbb{I}, \tag{2.1.27}$$
where μ and λ are positive and termed as the *shear* and *bulk* viscosity coefficients, respectively.

Entropy balance

Invoking the *second law of thermodynamics* we introduce the entropy $s = s(t, x)$. For the Boyle-Mariotte equation of state, the entropy with respect to the standard variables has the form
$$s(\varrho, \vartheta) = \log(\vartheta^{c_v}) - \log(\varrho).$$
For smooth solutions, the entropy equations (2.4.4) can be derived directly from the existing field equations. It is as follows
$$\partial_t(\varrho s) + \mathrm{div}_x(\varrho s \mathbf{u}) + \mathrm{div}_x \frac{\mathbf{q}}{\vartheta} = \frac{1}{\vartheta}\left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q}\cdot\nabla_x\vartheta}{\vartheta}\right). \tag{2.1.28}$$
Note that Fourier's law and the structural assumption about the viscous stress tensor $\mathbb{S}$ help us to conclude that the right hand side of the above equation is non-negative, which is necessary for any physically relevant process.

Furthermore, we note that the state variables satisfy the *Gibbs relation:*
$$\vartheta D s(\varrho, \vartheta) = De(\varrho, \vartheta) + p(\varrho, \vartheta) D\left(\frac{1}{\varrho}\right).$$

Remark 2.1.4. In a more general setup, one can choose any equation of state compatible with the Gibbs relation instead of the Boyle-Mariotte equation of state.

Remark 2.1.5. In the class of smooth solutions, the energy balance, the internal energy balance and the entropy balance are same.

Summarizing the above discussion we get the system
$$\begin{aligned}
&\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0,\\
&\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) = \mathrm{div}_x \mathbb{T} + \varrho \mathbf{f},\\
&\partial_t(\varrho s) + \mathrm{div}_x(\varrho s \mathbf{u}) + \mathrm{div}_x \frac{\mathbf{q}}{\vartheta} = \frac{1}{\vartheta}\left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q}\cdot\nabla_x\vartheta}{\vartheta}\right).
\end{aligned} \tag{2.1.29}$$
The above system is called the *Navier–Stokes–Fourier* system with the Boyle-Mariotte equation of state, where $\mathbb{S}$ and $\mathbf{q}$ follow (2.1.27) and (2.1.26) respectively.

2.1.3 Perfect fluid: Euler system

In the context of the perfect fluid, we have $\mathbf{q} = 0$ and $\mathbb{S} = 0$. Then the total energy balance is

$$\partial_t \mathrm{e} + \mathrm{div}_x \left((\mathrm{e} + p)\frac{\mathbf{m}}{\varrho} \right) = \mathbf{f} \cdot \mathbf{m}.$$

Moreover, we obtain the entropy equation in the following form:

$$\partial_t(\varrho s) + \mathrm{div}_x(s\mathbf{m}) = 0. \tag{2.1.30}$$

So we rewrite the system as

$$\begin{aligned}
&\partial_t \varrho + \mathrm{div}_x(\mathbf{m}) = 0, \\
&\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x \left(\mathbf{1}_{\{\varrho>0\}} \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) + \nabla_x p = \varrho \mathbf{f}, \\
&\partial_t(\varrho s) + \mathrm{div}_x(s\mathbf{m}) = 0.
\end{aligned} \tag{2.1.31}$$

Considering an appropriate equation of state, we call this system as *complete Euler system*.

The absence of the viscous term in the energy balance means that there is no *a priori* estimate for the velocity gradient. A weak formulation of the system is based on the energy inequality

$$\left[\int_\Omega \mathrm{e}(t) \,\mathrm{d}x \right]_{t=0}^{t=\tau} \leq 0,$$

for $\tau \in (0, T)$. Also, in the weak formulation we can relax the entropy balance and provide the entropy inequality, i.e.

$$\partial_t(\varrho s) + \mathrm{div}_x(s\mathbf{m}) \geq 0. \tag{2.1.32}$$

We introduce the total entropy S by $S = \varrho s$ and reformulate (2.1.32) as

$$\partial_t S + \mathrm{div}_x \left(\mathbf{1}_{\{\varrho>0\}} S \frac{\mathbf{m}}{\varrho} \right) \geq 0.$$

We recall that the total energy e of the fluid

$$\mathrm{e} = \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + \varrho e,$$

consists of the kinetic energy $\frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho}$ and the internal energy ϱe. If we consider $\mathbf{m} = \varrho \mathbf{u}$, then we find that the map $(\varrho, \mathbf{u}) \mapsto \frac{1}{2}\varrho|\mathbf{u}|^2$ is not convex, although it is continuous for $\varrho \geq 0$. On the other hand, we have the observation that the map $(\varrho, \mathbf{m}) \mapsto \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho}$ is convex for $\varrho > 0$. So, in the context of the energy inequality, it is better to think of the kinetic energy in terms of $\varrho, \mathbf{m}$ so that we can use the properties of a convex function if needed.

The total entropy S helps us to rewrite the pressure p and the specific internal energy e in terms of ϱ and S as

$$p = p(\varrho, S) = \varrho^\gamma \exp\left(\frac{S}{c_v \varrho}\right), \; e = e(\varrho, S) = \frac{1}{\gamma - 1}\varrho^{\gamma-1} \exp\left(\frac{S}{c_v \varrho}\right).$$

The advantage of the above way of writing is that $(\varrho, S) \mapsto \varrho^\gamma \exp\left(\frac{S}{c_v\varrho}\right)$ is a strictly convex function for $\varrho > 0$, see Breit et al. [16, Lemma 3.1]. This leads to a possible *energy extension* in $\mathbb{R}^{d+2}$:

$$(\varrho, \mathbf{m}, S) \mapsto \mathrm{e}(\varrho, \mathbf{m}, S) \equiv \begin{cases} \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + c_v \varrho^\gamma \exp\left(\frac{S}{c_v\varrho}\right), \text{ if } \varrho > 0, \\ 0, \text{ if } \varrho = \mathbf{m} = 0, \; S \leq 0, \\ \infty, \text{ otherwise} \end{cases} \tag{2.1.33}$$

We conclude that the map $(\varrho, \mathbf{m}, S) \mapsto \mathrm{e}(\varrho, \mathbf{m}, S)$ is a convex l.s.c. function and it is strictly convex in the domain of positivity, that is, at the points, where it is finite and positive.

Perfect fluid following barotropic pressure

Our next consideration is the Euler system with barotropic pressure, i.e. the pressure depends only on the density. Here we consider the pressure

$$p(\varrho) = a\varrho^\gamma, \; \gamma \geq 1, \; a > 0,$$

where γ is called the adiabatic constant.

In this case, we observe that the internal energy satisfies

$$\partial_t \mathrm{e}_{\mathrm{int}} + \mathrm{div}_x(\mathrm{e}_{\mathrm{int}}\mathbf{u}) = -p(\varrho)\mathrm{div}_x\mathbf{u},$$

We define a *pressure potential* (P) as

$$P(\varrho) = \varrho \int_1^\varrho \frac{p(z)}{z^2} \, \mathrm{d}z. \tag{2.1.34}$$

The pressure p and the pressure potential P satisfy the relation

$$\varrho P'(\varrho) - P(\varrho) = p(\varrho).$$

Thus, if we substitute P for b in the renormalized continuity equation (2.1.14), we get

$$\partial_t P(\varrho) + \mathrm{div}_x(P(\varrho)\mathbf{u}) = -p(\varrho)\mathrm{div}_x\mathbf{u}.$$

Therefore, the total energy e of the fluid is given by

$$\mathrm{e} = \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho).$$

This shows that the internal energy in the case of a barotropic fluid is represented by the pressure potential.

Remark 2.1.6. If $\gamma > 1$, we can consider a pressure potential as

$$P(\varrho) = \frac{a}{\gamma - 1}\varrho^\gamma.$$

2.1.4 Viscous fluid: Compressible Navier–Stokes system

We have discussed the Navier–Stokes–Fourier system (2.1.29) describing a heat conductive viscous fluid. Now we assume the barotropic pressure law, i.e. $p = p(\varrho)$, and then refer to the system as the Navier-Stokes system. Here $\mathbb{S}(\nabla_x \mathbf{u})$ is *Newtonian stress tensor* defined by

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu\left(\frac{\nabla_x \mathbf{u} + \nabla_x^T \mathbf{u}}{2} - \frac{1}{d}(\mathrm{div}_x \mathbf{u})\mathbb{I}\right) + \lambda(\mathrm{div}_x \mathbf{u})\mathbb{I},$$

where $\mu > 0$ and $\lambda \geq 0$ are constant. We define the pressure potential in a similar way as in (2.1.34). We consider a bounded domain Ω with Lipschitz boundary and impermeability boundary condition (2.1.9). Then we integrate the energy equation in space and get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mathrm{E}(t)\ \mathrm{d}x + \int_\Omega \mathbb{S}(\nabla_x \mathbf{u})\colon \nabla_x \mathbf{u}\ \mathrm{d}x = \int_\Omega \varrho \mathbf{f} \cdot \mathbf{u}\ \mathrm{d}x,$$

with the *total energy*

$$\mathrm{E}(t) = \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho)\right)(t)\ \mathrm{d}x.$$

If we assume $\mathbf{f} = 0$, we find that in this case the total energy is a non-increasing function of t. Once $\nabla_x \mathbf{u} \neq 0$, the total energy is not conserved. In the context of this simplified model, the resulting temperature changes and their influence on fluid motion are not considered, so that an exact energy balance is not possible. Instead, we will focus on the energy inequality for weak solutions.

Pressure Laws

In this thesis we are concerned with various barotropic pressure laws. In the introduction we mention some of them.

Isentropic equation of state: The pressure p and the density ϱ of the fluid are interrelated by :

$$p(\varrho) = a\varrho^\gamma, \text{ with } \gamma \geq 1, \ a > 0. \tag{2.1.35}$$

In a perturbation of the above setting, the pressure p and the density ϱ of the fluid are related by :

$$p(\varrho) = a\varrho^\gamma + q(\varrho), \text{ with } \gamma \geq 1, \ a > 0 \text{ and } q \in C[0,\infty). \tag{2.1.36}$$

Instead of considering $a\varrho^\gamma$, we can consider a more general barotropic equation of state

$$\begin{aligned} &p = p(\varrho), p \in C^1[0,\infty), \ p(0) = 0, \ p' > 0, \text{ in } (0,\infty), \\ &\text{and } \liminf_{\varrho\to\infty} \frac{p(\varrho)}{\varrho^\gamma} > 0 \text{ with } \ \gamma \geq 1, \end{aligned} \tag{2.1.37}$$

and its non-monotone counterpart as

$$\begin{aligned} &p(\varrho) = h(\varrho) + q(\varrho), \text{ with, } q \in C[0,\infty) \text{ such that } p > 0 \\ &h \in C^1[0,\infty), \ h(0) = 0, \ h' > 0, \text{ in } (0,\infty), \text{ and } \liminf_{\varrho\to\infty} \frac{h(\varrho)}{\varrho^\gamma} > 0 \text{ with } \ \gamma \geq 1. \end{aligned} \tag{2.1.38}$$

In both (2.1.36) and (2.1.38) cases, we focus on two possibilities of q :

- $q \in C^{0,1}[0,\infty)$ i.e. q is globally Lipschitz,
- $q \in C^1_c(0,\infty)$, i.e. q is compactly supported.

Singular pressure law: Finally, we consider a singular pressure law, where the pressure p and the density ϱ of the fluid are interrelated by a hard-sphere equation of state in the interval $[0,\bar\varrho)$:

$$p \in C^1[0,\bar\varrho), \ p(0) = 0, p' > 0 \text{ on } (0,\bar\varrho), \ \lim_{\varrho\to\bar\varrho} p(\varrho) = +\infty. \tag{2.1.39}$$

This is also known as *hard-sphere pressure law.* The pressure law (2.1.39) is motivated by two famous models for viscous fluids, namely Van Der Waal's equation of state and hard-sphere law, modeled by Carnahan-Sterling. Van Der Waal's equation is given by

$$p(\varrho) = C\frac{\bar p(\varrho)}{\bar\varrho - \varrho},$$

where $\bar p$ is an arbitrary polynomial and $\bar\varrho$, C are positive constants. In general, it describes a non-monotone pressure-density relation. The Carnahan-Sterling model reflects the hard-sphere model and is given by

$$p(\varrho) = C\frac{\tilde p(\varrho)}{(\bar\varrho - \varrho)^3},$$

with a polynomial $\tilde{p}$ and positive constants $\bar{\varrho}$, C . We therefore consider the non-monotone counterpart of (2.1.39) as

$$\begin{aligned} &p \in C^1[0, \bar{\varrho}),\ p(\varrho) = h(\varrho) + q(\varrho), h(0) = 0, \\ &h^{'} > 0 \text{ on } (0, \bar{\varrho}),\ \lim_{\varrho \to \bar{\varrho}} h(\varrho) = +\infty,\ q \in C^1_c(0, \bar{\varrho}). \end{aligned} \tag{2.1.40}$$

Pressure Potential

In general we define *pressure potential* (P) as

$$P(\varrho) = \varrho \int_1^{\varrho} \frac{p(z)}{z^2}\, \mathrm{d}z. \tag{2.1.41}$$

As a trivial consequence of above we obtain

$$\varrho P^{'}(\varrho) - P(\varrho) = p(\varrho) \text{ and } \varrho P^{''}(\varrho) = p'(\varrho) \text{ for } \varrho > 0, \tag{2.1.42}$$

Remark 2.1.7. We notice that instead of lower integral limit 1, we can consider any constant $b > 0$. In the context of hard-sphere pressure law (2.1.39) and (2.1.40) we choose $\frac{\bar{\varrho}}{2}$ instead of 1.

Remark 2.1.8. If $p(\varrho) = a\varrho^{\gamma}$, with $\gamma > 1,\ a > 0$ we consider $P(\varrho) = \frac{a}{\gamma - 1}\varrho^{\gamma}$. If $\gamma = 1,\ a > 0$, we consider pressure potential as $P(\varrho) = a\varrho \ln \varrho$.

2.2 Compressible Navier–Stokes system

We recall the compressible Navier–Stokes equation in the time-space cylinder $(0, T) \times \Omega$:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{2.2.1}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}) + \varrho \mathbf{f}. \tag{2.2.2}$$

Here $\mathbb{S}(\nabla_x \mathbf{u})$ is *Newtonian stress tensor* defined by

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\frac{\nabla_x \mathbf{u} + \nabla_x^T \mathbf{u}}{2} - \frac{1}{d}(\mathrm{div}_x \mathbf{u})\mathbb{I} \right) + \lambda(\mathrm{div}_x \mathbf{u})\mathbb{I}, \tag{2.2.3}$$

where $\mu > 0$ and $\lambda > 0$ are the *shear* and *bulk* viscosity coefficients, respectively, and $\varrho \mathbf{f}$ is a *source term.* Also the pressure p satisfies one of the following relations (2.1.35)-(2.1.40).

Boundary, far field and initial conditions

Boundary conditions: If Ω is a bounded domain then we invoke different boundary conditions:

- **Periodic boundary condition:**

$$\Omega = \mathbb{T}^d. \tag{2.2.4}$$

- **No-slip boundary condition:**

$$\mathbf{u} = 0 \text{ on } \partial\Omega. \tag{2.2.5}$$

- **Navier slip boundary condition:**

$$\mathbf{u} \cdot \mathbf{n} = 0 \text{ and } [\mathbb{S} \cdot \mathbf{n}]_{\tan} = 0 \text{ on } \partial\Omega, \tag{2.2.6}$$

where $\mathbf{n}$ is the outer normal vector on $\partial\Omega$.

Far field condition: For an unbounded domain, we introduce the *far field condition* as,

$$\varrho \to \varrho_\infty, \ \mathbf{u} \to \mathbf{u}_\infty \text{ as } x \in \Omega \text{ and } |x| \to \infty, \tag{2.2.7}$$

with $\varrho_\infty > 0, \text{ and } \mathbf{u}_\infty \in \mathbb{R}^d$.
Initial conditions: We supplement an initial data $(\varrho_0, (\varrho\mathbf{u})_0)$ as

$$\varrho(0, \cdot) = \varrho_0, \ (\varrho\mathbf{u})(0, \cdot) = (\varrho\mathbf{u})_0. \tag{2.2.8}$$

2.2.1 Strong solution

For the sake of simplicity we consider a bounded domain Ω with no slip boundary condition (2.2.5) and pressure law (2.1.35)-(2.1.38). We prescribe the initial state

$$\varrho(0, \cdot) = \varrho_0, \ \varrho\mathbf{u}(0, \cdot) = (\varrho\mathbf{u})_0, \tag{2.2.9}$$

with $\varrho_0, (\varrho\mathbf{u})_0$ are as smooth as needed and $\varrho_0 > 0$ to avoid the degenerate vacuum regime. Then the compressible Navier–Stokes system((2.2.1)-(2.2.3)) is locally well posed in the Sobolev space $W^{k,2}(\Omega)$ for large k, see e.g. Kazhikov and Shelukhin [90], Valli and Zajaczkowski [118], and Matsumara and Nishida[100]. We state the following theorem that ensures the local existence in a bounded domain Ω of class $C^{2+\sigma}$ for $\sigma > 0$, with no-slip boundary condition:

Theorem 2.2.1. *Let $p \in C^\infty(0, \infty), p(\varrho) > 0$ for $\varrho > 0$, $\mu > 0$, $\eta > 0$, and*

$$\varrho_0 > 0, \ \varrho_0 \in W^{k,2}(\Omega), \mathbf{u}_0 = \frac{(\varrho\mathbf{u})_0}{\varrho_0} \in W_0^{k,2}(\Omega) \ \textit{for } k > \left[\frac{d}{2}\right] + 1.$$

Moreover, the initial data satisfies the compatibility condition:

$$\begin{aligned} &\mathbf{u}_0 = 0 \ \textit{on } \partial\Omega, \\ &\nabla_x p(\varrho_0) = \mu\left(\frac{\nabla_x \mathbf{u}_0 + \nabla_x^T \mathbf{u}_0}{2} - \frac{1}{d}(div_x \mathbf{u}_0)\mathbb{I}\right) + \lambda(div_x \mathbf{u}_0)\mathbb{I} \ \textit{on } \partial\Omega. \end{aligned} \tag{2.2.10}$$

Then there exists a positive $T > 0$ such that the problem (2.2.1)-(2.2.3) *admits a strong solution* $(\varrho, \mathbf{u})$ *in* $(0,T) \times \Omega$*, unique in the class*

$$\begin{aligned} &\varrho \in C([0,T]; W^{k,2}(\Omega)),\ \partial_t \varrho \in C([0,T]; W^{k-1,2}(\Omega)), \\ &\mathbf{u} \in C([0,T]; W_0^{k,2}(\Omega;\mathbb{R}^d)), \partial_t \mathbf{u} \in C([0,T]; W^{k-2,2}(\Omega)). \end{aligned} \tag{2.2.11}$$

Remark 2.2.2. For a bounded domain, we use the Sobolev embedding theorem (1.1.1) to conclude

$$W^{k,2}(\Omega) \hookrightarrow C(\overline{\Omega}),\ \text{for } k > \left[\frac{d}{2}\right] + 1.$$

The assumption in the Theorem 2.2.1 ensures that $(\varrho, \mathbf{u})$ is continuous. If we consider large k, then we obtain more regular solution.

The Navier–Stokes system admits global-in-time solutions provided the initial data close enough to an equilibrium state. Here we state a result that was shown by Matsumara and Nishida [100] (cf. also Valli and Zajaczkowski[118])

Theorem 2.2.3. *Let* $p \in C^\infty(0,\infty), p'(\varrho) > 0$ *for* $\varrho > 0$, $\mu > 0$, $\eta > 0$. *Let a positive constant* $\bar{\varrho} > 0$ *be given. Then there exists* $\epsilon > 0$ *such that for any initial data*

$$\begin{aligned} \varrho_0 \in W^{3,2}(\Omega),\ \mathbf{u}_0 \in W_0^{3,2}(\Omega),\ \int_\Omega (\varrho_0 - \bar{\varrho})\ dx = 0, \\ \|\varrho_0 - \bar{\varrho}\|_{W^{3,2}(\Omega)} + \|\mathbf{u}_0\|_{W^{3,2}(\Omega;\mathbb{R}^d)} < \epsilon, \end{aligned}$$

with the compatibility condition (2.2.10)*, the Navier–Stokes system admits a unique strong solution* $[\varrho, \mathbf{u}]$ *defined on the time interval* $(0.\infty)$*,*

$$\begin{aligned} &\varrho \in C([0,T]; W^{3,2}(\Omega)),\ \partial_t \varrho \in C([0,T]; W^{2,2}(\Omega)), \\ &\mathbf{u} \in C([0,T]; W_0^{3,2}(\Omega;\mathbb{R}^d)), \partial_t \mathbf{u} \in C([0,T]; W^{1,2}(\Omega)). \end{aligned}$$

such that

$$\varrho(t,\cdot) \to \bar{\varrho}\ \textit{in}\ W^{3,2}(\Omega),\ \mathbf{u}(t,\cdot) \to 0\ \textit{in}\ W^{3,2}(\Omega;\mathbb{R}^d)\ \textit{as}\ t \to \infty.$$

Remark 2.2.4. We note that in the introduction we also give the term classical solution. Although various literature points out the subtle difference between classical and strong solution. In this thesis, when we speak of classical solution or strong solution, we imply that these solutions are regular enough and solve the system in a pointwise sense.

2.2.2 Weak solution

Here we discuss the weak solutions of the compressible Navier–Stokes system. In general, weak solutions satisfy the equations in terms of distributions. A single equation in the weak formulation is replaced by a family of integral identities satisfied for all sufficiently smooth test functions.

Isentropic type pressure laws

The isentropic equation of state is given in (2.1.35). A general pressure law of isentropic type and its non-monotone variants are described in (2.1.36)–(2.1.38).

Bounded domain: We give the definition of a *dissipative weak solution* of the system in a bounded domain Ω. First, we consider a no slip boundary condition (2.2.5).

We call an initial data $(\varrho_0, (\varrho\mathbf{u})_0)$ as *finite energy initial data*, if it satisfies the following conditions

$$0 \leq \varrho_0 \text{ in } \Omega, \text{ and } E_0 = \int_\Omega \left(\frac{|(\varrho\mathbf{u})_0|^2}{\varrho_0} + P(\varrho_0) \right) \,\mathrm{d}x < \infty. \tag{2.2.12}$$

Definition 2.2.5. We say that $(\varrho, \mathbf{u})$ is a *dissipative weak solution in* $(0,T)\times\Omega$ to the system of equations (2.2.1)-(2.2.3), with the no-slip condition (2.2.5), the finite energy initial data (2.2.12), and the source term $\mathbf{f} \in L^\infty((0,T)\times\Omega)$ if the following is satisfied:

- **Regularity class:** $0 \leq \varrho \in C_w([0,T]; L^\gamma(\Omega))$, $\mathbf{u} \in L^2(0,T; W_0^{1,2}(\Omega;\mathbb{R}^d))$, $\varrho\mathbf{u} \in C_w([0,T]; L^{\frac{2\gamma}{\gamma+1}}(\Omega;\mathbb{R}^d))$, $p(\varrho) \in L^\infty(0,T; L^1(\Omega))$.
- **Renormalized equation of continuity:** For any $\tau \in (0,T)$ and any $\varphi \in C_c^1([0,T]\times\overline{\Omega})$, it holds

$$\begin{aligned} &\left[\int_\Omega (\varrho + b(\varrho))\varphi \,\mathrm{d}x \right]_{t=0}^{t=\tau} \\ &= \int_0^\tau \int_\Omega [(\varrho + b(\varrho))\partial_t\varphi + (\varrho + b(\varrho))\mathbf{u}\cdot\nabla_x\varphi + (b(\varrho) - \varrho b'(\varrho))\mathrm{div}_x\mathbf{u}\,\varphi] \,\mathrm{d}x \,\mathrm{d}t, \end{aligned} \tag{2.2.13}$$

  where $b \in C^1[0,\infty)$ and there exists a $r_b > 0$ such that $b'(x) = 0$, for all $x > r_b$.
- **Momentum equation:** For any $\tau \in (0,T)$ and any $\boldsymbol{\varphi} \in C_c^1([0,T]\times\Omega;\mathbb{R}^d)$, it holds

$$\begin{aligned} &\left[\int_\Omega \varrho\mathbf{u}(\tau,\cdot)\cdot\boldsymbol{\varphi}(\tau,\cdot) \,\mathrm{d}x \right]_{t=0}^{t=\tau} \\ &= \int_0^\tau \int_\Omega [\varrho\mathbf{u}\cdot\partial_t\boldsymbol{\varphi} + \varrho\mathbf{u}\otimes\mathbf{u} : \nabla_x\boldsymbol{\varphi} + p(\varrho)\mathrm{div}_x\boldsymbol{\varphi} - \mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\boldsymbol{\varphi} - \varrho\mathbf{f}\cdot\boldsymbol{\varphi}] \mathrm{d}x\mathrm{d}t. \end{aligned} \tag{2.2.14}$$

- **Energy inequality:** The total energy E is defined as

$$E(\tau) = \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho) \right)(\tau,\cdot) \,\mathrm{d}x$$

for $\tau \in [0, T)$. It satisfies

$$E(\tau) + \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \, \mathrm{d}x \, \mathrm{d}t \leq E_0 + \int_0^\tau \int_\Omega \varrho \mathbf{u} \cdot \mathbf{f} \, \mathrm{d}x \, \mathrm{d}t \tag{2.2.15}$$

for a.e. $\tau > 0$.

Remark 2.2.6. A similar definition is possible for the periodic boundary condition(2.2.4). Here we have to consider $\mathbf{u} \in L^2(0, T; W^{1,2}(\mathbb{T}^d))$. The other state variables belong to the same regularity class and the identities and inequalities (2.2.13)-(2.2.15) remain the same. We also refer *dissipative weak solution* as *finite energy weak solution.*

Remark 2.2.7. The energy inequality can be rewritten using the following observation:

$$\mathbb{S}(\nabla_x \mathbf{u}) \colon \nabla_x \mathbf{u} = \mu \mathbb{D}_0(\nabla_x \mathbf{u}) \colon \mathbb{D}_0(\nabla_x \mathbf{u}) + \lambda |\mathrm{div}_x \mathbf{u}|^2.$$

If we consider the Navier slip boundary condition (2.1.20) to the system, the definition is slightly modified.

Definition 2.2.8. We say $(\varrho, \mathbf{u})$ is a dissipative weak solution of the system (2.2.1)-(2.2.3) with finite energy initial data (2.2.12) and boundary condition (2.2.6) if we have

$$\mathbf{u} \in L^2(0, T; W^{1,2}(\Omega; \mathbb{R}^d)) \text{ with } \mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega.$$

The other variables $\varrho, \varrho\mathbf{u}$ and $P(\varrho)$ belong to the same regularity class as in the Definition 2.2.5. The renormalized continuity equation (2.2.13) and the energy inequality (2.2.15) remain same. For the momentum equation, the integral identity

$$\begin{aligned} &\left[\int_\Omega \varrho \mathbf{u}(\tau, \cdot) \cdot \boldsymbol{\varphi}(\tau, \cdot) \, \mathrm{d}x \right]_{t=0}^{t=\tau} \\ &= \int_0^\tau \int_\Omega [\varrho \mathbf{u} \cdot \partial_t \boldsymbol{\varphi} + \varrho \mathbf{u} \otimes \mathbf{u} : \nabla_x \boldsymbol{\varphi} + p(\varrho) \mathrm{div}_x \boldsymbol{\varphi} - \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \boldsymbol{\varphi} - \varrho \mathbf{f} \cdot \mathbf{u}] \mathrm{d}x \, \mathrm{d}t. \end{aligned} \tag{2.2.16}$$

holds for any $\tau \in (0, T)$ and any $\boldsymbol{\varphi} \in C^1([0, T] \times \overline{\Omega}; \mathbb{R}^d)$ with $\boldsymbol{\varphi} \cdot \mathbf{n} = 0$ on $\partial\Omega$.

Definition in unbounded domain: Here we give the definition in the domain $\mathbb{R}^d$. First, we need to include the far field condition(2.2.7) suitably. For a simpler consideration we choose $\mathbf{f} = 0$. We choose $(\bar{\varrho}, \mathbf{0})$ as a static solution of the system with $\bar{\varrho}$ constant. The we perform the following modification of the system:

- **Initial data:** It satisfies $\varrho_0 \geq 0$, $\varrho_0 \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and

$$E_0 = \int_{\mathbb{R}^d} \left(\frac{1}{2} \frac{|(\varrho \mathbf{u})_0|^2}{\varrho_0} + (P(\varrho_0) - (\varrho_0 - \bar{\varrho}) P'(\bar{\varrho})) - P(\bar{\varrho}) \right) \mathrm{d}x < \infty. \tag{2.2.17}$$

- **Regularity class:** $\varrho - \bar{\varrho} \in C_w([0,T]; L^2 + L^\gamma(\mathbb{R}^d))$, $\mathbf{u} \in L^2(0,T; D_0^{1,2}(\mathbb{R}^d;\mathbb{R}^d))$, $\varrho\mathbf{u} \in C_w([0,T]; L^2 + L^{\frac{2\gamma}{\gamma+1}}(\mathbb{R}^d;\mathbb{R}^d))$.
- The renormalized continuity equation holds for the class of test functions is $C_c^1([0,T] \times \mathbb{R}^d)$. The momentum equation remains true for text functions in $C_c^1([0,T] \times \mathbb{R}^d;\mathbb{R}^d)$.
- The far field conditions are incorporated through the energy inequality. The total energy E is defined in $[0,T)$ as

$$E(\tau) = \int_{\mathbb{R}^d} \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + (P(\varrho) - (\varrho - \bar{\varrho})P'(\bar{\varrho})) - P(\bar{\varrho}) \right)(\tau,\cdot)\ \mathrm{d}x. \tag{2.2.18}$$

  It satisfies

$$E(\tau) + \int_0^\tau \int_{\mathbb{R}^d} \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u}\ \mathrm{d}x\ \mathrm{d}t \le E_0 \tag{2.2.19}$$

  for a.e. $\tau > 0$.

Here we notice a different form of total energy(2.2.18) in energy inequality(2.2.19). We now attempt to provide an informal justification for such a consideration.

To reduce the complexity, we assume a monotone pressure law (2.1.35) or (2.1.37) with pressure potential P. For $R > 0$, we consider $B(0,R) \subset \mathbb{R}^d$. We also assume the system is provided by no-slip boundary condition, i.e., $\mathbf{u}_R = 0$ on $\partial B(0,R)$ where $(\varrho_R, \mathbf{u}_R)$ denotes a weak solution following Definition 2.2.5 in $B(0,R)$ with initial data $(\varrho_{0,R}, (\varrho\mathbf{u})_{0,R} = (\mathbf{1}_{B(0,R)}\varrho_0, \mathbf{1}_{B(0,R)}(\varrho\mathbf{u})_0))$.

The energy inequality in $B(0,R)$ is given by

$$E_R(\tau) + \int_0^\tau \int_{B(0,R)} \mathbb{S}(\nabla_x \mathbf{u}_R) : \nabla_x \mathbf{u}_R\ \mathrm{d}x\ \mathrm{d}t \le E_{0,R}$$

for a.e. $\tau > 0$ with

$$E_R(\tau) = \int_{B(0,R)} \left(\frac{1}{2}\varrho_R|\mathbf{u}_R|^2 + P(\varrho_R) \right)(\tau,\cdot)\ \mathrm{d}x.$$

In $B(0,R)$, the conservation of mass yields

$$\int_{B(0,R)} (\varrho_R - \bar{\varrho})P'(\bar{\varrho})(\tau,\cdot)\ \mathrm{d}x = \int_{B(0,R)} (\varrho_{0,R} - \bar{\varrho})P'(\bar{\varrho})\ \mathrm{d}x.$$

Hence, we rewrite the energy inequality as

$$\begin{aligned} E_R(\tau) - \int_{B(0,R)} ((\varrho_R - \bar{\varrho})P'(\bar{\varrho}) - P(\bar{\varrho}))(\tau,\cdot)\ \mathrm{d}x + \int_0^\tau \int_{B(0,R)} \mathbb{S}(\nabla_x \mathbf{u}_R) : \nabla_x \mathbf{u}_R\ \mathrm{d}x\ \mathrm{d}t \\ \le E_{0,R} - \int_{B(0,R)} ((\varrho_{R,0} - \bar{\varrho})P'(\bar{\varrho}) - P(\bar{\varrho}))(\tau,\cdot)\ \mathrm{d}x. \end{aligned}$$

This motivates us to consider

$$\overline{E}_R(\tau) = \int_{B(0,R)} \left(\frac{1}{2}\varrho_R|\mathbf{u}_R|^2 + (P(\varrho_R) - (\varrho_R - \bar{\varrho})P'(\bar{\varrho}) - P(\bar{\varrho})) \right)(\tau, \cdot)\ \mathrm{d}x$$

Considering the following extension of $(\varrho_R, \mathbf{u}_R)$ in $\mathbb{R}^d$ as

$$\grave{\varrho}_R = \begin{cases} \varrho_R \text{ in } B(0,R) \\ \bar{\varrho} \text{ otherwise} \end{cases} \quad \text{and } \grave{\mathbf{u}}_R = \begin{cases} \mathbf{u}_R \text{ in } B(0,R) \\ 0 \text{ otherwise} \end{cases},$$

we obtain

$$\int_{\mathbb{R}^d} \left(\frac{1}{2}\grave{\varrho}_R|\grave{\mathbf{u}}_R|^2 + (P(\grave{\varrho}_R) - (\grave{\varrho}_R - \bar{\varrho})P'(\bar{\varrho}) - P(\bar{\varrho})) \right)(\tau, \cdot)\ \mathrm{d}x + \int_0^\tau \int_{\mathbb{R}^d} \mathbb{S}(\nabla_x \grave{\mathbf{u}}_R) : \nabla_x \grave{\mathbf{u}}_R\ \mathrm{d}x\ \mathrm{d}t \le E_0.$$

Passing limit $R \to \infty$, we expect the total energy (2.2.18) in $\mathbb{R}^d$.

Remark 2.2.9. Let $\mathbf{f}$ be a time independent function and of the form $\mathbf{f} = \nabla_x G$ with G a real valued function on $\mathbb{R}^d$. Then a static solution $(\tilde{\varrho}, 0)$ of the system satisfies

$$\nabla_x p(\tilde{\varrho}) = \tilde{\varrho} \nabla_x G.$$

Even if for a simple G, $\tilde{\varrho}$ may not be constant. In this case we consider a far field condition as

$$\varrho \to \tilde{\varrho},\ \mathbf{u} \to 0 \text{ as } |x| \to \infty. \tag{2.2.20}$$

Remark 2.2.10. For a general unbounded domain with boundary (like, exterior domain, $\mathbb{R}^d \setminus B(0,1)$), we need to implement the far field condition and the boundary condition accordingly.

The existence of the weak solution for the compressible system Navier–Stokes has been studied in the last decades.

- In $d = 1$, the existence of a global in time weak solution was proved by Antontsev et al. [5].
- In $d = 2$ with $\gamma \ge \frac{3}{2}$ and $d = 3$ with $\gamma \ge \frac{9}{5}$, P. L. Lions [96] proved the same.
- Feireisl in [50], improved the result for $d = 2$ with $\gamma > 1$ and for $d = 3$ with $\gamma > \frac{3}{2}$.
- For $d = 2$, there is a certain improvement by Vaigant and Kazhikhov [119], the existence is valid for $\gamma \ge 1$.

The above results mostly consider a bounded domain Ω with no-slip boundary condition. We state a theorem on the existence of a weak solution.

Theorem 2.2.11. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$ be a bounded domain with Lipschitz boundary and the initial data $(\varrho_0, (\varrho\mathbf{u})_0)$ satisfies (2.2.12). If the pressure law is given by (2.1.35) or (2.1.37) with $\gamma > \frac{d}{2}$, then there exists a weak solution in $(0,T)\times\Omega$ for the Navier–Stokes system (2.2.1)-(2.2.3) with no-slip boundary condition (2.2.5) following the Definition 2.2.5.*

There are a number of results related to various boundary conditions and general non-monotone pressure laws. Here we list some of them.

- Global in time weak solution exists for a periodic boundary condition, i.e. $\Omega = \mathbb{T}^d$. We also have a similar result for a bounded domain with the Navier slip boundary condition(2.2.6), see Novotný and Straškraba [105, Section 7.12].
- For an exterior domain, there is an existence result proved by Novotný and Pokorný [104].
- For a compactly supported perturbation of the pressure law (2.1.36), Feireisl [49] showed existence.
- Recently, Bresch and Jabin [19] proved the existence for a more general non-monotone pressure law.

Hard-sphere pressure law

We consider the system (2.2.1)-(2.2.3) in the domain $\mathbb{T}^d$, i.e. the state variables are endowed with periodic boundary conditions with the pressure law (2.1.39) or (2.1.40). A suitable modification of the Definition 2.2.5 is required.

First, we notice that the pressure laws (2.1.39) and (2.1.40) have the property

$$\lim_{\varrho\to\bar{\varrho}} p(\varrho) = +\infty.$$

Accordingly, we modify the hypothesis on the initial data as

$$\begin{aligned} &\varrho(0,\cdot) = \varrho_0(\cdot) \text{ with } 0 \leq \varrho_0 < \bar{\varrho} \text{ in } \mathbb{T}^d, \ \int_{\mathbb{T}^d} P(\varrho_0)\, \mathrm{d}x < \infty, \\ &\varrho\mathbf{u}(0,\cdot) = (\varrho\mathbf{u})_0, \text{ and } \int_{\mathbb{T}^d} \frac{|(\varrho\mathbf{u})_0|^2}{\varrho_0} < \infty, \end{aligned} \tag{2.2.21}$$

where P is given by,

$$P(\varrho) = \varrho \int_{\frac{\bar{\varrho}}{2}}^{\varrho} \frac{p(z)}{z^2}\, \mathrm{d}z. \tag{2.2.22}$$

Similarly, we can define H and Q. We now give the definition of a weak solution for such pressure laws in the periodic domain $\mathbb{T}^d$:

Definition 2.2.12. We say that $(\varrho, \mathbf{u})$ is a dissipative weak solution in $(0,T)\times\mathbb{T}^d$ to the system of equations ((2.2.1)-(2.2.3)), with the periodic boundary conditions (2.2.4), supplemented with initial data (2.2.21) if:

- **Regularity class:** For a.e. $(t,x) \in (0,T) \times \mathbb{T}^d$ we have $0 \le \varrho(t,x) < \bar{\varrho}$. It holds that

$$\begin{aligned} &\varrho \in C_w([0,T]; L^\gamma(\mathbb{T}^d)) \text{ for any } \gamma > 1,\ p(\varrho) \in L^1((0,T) \times \mathbb{T}^d), \\ &\mathbf{u} \in L^2(0,T; W^{1,2}(\mathbb{T}^d;\mathbb{R}^d)),\ \varrho\mathbf{u} \in C_w([0,T]; L^2(\mathbb{T}^d;\mathbb{R}^d)). \end{aligned} \tag{2.2.23}$$

- **Continuity equation:** For any $\tau \in (0,T)$ and any test function $\varphi \in C^\infty([0,T] \times \mathbb{T}^d)$, it holds

$$\int_0^\tau \int_{\mathbb{T}^d} [\varrho \partial_t \varphi + \varrho\mathbf{u} \cdot \nabla_x \varphi] \,\mathrm{d}x \,\mathrm{d}t = \int_{\mathbb{T}^d} \varrho(\tau,\cdot)\varphi(\tau,\cdot) - \int_{\mathbb{T}^d} \varrho_0 \varphi(0,\cdot) \,\mathrm{d}x. \tag{2.2.24}$$

- **Renormalized continuity equation:** The continuity equation also holds in the sense of renormalized solutions:

$$\begin{aligned} &\left[\int_{\mathbb{T}^d} (b(\varrho))\varphi \,\mathrm{d}x\right]_{t=0}^{t=\tau} \\ &= \int_0^\tau \int_{\mathbb{T}^d} [b(\varrho)\partial_t\varphi + b(\varrho)\mathbf{u} \cdot \nabla_x \varphi + (b(\varrho) - \varrho b'(\varrho))\mathrm{div}_x \mathbf{u}\, \varphi] \,\mathrm{d}x \,\mathrm{d}t\,, \end{aligned} \tag{2.2.25}$$

  where $\varphi \in C^1([0,T] \times \mathbb{T}^d)$ for any $b \in C^1[0,\bar{\varrho})$ satisfying

$$|b(s)|^2 + |b'(s)|^2 \le C(1 + h(s)) \text{ for some constant } C \text{ and any } s \in [0,\bar{\varrho}). \tag{2.2.26}$$

- **Momentum equation:** For any $\tau \in (0,T)$ and any test function $\boldsymbol{\varphi} \in C^1(0,T; C^2(\mathbb{T}^d;\mathbb{R}^d))$, it holds

$$\begin{aligned} &\int_0^\tau \int_{\mathbb{T}^d} [\varrho\mathbf{u} \cdot \partial_t \boldsymbol{\varphi} + (\varrho\mathbf{u} \otimes \mathbf{u} : \nabla_x \boldsymbol{\varphi} + p(\varrho)\mathrm{div}_x \boldsymbol{\varphi} - \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \boldsymbol{\varphi}] \,\mathrm{d}x \,\mathrm{d}t \\ &\quad = \int_{\mathbb{T}^d} \varrho\mathbf{u}(\tau,\cdot) \cdot \boldsymbol{\varphi}(\tau,\cdot) \,\mathrm{d}x - \int_{\mathbb{T}^d} \varrho_0 \mathbf{u}_0 \cdot \boldsymbol{\varphi}(0,\cdot) \,\mathrm{d}x. \end{aligned} \tag{2.2.27}$$

- **Energy inequality:** For a.e. $\tau \in (0,T)$, the following inequality holds

$$\begin{aligned} &\int_{\mathbb{T}^d} \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho)\right](\tau,\cdot) \,\mathrm{d}x + \int_0^\tau \int_{\mathbb{T}^d} \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \,\mathrm{d}x \,\mathrm{d}t \\ &\quad \le \int_{\mathbb{T}^d} \left[\frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + P(\varrho_0)\right] \mathrm{d}x, \end{aligned} \tag{2.2.28}$$

Concerning to the existence of a dissipate weak solution we have the following remark.

Remark 2.2.13.

- By applying the argument in Feireisl and Zhang [78] with the refined argument of Feireisl, Lu and Málek [63], it can be shown that, under the following assumption on the pressure near the singular point

$$\lim_{\varrho \to \bar{\varrho}} h(\varrho)(\bar{\varrho} - \varrho)^{\beta} > 0, \text{ for some } \beta > \frac{5}{2}, \tag{2.2.29}$$

  there exists a global in time weak solution following the Definition 2.2.12.

- Condition (2.2.26) on b ensures that $b(\varrho), b'(\varrho) \in L^2((0,T) \times \mathbb{T}^d)$.

- The pressure potential P defined by (2.2.22) is bounded below, i.e., there exists $C > 0$ such that $P(\varrho) + C \geq 0$ for all $\varrho \in [0, \bar{\varrho})$.

2.3 Compressible Euler system

As we mentioned earlier, the Euler system describes the inviscid fluid. Let $T > 0$ and $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ be the domain. We consider the *compressible Euler equation* in time-space cylinder $Q_T = (0, T) \times \Omega$ describing the time evolution of the mass density $\varrho = \varrho(t, x)$ and the momentum field $\mathbf{m} = \mathbf{m}(t, x)$

$$\partial_t \varrho + \mathrm{div}_x \mathbf{m} = 0, \tag{2.3.1}$$

$$\partial_t \mathbf{m} + \mathrm{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) + \nabla_x p(\varrho) = \varrho \mathbf{f}. \tag{2.3.2}$$

- **Pressure law:** The pressure p and the density ϱ of the fluid are interrelated by the standard isentropic law

$$p(\varrho) = a\varrho^{\gamma}, \ a > 0, \ \gamma > 1. \tag{2.3.3}$$

- **Boundary condition:** For a bounded domain we consider impermeability condition on the boundary, i.e.

$$\mathbf{m} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega, \ \mathbf{n} \text{ is the outer normal vector on the boundary .} \tag{2.3.4}$$

- **Far field condition:** For an unbounded or exterior domain we prescribe the *far field* condition as,

$$|\varrho - \tilde{\varrho}| \to 0, \ \mathbf{m} \to \mathbf{0} \text{ as } |x| \to \infty, \tag{2.3.5}$$

  with $\tilde{\varrho}$ satisfies

$$\nabla_x p(\tilde{\varrho}) = \tilde{\varrho} \mathbf{f}.$$

  For the sake of simplicity, we restrict ourselves only to the case where $\mathbf{f}$ is a time independent function. In this case, we have $\tilde{\varrho}$ is time independent too.

- **Initial data:** We supplement the initial data as

$$\varrho(0, \cdot) = \varrho_0, \ \mathbf{m}(0, \cdot) = \mathbf{m}_0. \tag{2.3.6}$$

2.3.1 Weak solution

The strong or classical solution for Euler system exists only in local in time. First we provide the definition for bounded domain Ω with impermeability boundary condition(2.3.4).

Definition 2.3.1. Let $\mathbf{f} \in L^\infty((0,T)\times\Omega)$. A pair $(\varrho, \mathbf{m})$ is called a *weak solution* of the Euler system with initial data $(\varrho_0, \mathbf{m}_0)$ satisfying

$$\varrho_0 \geq 0 \text{ a.e. in } \Omega \text{ and } \int_\Omega \left(\frac{|\mathbf{m}_0|^2}{\varrho_0} + P(\varrho_0) \right) \, \mathrm{d}x < \infty \tag{2.3.7}$$

if the following is true.

- **Measurability:** The variables $\varrho = \varrho(t,x)$, $\mathbf{m} = \mathbf{m}(t,x)$ are measurable function in $(0,T)\times\mathbb{R}^d$, $\varrho \geq 0$,
- **Continuity equation:** The integral identity
$$\int_0^T \int_\Omega \left[\varrho \partial_t \phi + \mathbf{m}\cdot\nabla_x \phi \right] \, \mathrm{d}x \, \mathrm{d}t = -\int_\Omega \varrho_0 \phi(0,\cdot) \, \mathrm{d}x \tag{2.3.8}$$
holds for any $\phi \in C_c^1([0,T)\times\overline{\Omega})$.
- **Momentum equation:** The integral identity
$$\begin{aligned} &\int_0^T \int_\Omega \left[\mathbf{m}\cdot\partial_t \boldsymbol{\varphi} + \mathbf{1}_{\{\varrho>0\}} \frac{\mathbf{m}\otimes\mathbf{m}}{\varrho} : \nabla_x \boldsymbol{\varphi} + p(\varrho) \mathrm{div}_x \boldsymbol{\varphi} - \varrho \mathbf{f}\cdot\boldsymbol{\varphi} \right] \mathrm{d}x \, \mathrm{d}t \\ &= -\int_{\mathbb{R}^d} \mathbf{m}_0 \cdot \boldsymbol{\varphi}(0.\cdot) \, \mathrm{d}x, \end{aligned} \tag{2.3.9}$$
holds for any $\boldsymbol{\varphi} \in C_c^1([0,T)\times\overline{\Omega};\mathbb{R}^d)$ with $\boldsymbol{\varphi}\cdot\mathbf{n}|_{\partial\Omega} = 0$.

A weak solution is called *admissible weak solution* if the *energy inequality* holds, i.e., for a.e. $0 \leq \tau \leq T$,

$$\int_\Omega \left(\frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \right)(\tau,\cdot) \, \mathrm{d}x \leq \int_\Omega \left(\frac{1}{2}\frac{|\mathbf{m}_0|^2}{\varrho_0} + P(\varrho_0) \right) \, \mathrm{d}x + \int_0^\tau \int_\Omega \mathbf{m}\cdot\mathbf{f} \, \mathrm{d}x. \tag{2.3.10}$$

Remark 2.3.2. Different forms of the energy inequality are available in literature, for instance

$$\int_0^T \int_\Omega \left(\frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \right) \partial_t \phi \, \mathrm{d}x \, \mathrm{d}t \leq -\int_\Omega \left(\frac{1}{2}\frac{|\mathbf{m}_0|^2}{\varrho_0} + P(\varrho_0) \right) \phi(0) \, \mathrm{d}x, \tag{2.3.11}$$

for any test function $\phi \in C_c^1[0,T)$, $\phi \geq 0$.

For this inequality (2.3.11), we assume $\mathbf{f} = 0$ in (2.3.2). This form of the energy inequality (2.3.11) provides more information about the distributional time derivative of the total energy. Although the former form of energy inequality is more general and allows a larger class for its solution.

Remark 2.3.3. Similar definition can be provided for domain $\mathbb{R}^d$ equipped with a *far field condition.*

2.4 Complete Euler system

Recall, the complete Euler system describes the time evolution of the density $\varrho = \varrho(t,x)$, the momentum $\mathbf{m} = \mathbf{m}(t,x)$ and the energy $\mathrm{e} = \mathrm{e}(t,x)$ of a compressible inviscid fluid in the time-space cylinder $Q_T = (0,T) \times \mathbb{R}^d$:

$$\partial_t \varrho + \mathrm{div}_x \mathbf{m} = 0, \tag{2.4.1}$$

$$\partial_t \mathbf{m} + \mathrm{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) + \nabla_x p = 0, \tag{2.4.2}$$

$$\partial_t \mathrm{e} + \mathrm{div}_x \left[(\mathrm{e} + p) \frac{\mathbf{m}}{\varrho} \right] = 0. \tag{2.4.3}$$

The total energy e of the fluid

$$\mathrm{e} = \frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + \varrho e,$$

consists of the kinetic energy $\mathrm{e}_{kin} (= \frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho})$ and the internal energy $\mathrm{e}_{int} (= \varrho e)$. Here, p is the pressure related to ϱ, e through Boyle-Mariotte equation of state (2.1.25). Also introducing entropy(s) as

$$s(\varrho, \vartheta) = \log(\vartheta^{c_v}) - \log(\varrho),$$

where ϑ is absolute temperature, we have the *entropy equation:*

$$\partial_t(\varrho s) + \mathrm{div}_x(s\mathbf{m}) = 0. \tag{2.4.4}$$

Now with the introduction of the total entropy S by $S = \varrho s$, we rephrase (2.4.4) as

$$\partial_t S + \mathrm{div}_x \left(S \frac{\mathbf{m}}{\varrho} \right) = 0. \tag{2.4.5}$$

The total entropy helps us to rewrite the pressure p and e in terms of ϱ and S as

$$p = p(\varrho, S) = \varrho^\gamma \exp\left(\frac{S}{c_v \varrho} \right), \quad e = e(\varrho, S) = \frac{1}{\gamma - 1} \varrho^{\gamma - 1} \exp\left(\frac{S}{c_v \varrho} \right),$$

and, $(\varrho, S) \mapsto \varrho^\gamma \exp\left(\frac{S}{c_v \varrho} \right)$ is a strictly convex function in the domain of positivity, meaning at points, where it is finite and positive. In the context of weak solution we allow an inequality instead of (2.4.5) as

$$\partial_t S + \mathrm{div}_x \left(S \frac{\mathbf{m}}{\varrho} \right) \geq 0.$$

Let us complete the formulation of the complete Euler system by imposing the initial and far field conditions:

- **Initial data:** The initial state of the fluid is given through the conditions

$$\varrho(0,\cdot) = \varrho_0,\ \mathbf{m}(0,\cdot) = \mathbf{m}_0,\ S(0,\cdot) = S_0. \tag{2.4.6}$$

- **Far field condition:** We introduce the *far field condition* as,

$$\varrho \to \varrho_\infty,\ \mathbf{m} \to \mathbf{m}_\infty,\ S \to S_\infty \text{ as } |x| \to \infty, \tag{2.4.7}$$

  with $\varrho_\infty > 0,\ \mathbf{m}_\infty \in \mathbb{R}^d$ and $S_\infty \in \mathbb{R}$.

Let us fix some notations for this case:

- Let $(\varrho_\infty, \mathbf{m}_\infty, S_\infty) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}$ such that $\varrho_\infty > 0$. We define the relative energy with respect to $(\varrho_\infty, \mathbf{m}_\infty, S_\infty)$as,

$$\mathrm{e}(\varrho, \mathbf{m}, S | \varrho_\infty, \mathbf{m}_\infty, S_\infty) = \mathrm{e}_{\mathrm{int}}(\varrho, S | \varrho_\infty, S_\infty) + \mathrm{e}_{\mathrm{kin}}(\varrho, \mathbf{m} | \varrho_\infty, \mathbf{m}_\infty),$$

  with

$$\begin{aligned}\mathrm{e}_{\mathrm{int}}(\varrho, S|\varrho_\infty, S_\infty) = \mathrm{e}_{\mathrm{int}}(\varrho, S) &- \frac{\partial \mathrm{e}_{\mathrm{int}}}{\partial \varrho}(\varrho_\infty, S_\infty)(\varrho - \varrho_\infty) \\ &- \frac{\partial \mathrm{e}_{\mathrm{int}}}{\partial S}(\varrho_\infty, S_\infty)(S - S_\infty) - \mathrm{e}_{\mathrm{int}}(\varrho_\infty, S_\infty)\end{aligned}$$

  and

$$\begin{aligned}\mathrm{e}_{\mathrm{kin}}(\varrho, \mathbf{m}|\varrho_\infty, \mathbf{m}_\infty) =& \mathrm{e}_{\mathrm{kin}}(\varrho, \mathbf{m}) - \frac{\partial \mathrm{e}_{\mathrm{kin}}}{\partial \varrho}(\varrho_\infty, \mathbf{m}_\infty)(\varrho - \varrho_\infty) \\ &- \frac{\partial \mathrm{e}_{\mathrm{kin}}}{\partial \mathbf{m}}(\varrho_\infty, \mathbf{m}_\infty) \cdot (\mathbf{m} - \mathbf{m}_\infty) - \mathrm{e}_{\mathrm{kin}}(\varrho_\infty, \mathbf{m}_\infty).\end{aligned}$$

  Introducing the velocity fields $\mathbf{u}$, $\mathbf{u}_\infty$ as $\mathbf{m} = \varrho\mathbf{u}$ and $\mathbf{m}_\infty = \varrho_\infty\mathbf{u}_\infty$, respectively we observe

$$\mathrm{e}_{\mathrm{kin}}(\varrho, \mathbf{u}|\varrho_\infty, \mathbf{u}_\infty) = \frac{1}{2}\varrho|\mathbf{u} - \mathbf{u}_\infty|^2.$$

- In a more precise notation we write

$$\begin{aligned}&\mathrm{e}(\varrho, \mathbf{m}, S|\varrho_\infty, \mathbf{m}_\infty, S_\infty) \\ &= \mathrm{e}(\varrho, \mathbf{m}, S) - \partial\mathrm{e}(\varrho_\infty, \mathbf{m}_\infty, S_\infty) \cdot [(\varrho, \mathbf{m}, S) - (\varrho_\infty, \mathbf{m}_\infty, S_\infty)] \\ &\quad - \mathrm{e}(\varrho_\infty, \mathbf{m}_\infty, S_\infty).\end{aligned}$$

We assume an initial data $(\varrho_0, \mathbf{m}_0, S_0)$ satisfies

$$0 \leq \varrho_0 \text{ in } \mathbb{R}^d \text{ and } \int_{\mathbb{R}^d} \mathrm{e}(\varrho_0, \mathbf{m}_0, S_0|\varrho_\infty, \mathbf{m}_\infty, S_\infty)\ \mathrm{d}x < \infty. \tag{2.4.8}$$

2.4.1 Weak solution

The definition of the weak solutions is as follows:

Definition 2.4.1. Let $(\varrho_\infty, \mathbf{m}_\infty, S_\infty) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}$ such that $\varrho_\infty > 0$. The triplet $(\varrho, \mathbf{m}, S)$ is called an *admissible weak solution* of the complete Euler system with an initial data $(\varrho_0, \mathbf{m}_0, S_0)$ which follows (2.4.8), if the following is true:

- **Measurability:** The variables $\varrho = \varrho(t,x)$, $\mathbf{m} = \mathbf{m}(t,x)$ $S = S(t,x)$ are measurable function in $(0,T) \times \mathbb{R}^d$ and $\varrho \geq 0$ for a.e. $(0,T) \times \mathbb{R}^d$.

- **Continuity equation:** The integral identity

$$\int_0^T \int_{\mathbb{R}^d} \left[\varrho \partial_t \phi + \mathbf{m} \cdot \nabla_x \phi \right] \, \mathrm{d}x \, \mathrm{d}t = - \int_{\mathbb{R}^d} \varrho_0 \phi(0, \cdot) \, \mathrm{d}x \tag{2.4.9}$$

  holds for any $\phi \in C_c^1([0,T) \times \mathbb{R}^d)$.

- **Momentum equation:** The integral identity

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^d} \left[\mathbf{m} \cdot \partial_t \boldsymbol{\varphi} + \mathbf{1}_{\{\varrho > 0\}} \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} : \nabla_x \boldsymbol{\varphi} + \mathbf{1}_{\{\varrho > 0\}} p(\varrho, S) \mathrm{div}_x \boldsymbol{\varphi} \right] \mathrm{d}x \, \mathrm{d}t \\ &= - \int_{\mathbb{R}^d} \mathbf{m}_0 \cdot \boldsymbol{\varphi}(0 . \cdot) \, \mathrm{d}x \end{aligned} \tag{2.4.10}$$

  holds for any $\boldsymbol{\varphi} \in C_c^1([0,T) \times \mathbb{R}^d; \mathbb{R}^d)$.

- **Energy inequality:** The satisfaction of the far field conditions is enforced through the energy inequality in the following form :

$$\int_{\mathbb{R}^d} \mathrm{e}(\varrho, \mathbf{m}, S | \varrho_\infty, \mathbf{m}_\infty, S_\infty) \, (\tau, \cdot) \, \mathrm{d}x \leq \int_{\mathbb{R}^d} \mathrm{e}(\varrho_0, \mathbf{m}_0, S_0 | \varrho_\infty, \mathbf{m}_\infty, S_\infty) \, \mathrm{d}x, \tag{2.4.11}$$

  for a.e. $\tau \in (0,T)$.

- **Entropy inequality:** The integral inequality

$$\int_0^T \int_{\mathbb{R}^d} \left[S \, \partial_t \phi + \mathbf{1}_{\{\varrho > 0\}} \frac{S}{\varrho} \mathbf{m} \cdot \nabla_x \phi \right] \mathrm{d}x \, \mathrm{d}t \leq 0 \tag{2.4.12}$$

  holds for any $\phi \in C_c^1((0,T) \times \mathbb{R}^d)$ with $\phi \geq 0$.

Remark 2.4.2. A definition for bounded domain is similar.

We refer the following articles and monographs for the local in time strong solutions for bounded as well as unbounded domains [97], [110] and [12]. For Euler system a finite time blow-up for strong solutions has been observed by Smoller[111]. We already mention in the introduction that there are several results indicating the ill-posedness of weak solutions for Euler system for a large class of data, see [33], [35], [61], [36].

2.5 Measure-valued solutions

There are several numerical schemes for solving compressible fluid models. Unfortunately, it is difficult to establish that a given numerical scheme converges to a weak solution for the general case. In the context of the isentropic Navier-Stokes system, the convergence of some schemes has a restriction to the adiabatic exponent γ. For physically relevant adiabatic exponents, $1 \leq \gamma \leq \frac{3}{2}$, both the existence of weak solutions and the convergence of numerical schemes in space dimension three have not yet been proved. There are some results of Karper [88], Feireisl, Karper and Pokorný [60], for large adiabatic exponents.

The concept of measure-valued solutions was introduced by DiPerna [41] in the context of hyparbolic conservation laws. For incompressible fluids, there is a similar consideration by DiPerna and Majda [43], in particular, focusing on the incompressible Euler system and other related models for inviscid fluids. For the viscous counterpart, measure-valued solutions are given by Málek et al. [98], Neustupa [102] etc. Unfortunately, the generalized weak-strong uniqueness property is missing in the class of solutions considered by DiPerna and Majda in [43].

Recently, the concept of measure-valued solution is revisited by Fjordholm et al. [80], [81], [79] through certain numerical experiments with oscillatory solutions. On the other hand, more appropriate definition of measure-valued solutions is given by Gwiazda et al. [86] for the barotropic Euler system, Feireisl et al. [56] for the Navier–Stokes system, Březina and Feireisl [20] for the complete Euler system, Březina, Feireisl and Novotný [27] for the Navier-Stokes-Fourier system. These measure-valued solutions satisfy the desired weak-strong uniqueness property.

In addition, there are several results on identifying the limit of numerical schemes as a measure-valued solution. For the Navier–Stokes system, there are results for the physically admissible adiabatic exponent, see Feireisl and Lukáčová-Medvidova [65] and Feireisl et al. [68]. In the context of Euler, there are results by Feireisl, Lukáčová-Medvidova and Mizerova [69], [70]. Furthermore, the availability of the generalized weak-strong uniqueness principle ensures a strong convergence of the numerical solutions to the strong (classical) solution in the lifespan of the strong solution.

2.5.1 A general approach

Let Ω ($\subset \mathbb{R}^d$) be a bounded domain . For each $n \in \mathbb{N}$, we consider a map $\mathbf{w}_n \colon \mathbb{R}^d \to \mathbb{R}^d$. Taking motivation from (2.1.4), we consider an approximation of a general balance law with $\mathbf{F}_n = \mathbf{F}(d_n, \mathbf{w}_n)$ such that

$$\begin{aligned} &\int_0^T \int_\Omega d_n(t,x) \partial_t \phi(t,x) \ \mathrm{d}x \ \mathrm{d}t + \int_0^T \int_\Omega \mathbf{F}_n \cdot \nabla_x \phi(t,x) \ \mathrm{d}x \ \mathrm{d}t \\ &= - \int_0^T \int_\Omega \sigma_n(t,x) \phi(t,x) \ \mathrm{d}x \ \mathrm{d}t + \int_0^T \mathfrak{E}_n[\phi] \ \mathrm{d}t, \end{aligned} \tag{2.5.1}$$

with $\phi \in C_c^1((0,T) \times \Omega)$. Furthermore, We have the following assumptions:

- Variables $d_n \in L^\infty(0,T;L^p(\Omega))$ and $\mathbf{w}_n \in L^\infty(0,T;L^q(\Omega))$ are bounded independently of n, for $p,q > 1$, and $\mathbf{F} : \mathbb{R}\times\mathbb{R}^d \to \mathbb{R}^d$ is a continuous function with $\mathbf{F}_n \in L^1((0,T)\times\Omega)$ bounded uniformly.

- The source term $\sigma_n \to \sigma$ weakly in $L^r((0,T)\times\Omega)$ for $r \geq 1$.

- The term $\mathfrak{E}_n$ represents an error. Moreover, if $\mathfrak{E}_n[\phi] \to 0$ in $L^1(0,T)$, then we call the above approximation *consistent*.

The scheme involving the approximate sequence $d_n, \mathbf{w}_n$ together with the equation and consistency error is called the *consistent approximation scheme*.

From the bounds of d_n, $\mathbf{F}_n$ and σ_n we infer

$$\begin{aligned}
&d_n \to \bar{d} \text{ weak-(*)ly in } L^\infty_{\text{weak-(*)}}(0,T;L^p(\Omega)),\\
&\mathbf{w}_n \to \bar{\mathbf{w}} \text{ weak-(*)ly in } L^\infty_{\text{weak-(*)}}(0,T;L^q(\Omega)),\\
&\mathbf{F}_n \to \bar{\mathbf{F}} \text{ weak-(*)ly in } \mathcal{M}((0,T)\times\Omega).
\end{aligned}$$

On the other hand, we can conclude that there exists a Young measure

$$\mathcal{V} \in L^\infty_{\text{weak-(*)}}((0,T)\times\Omega;\mathcal{P}(\mathbb{R}\times\mathbb{R}^d))$$

such that the weak-(*) limits of $d_n, \mathbf{w}_n$ coincides with the barycenter of the Young measure, i.e., $\bar{d} = \langle \mathcal{V}; \tilde{d}\rangle$ and $\bar{\mathbf{w}} = \langle \mathcal{V}; \tilde{\mathbf{w}}\rangle$. But for the term $\mathbf{F}(d_n, \mathbf{w_n})$ we introduce

$$\mathfrak{R} = \bar{\mathbf{F}} - \left\langle \mathcal{V}; \mathbf{F}(\tilde{d},\tilde{\mathbf{w}})\right\rangle,$$

the *defect measure*. Thus in terms of the Young measure we rewrite the balance law with defect as

$$\begin{aligned}
&\int_0^T \int_\Omega \langle \mathcal{V};\tilde{d}\rangle \partial_t\phi(t,x)\ \mathrm{d}x\ \mathrm{d}t + \int_0^T\int_\Omega \left\langle \mathcal{V}; \mathbf{F}(\tilde{d},\tilde{\mathbf{w}})\right\rangle \cdot \nabla_x\phi(t,x)\ \mathrm{d}x\ \mathrm{d}t\\
&= -\int_0^T\int_\Omega \sigma(t,x)\phi(t,x)\ \mathrm{d}x\ \mathrm{d}t + \langle \mathfrak{R},\phi\rangle_{\mathcal{M}((0,T)\times\Omega),C_c((0,T)\times\Omega)},
\end{aligned} \tag{2.5.2}$$

with $\phi \in C^1_c((0,T)\times\Omega)$.

2.5.2 Definition of measure-valued solutions

In the last subsection we give an idea of how to define the measure-valued solution of a balance law. Similarly, we consider a set of consistent approximate solutions for the continuity and the momentum equation and observe that the limiting behavior leads to a measure-valued solution. For the barotropic Navier–Stokes system, Feireisl et al. [56] give a definition of a measure-valued solution for a monotone pressure law (2.1.35) or (2.1.37) with no-slip boundary condition. We state the definition below.

Definition 2.5.1. We say that a parameterized measure $\{\nu_{t,x}\}_{(t,x)\in(0,T)\times\Omega}$,

$$\nu \in L^\infty_{\rm weak}\big((0,T)\times\Omega;\mathcal{P}([0,\infty)\times\mathbb{R}^d)\big),\ \langle \nu_{\tau,x};\tilde{\varrho}\rangle \equiv \varrho,\ \langle \nu_{\tau,x};\tilde{\mathbf{u}}\rangle \equiv \mathbf{u}$$

is a *dissipative measure-valued solution* of the Navier–Stokes system in $(0,T)\times\Omega$, with the initial condition ν_0 and dissipation defect $\mathcal{D}$,

$$\mathcal{D}\in L^\infty(0,T),\ \mathcal{D}\geq 0,$$

if the following holds.

- **Continuity equation:** For a.e. $\tau\in(0,T)$ and $\psi\in C^1([0,T]\times\overline{\Omega})$ it holds

$$\begin{aligned}\int_\Omega \langle \nu_{\tau,x};\tilde{\varrho}\rangle\psi(\tau,\cdot)\ {\rm d}x - \int_\Omega \langle \nu_0;\tilde{\varrho}\rangle\psi(0,\cdot)\ {\rm d}x \\ = \int_0^\tau\int_\Omega \big[\langle \nu_{t,x};\tilde{\varrho}\rangle\partial_t\psi + \langle \nu_{t,x};\tilde{\varrho}\tilde{\mathbf{u}}\rangle\cdot\nabla_x\psi\big]\ {\rm d}x\ {\rm d}t.\end{aligned} \tag{2.5.3}$$

- **Momentum equation:** Let

$$\mathbf{u} = \langle \nu_{t,x};\tilde{\mathbf{u}}\rangle \in L^2(0,T;W^{1,2}_0(\Omega;\mathbb{R}^d)),$$

and there exists a measure $r^M\in L^\infty_{\text{weak-(*)}}(0,T;\mathcal{M}(\overline{\Omega}))$ and $\xi\in L^1(0,T)$ such that for a.e. $\tau\in(0,T)$ and every $\boldsymbol{\varphi}\in C^1([0,T]\times\overline{\Omega};\mathbb{R}^d)$, $\boldsymbol{\varphi}|_{\partial\Omega}=0$,

$$|\langle r^M;\nabla_x\boldsymbol{\varphi}\rangle| \leq \xi(\tau)\mathcal{D}(\tau)\|\boldsymbol{\varphi}\|_{C^1(\overline{\Omega})} \tag{2.5.4}$$

and

$$\begin{aligned}&\int_\Omega \langle \nu_{\tau,x};\tilde{\varrho}\tilde{\mathbf{u}}\rangle\cdot\boldsymbol{\varphi}(\tau,\cdot)\ {\rm d}x - \int_\Omega \langle \nu_0;\tilde{\varrho}\tilde{\mathbf{u}}\rangle\cdot\boldsymbol{\varphi}(0,\cdot)\ {\rm d}x \\ &= \int_0^\tau\int_\Omega \big[\langle \nu_{t,x};\tilde{\varrho}\tilde{\mathbf{u}}\rangle\cdot\partial_t\boldsymbol{\varphi} + \langle \nu_{t,x};\tilde{\varrho}\tilde{\mathbf{u}}\otimes\tilde{\mathbf{u}}\rangle : \nabla_x\boldsymbol{\varphi} + \langle \nu_{t,x};p(\tilde{\varrho})\rangle{\rm div}_x\boldsymbol{\varphi}\big]\ {\rm d}x\ {\rm d}t \\ &- \int_0^\tau\int_\Omega \mathbb{S}(\nabla_x\mathbf{u}):\nabla_x\boldsymbol{\varphi}\ {\rm d}x\ {\rm d}t + \int_0^\tau \langle r^M;\nabla_x\boldsymbol{\varphi}\rangle\ {\rm d}t.\end{aligned} \tag{2.5.5}$$

- **Energy inequality:** The integral inequality

$$\begin{aligned}\int_\Omega \left\langle \nu_{t,x};\left(\frac{1}{2}\tilde{\varrho}|\tilde{\mathbf{u}}|^2 + P(\tilde{\varrho})\right)\right\rangle {\rm d}x + \int_0^\tau\int_\Omega \mathbb{S}(\nabla_x\mathbf{u}):\nabla_x\mathbf{u}\ {\rm d}x\ {\rm d}t + \mathcal{D}(\tau) \\ \leq \int_\Omega \left\langle \nu_0;\left(\frac{1}{2}\tilde{\varrho}|\tilde{\mathbf{u}}|^2 + P(\tilde{\varrho})\right)\right\rangle {\rm d}x\end{aligned} \tag{2.5.6}$$

holds for a.e. $\tau\in(0,T)$. In addition, the following version of 'Poincaré's inequality' holds for a.e $\tau\in(0,T)$:

$$\int_0^\tau\int_\Omega \langle \nu_{t,x};|\tilde{\mathbf{u}}-\mathbf{u}|^2\rangle\ {\rm d}x\ {\rm d}t \leq c_P\mathcal{D}(\tau) \tag{2.5.7}$$

Remark 2.5.2. In [56, Section 2.2], there is an existence result of such measure-valued solution. One can clearly observe that there are more unknowns than the number of equations and constraints. Thus, uniqueness for such solutions cannot be expected. A positive fact is that these solutions satisfy the compatibility and the generalized weak-strong uniqueness property, as mentioned in the introduction.

Renomalized dissipative measure-valued solution (rDMV solution)

In particular, the above definition applies to a monotone pressure law. Our goal is to establish a weak-strong uniqueness property for a non-monotone pressure law. The renormalized equation plays an important role in the context of a non-monotone pressure. Unfortunately, the above definition is not sufficient to conclude that the renormalized continuity equation holds, at least in the measure-valued sense. This leads us to consider a more general class of solutions, namely *renormalized dissipative measure-valued solution*. Here, the velocity gradient$(\nabla_x \mathbf{u})$ has been incorporated as part of the Young measure along with natural candidates for the phase space e.g. density and velocity $(\varrho, \mathbf{u})$.

Phase Space: Therefore, a suitable phase space framework for the measure–valued solution is

$$\mathcal{F} = \left\{ \left(\tilde{\varrho}, \tilde{\mathbf{u}}, \widetilde{\mathbb{D}_{\mathbf{v}}} \right) \middle| \tilde{\varrho} \in [0,\infty),\ \tilde{\mathbf{u}} \in \mathbb{R}^d,\ \widetilde{\mathbb{D}_{\mathbf{v}}} \in \mathbb{R}^{d\times d}_{\mathrm{sym}} \right\}. \tag{2.5.8}$$

Definition 2.5.3. Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. We say that a parameterized measure $\{\mathcal{V}_{t,x}\}_{(t,x)\in(0,T)\times\Omega}$,

$$\mathcal{V} \in L^\infty_{\text{weak-}(*)}\big((0,T)\times\Omega; \mathcal{P}(\mathcal{F})\big),$$

is a renormalized dissipative measure-valued (rDMV) solution of the Navier–Stokes system (2.2.1)-(2.2.3) in $(0,T)\times\Omega$ with the no-slip boundary condition (2.2.5), the pressure-density relation (2.1.36) or (2.1.38), the initial condition

$$\mathcal{V}_0 \in L^\infty_{\text{weak-}(*)}\big(\Omega; \mathcal{P}([0,\infty)\times\mathbb{R}^d)\big)$$

and, a *dissipation defect* $\mathcal{D}$,

$$\mathcal{D} \in L^\infty(0,T),\ \mathcal{D} \geq 0,$$

if the following holds:

- **Equation of continuity:** For a.e. $\tau \in (0,T)$ and $\psi \in C^1([0,T]\times\overline{\Omega})$, we have

$$\begin{aligned} &\int_\Omega \langle \mathcal{V}_{\tau,x}; \tilde{\varrho}\rangle \psi(\tau,\cdot)\ \mathrm{d}x - \int_\Omega \langle \mathcal{V}_{0,x}; \tilde{\varrho}\rangle \psi(0,\cdot)\ \mathrm{d}x \\ &\quad = \int_0^\tau \int_\Omega \left[\langle \mathcal{V}_{t,x}; \tilde{\varrho}\rangle \partial_t \psi + \langle \mathcal{V}_{t,x}; \tilde{\varrho}\tilde{\mathbf{u}}\rangle \cdot \nabla_x \psi \right]\ \mathrm{d}x\ \mathrm{d}t. \end{aligned} \tag{2.5.9}$$

- **Renormalized equation of continuity :** For a.e. $\tau \in (0,T)$ and a test function $\psi \in C^1([0,T]\times\overline{\Omega})$, we have

$$\begin{aligned}
&\int_\Omega \langle \mathcal{V}_{\tau,x}; b(\tilde\varrho)\rangle \psi(\tau,\cdot)\ \mathrm{d}x - \int_\Omega \langle \mathcal{V}_{0,x}; b(\tilde\varrho)\rangle \psi(0,\cdot)\ \mathrm{d}x \\
&\quad = \int_0^\tau \int_\Omega \left[\langle \mathcal{V}_{t,x}; b(\tilde\varrho)\rangle \partial_t \psi + \langle \mathcal{V}_{t,x}; b(\tilde\varrho)\tilde{\mathbf{u}}\rangle \cdot \nabla_x \psi\right]\ \mathrm{d}x\ \mathrm{d}t \\
&\quad\quad - \int_0^\tau \int_\Omega \langle \mathcal{V}_{t,x}; (\tilde\varrho b'(\tilde\varrho) - b(\tilde\varrho))\mathrm{tr}(\widetilde{\mathbb{D}_{\mathbf{v}}})\rangle \cdot \psi\ \mathrm{d}x\ \mathrm{d}t,
\end{aligned} \tag{2.5.10}$$

where $b \in C^1[0,\infty)$, $\exists r_b > 0$ such that $b'(x) = 0$, $\forall x > r_b$.

- **Momentum equation:** There exists a measure

$$r^M \in L^\infty_{\text{weak-}(*)}(0,T; \mathcal{M}(\overline{\Omega}; \mathbb{R}^{d\times d})),$$

and $\xi \in L^1(0,T)$ such that for a.e. $\tau \in (0,T)$ and for any $\boldsymbol{\varphi} \in C^1([0,T]\times\overline{\Omega};\mathbb{R}^d)$ with $\boldsymbol{\varphi}|_{\partial\Omega} = 0$, we have

$$|\langle r^M(\tau); \nabla_x \boldsymbol{\varphi}\rangle_{\{\mathcal{M}(\overline{\Omega};\mathbb{R}^{d\times d}), C(\overline{\Omega};\mathbb{R}^{d\times d})\}}| \le \xi(\tau)\mathcal{D}(\tau)\|\boldsymbol{\varphi}\|_{C^1(\overline{\Omega})} \tag{2.5.11}$$

and

$$\begin{aligned}
&\int_\Omega \langle \mathcal{V}_{\tau,x}; \tilde\varrho\tilde{\mathbf{u}}\rangle \cdot \boldsymbol{\varphi}(\tau,\cdot)\ \mathrm{d}x - \int_\Omega \langle \mathcal{V}_{0,x}; \tilde\varrho\tilde{\mathbf{u}}\rangle \cdot \boldsymbol{\varphi}(0,\cdot)\ \mathrm{d}x \\
&= \int_0^\tau \int_\Omega \left[\langle \mathcal{V}_{t,x}; \tilde\varrho\tilde{\mathbf{u}}\rangle \cdot \partial_t \boldsymbol{\varphi} + \langle \mathcal{V}_{t,x}; \tilde\varrho(\tilde{\mathbf{u}}\otimes\tilde{\mathbf{u}})\rangle : \nabla_x \boldsymbol{\varphi} + \langle \mathcal{V}_{t,x}; p(\tilde\varrho)\rangle \mathrm{div}_x \boldsymbol{\varphi}\right]\ \mathrm{d}x\ \mathrm{d}t \\
&- \int_0^\tau \int_\Omega \langle \mathcal{V}_{t,x}; \mathbb{S}(\widetilde{\mathbb{D}_{\mathbf{v}}})\rangle : \nabla_x \boldsymbol{\varphi}\ \mathrm{d}x\ \mathrm{d}t + \int_0^\tau \langle r^M; \nabla_x \boldsymbol{\varphi}\rangle_{\{\mathcal{M}(\overline{\Omega};\mathbb{R}^{d\times d}), C(\overline{\Omega};\mathbb{R}^{d\times d})\}}\ \mathrm{d}t.
\end{aligned} \tag{2.5.12}$$

- **Momentum compatibility:** The following compatibility condition remains true:

$$- \int_0^\tau \int_\Omega \langle \mathcal{V}_{t,x}; \tilde{\mathbf{u}}\rangle \cdot \mathrm{div}_x \mathbb{M}\ \mathrm{d}x\ \mathrm{d}t = \int_0^\tau \int_\Omega \langle \mathcal{V}_{t,x}; \widetilde{\mathbb{D}_{\mathbf{v}}}\rangle : \mathbb{M}\ \mathrm{d}x\ \mathrm{d}t \quad \text{for any } \mathbb{M} \in C^1(\bar{Q}_T; \mathbb{R}^{d\times d}_{\text{sym}}). \tag{2.5.13}$$

- **Energy inequality:** The energy inequality

$$\begin{aligned}
&\int_\Omega \left\langle \mathcal{V}_{t,x}; \left(\frac{1}{2}\tilde\varrho|\tilde{\mathbf{u}}|^2 + P(\tilde\varrho)\right)\right\rangle \mathrm{d}x + \int_0^\tau \int_\Omega \langle \mathcal{V}_{t,x}; \mathbb{S}(\widetilde{\mathbb{D}_{\mathbf{v}}}) : \widetilde{\mathbb{D}_{\mathbf{v}}}\rangle\ \mathrm{d}x\ \mathrm{d}t + \mathcal{D}(\tau) \\
&\qquad \le \int_\Omega \left\langle \mathcal{V}_{0,x}; \left(\frac{1}{2}\tilde\varrho|\tilde{\mathbf{u}}|^2 + P(\tilde\varrho)\right)\right\rangle \mathrm{d}x
\end{aligned} \tag{2.5.14}$$

holds for a.e. $\tau \in (0,T)$.

- **Generalized Korn-Poincaré inequality:** For $\mathbf{v} \in L^2(0,T;H_0^1(\Omega;\mathbb{R}^d))$, the following inequality is true

$$\int_0^\tau \int_\Omega \langle \mathcal{V}_{t,x}; |\tilde{\mathbf{u}} - \mathbf{v}|^2 \rangle \,\mathrm{d}x\,\mathrm{d}t \le c_P \int_0^\tau \int_\Omega \langle \mathcal{V}_{t,x}; |\mathbb{D}_0(\widetilde{\mathbb{D}_{\mathbf{v}}}) - \mathbb{D}_0(\nabla_x \mathbf{v})|^2 \rangle \,\mathrm{d}x\,\mathrm{d}t. \tag{2.5.15}$$

Remark 2.5.4. In all the above expressions, $\mathcal{V}_{0,x} = \mathcal{V}_0(x)$ for a.e. $x \in \Omega$.

Remark 2.5.5. Here one can consider an initial data $\mathcal{V}_0$ as

$$\int_\Omega \left\langle \mathcal{V}_{0,x} \frac{1}{2} \tilde{\varrho} |\tilde{\mathbf{u}}|^2 + P(\tilde{\varrho}) \right\rangle \,\mathrm{d}x < \infty. \tag{2.5.16}$$

Our main goal is to prove weak-strong uniqueness, given sufficiently smooth initial data. Therefore, instead of considering initial conditions as a measure $\mathcal{V}_0$, we can consider *finite energy initial data*. This means $(\langle \mathcal{V}_0; \tilde{\varrho} \rangle, \langle \mathcal{V}_0; \tilde{\varrho}\mathbf{v} \rangle) = (\varrho_0, (\varrho\mathbf{u})_0)$ are functions with $\varrho_0 \ge 0$, $(\varrho\mathbf{u})_0 = 0$ on the set $\{x \in \Omega \,|\, \varrho_0(x) = 0\}$ and

$$\int_\Omega \left(\frac{1}{2} \frac{|(\varrho\mathbf{u})_0|^2}{\varrho_0} + P(\varrho_0) \right)(t,\cdot) \,\mathrm{d}x < \infty. \tag{2.5.17}$$

Remark 2.5.6. As a consequence of the above definition, we have

$$\left[\int_\Omega \langle \mathcal{V}_{t,x}; Q(\tilde{\varrho}) \rangle (t,\cdot) \,\mathrm{d}x \right]_{t=0}^{t=\tau} = - \int_0^\tau \int_\Omega \langle \mathcal{V}_{t,x}; q(\tilde{\varrho}) \mathrm{tr}(\widetilde{\mathbb{D}_{\mathbf{v}}}) \rangle \,\mathrm{d}x\,\mathrm{d}t, \tag{2.5.18}$$

where $q \in C_c^1(0,\infty)$ and $Q(s) = s \int_1^s \frac{q(\xi)}{\xi^2} \mathrm{d}\xi$, for $s > 0$.

A possible extension to the Navier slip boundary condition

In the case of Navier slip boundary condition (2.2.6), we modify the Definition 2.5.3 in the following way:

Definition 2.5.7. We say that a parameterized measure $\{\mathcal{V}_{t,x}\}_{(t,x)\in(0,T)\times\Omega}$,

$$\mathcal{V} \in L^\infty_{\text{weak-(*)}}\big((0,T)\times\Omega; \mathcal{P}(\mathcal{F})\big),$$

is a renormalized dissipative measure–valued (rDMV) solution of the Navier–Stokes system (2.2.1)-(2.2.3) in $(0,T)\times\Omega$, with the pressure-density relation (2.1.36) or (2.1.38), the initial condition

$$\mathcal{V}_0 \in L^\infty_{\text{weak-(*)}}\big(\Omega; \mathcal{P}([0,\infty)\times\mathbb{R}^d)\big)$$

and a *dissipation defect* $\mathcal{D}$,

$$\mathcal{D} \in L^\infty(0,T),\ \mathcal{D} \ge 0,$$

if the following holds.

- The *continuity equation* and the *renormalized continuity equation* hold as in (2.5.9) and (2.5.10).

- **Momentum equation:** There exists a measure

$$r^M \in L^\infty_{\text{weak-}(*)}(0,T;\mathcal{M}(\overline{\Omega};\mathbb{R}^{d\times d}))$$

and $\xi \in L^1(0,T)$ such that for a.e. $\tau \in (0,T)$ and every $\boldsymbol{\varphi} \in C^1([0,T]\times\overline{\Omega};\mathbb{R}^d)$, $\boldsymbol{\varphi}\cdot\mathbf{n}|_{\partial\Omega} = 0$, the following holds:

$$|\langle r^M(\tau); \nabla_x \boldsymbol{\varphi}\rangle_{\{\mathcal{M}(\overline{\Omega};\mathbb{R}^{d\times d}),C(\overline{\Omega};\mathbb{R}^{d\times d})\}}| \leq \xi(\tau)\mathcal{D}(\tau)\|\boldsymbol{\varphi}\|_{C^1(\overline{\Omega})} \tag{2.5.19}$$

and

$$\begin{aligned}
&\int_\Omega \langle \mathcal{V}_{\tau,x}; \tilde{\varrho}\tilde{\mathbf{u}}\rangle \cdot \boldsymbol{\varphi}(\tau,\cdot)\ \mathrm{d}x - \int_\Omega \langle \mathcal{V}_{0,x}; \tilde{\varrho}\tilde{\mathbf{u}}\rangle \cdot \boldsymbol{\varphi}(0,\cdot)\ \mathrm{d}x \\
&= \int_0^\tau \int_\Omega \left[\langle \mathcal{V}_{t,x}; \tilde{\varrho}\tilde{\mathbf{u}}\rangle \cdot \partial_t \boldsymbol{\varphi} + \langle \mathcal{V}_{t,x}; \tilde{\varrho}(\tilde{\mathbf{u}}\otimes\tilde{\mathbf{u}})\rangle : \nabla_x \boldsymbol{\varphi} + \langle \mathcal{V}_{t,x}; p(\tilde{\varrho})\rangle \mathrm{div}_x \boldsymbol{\varphi}\right]\ \mathrm{d}x\ \mathrm{d}t \\
&- \int_0^\tau \int_\Omega \langle \mathcal{V}_{t,x}; \mathbb{S}(\widetilde{\mathbb{D}_{\mathbf{v}}})\rangle : \nabla_x \boldsymbol{\varphi}\ \mathrm{d}x\ \mathrm{d}t + \int_0^\tau \langle r^M; \nabla_x \boldsymbol{\varphi}\rangle_{\{\mathcal{M}(\overline{\Omega};\mathbb{R}^{d\times d}),C(\overline{\Omega};\mathbb{R}^{d\times d})\}}\ \mathrm{d}t.
\end{aligned} \tag{2.5.20}$$

- *Energy inequality* remains unchanged as in (2.5.14).

- *Momentum compatibility* condition satisfy as it is, (2.5.13).

- *Generalized Korn-Poincaré inequality* holds as

$$\begin{aligned}
\int_0^\tau \int_\Omega \left\langle \mathcal{V}_{t,x}; |\tilde{\mathbf{u}} - \mathbf{v}|^2 \right\rangle\ \mathrm{d}x\ \mathrm{d}t \leq & c_P \int_0^\tau \int_\Omega \left\langle \mathcal{V}_{t,x}; |\mathbb{D}_0(\widetilde{\mathbb{D}_{\mathbf{v}}}) - \mathbb{D}_0(\nabla_x \mathbf{v})|^2 \right\rangle\ \mathrm{d}x\ \mathrm{d}t \\
& + \int_0^\tau \int_\Omega \left\langle \mathcal{V}_{t,x}; \tilde{\varrho}|\tilde{\mathbf{u}} - \mathbf{v}|^2 \right\rangle\ \mathrm{d}x\ \mathrm{d}t,
\end{aligned} \tag{2.5.21}$$

for $\mathbf{v} \in L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^d))$.

2.5.3 Existence of a rDMV solution

Here we give an overview of the existence of a rDMV solution that follows the Definition 2.5.3 or 2.5.7, depending on the boundary condition. For now, we stick to no-slip boundary condition (2.2.5). First, we consider an approximation problem of the Navier–Stokes system (2.2.1)-(2.2.3) with the pressure law (2.1.36) or (2.1.38) and the adiabatic exponent $\gamma \geq 1$. In particular, we consider an *artificial pressure* approximation by modifying the pressure as

$$p_{\mathrm{mod}}(\varrho) = p(\varrho) + \delta\varrho^\Gamma, \tag{2.5.22}$$

with $\Gamma \geq 2$ and $\delta > 0$ is a small parameter. Hence the following approximate problem reads as

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{2.5.23}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) + \delta \nabla_x \varrho^\Gamma = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}), \tag{2.5.24}$$

with no-slip boundary condition

$$u = 0 \text{ on } \partial\Omega. \tag{2.5.25}$$

Remark 2.5.8. One can consider any general consistent approximation, of the problem. In Chapter 5 we will discuss about consistent approximation.

First we note that, for a fixed $\delta > 0$, the existence of a renormalized weak solution of the Navier–Stokes system, which follows the Definition 2.2.5 is known if pressure follows, (2.5.22) with $\Gamma \geq 2$, see, Feireisl [49] and Bresch and Jabin [19].

For each $\delta > 0$, we denote the weak solution as $(\varrho_\delta, \mathbf{u}_\delta)$. We need some additional hypothesis on initial energy. For the sake of simplicity, we choose an initial data

$$\mathcal{V}_0 = \delta_{(\varrho_0, (\varrho \mathbf{u})_0)},$$

such that

$$\int_\Omega \left(\frac{1}{2} \frac{|(\varrho \mathbf{u})_0|^2}{\varrho_0} + P(\varrho_0) \right) \, \mathrm{d}x < \infty. \tag{2.5.26}$$

For the approximate problem (2.5.23)-(2.5.25), we consider initial data $\{\varrho_{\delta,0}, (\varrho \mathbf{u})_{\delta,0}\}$ belong to a certain regularity class for which a weak solution exists and it satisfies

$$\frac{1}{2} \varrho_{\delta,0} |\mathbf{u}_{\delta,0}|^2 + P(\varrho_{\delta,0}) + \frac{\delta}{\Gamma - 1} \varrho_{\delta,0}^\Gamma \to \frac{1}{2} \frac{|(\varrho \mathbf{u})_0|^2}{\varrho_0} + P(\varrho_0) \text{ in } L^1(\Omega), \tag{2.5.27}$$

as $\delta \to 0$.

Our goal is to verify that the family of weak solutions $\{\varrho_\delta, \mathbf{u}_\delta, \nabla_x \mathbf{u}_\delta\}_{\delta > 0}$ generates a renormalized dissipative measure-valued solution(rDMV), that follows the Definition 2.5.3.

Apriori estimates:

From (2.5.27) we obtain

$$\int_\Omega \left(\frac{1}{2} \varrho_{\delta,0} |\mathbf{u}_{\delta,0}|^2 + P(\varrho_{\delta,0}) + \frac{\delta}{\Gamma - 1} \varrho_{\delta,0}^\Gamma \right) \, \mathrm{d}x \leq C,$$

where C is independent of δ. From the energy inequality (2.2.15) of a finite energy weak solution we have the following uniform estimates:

$$\sup_{t\in[0,T]} \int_\Omega H(\varrho_\delta)(t,\cdot)\ \mathrm{d}x \le C,$$
$$\sup_{t\in[0,T]} \int_\Omega \varrho_\delta |\mathbf{u}_\delta|^2(t,\cdot)\ \mathrm{d}x \le C,$$
$$\int_0^T \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}_\delta) : \nabla_x \mathbf{u}_\delta\ \mathrm{d}x\ \mathrm{d}t \le C,$$
$$\sup_{t\in[0,T]} \frac{\delta}{\Gamma - 1} \int_\Omega \varrho_\delta^\Gamma (t,\cdot)\ \mathrm{d}x \le C.$$

By Korn and Poincaré inequality we have that $\mathbf{u}_\delta$ is bounded in $L^2(0,T;W_0^{1,2}(\Omega))$. We also get that $\{\varrho_\delta\}$ is bounded in $L^\infty(0,T;L^\gamma(\Omega))$ for $\gamma > 1$ and $\{\varrho_\delta \log \varrho_\delta\}$ is bounded in $L^\infty(0,T;L^1(\Omega))$ for $\gamma = 1$. From our assumption $q \in C_c^1[0,\infty)$, we have $Q(\varrho) \approx \varrho$. Hence we can conclude that

$$\Big[\frac{1}{2}\varrho_\delta|\mathbf{u}_\delta|^2 + P(\varrho_\delta)\Big](t,\cdot) \in \mathcal{M}(\overline{\Omega}) \text{ is bounded uniformly for } t \in (0,T),$$
$$\Big[\mu|\nabla_x \mathbf{u}_\delta|^2 + (\lambda - \frac{\mu}{d})|\mathrm{div}_x \mathbf{u}_\delta|^2\Big] \text{ is bounded in } \mathcal{M}^+([0,T]\times\overline{\Omega}),$$
$$\delta \varrho_\delta^\Gamma(t,\cdot) \in \mathcal{M}^+(\overline{\Omega}) \text{ is bounded uniformly for } t \in (0,T).$$

Thus passing to a subsequence, we obtain

$$\Big[\frac{1}{2}\varrho_\delta|\mathbf{u}_\delta|^2 + P(\varrho_\delta)\Big](t,\cdot) \to E \text{ weakly-(*) in } L^\infty_{\text{weak}}(0,T;\mathcal{M}(\overline{\Omega})),$$
$$\Big[\mu|\nabla_x \mathbf{u}_\delta|^2 + (\lambda - \frac{\mu}{d})|\mathrm{div}_x \mathbf{u}_\delta|^2\Big] \to \sigma \text{ weakly-(*) in } \mathcal{M}^+([0,T]\times\overline{\Omega}),$$
$$\delta \varrho_\delta^\Gamma(t,\cdot) \to \zeta \text{ weakly-(*) in } L^\infty_{\text{weak}}(0,T;\mathcal{M}^+(\overline{\Omega})).$$

From the fundamental theorem of Young measure (1.3.1), we ensure the existence of a Young measure $\mathcal{V}$, generated by $\Big\{\varrho_\delta, \mathbf{u}_\delta, \mathbb{D}_{\mathbf{u}_\delta} = \frac{\nabla_x \mathbf{u}_\delta + \nabla_x^T \mathbf{u}_\delta}{2}\Big\}_{\delta>0}$.

Now we introduce two non-negative measures

$$E_\infty = E - \Big\langle \mathcal{V}_{t,x}; \frac{1}{2}s|\mathbf{v}|^2 + P(s)\Big\rangle, \tag{2.5.28}$$
$$\sigma_\infty = \sigma - \langle \mathcal{V}_{t,x}; \mathbb{S}(\widetilde{\mathbb{D}_{\mathbf{v}}}) : \widetilde{\mathbb{D}_{\mathbf{v}}}\rangle. \tag{2.5.29}$$

With the help of Lemma 1.3.14, we claim that

$$E_\infty \in L^\infty_{\text{weak-(*)}}(0,T;\mathcal{M}^+(\overline{\Omega})) \text{ and } \sigma_\infty \in \mathcal{M}^+([0,T]\times\overline{\Omega}).$$

Passage to limit in energy inequality:

The energy inequality of the approximate problem (2.5.23)-(2.5.25) reads as

$$\left[\int_\Omega \left(\frac{1}{2}\varrho_\delta|\mathbf{u}_\delta|^2 + P(\varrho_\delta) + \frac{\delta}{\Gamma-1}\varrho_\delta^\Gamma\right)(t,\cdot)\ \mathrm{d}x\right]_{t=0}^{t=\tau} + \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}_\delta) : \nabla_x \mathbf{u}_\delta\ \mathrm{d}x\ \mathrm{d}t \leq 0.$$

We perform the passage of limit in the energy inequality and, we obtain

$$\int_\Omega \left\langle \mathcal{V}_{\tau,x}; \left(\frac{1}{2}\tilde{\varrho}|\tilde{\mathbf{u}}|^2 + P(\tilde{\varrho})\right)\right\rangle \mathrm{d}x + \int_0^\tau \int_\Omega \langle \mathcal{V}_{t,x}; \mathbb{S}(\widetilde{\mathbb{D}_\mathbf{v}}) : \widetilde{\mathbb{D}_\mathbf{v}}\rangle\ \mathrm{d}x\ \mathrm{d}t$$
$$+ E_\infty(\tau)[\overline{\Omega}] + C\zeta(\tau)[\overline{\Omega}] + \sigma_\infty[[0,\tau]\times\overline{\Omega}] \leq \int_\Omega \left\langle \mathcal{V}_{0,x}; \left(\frac{1}{2}\tilde{\varrho}|\mathbf{v}|^2 + P(\tilde{\varrho})\right)\right\rangle \mathrm{d}x,$$

where $C > 0$ is a constant. We consider

$$\mathcal{D}(\tau) = E_\infty(\tau)[\overline{\Omega}] + C\zeta(\tau)[\overline{\Omega}] + \sigma_\infty[[0,\tau]\times\overline{\Omega}]. \tag{2.5.30}$$

Passage to limit in renormalized continuity equation:

We have,

$$\left[\int_\Omega (\varrho_\delta + b(\varrho_\delta))\varphi\ \mathrm{d}x\right]_{t=0}^{t=\tau}$$
$$= \int_0^\tau \int_\Omega \big[(\varrho_\delta + b(\varrho_\delta))\partial_t\varphi + (\varrho_\delta + b(\varrho_\delta))\mathbf{u}_\delta \cdot \nabla_x\varphi + (b(\varrho_\delta) - \varrho_\delta b'(\varrho_\delta))\mathrm{div}_x\mathbf{u}_\delta\varphi\big]\mathrm{d}x\,\mathrm{d}t,$$

where $b \in C^1[0,\infty)$ and there exists $r_b > 0$ such that $b'(x) = 0,\ \forall x > r_b$. This choice of b implies

$$(b(\varrho_\delta) - \varrho_\delta b'(\varrho_\delta))\mathrm{div}_x\mathbf{u}_\delta \in L^1((0,T)\times\Omega) \text{ is uniformly bounded.} \tag{2.5.31}$$

From this it follows

$$\int_\Omega \langle \mathcal{V}_{\tau,x}; \tilde{\varrho}\rangle\psi(\tau,\cdot)\ \mathrm{d}x - \int_\Omega \langle \mathcal{V}_0; \tilde{\varrho}\rangle\psi(0,\cdot)\ \mathrm{d}x$$
$$= \int_0^\tau \int_\Omega \big[\langle \mathcal{V}_{t,x}; \tilde{\varrho}\rangle\partial_t\psi + \langle \mathcal{V}_{t,x}; \tilde{\varrho}\tilde{\mathbf{u}}\rangle \cdot \nabla_x\psi\big]\ \mathrm{d}x\ \mathrm{d}t$$

and

$$\int_\Omega \langle \mathcal{V}_{\tau,x}; b(\tilde{\varrho})\rangle\psi(\tau,\cdot)\ \mathrm{d}x - \int_\Omega \langle \mathcal{V}_0; b(\tilde{\varrho})\rangle\psi(0,\cdot)\ \mathrm{d}x$$
$$= \int_0^\tau \int_\Omega \big[\langle \mathcal{V}_{t,x}; b(\tilde{\varrho})\rangle\partial_t\psi + \langle \mathcal{V}_{t,x}; b(\tilde{\varrho})\tilde{\mathbf{u}}\rangle \cdot \nabla_x\psi\big]\ \mathrm{d}x\ \mathrm{d}t$$
$$- \int_0^\tau \int_\Omega \langle \mathcal{V}_{t,x}; (\tilde{\varrho}b'(\tilde{\varrho}) - b(\tilde{\varrho}))\mathrm{tr}(\widetilde{\mathbb{D}_\mathbf{v}})\rangle\psi\ \mathrm{d}x\ \mathrm{d}t.$$

Passage to limit in momentum equation:

We have

$$\begin{aligned}
&\left[\int_\Omega \varrho_\delta \mathbf{u}_\delta(\tau,\cdot)\cdot\boldsymbol{\varphi}(\tau,\cdot)\ \mathrm{d}x\right]_{t=0}^{t=\tau} \\
&= \int_0^\tau\int_\Omega \big[\varrho_\delta \mathbf{u}_\delta\cdot\partial_t\boldsymbol{\varphi} + (\varrho_\delta\mathbf{u}_\delta\otimes\mathbf{u}_\delta) : \nabla_x\boldsymbol{\varphi} + (p(\varrho_\delta)+\delta\varrho_\delta^\Gamma)\mathrm{div}_x\boldsymbol{\varphi} \\
&\qquad - \mathbb{S}(\nabla_x\mathbf{u}_\delta) : \nabla_x\boldsymbol{\varphi}\big]\ \mathrm{d}x\ \mathrm{d}t.
\end{aligned}$$

Using $|(\varrho_\delta\mathbf{u}_\delta\otimes\mathbf{u}_\delta)_{ij}| \le \varrho_\delta|\mathbf{u}_\delta|^2$ for $i,j=1,\cdots,d$ and Lemma 2.1 from Feireisl et al. [56] we obtain

$$\begin{aligned}
&\int_\Omega \langle\mathcal{V}_{\tau,x};\tilde\varrho\tilde{\mathbf{u}}\rangle\cdot\boldsymbol{\varphi}(\tau,\cdot)\ \mathrm{d}x - \int_\Omega\langle\mathcal{V}_0;\tilde\varrho\tilde{\mathbf{u}}\rangle\cdot\boldsymbol{\varphi}(0,\cdot)\ \mathrm{d}x \\
&= \int_0^\tau\int_\Omega \big[\langle\mathcal{V}_{t,x};\tilde\varrho\tilde{\mathbf{u}}\rangle\cdot\partial_t\boldsymbol{\varphi} + \langle\mathcal{V}_{t,x};\tilde\varrho(\tilde{\mathbf{u}}\otimes\tilde{\mathbf{u}})\rangle : \nabla_x\boldsymbol{\varphi} + \langle\mathcal{V}_{t,x};p(\tilde\varrho)\rangle\mathrm{div}_x\boldsymbol{\varphi}\big]\ \mathrm{d}x\ \mathrm{d}t \\
&- \int_0^\tau\int_\Omega\langle\mathcal{V}_{t,x};\mathbb{S}(\widetilde{\mathbb{D}_{\mathbf{v}}})\rangle : \nabla_x\boldsymbol{\varphi}\ \mathrm{d}x\ \mathrm{d}t + \int_0^\tau\langle r^M;\nabla_x\boldsymbol{\varphi}\rangle\ \mathrm{d}t + \int_0^\tau\langle r^L;\mathrm{div}_x\boldsymbol{\varphi}\rangle\ \mathrm{d}t,
\end{aligned}$$

where $r^M = \{r^M_{i,j}\}_{i,j=1}^d, r^M_{i,j}\in L^\infty_{\text{weak}}(0,T;\mathcal{M}(\overline{\Omega}))$ and $r^L\in L^\infty_{\text{weak}}(0,T;\mathcal{M}(\overline{\Omega}))$ such that

$$|r^M_{i,j}(\tau)| \le E_\infty(\tau) \text{ and } |r^L(\tau)| \le \zeta(\tau).$$

The defect measures r^M and r^L contain the concentration defect of the terms $\varrho_\delta\mathbf{u}_\delta\otimes\mathbf{u}_\delta$, $p(\varrho_\delta)$ and $\delta\varrho_\delta^\Gamma$. Due to (2.5.30), r^M and r^L are controlled by $\mathcal{D}$.

Verification of momentum compatibility:

Since $\mathbf{u}_\delta$ is bounded in $L^2(0,T;W_0^{1,2}(\Omega))$, in this case we can check the relation easily.

Verification of Generalized Korn–Poincaré inequality:

This follows from the inequality

$$\int_0^\tau\int_\Omega |\mathbf{u}_\delta|^2\ \mathrm{d}x\ \mathrm{d}t \le C_p\int_0^\tau\int_\Omega \mathbb{D}_0(\nabla_x\mathbf{u}_\delta) \colon \mathbb{D}_0(\nabla_x\mathbf{u}_\delta)\ \mathrm{d}x\ \mathrm{d}t.$$

Thus we summarize the above discussion in the following theorem.

Theorem 2.5.9. *Suppose Ω be a Lipschitz bounded domain in $\mathbb{R}^d$ with $d=1,2,3$ and suppose the pressure satisfies* (2.1.36) *or* (2.1.38). *If $(\varrho_0,(\varrho\mathbf{u})_0)$ satisfies* (2.5.26), *then there exists a renormalized dissipative measure-valued solution(rDMV) as defined in Definition 2.5.3 with initial data $\mathcal{V}_0 = \delta_{\{\varrho_0,(\varrho\mathbf{u})_0\}}$.*

Navier Slip boundary condition: We can use a similar technique to prove the existence for Navier slip boundary condition. The existence of weak solutions for this boundary condition can be found in Novotný-Straškraba [105].

Here we have to use generalized Korn-Poincaré inequality to obtain an uniform bound for $\|\mathbf{u}_\delta\|_{L^2(0,T;W^{1,2}(\Omega:\mathbb{R}^3)}$. We use the fact $\int_\Omega \varrho_\delta(t,x)\ \mathrm{d}x = \int_\Omega \varrho_{0,\delta}\ \mathrm{d}x > 0$ for a.e. t in $(0,T)$ to obtain

$$\int_0^\tau \int_\Omega |\mathbf{u}_\delta|^2\ \mathrm{d}x\ \mathrm{d}t$$
$$\leq C_p \left(\int_0^\tau \int_\Omega \mathbb{D}_0(\nabla_x \mathbf{u}_\delta) \colon \mathbb{D}_0(\nabla_x \mathbf{u}_\delta)\ \mathrm{d}x\ \mathrm{d}t + \int_0^\tau \int_\Omega \varrho_\delta |\mathbf{u}_\delta|^2\ \mathrm{d}x\ \mathrm{d}t \right).$$

This gives the required bound of the norm $\|\mathbf{u}_\delta\|_{L^2(0,T;W^{1,2}(\Omega:\mathbb{R}^3)}$. We obtain uniform bounds on other variables, as we obtained in the case of no-slip boundary condition.

We have the test function class for the momentum equation is

$$\{\boldsymbol{\varphi} \in C^1([0,T]; C^1(\overline{\Omega}; \mathbb{R}^d)) \text{ with } \varphi \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}.$$

On the other hand, we obtain that the defect measure r^M belongs to the space $L^\infty_{\text{weak-}(*)}(0,T; \mathcal{M}(\overline{\Omega}; \mathbb{R}^{d\times d}))$.

Thus we obtain a similar results and verify all the desires properties including the Generalized Korn-Poincaré inequality(2.5.21). Thus for the Navier slip condition a similar theorem is also true.

Theorem 2.5.10. *Suppose Ω be a Lipschitz bounded domain in $\mathbb{R}^d$ with $d = 1,2,3$ and suppose the pressure satisfies* (2.1.36) *or* (2.1.38). *If $(\varrho_0, (\varrho\mathbf{u})_0)$ satisfies* (2.5.26), *then there exists a renormalized dissipative measure-valued solution(rDMV) as defined in Definition 2.5.7 with initial data $\mathcal{V}_0 = \delta_{\{\varrho_0, (\varrho\mathbf{u})_0\}}$.*

2.6 Dissipative solutions of compressible fluids

Let us first clarify an ambiguity for the term 'dissipative solutions'. P. L. Lions [95]-[96], first coined this term to refer to a class of generalized solutions that satisfy a certain relative energy inequality, which we will introduce in the next chapter. In other words these dissipative solutions should satisfy the weak-strong uniqueness property. Here we consider a class of generalized solutions and call them as dissipative solutions since they satisfy the weak-strong uniqueness property.

2.6.1 Definition of a dissipative solutions for the Navier-Stokes system

When considering a monotone pressure law (2.1.35), we can simplify the notion of measure-valued solutions. We refer to this type of solutions as *dissipative solutions.*

Suppose we consider the artificial pressure approximation of the Navier–Stokes system, i.e., (2.5.23)-(2.5.25) with a monotone pressure law (2.1.35). We recall the uniform bounds:

$$\begin{aligned}
&\|\varrho_\delta\|_{L^\infty(0,T;L^\gamma(\Omega))} \leq C,\\
&\|\sqrt{\varrho_\delta}\mathbf{u}_\delta\|_{L^2(\Omega;\mathbb{R}^3)} \leq C,\\
&\|\mathbf{u}_\delta\|_{L^2(0,T;W^{1,2}(\Omega:\mathbb{R}^3)} \leq C,\\
&\|H(\varrho_\delta)\|_{L^\infty(0,T;L^1(\Omega))} \leq C.
\end{aligned}$$

As a consequence of it, we obtain

$$\begin{aligned}
&\|\varrho_\delta\mathbf{u}_\delta\|_{L^\infty(0,T;L^{2\gamma/\gamma+1}(\Omega;\mathbb{R}^3))} \leq C,\\
&\operatorname*{ess\,sup}_{t\in[0,T]}\int_\Omega \delta\varrho_\delta^\Gamma \,\mathrm{d}x \leq C.
\end{aligned}$$

From the above bounds, we obtain the following weak and weak-(*) convergence

$$\begin{aligned}
&\varrho_\delta \to \varrho \text{ weak-(*)ly in } L^\infty(0,T;L^\gamma(\Omega)),\\
&\mathbf{u}_\delta \to \mathbf{u} \text{ weakly in } L^2(0,T;W^{1,2}(\Omega:\mathbb{R}^3).
\end{aligned}$$

Also with the help of Lemma 8.1 (Appendix) of Abbatiello et al. [2], we obtain

$$\varrho_\delta\mathbf{u}_\delta \to \varrho\mathbf{u} \text{ weak-(*)ly in } L^\infty(0,T;L^{2\gamma/\gamma+1}(\Omega)).$$

Let us introduce the conservative variable $\mathbf{m}_\delta = \varrho_\delta\mathbf{u}_\delta$. Also we notice that the sequence $\{\varrho_\delta,\mathbf{m}_\delta\}_{\delta>0}$ generates a Young measure $\{\mathcal{V}_{t,x}\}_{(t,x)\in(0,T)\times\Omega}$. Moreover, we have

$$(\varrho,\mathbf{m}) = (\{(t,x)\mapsto\langle\mathcal{V}_{t,x};\tilde{\varrho}\rangle\},\{(t,x)\mapsto\langle\mathcal{V}_{t,x};\tilde{\mathbf{m}}\rangle\})$$

In terms of momentum, we rewrite kinetic energy with a possible extension:

$$(\varrho_\delta,\mathbf{m}_\delta)\mapsto\frac{1}{2}\frac{|\mathbf{m}_\delta|^2}{\varrho_\delta} = \begin{cases}\frac{1}{2}\frac{|(\varrho_\delta\mathbf{u}_\delta)|^2}{\varrho_\delta} & \text{if },\varrho_\delta\neq 0,\mathbf{m}_\delta\neq 0,\\ 0 & \text{if } \varrho_\delta = 0,\ \mathbf{m}_\delta = 0,\\ \infty & \text{if } \varrho_\delta = 0,\ \mathbf{m}_\delta\neq 0.\end{cases} \tag{2.6.1}$$

It is easy to verify that the map $(\varrho,\mathbf{m})\mapsto\frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho}$ is convex l.s.c in $(0,\infty)\times\mathbb{R}^d$. From energy inequality, it is worth to notice that it equals ∞ only on a set of zero measure in $(0,T)\times\Omega$. We have the following convergence:

$$\begin{aligned}
&\frac{1}{2}\frac{|\mathbf{m}_\delta|^2}{\varrho_\delta} \to \frac{1}{2}\overline{\frac{|\mathbf{m}|^2}{\varrho}} \text{ weakly-(*) in } L^\infty(0,T;\mathcal{M}(\overline{\Omega};\mathbb{R}^{d\times d}_{\mathrm{sym}})),\\
&P(\varrho_\delta)\to\overline{P(\varrho)} \text{ weakly-(*) in } L^\infty(0,T;\mathcal{M}(\overline{\Omega})),\\
&\delta\varrho_\delta^\Gamma(t,\cdot)\to\zeta \text{ weakly-(*) in } L^\infty(0,T;\mathcal{M}(\overline{\Omega})).
\end{aligned}$$

Hence we define a measure

$$\begin{aligned}
\mathfrak{R}_e^{\mathrm{kin}} =& \mathfrak{R}_{e,1}^{\mathrm{kin}} + \mathfrak{R}_{e,1}^{\mathrm{kin}}\\
=& \left(\frac{1}{2}\overline{\frac{|\mathbf{m}|^2}{\varrho}} - \left\langle\mathcal{V}_{t,x};\frac{1}{2}\frac{|\tilde{\mathbf{m}}|^2}{\tilde{\varrho}}\right\rangle\right) + \left(\left\langle\mathcal{V}_{t,x};\frac{1}{2}\frac{|\tilde{\mathbf{m}}|^2}{\tilde{\varrho}}\right\rangle - \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho}\right).
\end{aligned}$$

Using (1.3.13), we obtain $\mathfrak{R}_{e,1}^{\text{kin}} \in L^\infty(0,T;\mathcal{M}^+(\overline{\Omega}))$. Also from Jensen's inequality we have $0 \le \mathfrak{R}_{e,1}^{\text{kin}} \in L^\infty(0,T;L^1(\Omega))$. Now we consider the *total energy defect* as

$$\mathfrak{R}_e = \frac{1}{2}\overline{\frac{|\mathbf{m}|^2}{\varrho}} - \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + \overline{P(\varrho)} - P(\varrho) + \frac{1}{\Gamma-1}\zeta.$$

Hence, we have$\mathfrak{R}_e \in L^\infty(0,T;\mathcal{M}^+(\overline{\Omega}))$.

Also, we have the following convergence:

$$\begin{aligned}
&\frac{\mathbf{m}_\delta \otimes \mathbf{m}_\delta}{\varrho_\delta} \to \overline{\frac{\mathbf{m}\otimes\mathbf{m}}{\varrho}} \text{ weakly-(*) in } L^\infty(0,T;\mathcal{M}(\overline{\Omega};\mathbb{R}^{d\times d}_{\text{sym}})),\\
&p(\varrho_\delta) \to \overline{p(\varrho)} \text{ weakly-(*) in } L^\infty(0,T;\mathcal{M}(\overline{\Omega})),\\
&\delta\varrho_\delta^\Gamma(t,\cdot) \to \zeta \text{ weakly-(*) in } L^\infty(0,T;\mathcal{M}(\overline{\Omega})).
\end{aligned}$$

We consider

$$\mathfrak{R}_m = \left[\overline{\frac{\mathbf{m}\times\mathbf{m}}{\varrho}} - \frac{\mathbf{m}\times\mathbf{m}}{\varrho}\right] + \left[\overline{p(\varrho)} - p(\varrho) + \zeta\right]\mathbb{I},$$

We notice that for any $\xi \in \mathbb{R}^d$, the map $(\varrho,\mathbf{m}) \mapsto \frac{|\mathbf{m}\cdot\xi|^2}{\varrho}$, with a similar extension like (2.6.1), is convex l.s.c. This helps to conclude that

$$\text{For any } \xi \in \mathbb{R}^d, \mathfrak{R}_m \colon \xi\otimes\xi \in \mathfrak{R}_e \in L^\infty(0,T;\mathcal{M}^+(\overline{\Omega})).$$

This implies

$$\mathfrak{R}_m \in L^\infty(0,T;\mathcal{M}^+(\overline{\Omega};\mathbb{R}^{d\times d}_{\text{sym}})).$$

Furthermore, we observe

$$\lambda_1 \mathrm{Tr}(\mathfrak{R}_m) \le \mathfrak{R}_e \le \lambda_2 \mathrm{Tr}(\mathfrak{R}_m),\ \lambda_1,\lambda_2 > 0. \tag{2.6.2}$$

Defect measures $\mathfrak{R}_m, \mathfrak{R}_e$ are called *turbulent defect measures*. Now we need to perform the limit passage in the approximate continuity equation, the approximate momentum equation and the approximate energy inequality and summarize the above discussion to give the definition of the dissipative solutions which is as follows:

Definition 2.6.1. Let $\gamma \ge 1$. We say that $(\varrho,\mathbf{u})$ with

$$\begin{aligned}
\varrho \in &C_{\text{weak}}([0,T];L^\gamma(\Omega)),\ \varrho \ge 0,\ \varrho\mathbf{u} \in C_{\text{weak}}([0,T];L^{\frac{2\gamma}{\gamma+1}}(\Omega)),\\
&\mathbf{u} \in L^2(0,T;W_0^{1,2}(\Omega)),
\end{aligned}$$

is a *dissipative solution* of the system (2.2.1)-(2.2.3) with no-slip boundary condition (2.2.5) and initial data $(\varrho_0,(\varrho\mathbf{u})_0)$ which satisfies

$$\varrho_0 \ge 0,\ E_0 = \int_\Omega \left(\frac{1}{2}\frac{|(\varrho\mathbf{u})_0|^2}{\varrho_0} + P(\varrho_0)\right)\,\mathrm{d}x < \infty, \tag{2.6.3}$$

if there exist *turbulent defect measures*

$$\mathfrak{R}_m \in L^\infty(0,T;\mathcal{M}^+(\overline{\Omega};\mathbb{R}^{d\times d}_{\rm sym})) \text{ and } \mathfrak{R}_e \in L^\infty(0,T;\mathcal{M}^+(\overline{\Omega})), \tag{2.6.4}$$

satisfying the compatibility condition

$$\lambda_1 \mathrm{Tr}(\mathfrak{R}_m) \le \mathfrak{R}_e \le \lambda_2 \mathrm{Tr}(\mathfrak{R}_m), \quad \text{for some } \lambda_1, \lambda_2 > 0, \tag{2.6.5}$$

such that the following is satisfied:

- **Equation of continuity:** For any $\tau \in (0,T)$ and any $\varphi \in C^1_c([0,T]\times\overline{\Omega})$, it holds

$$\left[\int_\Omega \varrho\varphi \,\mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega [\varrho\partial_t\varphi + \varrho\mathbf{u}\cdot\nabla_x\varphi] \,\mathrm{d}x\,\mathrm{d}t. \tag{2.6.6}$$

- **Momentum equation:** For any $\tau \in (0,T)$ and any $\boldsymbol{\varphi} \in C^1_c([0,T]\times\Omega;\mathbb{R}^d)$ with $\boldsymbol{\varphi}\cdot\mathbf{n}|_{\partial\Omega} = 0$, it holds

$$\begin{aligned}\left[\int_\Omega \varrho\mathbf{u}(\tau,\cdot)\cdot\boldsymbol{\varphi}(\tau,\cdot) \,\mathrm{d}x\right]_{t=0}^{t=\tau} &= \int_0^\tau \int_\Omega [\varrho\mathbf{u}\cdot\partial_t\boldsymbol{\varphi} + \varrho\mathbf{u}\otimes\mathbf{u} : \nabla_x\boldsymbol{\varphi} + p(\varrho)\mathrm{div}_x\boldsymbol{\varphi}] \,\mathrm{d}x\,\mathrm{d}t \\ &\quad - \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\boldsymbol{\varphi} \,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau \int_{\overline{\Omega}} \nabla_x\boldsymbol{\varphi} : \mathrm{d}\mathfrak{R}_m\,\mathrm{d}t.\end{aligned} \tag{2.6.7}$$

- **Energy inequality:** The total energy E is defined in $[0,T)$ as

$$E(\tau) = \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho)\right)(\tau,\cdot) \,\mathrm{d}x.$$

  It satisfies,

$$E(\tau) + \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\mathbf{u} \,\mathrm{d}x\,\mathrm{d}t + \int_{\overline{\Omega}} \mathrm{d}\,\mathfrak{R}_e(\tau,\cdot) \le E_0 \tag{2.6.8}$$

  for a.e. $\tau > 0$.

From the discussion at the beginning of the subsection, we establish the existence of a dissipative solution.

Theorem 2.6.2. *Suppose Ω be a bounded domain and pressure follows* (2.1.35) *with $\gamma > 1$. If $(\varrho_0, (\varrho\mathbf{u})_0)$ satisfies* (2.6.3)*, then there exists a* dissipative solution *of the compressible Navier–Stokes system, that follows the Definition 2.6.1.*

Remark 2.6.3. Instead of pressure law (2.1.35) we can consider a general pressure as described in (2.1.37).

2.6.2 Dissipative solution: Barotropic Euler system

Similarly, for the Euler system (2.3.1)-(2.3.2) with pressure law (2.3.3) we provide a definition of the *dissipative solution* in a bounded domain Ω with impermeability boundary condition (2.3.4).

Definition 2.6.4. We say that $(\varrho, \mathbf{m})$ with

$$\varrho \in C_{\text{weak}}([0,T]; L^{\gamma}(\Omega)),\ \varrho \geq 0,\ \mathbf{m} \in C_{\text{weak}}([0,T]; L^{\frac{2\gamma}{\gamma+1}}(\Omega)),$$

is a *dissipative solution* to the compressible Euler equation with the boundary condition (2.3.4) and the initial data $(\varrho_0, \mathbf{m}_0)$ satisfying

$$\varrho_0 \geq 0,\ E_0 = \int_\Omega \left(\frac{1}{2} \frac{|\mathbf{m}_0|^2}{\varrho_0} + P(\varrho_0) \right) \mathrm{d}x < \infty, \tag{2.6.9}$$

if there exist the *turbulent defect measures*

$$\mathfrak{R}_m \in L^\infty(0,T; \mathcal{M}^+(\overline{\Omega}; \mathbb{R}^{d\times d}_{\text{sym}})),\ \mathfrak{R}_e \in L^\infty(0,T; \mathcal{M}^+(\overline{\Omega})),$$

satisfying the compatibility condition

$$\lambda_1 \mathrm{Tr}(\mathfrak{R}_m) \leq \mathfrak{R}_e \leq \lambda_2 \mathrm{Tr}(\mathfrak{R}_m),\ \lambda_1, \lambda_2 > 0, \tag{2.6.10}$$

such that the following holds:

- **Equation of continuity:** For any $\tau \in (0,T)$ and any $\varphi \in C^1_c([0,T) \times \overline{\Omega})$ it holds

$$\left[\int_\Omega \varrho \varphi \ \mathrm{d}x \right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega [\varrho \partial_t \varphi + \mathbf{m} \cdot \nabla_x \varphi] \ \mathrm{d}x \ \mathrm{d}t; \tag{2.6.11}$$

- **Momentum equation:** For any $\tau \in (0,T)$ and any $\boldsymbol{\varphi} \in C^1_c([0,T) \times \overline{\Omega}; \mathbb{R}^d)$ with $\boldsymbol{\varphi} \cdot \mathbf{n}|_{\partial\Omega} = 0$, it holds

$$\begin{aligned} &\left[\int_\Omega \mathbf{m}(\tau, \cdot) \cdot \boldsymbol{\varphi}(\tau, \cdot) \ \mathrm{d}x \right]_{t=0}^{t=\tau} \\ &= \int_0^\tau \int_\Omega \left[\mathbf{m} \cdot \partial_t \boldsymbol{\varphi} + \mathbf{1}_{\{\varrho>0\}} \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) : \nabla_x \boldsymbol{\varphi} + p(\varrho) \mathrm{div}_x \boldsymbol{\varphi} \right] \mathrm{d}x \ \mathrm{d}t \\ &+ \int_0^\tau \int_{\overline{\Omega}} \nabla_x \boldsymbol{\varphi} : \mathrm{d}\mathfrak{R}_m \ \mathrm{d}t; \end{aligned} \tag{2.6.12}$$

- **Energy inequality:** The total energy E is defined in $[0,T)$ as

$$E(\tau) = \int_\Omega \left(\frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + (P(\varrho) \right) (\tau, \cdot) \ \mathrm{d}x$$

  and it satisfies

$$E(\tau) + \int_{\overline{\Omega}} \mathrm{d}\ \mathfrak{R}_e(\tau, \cdot) \leq E_0 \tag{2.6.13}$$

  for a.e. $\tau > 0$;

Remark 2.6.5. An existence of a dissipative solution for the Euler system is found in Breit, Feireisl and Hofmanová [17, Section 5]

The definition in the full domain $\Omega = \mathbb{R}^d$, equipped with far field condition is as follows:

Definition 2.6.6. We say that $(\varrho, \mathbf{m})$ with

$$\varrho \in C_{\text{weak}}([0,T]; L^2 + L^\gamma(\mathbb{R}^d)),\ \varrho \geq 0,\ \mathbf{m} \in C_{\text{weak}}([0,T]; L^2 + L^{\frac{2\gamma}{\gamma+1}}(\mathbb{R}^d)),$$

is a *dissipative solution* to the compressible Euler equation (2.3.1)-(2.3.2) with initial data $(\varrho_0, \mathbf{m}_0)$ satisfying

$$\varrho_0 \geq 0,\ E_0 = \int_{\mathbb{R}^d} \left(\frac{1}{2} \frac{|\mathbf{m}_0|^2}{\varrho_0} + P(\varrho_0) - (\varrho_0 - \bar{\varrho}) P'(\bar{\varrho}) - P(\bar{\varrho}) \right) \mathrm{d}x < \infty, \tag{2.6.14}$$

if there exist the *turbulent defect measures*

$$\mathfrak{R}_m \in L^\infty(0,T; \mathcal{M}^+(\mathbb{R}^d; \mathbb{R}^{d\times d}_{\text{sym}})),\ \mathfrak{R}_e \in L^\infty(0,T; \mathcal{M}^+(\mathbb{R}^d)),$$

satisfying the compatibility condition

$$\lambda_1 \mathrm{Tr}(\mathfrak{R}_m) \leq \mathfrak{R}_e \leq \lambda_2 \mathrm{Tr}(\mathfrak{R}_m),\ \lambda_1, \lambda_2 > 0, \tag{2.6.15}$$

such that the following holds:

- **Equation of continuity:** The integral identity

$$\left[\int_{\mathbb{R}^d} \varrho \varphi \ \mathrm{d}x \right]_{t=0}^{t=\tau} = \int_0^\tau \int_{\mathbb{R}^d} [\varrho \partial_t \varphi + \mathbf{m} \cdot \nabla_x \varphi] \ \mathrm{d}x \ \mathrm{d}t \tag{2.6.16}$$

  holds for any $\tau \in (0,T)$ and any $\varphi \in C_c^1([0,T) \times \mathbb{R}^d)$.

- **Momentum equation:** The integral identity

$$\begin{aligned} &\left[\int_{\mathbb{R}^d} \mathbf{m}(\tau,\cdot) \cdot \boldsymbol{\varphi}(\tau,\cdot) \ \mathrm{d}x \right]_{t=0}^{t=\tau} \\ &= \int_0^\tau \int_{\mathbb{R}^d} \left[\mathbf{m} \cdot \partial_t \boldsymbol{\varphi} + \mathbf{1}_{\{\varrho>0\}} \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) : \nabla_x \boldsymbol{\varphi} + p(\varrho) \mathrm{div}_x \boldsymbol{\varphi} \right] \mathrm{d}x \ \mathrm{d}t \\ &+ \int_0^\tau \int_{\mathbb{R}^d} \nabla_x \boldsymbol{\varphi} : \mathrm{d}\mathfrak{R}_m \ \mathrm{d}t \end{aligned} \tag{2.6.17}$$

  holds for any $\tau \in (0,T)$ and any $\boldsymbol{\varphi} \in C_c^1([0,T) \times \mathbb{R}^d; \mathbb{R}^d)$.

- **Energy inequality:** The total energy E is defined for $\tau \in (0,T)$ as

$$E(\tau) = \int_{\mathbb{R}^d} \left(\frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + (P(\varrho) - (\varrho - \bar{\varrho}) P'(\bar{\varrho}) - P(\bar{\varrho})) \right) (\tau,\cdot) \ \mathrm{d}x.$$

  It satisfies

$$E(\tau) + \int_{\mathbb{R}^d} \mathrm{d}\, \mathfrak{R}_e(\tau,\cdot) \leq E_0 \tag{2.6.18}$$

  for a.e. $\tau > 0$.

2.6.3 A possible adaptation for a special domain

In the previous sections we have given the definition of a *weak* and a *dissipative* solution in the bounded domain and the full domain. Now we consider the domain $\Omega = \mathbb{R}^2 \times (0,1)$, an infinite slab.

Navier stokes system:

For the compressible Navier–Stokes system (2.2.1)-(2.2.3) with a monotone isentropic pressure law (2.1.35) and finite energy initial data, we assume a far field condition,

$$|\varrho - \tilde{\varrho}| \to 0, \ \mathbf{u} \to \mathbf{0} \text{ as } |x_h| \to \infty, \tag{2.6.19}$$

where $(\tilde{\varrho}, 0)$ is a static solution, and, a boundary condition

$$\mathbf{u} \cdot \mathbf{n} = 0 \text{ and } [\mathbb{S} \cdot \mathbf{n}]_{\tan} = 0 \text{ on } \partial\Omega. \tag{2.6.20}$$

In the presence of an external force $\mathbf{f}$ in the momentum equation we observe that the static solutions $\tilde{\varrho}$ satisfy

$$\nabla_x p(\tilde{\varrho}) = \tilde{\varrho}\mathbf{f},$$

for a time independent function $\mathbf{f}$.

Weak solution: First we give the definition of weak solution in this domain from Feireisl and Novotný [75, Section 2.2]. We consider a *finite energy initial data*, i.e., $\varrho_0 \geq 0$, $\varrho_0 \in L^1_{\text{loc}}(\mathbb{R}^d)$ and

$$E_0 = \int_\Omega \left(\frac{1}{2} \frac{|(\varrho\mathbf{u})_0|^2}{\varrho_0} + (P(\varrho_0) - (\varrho_0 - \tilde{\varrho})P'(\tilde{\varrho})) - P(\tilde{\varrho}) \right) \mathrm{d}x < \infty. \tag{2.6.21}$$

Definition 2.6.7. Let $\gamma \geq 1$ and $(\varrho_0, (\varrho\mathbf{u})_0)$ be a finite energy initial data. We say that $(\varrho, \mathbf{u})$ is a weak solution of the Navier–Stokes system with pressure law(2.1.35) in $\Omega = \mathbb{R}^2 \times (0,1)$, if the following is true.

- **Regularity class:** We have $0 \leq \varrho$, $\varrho - \tilde{\varrho} \in C_{\text{weak}}(0,T; L^2 + L^\gamma(\Omega))$, $\mathbf{u} \in L^2(0,T; W^{1,2}(\Omega; \mathbb{R}^d))$ and $\varrho\mathbf{u} \in C_{\text{weak}}(0,T; L^2 + L^{\frac{2\gamma}{\gamma+1}}(\Omega))$.
- The renormalized continuity equation holds in weak sense for the class of test functions is $C_c^\infty([0,T) \times \overline{\Omega})$. The momentum equation remains true in weak sense for text function class $\{\boldsymbol{\phi} \in C_c^\infty([0,T] \times \overline{\Omega}; \mathbb{R}^d) \,|\, \boldsymbol{\phi} \cdot \mathbf{n} = 0 \text{ in } \Omega\}$.
- The far field conditions are incorporated through the energy inequality. The total energy E is defined in $[0,T)$ as,

$$E(\tau) = \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + (P(\varrho) - (\varrho - \tilde{\varrho})P'(\tilde{\varrho})) - P(\tilde{\varrho}) \right)(\tau, \cdot) \, \mathrm{d}x$$

It satisfies,

$$E(\tau) + \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \, \mathrm{d}x \, \mathrm{d}t \leq E_0 \tag{2.6.22}$$

for a.e. $\tau > 0$.

Remark 2.6.8. In general one can assume $\mathbf{f} \in L^\infty((0,T) \times \Omega)$. Here we consider $\mathbf{f} = \nabla G$ with G is independent of time and $G \in C^1(\Omega)$ with $\nabla_x G \in L^\infty(\Omega)$. A simple example is $G(x_h, x_3) = -x_3$ which resemblances the simplest form of the *gravitational potential*, as a consequence of such choice of G, one can choose $\tilde{\varrho} \in C^2(\Omega) \cap L^\infty(\Omega)$. Later In our application we consider this particular form of G. Hence we investigate on this particular G.

Next, we give the definition of a dissipative solution.

Definition 2.6.9. Let $\mathbf{f} \in L^\infty(\Omega)$ and Let $0 < \tilde{\varrho} \in W^{1,\infty}(\Omega)$ and it satisfies $\nabla_x p(\tilde{\varrho}) = \tilde{\varrho}\mathbf{f}$. We say that $(\varrho, \mathbf{u})$ with

$$\begin{aligned} &\varrho - \tilde{\varrho} \in C_{\text{weak}}([0,T]; L^2 + L^\gamma(\Omega)), \ \varrho \geq 0, \ \varrho\mathbf{u} \in C_{\text{weak}}([0,T]; L^2 + L^{\frac{2\gamma}{\gamma+1}}(\Omega)), \\ &\text{and } \mathbf{u} \in L^2(0,T; W^{1,2}(\Omega)), \end{aligned}$$

is a *dissipative solution* to (2.2.1)-(2.2.3) with boundary condition (2.6.20), initial data $(\varrho_0, (\varrho\mathbf{u})_0)$ and far field condition (2.6.19) satisfying

$$\varrho_0 \geq 0, \ E_0 = \int_\Omega \left(\frac{1}{2} \frac{|(\varrho\mathbf{u})_0|^2}{\varrho_0} + P(\varrho_0) - (\varrho_0 - \tilde{\varrho})P'(\tilde{\varrho}) - P(\tilde{\varrho}) \right) \mathrm{d}x < \infty, \tag{2.6.23}$$

if there exist the *turbulent defect measures*

$$\mathfrak{R}_m \in L^\infty(0,T; \mathcal{M}^+(\overline{\Omega}; \mathbb{R}^{d\times d}_{\text{sym}})), \ \mathfrak{R}_e \in L^\infty(0,T; \mathcal{M}^+(\overline{\Omega})),$$

satisfying the compatibility condition

$$\lambda_1 \operatorname{Tr}(\mathfrak{R}_m) \leq \mathfrak{R}_e \leq \lambda_2 \operatorname{Tr}(\mathfrak{R}_m), \ \lambda_1, \lambda_2 > 0, \tag{2.6.24}$$

such that the following holds:

- **Equation of continuity:** For any $\varphi \in C_c^\infty([0,T] \times \overline{\Omega})$, it holds

$$\left[\int_\Omega \varrho\varphi \, \mathrm{d}x \right]_{t=0}^{t=T} = \int_0^\tau \int_\Omega [\varrho \partial_t \varphi + \varrho\mathbf{u} \cdot \nabla_x \varphi] \, \mathrm{d}x \, \mathrm{d}t. \tag{2.6.25}$$

- **Momentum equation:** For any $\boldsymbol{\varphi} \in C_c^\infty([0,T] \times \Omega; \mathbb{R}^d)$, it holds

$$\begin{aligned} &\left[\int_\Omega \varrho\mathbf{u}(\tau,\cdot) \cdot \boldsymbol{\varphi}(\tau,\cdot) \, \mathrm{d}x \right]_{t=0}^{t=\tau} \\ &= \int_0^\tau \int_\Omega [\varrho\mathbf{u} \cdot \partial_t \boldsymbol{\varphi} + \varrho\mathbf{u} \otimes \mathbf{u} : \nabla_x \boldsymbol{\varphi} + p(\varrho_\epsilon) \operatorname{div}_x \boldsymbol{\varphi}] \, \mathrm{d}x \, \mathrm{d}t \\ &- \int_0^\tau \int_\Omega [\mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \boldsymbol{\varphi} - \varrho\mathbf{f} \cdot \boldsymbol{\varphi}] \, \mathrm{d}x \, \mathrm{d}t + \int_0^\tau \int_{\overline{\Omega}} \nabla_x \boldsymbol{\varphi} : \mathrm{d}\mathfrak{R}_m \, \mathrm{d}t. \end{aligned} \tag{2.6.26}$$

- **Energy inequality:** The total energy E is defined in $[0,T)$ as

$$E(\tau) = \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + (P(\varrho) - (\varrho - \tilde{\varrho}P'(\tilde{\varrho})) - P(\tilde{\varrho}) \right)(\tau,\cdot)\ \mathrm{d}x.$$

It satisfies

$$E(\tau) + \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u}\ \mathrm{d}x\ \mathrm{d}t + \int_{\overline{\Omega}} \mathrm{d}\,\mathfrak{R}_e(\tau,\cdot) \leq E_0 \tag{2.6.27}$$

for a.e. $\tau > 0$.

In the Definition 2.6.7 as well as in Definition 2.6.9, we notice a different form of the energy inequality, (2.6.22) and (2.6.27). In the sub-section 2.2.2, we discussed informally the invading domain technique for $\mathbb{R}^d$, and how the far field conditions are incorporated through energy inequality.

In this case, the situation is a bit more delicate. We again try to justify informally how we obtain such energy inequality as described in (2.6.22) and (2.6.27).

First,we assume $\mathbf{f} = \nabla_x G$ with $G \in W^{1,\infty}(\Omega)$ and the initial data $(\varrho_0, (\varrho\mathbf{u})_0)$ satisfies (2.6.21). We consider $\Omega_R = B(0,R) \times (0,1)$ where $B(0,R)$ is a ball of radius R in $\mathbb{R}^2$, also assume the system is provided by no-slip boundary condition, i.e., $\mathbf{u}_R = 0$ on $\partial\Omega_R$, where $(\varrho_R, \mathbf{R})$ denotes a finite energy weak solution in Ω_R. We recall the energy inequality in Ω_R:

$$E_R(\tau) + \int_0^\tau \int_{\Omega_R} \mathbb{S}(\nabla_x \mathbf{u}_R) : \nabla_x \mathbf{u}_R\ \mathrm{d}x\ \mathrm{d}t \leq E_0 + \int_0^\tau \int_{\Omega_R} \varrho_R \mathbf{u}_R \cdot \mathbf{f}\ \mathrm{d}x\ \mathrm{d}t\ .$$

for a.e. $\tau > 0$ with

$$E_R(\tau) = \int_{\Omega_R} \left(\frac{1}{2}\varrho_R|\mathbf{u}_R|^2 + P(\varrho_R) \right)(\tau,\cdot)\ \mathrm{d}x,$$

and $\varrho_{0,R} = \mathbf{1}_{\Omega_R}\varrho_0, (\varrho\mathbf{u})_{0,R} = \mathbf{1}_{\Omega_R}(\varrho\mathbf{u})_0$. Using the fact that $\tilde{\varrho}$ is independent of time from the continuity equation we have

$$\left[\int_{\Omega_R} (\varrho_R - \tilde{\varrho})P'(\tilde{\varrho})\ \mathrm{d}x \right]_{t=0}^{t=\tau} = \int_0^\tau \int_{\Omega_R} \varrho_R \mathbf{u}_R \cdot \nabla_x (P'(\tilde{\varrho}))\ \mathrm{d}x\ \mathrm{d}t.$$

Using the property of a static solution that it satisfies $\nabla_x P'(\tilde{\varrho}) = \mathbf{f}$, we observe

$$\begin{aligned} E_R(\tau) - \int_{\Omega_R} ((\varrho_R - \overline{\varrho})P'(\overline{\varrho}) - P(\overline{\varrho}))(\tau,\cdot)\ \mathrm{d}x + \int_0^\tau \int_{\Omega_R} \mathbb{S}(\nabla_x \mathbf{u}_R) : \nabla_x \mathbf{u}_R\ \mathrm{d}x\ \mathrm{d}t \\ \leq E_{0,R} - \int_{\Omega_R} ((\varrho_{R,0} - \overline{\varrho})P'(\overline{\varrho}) - P(\overline{\varrho}))(\tau,\cdot)\ \mathrm{d}x. \end{aligned}$$

This motivates to consider

$$\tilde{E}_R(\tau) = \int_{\Omega_R} \left(\frac{1}{2}\varrho_R|\mathbf{u}_R|^2 + (P(\varrho) - (\varrho_R - \tilde{\varrho})P'(\tilde{\varrho})) - P(\tilde{\varrho}) \right)(\tau,\cdot)\ \mathrm{d}x.$$

Using this we rewrite energy inequality as

$$\tilde{E}_R(\tau) + \int_0^\tau \int_{\Omega_R} \mathbb{S}(\nabla_x \mathbf{u}_R) : \nabla_x \mathbf{u}_R \, \mathrm{d}x \, \mathrm{d}t \leq \tilde{E}_{0,R} \tag{2.6.28}$$

Next, we consider a possible extension in $\mathbb{R}^d$ as

$$\grave{\varrho}_R = \begin{cases} \varrho_R \text{ in } B(0,R) \\ \tilde{\varrho} \text{ otherwise} \end{cases} \quad \text{and } \grave{\mathbf{u}}_R = \begin{cases} \mathbf{u}_R \text{ in } B(0,R) \\ 0 \text{ otherwise} \end{cases} .$$

This helps us to extend the inequality (2.6.28) in $\mathbb{R}^d$. We have $E_{0,R} \leq E_0$, where E_0 is independent of R. Using some structural property of kinetic energy and pressure, we obtain an uniform bound for $\grave{\varrho}_R - \tilde{\varrho}$, and similarly for other variables. Finally a suitable limiting process gives the precise energy inequality as in (2.6.22) or (2.6.27).

The above discussion is too informal, it is just to give an idea how we get the energy inequality. It is mathematically incorrect formulation for our problem as we consider no-slip boundary condition on $\partial(B(0,R) \times (0,1))$ instead of proposed Navier slip boundary. One can consider the weak solutions with Navier-slip boundary condition in bounded domain but this leads some other problem of possible zero extensions of $\mathbf{u}_R$. Although there is a standard approach to deal this difficulty is by introducing a suitable symmetry class.

Symmetry Class: Ebin[46] described that the slip boundary condition (impermeability boundary condition) in $\mathbb{R}^2 \times (0,1)$ can be transformed into periodic ones by considering the space of symmetric functions. Here $\varrho, \mathbf{u}_h (= u_1, u_2)$ were extended as even functions in the x_3-variable defined on $\mathbb{R}^2 \times \mathbb{T}^1$, while u_3 is extended as an odd function in x_3 on the same set, i.e.,

$$\begin{aligned} &\varrho(t, x_h, -x_3) = \varrho(t, x_h, x_3), \ \mathbf{u}_h(t, x_h, -x_3) = \mathbf{u}_h(t, x_h, x_3), \\ &u_3(t, x_h, -x_3) = -u_3(t, x_h, x_3). \end{aligned} \tag{2.6.29}$$

for all $t \in (0,T), \ x_h \in \mathbb{R}^2, \ x_3 \in \mathbb{T}^1$. A similar convention is adopted for the initial data.

Hence, the consideration of the domain $\mathbb{R}^2 \times (0,1)$ with slip boundary condition is equivalent to $\mathbb{R}^2 \times \mathbb{T}^1$. We have to consider solutions in the class (2.6.29). Just a small remark that, now we can justify the consideration of no-slip boundary condition on $\partial(B(0,R) \times \mathbb{T}^1)$ in the informal justification of the energy inequality above and a possible extension of $\mathbf{u}_R$ in whole Ω by zero outside $B(0,R) \times \mathbb{T}^1$. Here also we have a similar definition of weak solution in domain $\mathbb{R}^2 \times \mathbb{T}^1$

Definition 2.6.10. Let $\gamma \geq 1$, $(\varrho_0, (\varrho\mathbf{u})_0)$ is a finite energy initial data, then we say $(\varrho, \mathbf{u})$ solves the Navier–Stokes system with pressure law in $\mathbb{R}^2 \times \mathbb{T}^1$, if

- **Regularity class:** $0 \leq \varrho$, $\varrho - \tilde{\varrho} \in C_{\text{weak}}(0,T; L^2 + L^\gamma(\Omega))$, $\mathbf{u} \in L^2(0,T; W^{1,2}(\Omega;\mathbb{R}^d))$ and $\varrho\mathbf{u} \in C_{\text{weak}}(0,T; L^2 + L^{\frac{2\gamma}{\gamma+1}}(\Omega))$.

- The renormalized continuity equation holds for the class of test functions is $C^1_c([0,T] \times \mathbb{R}^2 \times \mathbb{T}^1)$. The momentum equation remains true for text functions in $C^1_c([0,T] \times \mathbb{R}^2 \times \mathbb{T}^1; \mathbb{R}^d)$.
- The energy inequality is similar to that in (2.6.27).

Dissipative solution: Similarly, we can provide the definition of a dissipative solution in $\Omega = \mathbb{R}^2 \times \mathbb{T}^1$. The definition is similar to the Definition 2.6.9,

Definition 2.6.11. Let $\gamma \geq 1$, $(\varrho_0, (\varrho \mathbf{u})_0)$ is a finite energy initial data, then we say $(\varrho, \mathbf{u})$ solves the Navier–Stokes system with pressure law in $\mathbb{R}^2 \times \mathbb{T}^1$, if

- **Regularity class:** $0 \leq \varrho,\ \ \varrho - \tilde{\varrho} \in C_{\text{weak}}(0,T; L^2 + L^\gamma(\Omega))$, $\mathbf{u} \in L^2(0,T; W^{1,2}(\Omega;\mathbb{R}^d))$ and $\varrho\mathbf{u} \in C_{\text{weak}}(0,T; L^2 + L^{\frac{2\gamma}{\gamma+1}}(\Omega))$.
- The renormalized continuity equation holds for the class of test functions is $C^1_c([0,T] \times \mathbb{R}^2 \times \mathbb{T}^1)$. The momentum equation remains true for text functions in $C^1_c([0,T] \times \mathbb{R}^2 \times \mathbb{T}^1; \mathbb{R}^d)$.
- The far field conditions are incorporated through the energy inequality. The total energy E is defined in $[0,T)$ as,

$$E(\tau) = \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + (P(\varrho) - (\varrho - \tilde{\varrho})P'(\tilde{\varrho})) - P(\tilde{\varrho}) \right)(\tau, \cdot)\ \mathrm{d}x$$

  It satisfies,

$$E(\tau) + \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u}\ \mathrm{d}x\ \mathrm{d}t \leq E_0 \tag{2.6.30}$$

  for a.e. $\tau > 0$.

Euler System

For the compressible Euler system (2.2.1)-(2.2.3) with a monotone isentropic pressure law (2.1.35) and finite energy initial data, we assume a far field and boundary condition as follows:

- **Boundary condition:** The impermeability or slip boundary condition on $\partial\Omega$ is given by

$$\mathbf{m} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega.$$

- **Far field condition:** Considering $x = (x_h, x_3)$, the conditions read as

$$|\varrho - \tilde{\varrho}| \to 0,\ \mathbf{u} \to \mathbf{0} \text{ as } |x_h| \to \infty, \tag{2.6.31}$$

where a static solution$(\tilde{\varrho}, 0)$ satisfies $\nabla_x p(\tilde{\varrho}) = \tilde{\varrho}\mathbf{f}$ in Ω. Now we provide the definition.

Definition 2.6.12. We say functions $\varrho, \mathbf{u}$ with

$$\varrho - \tilde{\varrho} \in C_{\text{weak}}([0,T]; L^2 + L^{\gamma}(\Omega)),\ \varrho \geq 0,\ \mathbf{m} \in C_{\text{weak}}([0,T]; L^2 + L^{\frac{2\gamma}{\gamma+1}}(\Omega)),$$

are a *dissipative solution* to the compressible Euler equation (2.3.1)-(2.3.2) with initial data $(\varrho_0, (\varrho\mathbf{u})_0)$ satisfying,

$$\varrho_0 \geq 0,\ E_0 = \int_{\Omega} \left(\frac{1}{2} \frac{|\mathbf{m}_0|^2}{\varrho_0} + P(\varrho_0) - (\varrho_0 - \tilde{\varrho})P'(\tilde{\varrho}) - P(\tilde{\varrho}) \right) \mathrm{d}x < \infty, \tag{2.6.32}$$

if there exist the *turbulent defect measures*

$$\mathfrak{R}_m \in L^\infty(0,T; \mathcal{M}^+(\overline{\Omega}; \mathbb{R}^{d\times d}_{\text{sym}})),\ \mathfrak{R}_e \in L^\infty(0,T; \mathcal{M}^+(\overline{\Omega})),$$

satisfying compatibility condition

$$\lambda_1 \mathrm{Tr}(\mathfrak{R}_m) \leq \mathfrak{R}_e \leq \lambda_2 \mathrm{Tr}(\mathfrak{R}_m),\ \lambda_1, \lambda_2 > 0, \tag{2.6.33}$$

such that the following holds:

- **Equation of continuity:** For any $\tau \in (0,T)$ and any $\varphi \in C^1_c([0,T) \times \overline{\Omega})$ it holds

$$\left[\int_{\Omega} \varrho\varphi \ \mathrm{d}x \right]_{t=0}^{t=\tau} = \int_0^\tau \int_{\Omega} \left[\varrho \partial_t \varphi + \mathbf{m} \cdot \nabla_x \varphi \right] \ \mathrm{d}x \ \mathrm{d}t; \tag{2.6.34}$$

- **Momentum equation:** For any $\tau \in (0,T)$ and any $\boldsymbol{\varphi} \in C^1_c([0,T) \times \overline{\Omega}; \mathbb{R}^d)$ with $\boldsymbol{\varphi} \cdot \mathbf{n}|_{\partial\Omega} = 0$, it holds

$$\begin{aligned} &\left[\int_{\Omega} \mathbf{m}(\tau, \cdot) \cdot \boldsymbol{\varphi}(\tau, \cdot) \ \mathrm{d}x \right]_{t=0}^{t=\tau} \\ &= \int_0^\tau \int_{\Omega} \left[\mathbf{m} \cdot \partial_t \boldsymbol{\varphi} + \mathbf{1}_{\{\varrho > 0\}} \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) : \nabla_x \boldsymbol{\varphi} + p(\varrho) \mathrm{div}_x \boldsymbol{\varphi} \right] \ \mathrm{d}x \ \mathrm{d}t \\ &+ \int_0^\tau \int_{\Omega} \mathbf{f} \cdot \boldsymbol{\varphi} \ \mathrm{d}x \ \mathrm{d}t + \int_0^\tau \int_{\overline{\Omega}} \nabla_x \boldsymbol{\varphi} : \mathrm{d}\mathfrak{R}_m \ \mathrm{d}t; \end{aligned} \tag{2.6.35}$$

- **Energy inequality:** The total energy E is defined in $[0,T)$ as

$$E(\tau) = \int_{\Omega} \left(\frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + (P(\varrho) - (\varrho - \tilde{\varrho})P'(\tilde{\varrho})) - P(\tilde{\varrho}) \right)(\tau, \cdot) \ \mathrm{d}x,$$

  and it satisfies

$$E(\tau) + \int_{\overline{\Omega}} \mathrm{d}\, \mathfrak{R}_e(\tau, \cdot) \leq E_0 \tag{2.6.36}$$

  for a.e. $\tau > 0$;

Remark 2.6.13. If we want to provide the definition in $\mathbb{R}^2 \times \mathbb{T}^1$. Then we have to mention $\varrho, \mathbf{m}$ belong to certain symmetry class along with $\mathbf{f}$. The test function in momentum equation belongs to the class $\{\boldsymbol{\varphi} \in C_c^1([0,T) \times \Omega; \mathbb{R}^d)\}$.

Remark 2.6.14. Euler system is equipped with impermeability boundary condition. Existence of s dissipative solution of Euler system can be proved by taking a vanishing viscosity limit of the Navier–Stokes system equipped with Navier slip boundary condition.

Remark 2.6.15. We can repeat our 'informal justification' in this context to legitimize the choice of total energy and energy inequality (2.6.36).

Remark 2.6.16. The purpose of this definition is to use it in the case of rotating fluids which we consider in the Chapter 4.

2.7 Concluding remark

We have omitted here a very detailed description of the derivation of the system; interested readers may follow Trusendal and Rajagopal [117]. We mainly follow the derivation of Feireisl, Karper and Pokorný [60] and Feireisl [50]. For compressible Navier–Stokes we recommend the following monograph for a detailed discussion and proof of weak solutions, P.L. Lions [96], Feireisl [50], Novotný and Straškaba [105]. In connection with the generalized solution, we refer the reader to Feireisl et al. [67].

After the introduction of the *dissipative* solution for the Navier–Stokes system, it looks as if the meaning of the *measure-valued* solution is less relevant. However, it should be noted that the dissipative solution is only available for a monotone pressure, while the *renormalized dissipative measure-valued* (rDMV) solution is for a general non-monotone pressure. So it is worth considering as long as no suitable definition of the dissipative solution is available for a non-monotone pressure.

Chapter 3

Generalized weak–strong uniqueness property for a viscous fluid

3.1 Introduction

We discussed the limitations of the *classical* or *strong* solution and the need to consider a generalized solution. Also, the importance of *compatibility* and *generalized weak–strong uniqueness* for generalized solutions was pointed out in the introduction. We recall the generalized weak–strong uniqueness property which asserts that *a generalized solution and the strong solution emanating for the same initial data coincide as long as the strong solution exists.*

In this chapter we focus on generalized weak-strong uniqueness results for the compressible system Navier-Stokes with non-monotone pressure laws. In the last chapter, we introduced several non-monotone pressure laws, see (2.1.36), (2.1.38), and (2.1.40).

In the context of the Navier–Stokes system for a monotone pressure, Germain [85] showed weak–strong uniqueness in a class of weak solutions that enjoys additional regularity properties. Unfortunately, the existence of weak solutions in his class is still an open problem.

Feireisl, Novotný and Sun [76] and Feireisl, Jin and Novotný [59] showed the weak–strong uniqueness result in the existence class for an isentropic (barotropic) pressure equation of state with strictly increasing pressure. They consider the finite energy weak solutions of the compressible Navier–Stokes system. These results were extended by Feireisl et al. [56] to the class of the so–called dissipative measure–valued solutions .

Feireisl, Lu and Novotný [64] extended the weak–strong uniqueness principle to the hard–sphere pressure type equation of state, still with strictly monotone pressure–density relation.

Recently, Feireisl [53] proved weak–strong uniqueness in the class of weak solutions, with a non–monotone compactly supported perturbation of the isentropic equation of state.

Our basic goal is first to extend the weak-strong uniqueness property for a more general non-monotone Lipschitz perturbation of isentropic pressure. We also try to verify the similar results for different boundary conditions and for measure-valued solutions. Similarly, for the hard-sphere type pressure law, we consider a compactly supported perturbation of the pressure and verify the weak-strong uniqueness property.

The plan for this chapter is as follows:

- First, we derive the *relative energy inequality* for various pressure laws and boundary conditions. We do not need a strong solution to derive the relative energy inequality. We just need appropriate test functions.
- Next, we prove the weak-strong uniqueness property for non–monotone barotropic pressures, and we discuss weak-strong uniqueness when the pressure is given by a hard-sphere type pressure law. For the barotropic case, we discuss two different boundary conditions, no-slip and Navier slip. In the context of hard-sphere pressure, we consider a periodic boundary condition.
- The last part of this chapter is devoted to the generalized weak–strong uniqueness. Here we consider a renormalized dissipative measure-valued(rDMV) solution of the system with a non-monotone compactly supported perturbation of the barotropic pressure.

3.2 Relative Energy

In the introduction we discussed the relative energy and its importance. Here we provide a relative energy for compressible Navier-Stokes system following Feireisl, Novotný and Sun [76] and Feireisl, Jin and Novotný [59]. Their consideration is valid for monotone pressures (2.1.35) and (2.1.37), and the *relative energy* is given by

$$\begin{aligned}\mathcal{E}(\tau) &= \mathcal{E}(\varrho, \mathbf{u} | r, \mathbf{U})(\tau) \\ &:= \int_{\Omega} \left(\frac{1}{2} \varrho |\mathbf{u} - \mathbf{U}|^2 + (P(\varrho) - P(r) - P^{'}(r)(\varrho - r)) \right) (\tau, \cdot) \mathrm{d}x,\end{aligned} \tag{3.2.1}$$

where P is Pressure potential, $(\varrho, \mathbf{u})$ is a weak solutions of compressible Navier–Stokes system, $r, \mathbf{U}$ are two arbitrary test functions, and $\tau > 0$. The choice of monotone pressure law (2.1.35) or (2.1.37) ensures that P is a convex function.

A suitable modification for the non-monotone pressure

Our main goal is to deal with general non-monotone pressure-density relations (2.1.36), (2.1.38) and (2.1.40). When the pressure is non-monotone, the convexity for the pressure potential P is absent. As an immediate effect, we cannot infer the *distance property* of relative energy. To overcome this difficulty, Feireisl in [52] proposes a modified version of (3.2.1) by considering only its monotone part. It describes as

$$\begin{aligned}\mathcal{E}(\tau) &= \mathcal{E}(\varrho, \mathbf{u}|r, \mathbf{U})(\tau) \\ &:= \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}-\mathbf{U}|^2 + (H(\varrho) - H(r) - H'(r)(\varrho - r))\right)(\tau, \cdot)\ \mathrm{d}x,\end{aligned} \tag{3.2.2}$$

for $\tau > 0$, and $r, \mathbf{U}$ are arbitrary test functions and $(\varrho, \mathbf{u})$ is weak solution of the compressible Navier–Stokes system and H is the pressure potential related to h as in (2.1.36), (2.1.38) and (2.1.40).

3.2.1 Relative energy inequality for weak solutions

Only by assuming that $(\varrho, \mathbf{u})$ is a weak solution and $(r, \mathbf{U})$ is a suitable test function, we can derive an inequality which we call the *relative energy inequality.* We say the term 'suitable test functions' because we consider the following cases:

- Non-monotone isentropic pressure law and no-slip boundary condition,
- Non-monotone isentropic pressure law and Navier slip boundary condition,
- Non-monotone hard-sphere pressure law and periodic boundary condition.

In the subsection 2.2.2, we discuss the choice of test functions in momentum equation and their dependence on the boundary conditions.

Non-monotone isentropic pressure law and no-slip boundary condition

We consider the non-monotone pressure law (2.1.36) or (2.1.38) and the boundary condition for the velocity is (2.2.5). We assume that the test functions $(r, \mathbf{U})$ satisfy

$$r \in C_c^\infty([0,T] \times \Omega) \text{ and } \mathbf{U} \in C_c^\infty([0,T] \times \Omega; \mathbb{R}^d). \tag{3.2.3}$$

Then for a.e. $\tau \in (0, T)$, we have

$$\begin{aligned}\mathcal{E}(\tau) = &\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + H(\varrho)\right)(\tau, \cdot)\ \mathrm{d}x - \int_\Omega \varrho\mathbf{u} \cdot \mathbf{U}(\tau, \cdot)\ \mathrm{d}x \\ &+ \int_\Omega \varrho\left(\frac{1}{2}|\mathbf{U}|^2\ \mathrm{d}x - H'(r)\right)(\tau, \cdot)\ \mathrm{d}x + \int_\Omega h(r)(\tau, \cdot)\ \mathrm{d}x = \Sigma_{i=1}^4 K_i.\end{aligned}$$

The first term K_1 is related to the energy inequality and satisfies the following inequality:

$$
\begin{aligned}
&\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + H(\varrho)\right) \,\mathrm{d}x \\
&= \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho)\right)(\tau) \,\mathrm{d}x - \int_\Omega Q(\varrho)(\tau,\cdot) \,\mathrm{d}x \\
&\leq \int_\Omega \left(\frac{1}{2}\frac{|(\varrho\mathbf{u})_0|^2}{\varrho_0} + P(\varrho_0)\right) \,\mathrm{d}x - \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x\mathbf{u}) \colon \nabla_x\mathbf{u} \,\mathrm{d}x \,\mathrm{d}t \; - \int_\Omega Q(\varrho)(\tau,\cdot) \,\mathrm{d}x \\
&\leq \int_\Omega \left(\frac{1}{2}\frac{|(\varrho\mathbf{u})_0|^2}{\varrho_0} + H(\varrho_0)\right) \,\mathrm{d}x - \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x\mathbf{u}) \colon \nabla_x\mathbf{u} \,\mathrm{d}x \,\mathrm{d}t \\
&\qquad - \int_\Omega Q(\varrho)(\tau,\cdot) \,\mathrm{d}x + \int_\Omega Q(\varrho_0) \,\mathrm{d}x.
\end{aligned}
$$

If the non-monotone part of the pressure in compactly supported, i.e., $q \in C_c^1(0,\infty)$, Then corresponding pressure potential $Q \in C^1(0,\infty)$ with $Q(s) \approx s$. Noticing that the renormalized equation of continuity (2.2.13) makes sense for a function b such that $b(\varrho)$, $\varrho b'(\varrho) \in L^\infty(0,T;L^2(\Omega))$, we can substitute Q as b in the renormalized equation of continuity and obtain

$$
\int_\Omega Q(\varrho)(\tau) \,\mathrm{d}x - \int_\Omega Q(\varrho_0) \,\mathrm{d}x = -\int_0^\tau \int_\Omega q(\varrho)\mathrm{div}_x\mathbf{u} \,\mathrm{d}x \,\mathrm{d}t \,.
$$

Consequently, we have

$$
\begin{aligned}
&\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + H(\varrho)\right) \,\mathrm{d}x \\
&\leq \int_\Omega \left(\frac{1}{2}\frac{|(\varrho\mathbf{u})_0|^2}{\varrho_0} + H(\varrho_0)\right) \,\mathrm{d}x - \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x\mathbf{u}) \colon \nabla_x\mathbf{u} \,\mathrm{d}x \,\mathrm{d}t \\
&\qquad + \int_0^\tau \int_\Omega q(\varrho)\mathrm{div}_x\mathbf{u} \,\mathrm{d}x \,\mathrm{d}t \,.
\end{aligned}
$$

Remark 3.2.1. If we consider $q \in C^{0,1}[0,\infty)$, then with an additional assumption on the adiabatic exponent, which is $\gamma \geq 2$, we can perform a similar analysis.

For K_2 and K_3 we use the momentum equation (2.2.14) and the continuity equation (2.2.13), respectively. To compute the term K_5 we use the following identity:

$$
\int_0^\tau \int_\Omega h'(r)\partial_t r \,\mathrm{d}x \,\mathrm{d}t \; = \left[\int_\Omega h(r)(\tau) \,\mathrm{d}x\right]_{t=0}^{t=\tau}.
$$

Hence, we obtain

$$
\begin{aligned}
&\mathcal{E}(\tau) + \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x\mathbf{u}) \colon (\nabla_x\mathbf{u} - \nabla_x\mathbf{U}) \,\mathrm{d}x \,\mathrm{d}t \\
&\leq \int_\Omega \left(\frac{|(\varrho\mathbf{u})_0|^2}{\varrho_0} + H(\varrho_0)\right) \,\mathrm{d}x - \int_\Omega (\varrho\mathbf{u})_0 \cdot \mathbf{U}_0 \,\mathrm{d}x \\
&\quad + \int_\Omega \varrho_0 \left(\frac{1}{2}|\mathbf{U}_0|^2 - H'(r_0)\right) \,\mathrm{d}x + \int_\Omega h(r_0) \,\mathrm{d}x + \int_0^\tau \mathcal{R}(t) dt,
\end{aligned}
$$

where $r_0 = r(0,\cdot)$ and $\mathbf{U}_0 = \mathbf{U}(0,\cdot)$ in Ω and

$$\begin{aligned}\mathcal{R}(t) = &-\int_\Omega \left(\varrho\mathbf{u}\cdot\partial_t\mathbf{U} + \varrho\mathbf{u}\otimes\mathbf{u}\colon\nabla_x\mathbf{U} + p(\varrho)\mathrm{div}_x\mathbf{U}\right)\,\mathrm{d}x\\ &+\int_\Omega \left(\varrho\,\partial_t\left(\frac{1}{2}|\mathbf{U}|^2 - H'(r)\right) + \varrho\mathbf{u}\cdot\nabla_x\left(\frac{1}{2}|\mathbf{U}|^2 - H'(r)\right)\right)\,\mathrm{d}x\\ &+\int_\Omega h'(r)\partial_t r\,\mathrm{d}x + \int_\Omega q(\varrho)\mathrm{div}_x\mathbf{u}\,\mathrm{d}x.\end{aligned}$$

Let us introduce initial relative energy $\mathcal{E}_0$ as

$$\mathcal{E}_0 = \int_\Omega \frac{1}{2}\varrho_0\left|\frac{|(\varrho\mathbf{u})_0|^2}{\varrho_0} - \mathbf{U}_0\right|^2 + \big(H(\varrho_0) - H(r_0) - H'(r_0)(\varrho_0 - r_0)\big)\big)\,\mathrm{d}x.$$

By regrouping the terms in the above inequality, we deduce that

$$\begin{aligned}\mathcal{E}(\tau)+&\int_0^\tau\int_\Omega \mathbb{S}(\nabla_x\mathbf{u}) : (\nabla_x\mathbf{u} - \nabla_x\mathbf{U})\,\mathrm{d}x\,\mathrm{d}t\\ \le\ &\mathcal{E}_0 - \int_0^\tau\int_\Omega (\varrho\mathbf{u} - \varrho\mathbf{U})\cdot\partial_t\mathbf{U}\,\mathrm{d}x\,\mathrm{d}t\\ &-\int_0^\tau\int_\Omega \left(\varrho\mathbf{u}\otimes\mathbf{u}\colon\nabla_x\mathbf{U} - \varrho\mathbf{u}\cdot\nabla_x\left(\frac{1}{2}|\mathbf{U}|^2\right)\right)\,\mathrm{d}x\,\mathrm{d}t\\ &-\int_0^\tau\int_\Omega (h'(r) + \varrho H''(r))\partial_t r\,\mathrm{d}x\,\mathrm{d}t\\ &-\int_0^\tau\int_\Omega (h(\varrho)\mathrm{div}_x\mathbf{U} + \varrho\mathbf{u}H''(r)\nabla_x r)\,\mathrm{d}x\,\mathrm{d}t\\ &+\int_0^\tau\int_\Omega q(\varrho)(\mathrm{div}_x\mathbf{u} - \mathrm{div}_x\mathbf{U})\,\mathrm{d}x\,\mathrm{d}t\,.\end{aligned}$$

Further, we assume $0 < r$ in $(0,T)\times\Omega$. Eventually, by rearranging the terms we obtain the *relative energy inequality*

$$\begin{aligned}\mathcal{E}(\tau)+&\int_0^\tau\int_\Omega \mathbb{S}(\nabla_x\mathbf{u}) : (\nabla_x\mathbf{u} - \nabla_x\mathbf{U})\,\mathrm{d}x\,\mathrm{d}t\\ \le\ &\mathcal{E}_0 - \int_0^\tau\int_\Omega \varrho(\mathbf{u} - \mathbf{U})\cdot\partial_t\mathbf{U}\,\mathrm{d}x\,\mathrm{d}t\\ &-\int_0^\tau\int_\Omega [\varrho\mathbf{u}\otimes\mathbf{u} : \nabla_x\mathbf{U} - \varrho\mathbf{u}\cdot(\mathbf{U}\cdot\nabla_x)\mathbf{U} + h(\varrho)\mathrm{div}_x\mathbf{U}]\,\mathrm{d}x\,\mathrm{d}t\\ &+\int_0^\tau\int_\Omega \left(\left(1 - \frac{\varrho}{r}\right)h'(r)\partial_t r - \varrho\mathbf{u}\cdot\frac{h'(r)}{r}\nabla_x r\right)\,\mathrm{d}x\,\mathrm{d}t\\ &+\int_0^\tau\int_\Omega q(\varrho)(\mathrm{div}_x\mathbf{u} - \mathrm{div}_x\mathbf{U})\,\mathrm{d}x\,\mathrm{d}t\,.\end{aligned}\tag{3.2.4}$$

Suppose p follows a pressure law (2.1.36), (2.1.38). We consider the following hypothesis:

$$\begin{aligned}&\text{For } q\in C_c^1(0,\infty),\ \text{ we consider } \gamma\ge 1, \textit{and}\\ &\text{for } q\in C^{0,1}[0,\infty),\ \text{ we consider } \gamma\ge 2.\end{aligned}\tag{3.2.5}$$

We summarize the above discussion in the following lemma:

Lemma 3.2.2. *Suppose* $(\varrho, \mathbf{u})$ *is a weak solution of the Navier–Stokes system following the Definition 2.2.5 with pressure constraint* (3.2.5) *and* $(r, \mathbf{U})$ *is an arbitrary test function with* $0 < r \in C_c^\infty([0,T] \times \Omega)$ *and* $\mathbf{U} \in C_c^\infty([0,T] \times \Omega; \mathbb{R}^d)$*, then the inequality* (3.2.4) *holds.*

Non-monotone isentropic pressure law and Navier slip boundary condition

In the case of the Navier slip boundary condition, we note that the test function for the momentum equation belongs to the class $\{\boldsymbol{\varphi} \in C^\infty([0,T] \times \overline{\Omega}) \,|\, \boldsymbol{\varphi} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}$. So we choose test functions as

$$0 < r \in C^\infty([0,T] \times \overline{\Omega}) \text{ and } \mathbf{U} \in C^\infty([0,T] \times \overline{\Omega}; \mathbb{R}^d) \text{ with } \mathbf{U} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega. \tag{3.2.6}$$

In this case, we obtain a similar relative energy inequality and the statement follows as

Lemma 3.2.3. *Suppose* $(\varrho, \mathbf{u})$ *is a weak solution of the Navier–Stokes system following the Definition 2.2.8 with* (3.2.5) *and* $(r, \mathbf{U})$ *with* (2.1.20) *be any test function* $(r, \mathbf{U})$ *satisfying* (3.2.6) *then the inequality* (3.2.4) *holds.*

Non-monotone hard-sphere pressure law and periodic boundary condition

Considering the hard-sphere non-monotone pressure (2.1.40) in the domain flat torus i.e.$\mathbb{T}^d$, we can derive a similar relative energy inequality:

Lemma 3.2.4. *Suppose* $(\varrho, \mathbf{u})$ *is a weak solution of the Navier–Stokes system following the Definition 2.2.12 in* $(0,T) \times \mathbb{T}^d$ *and* $(r, \mathbf{U})$ *be any test function with* $0 < r \in C_c^\infty([0,T] \times \mathbb{T}^d)$ *and* $\mathbf{U} \in C_c^\infty([0,T] \times \mathbb{T}^d; \mathbb{R}^d)$ *then* (3.2.4) *is true.*

Remark 3.2.5. We notice a small difference in the renormalized continuity equation (2.2.25), where b has to satisfy certain structural assumption. The assumption q is compactly supported allows us to consider Q as b. Furthermore, in the Definition 2.2.12, we observe that the class of test function for the renormalized continuity equation and the momentum equation is $\{(\varphi, \boldsymbol{\varphi}) \in C^1([0,T] \times \mathbb{T}^d) \times C^1([0,T]; C^2(\mathbb{T}^d; \mathbb{R}^d))\}$. Hence the Lemma 3.2.4 remains trues in this class of test function.

3.2.2 Relative energy inequality for measure-valued solutions

In the context of measured-valued solution, we consider the relative energy as

$$\begin{aligned} \mathcal{E}_{mv}(t) &= \mathcal{E}_{mv}(\mathcal{V}|r, \mathbf{U})(t) \\ &:= \int_\Omega \left[\left\langle \mathcal{V}_{t,x}; \frac{1}{2}\tilde{\varrho}|\tilde{\mathbf{u}} - \mathbf{U}|^2 + H(\tilde{\varrho}) - H(r) - H'(r)(\tilde{\varrho} - r) \right\rangle \right](t,x) \, \mathrm{d}x, \end{aligned} \tag{3.2.7}$$

where $t > 0$, $(r, \mathbf{U})$ is an appropriate test function and $\mathcal{V}$ is a solution as defined in Definition 2.5.3 or 2.5.7 depending on boundary conditions.

Non-monotone isentropic pressure law and no-slip boundary condition

Lemma 3.2.6. *Let* $(\mathcal{V},\mathcal{D})$ *be a renormalized measure–valued solution(rDMV) of the Navier–Stokes system* (2.2.1)-(2.2.3) *for initial data* $\mathcal{V}_0$ *and boundary condition* (2.2.5) *following Definition 2.5.3. Furthermore, we assume a pressure follows* (2.1.36) *or* (2.1.38) *with* $q \in C_c^1(0,\infty)$ *and* $\gamma \geq 1$*. Then for* $0 < r \in C_c^\infty([0,T]\times\Omega)$ *and* $\mathbf{U} \in C_c^\infty([0,T]\times\Omega;\mathbb{R}^d)$*, we have the following relative energy inequality:*

$$\begin{aligned}\mathcal{E}_{mv}(\tau)+&\int_0^\tau\int_\Omega\langle\mathcal{V}_{t,x};\mathbb{S}(\widetilde{\mathbb{D}_{\mathbf{v}}}):\widetilde{\mathbb{D}_{\mathbf{v}}}\rangle\ dx\ dt\ -\int_0^\tau\int_\Omega\langle\mathcal{V}_{t,x};\mathbb{S}(\widetilde{\mathbb{D}_{\mathbf{v}}})\rangle:\nabla_x\mathbf{U}\ dx\ dt\ +\mathcal{D}(\tau)\\ &\leq\int_\Omega\left[\Big\langle\mathcal{V}_{0,x};\frac{1}{2}\tilde{\varrho}|\tilde{\mathbf{u}}-\mathbf{U}_0|^2+H(\tilde{\varrho})-H(r_0)-H'(r_0)(\tilde{\varrho}-r_0)\Big\rangle\right]\ dx\\ &\quad+\int_0^\tau\mathcal{R}_{mv}(t)\ dt\ ,\end{aligned}\tag{3.2.8}$$

where $\mathbf{U}_0(x)=\mathbf{U}(0,x)$*,* $r_0(x)=r(0,x)$ *for* $x\in\Omega$ *and for a.e.* $t\in(0,T)$ *the remainder term* $\mathcal{R}_{mv}(t)$ *is given by*

$$\begin{aligned}\mathcal{R}_{mv}(t)=&-\int_\Omega\langle\mathcal{V}_{\tau,x};\tilde{\varrho}\tilde{\mathbf{u}}\rangle\cdot\partial_t\mathbf{U}\ dx-\langle r^M(t);\nabla_x\mathbf{U}\rangle\\ &-\int_\Omega[\langle\mathcal{V}_{t,x};\tilde{\varrho}\tilde{\mathbf{u}}\otimes\tilde{\mathbf{u}}\rangle:\nabla_x\mathbf{U}+\langle\mathcal{V}_{t,x};h(\tilde{\varrho})\rangle div_x\mathbf{U}]\ dx\\ &+\int_\Omega[\langle\mathcal{V}_{t,x};\tilde{\varrho}\rangle\mathbf{U}\cdot\partial_t\mathbf{U}+\langle\mathcal{V}_{t,x};\tilde{\varrho}\tilde{\mathbf{u}}\rangle\cdot(\mathbf{U}\cdot\nabla_x)\mathbf{U}]\ dx\\ &+\int_\Omega\left[\langle\mathcal{V}_{t,x};\left(1-\frac{\tilde{\varrho}}{r}\right)\rangle h'(r)\partial_t r-\langle\mathcal{V}_{t,x};\tilde{\varrho}\tilde{\mathbf{u}}\rangle\cdot\frac{h'(r)}{r}\nabla_x r\right]\ dx\\ &-\int_\Omega\langle\mathcal{V}_{t,x};q(\tilde{\varrho})\rangle\ div_x\mathbf{U}\ dx+\int_\Omega\Big\langle\mathcal{V}_{t,x};q(\tilde{\varrho})\,Tr(\widetilde{\mathbb{D}_{\mathbf{v}}})\Big\rangle\ dx\end{aligned}$$

The proof is based on the similar approach as in the case of weak solutions. Here we have

$$\begin{aligned}\mathcal{E}_{mv}(\tau)=&\int_\Omega\Big\langle\mathcal{V}_{\tau,x};\frac{1}{2}\tilde{\varrho}|\tilde{\mathbf{u}}|^2+H(\tilde{\varrho})\Big\rangle\ \mathrm{d}x-\int_\Omega\langle\mathcal{V}_{\tau,x};\tilde{\varrho}\tilde{\mathbf{u}}\rangle\cdot\mathbf{U}\ \mathrm{d}x\\ &+\int_\Omega\frac{1}{2}\langle\mathcal{V}_{\tau,x};\tilde{\varrho}\rangle|\mathbf{U}|^2\ \mathrm{d}x-\int_\Omega\langle\mathcal{V}_{\tau,x};\tilde{\varrho}\rangle H'(r)\ \mathrm{d}x+\int_\Omega h(r)\ \mathrm{d}x.\end{aligned}$$

Now we need to apply the Definition 2.5.3 suitably to obtain the desired result.

Non-monotone isentropic pressure law and Navier slip boundary condition

A similar lemma is true for the Navier slip boundary condition when we consider appropriate test functions.

Lemma 3.2.7. *Let* $(\mathcal{V},\mathcal{D})$ *be a renormalized measure–valued solution(rDMV) of the Navier–Stokes system* (2.2.1)-(2.2.3) *for initial data* $\mathcal{V}_0$ *and boundary condition* (2.2.5) *following Definition* (2.5.7)*. Furthermore, we assume a pressure follows* (2.1.36) *or* (2.1.38) *with* $q \in C_c^1(0,\infty)$ *and* $\gamma \geq 1$*. Let* $(r, \mathbf{U})$ *be a test function class which satisfy* (3.2.6)*. Then* (3.2.8) *is true.*

Remark on a possible extension of the relative energy inequalities

We notice that we can extend both the inequalities (3.2.4) and (3.2.8), for a class of test functions in appropriate Sobolev spaces. In the Lemma 3.2.2, we consider the test functions in

$$0 < r \in C_c^\infty([0,T] \times \Omega) \text{ and } \mathbf{U} \in C_c^\infty([0,T] \times \Omega; \mathbb{R}^d).$$

The main idea is to use the density of the above functions in Sobolev space. However, it applies only to those Sobolev functions for which the integrals on the right-hand side of the relative energy inequality are well-defined. We can state a modified version of Lemma 3.2.2 in the following way:

Proposition 3.2.8. *Suppose* $(\varrho, \mathbf{u})$ *is a weak solution of the Navier–Stokes system following the Definition 2.2.5 with pressure constraint* (3.2.5) *and* $(r, \mathbf{U})$ *is an arbitrary test function with*

$$0 < r \in C^1([0,T]; W^{k,2}(\Omega)) \text{ and } \mathbf{U} \in C^1([0,T]; W_0^{k,2}(\Omega; \mathbb{R}^d)). \text{ with } k > \left[\frac{d}{2}\right] + 1,$$

then the inequality (3.2.4) *holds.*

The proof is direct but lengthy, using mainly the density of smooth functions in Sobolev space. We skip the proof here, an interested reader can look it up in Basaric [11, Section 2] for a detailed discussion.

3.3 Results on weak-strong uniqueness and generalized weak-strong uniqueness property

3.3.1 Weak-strong unqiueness for the compressible Navier–Stokes system with a non-monotone isentropic pressure law

In the first part of this section, we consider the *no-slip* boundary condition. We provide the main result and the proof of it. A similar result can be proved for the *Navier slip* boundary condition.

At first we assume that $(r, \mathbf{U})$ is a strong solution of the Navier–Stokes system (2.2.1)-(2.2.3) with initial data $(r_0, \mathbf{U}_0)$ with $r_0 > 0$ in the class as mentioned in (2.2.11). Then we have the following lemma:

Lemma 3.3.1. *Let* $(\varrho, \mathbf{u})$ *be a weak solution of the compressible Navier–Stokes system with pressure law* (2.1.36) *or* (2.1.38)*, and* $(r, \mathbf{U})$ *a strong solution of the same system.*

We further assume that a strong solution belongs to the class (2.2.11) *with initial condition* $(0 < r(0,\cdot), \mathbf{U}_0)$ *in* Ω. *Then the following inequality holds*

$$\begin{aligned}
\mathcal{E}(\tau)+&\int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) : (\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) \ \ dx \ dt \\
&\leq \mathcal{E}_0 + \int_0^\tau \int_\Omega \left(\frac{\varrho}{r} - 1\right) (\mathbf{U} - \mathbf{u}) \cdot (div_x \mathbb{S}(\nabla_x \mathbf{U}) - \nabla_x q(r)) \ \ dx \ dt \\
&\quad + \int_0^\tau \int_\Omega \varrho(\mathbf{u} - \mathbf{U}) \cdot ((\mathbf{U} - \mathbf{u}) \cdot \nabla_x)\mathbf{U} \ \ dx \ dt \\
&\quad + \int_0^\tau \int_\Omega (-h(\varrho) + h(r) + h'(r)(\varrho - r)) \ div_x \mathbf{U} \ \ dx \ dt \\
&\quad + \int_0^\tau \int_\Omega (div_x \mathbf{u} - div_x \mathbf{U})(q(\varrho) - q(r)) \ \ dx \ dt \ ,
\end{aligned} \tag{3.3.1}$$

for a.e. $\tau \in (0, T)$.

Proof. We assume that $(r, \mathbf{U})$ is a strong solution and the initial data $r(0,\cdot) > 0$ in Ω. Moreover, we have that the strong solution belongs to the class (2.2.11). Hence, we conclude that $r > 0$ in $(0,T) \times \Omega$. Thus, we obtain

$$\begin{aligned}
&\partial_t r + \mathrm{div}_x(r\mathbf{U}) = 0, \\
&r(\partial_t \mathbf{U} + (\mathbf{U} \cdot \nabla_x)\mathbf{U}) + \nabla_x p(r) = \mathrm{div}_x(\mathbb{S}(\nabla_x \mathbf{U})).
\end{aligned}$$

We use the above equations to the relative energy (3.2.4) and deduce the inequality (3.3.1). $\square$

Relation between pressure and pressure potential

We take the monotone part of the pressure, i.e. h, where the pressure p follows (2.1.36) or (2.1.38). We recall the pressure potential H when p follows (2.1.38). It is given by

$$H(\varrho) = \varrho \int_0^\varrho \frac{h(\xi)}{\xi^2} \,\mathrm{d}\xi, \ \ \varrho > 0.$$

If p follows (2.1.36) then we consider

$$H(\varrho) = \begin{cases} \frac{a}{\gamma-1}\varrho^\gamma, \ \gamma > 1, \\ a\varrho \log \varrho, \gamma = 1, \end{cases} .$$

Next we state an important lemma for H.

Lemma 3.3.2. *Let* $a, b > 0$. *Suppose* r *lies on a compact subset* $[a,b]$ *of* $(0,\infty)$. *Then there exists* $0 < r_1 < r_2$ *depending on* r, *such that the following relation holds:*

$$H(\varrho) - H(r) - H'(r)(\varrho - r) \geq c(r) \begin{cases} (\varrho - r)^2 \ \textit{for} \ r_1 \leq \varrho \leq r_2, \\ (1 + \varrho^\gamma) \ \textit{otherwise} \end{cases} , \tag{3.3.2}$$

where $c(r)$ *is uniformly bounded and depends on* a, b.

Proof. We give an idea of the proof for $h(\varrho) = a\varrho^\gamma$ with $\gamma > 1$. First we notice that

$$H(\varrho) - H(r) - H'(r)(\varrho - r) = H(\varrho) - H'(r)\varrho + h(r).$$

Since $r \in [a, b]$, thus $H'(r)$ and $h(r)$ is uniformly bounded. Hence for large ϱ, $H(\varrho)$ is the dominating term.

On the other hand we notice when $\varrho \to 0$, $H(\varrho) - H(r) - H'(r)(\varrho - r) \to h(r)$. Using the fact that h is monotone we have $h(r) \geq h(a)$, a fixed quantity. Hence we can choose r_1, r_2 such that the inequality holds when $\varrho < r_1$ or $\varrho > r_2$. Finally, using Taylor's formula we obtain (3.3.2).

For a general pressure we have to use the condition $\liminf_{\varrho\to\infty} \frac{p(\varrho)}{\varrho^\gamma} > 0$, to complete the proof. □

As a corollary of the above lemma we have the following:

Lemma 3.3.3. *For $\varrho \geq 0$, it holds that*

$$|h(\varrho) - h(r) - h'(r)(\varrho - r)| \leq C(r)(H(\varrho) - H(r) - H'(r)(\varrho - r)),$$

where $C(r)$ is uniformly bounded if r lies in some compact subset of $(0, \infty)$.

Proof. We observe that there exists r_3 and r_4 depending on r such that the following inequality holds:

$$h(\varrho) - h(r) - h'(r)(\varrho - r) \leq c_1(r) \begin{cases} (\varrho - r)^2 \text{ for } r_3 \leq \varrho \leq r_4, \\ (1 + \varrho^\gamma) \text{ otherwise} \end{cases},$$

where $c_1(r)$ is uniformly bounded for r belonging to compact subsets of $(0, \infty)$. Lemma 3.3.2 together with a suitable choice of r_1, r_2 helps us to conclude the proof. □

Now we state the main theorem.

Theorem 3.3.4. *Let $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$, be a bounded Lipschitz domain. Let the pressure be given by (2.1.36) or (2.1.38), with $\gamma \geq 2$ and $q \in C^{0,1}[0, \infty)$. We further assume that*

$$0 < r_0 = r(0, \cdot) \in W^{k,2}(\Omega) \text{ and } \mathbf{U}_0 = \mathbf{U}(0, \cdot) \in W^{k,2}(\Omega; \mathbb{R}^d) \text{ with } k > \left[\frac{d}{2}\right] + 2.$$

Suppose that $(\varrho, \mathbf{u})$ is a dissipative weak solution following the Definition 2.2.5 and $(r, \mathbf{U})$ is a strong solution of the problem (2.2.1)-(2.2.3) with no slip boundary condition (2.2.5) on the time interval $[0, T)$ such that the initial data satisfies

$$\varrho(0, \cdot) = r(0, \cdot) > 0, \ \varrho\mathbf{u}(0, \cdot) = r(0, \cdot)\mathbf{U}(0, \cdot).$$

Then

$$\varrho = r, \ \mathbf{u} = \mathbf{U} \text{ in } (0, T) \times \Omega.$$

Proof of the Theorem 3.3.4:

First we recall the relative energy inequality from the Lemma 3.3.1:

$$\begin{aligned}
\mathcal{E}(\tau)+&\int_0^\tau\int_\Omega \mathbb{S}(\nabla_x\mathbf{u}-\nabla_x\mathbf{U}):(\nabla_x\mathbf{u}-\nabla_x\mathbf{U})\ \mathrm{d}x\ \mathrm{d}t\\
\leq \mathcal{E}_0 &+\int_0^\tau\int_\Omega \left(\frac{\varrho}{r}-1\right)(\mathbf{U}-\mathbf{u})\ (\mathrm{div}_x\mathbb{S}(\nabla_x\mathbf{U})-\nabla_x q(r))\ \mathrm{d}x\ \mathrm{d}t\\
&+\int_0^\tau\int_\Omega \varrho(\mathbf{u}-\mathbf{U})\cdot((\mathbf{U}-\mathbf{u})\cdot\nabla_x)\mathbf{U}\ \mathrm{d}x\ \mathrm{d}t\\
&+\int_0^\tau\int_\Omega (-h(\varrho)+h(r)+h'(r)(\varrho-r))\ \mathrm{div}_x\mathbf{U}\ \mathrm{d}x\ \mathrm{d}t\\
&+\int_0^\tau\int_\Omega (\mathrm{div}_x\mathbf{u}-\mathrm{div}_x\mathbf{U})(q(\varrho)-q(r))\ \mathrm{d}x\ \mathrm{d}t\,.
\end{aligned}$$

Our goal is to estimate the terms on right hand side of the above inequality. We observe that the hypothesis about the initial data ensures a strong solution belongs to a regularity class (2.2.11).

Thus, we get $\|\nabla_x\mathbf{U}\|_{C([0,T]\times\Omega;\mathbb{R}^{d\times d})}$ is finite, $\inf_{[0,T]\times\Omega} r>0$ and r lies in a compact subset of $(0,\infty)$. These along with Lemma 3.3.3 yield

$$\int_0^\tau\int_\Omega \varrho(\mathbf{u}-\mathbf{U})\cdot((\mathbf{U}-\mathbf{u})\cdot\nabla_x)\mathbf{U}\ \mathrm{d}x\ \mathrm{d}t\ \leq \|\nabla_x\mathbf{U}\|_{C([0,T]\times\Omega)}\int_0^\tau \mathcal{E}(t)\ \mathrm{d}t \tag{3.3.3}$$

and

$$\int_0^\tau\int_\Omega (-h(\varrho)+h(r)+h'(r)(\varrho-r))\ \mathrm{div}_x\mathbf{U}\ \mathrm{d}x\ \mathrm{d}t\ \leq \|\nabla_x\mathbf{U}\|_{C([0,T]\times\Omega)}\int_0^\tau \mathcal{E}(t)\ \mathrm{d}t. \tag{3.3.4}$$

To compute the term

$$\int_0^\tau\int_\Omega \left(\frac{\varrho}{r}-1\right)(\mathbf{U}-\mathbf{u})\ (\mathrm{div}_x\mathbb{S}(\nabla_x\mathbf{U})-\nabla_x q(r))\ \mathrm{d}x\ \mathrm{d}t\ ,$$

we choose $r_1, r_2>0$ in the Lemma 3.3.2 such that they satisfy

$$r_1<\frac{\inf\limits_{(x,t)\in(0,T)\times\Omega} r(x,t)}{2},\ r_2>2\times \sup_{(x,t)\in(0,T)\times\Omega} r(x,t)$$
$$\text{and } 1+\varrho^\gamma\geq \max\{\varrho,\varrho^2\},\ \forall\varrho\geq r_2.$$

Since q is globally Lipschitz by *Rademacher's theorem* [48, Theorem 3.2], q is almost everywhere differentiable and its derivative is bounded by the Lipschitz constant L_q. Hence we obtain,

$$\left|\frac{1}{r}\nabla_x q(r)\right|\leq \frac{L_q}{\inf r}\|r\|_{C^1([0,T]\times\overline{\Omega})}.$$

Consider a cut-off function

$$\psi \in C_c^\infty(0,\infty) \text{ such that } 0 \le \psi \le 1 \text{ and } \psi(s) = 1 \text{ for } s \in (r_1, r_2), \tag{3.3.5}$$

and rewrite $(\varrho - r)(\mathbf{U} - \mathbf{u})$ as

$$(\varrho - r)(\mathbf{U} - \mathbf{u}) = \psi(\varrho)(\varrho - r)(\mathbf{U} - \mathbf{u}) + (1 - \psi(\varrho))(\varrho - r)(\mathbf{U} - \mathbf{u}).$$

Consequently, using Young's inequality(1.1.1), we obtain

$$\psi(\varrho)(\varrho - r)(\mathbf{U} - \mathbf{u}) \le \frac{1}{2}\frac{\psi^2(\varrho)}{\sqrt{\varrho}}(\varrho - r)^2 + \frac{1}{2}\frac{\psi^2(\varrho)}{\sqrt{\varrho}}\varrho|\mathbf{U} - \mathbf{u}|^2.$$

From the fact that ψ is compactly supported in $(0,\infty)$, and the Lemma 3.3.2, we have the following estimate:

$$\begin{aligned}
&\int_0^\tau \int_\Omega \psi(\varrho)(\varrho - r)(\mathbf{U} - \mathbf{u}) \cdot \frac{1}{r}\big(\mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{U}) - \nabla_x q(r)\big) \, \mathrm{d}x \, \mathrm{d}t \\
&\le \left(\left\| \frac{1}{r}\big(\mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{U})\big) \right\|_{C([0,T]\times\overline{\Omega};\mathbb{R}^d)} + \frac{L_q}{\inf r}\|r\|_{C^1([0,T]\times\overline{\Omega};\mathbb{R}^d)} \right) \int_0^\tau \mathcal{E}(t) \, \mathrm{d}t \\
&\le C(r, \mathbf{U}, q) \int_0^\tau \mathcal{E}(t) \, \mathrm{d}t \, .
\end{aligned} \tag{3.3.6}$$

Now, we consider

$$(1 - \psi(\varrho))(\varrho - r)(\mathbf{U} - \mathbf{u}) = (w_1(\varrho) + w_2(\varrho))(\varrho - r)(\mathbf{U} - \mathbf{u}),$$

where $\mathrm{supp}(w_1) \subset [0, r_1)$ and $\mathrm{supp}(w_2) \subset (r_2, \infty)$.
For any $\delta > 0$, using Young's inequality (1.1.1) again, we deduce that

$$w_1(\varrho)(\varrho - r)(\mathbf{U} - \mathbf{u}) \le C(\delta) w_1^2(\varrho)(\varrho - r)^2 + \delta|\mathbf{U} - \mathbf{u}|^2.$$

Thus using Poincaré inequality(1.1.8) and Korn inequality (1.1.11) we have

$$\begin{aligned}
&\int_0^\tau \int_\Omega w_1(\varrho)(\varrho - r)(\mathbf{U} - \mathbf{u}) \cdot \frac{1}{r}\big(\mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{U}) - \nabla_x q(r)\big) \, \mathrm{d}x \, \mathrm{d}t \\
&\le C(r, \mathbf{U}, q, \delta) \int_0^\tau \mathcal{E}(t) \, \mathrm{d}t \; + \delta \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) : (\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) \; \mathrm{d}x \, \mathrm{d}t \, .
\end{aligned} \tag{3.3.7}$$

Similarly, the following observation

$$w_2(\varrho)(\varrho - r)(\mathbf{U} - \tilde{\mathbf{u}}) \le C(r)(\varrho + \varrho|\mathbf{U} - \mathbf{u}|^2),$$

helps us to deduce

$$\begin{aligned}
&\int_0^\tau \int_\Omega w_2(\varrho)(\varrho - r)(\mathbf{U} - \mathbf{u}) \cdot \frac{1}{r}\big(\mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{U}) - \nabla_x q(r)\big) \, \mathrm{d}x \, \mathrm{d}t \\
&\le C(r, \mathbf{U}, q) \int_0^\tau \mathcal{E}(t) \, \mathrm{d}t \, .
\end{aligned} \tag{3.3.8}$$

Combining (3.3.6), (3.3.7) and (3.3.8) we obtain

$$\int_0^\tau \int_\Omega (\varrho - r)(\mathbf{U} - \mathbf{u}) \cdot \frac{1}{r}\big(\mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{U}) - \nabla_x q(r)\big) \; \mathrm{d}x \; \mathrm{d}t$$
$$\leq C(\delta, r, \mathbf{U}, q) \int_0^\tau \mathcal{E}(t) \; \mathrm{d}t \; + \delta \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) : (\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) \;\; \mathrm{d}x \; \mathrm{d}t \, . \tag{3.3.9}$$

First from Young's inequality we have

$$\int_0^\tau \int_\Omega (\mathrm{div}_x \mathbf{u} - \mathrm{div}_x \mathbf{U})(q(\varrho) - q(r)) \;\; \mathrm{d}x \; \mathrm{d}t$$
$$\leq C(\delta) L_q \int_0^\tau \int_\Omega (\varrho - r)^2 \; \mathrm{d}x \; \mathrm{d}t \; + \delta \int_0^\tau \int_\Omega |\mathrm{div}_x \mathbf{U} - \mathrm{div}_x \mathbf{u}|^2 \; \mathrm{d}x \; \mathrm{d}t \, .$$

We note that, with the help of Korn inequality (1.1.11), the last integral is controlled by

$$\int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) : (\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) \;\; \mathrm{d}x \; \mathrm{d}t \, .$$

We use the assumption $\gamma \geq 2$ to conclude

$$\int_0^\tau \int_\Omega (\mathrm{div}_x \mathbf{u} - \mathrm{div}_x \mathbf{U})(q(\varrho) - q(r)) \;\; \mathrm{d}x \; \mathrm{d}t$$
$$\leq C(\delta, r, q) \int_0^\tau \mathcal{E}(t) \; \mathrm{d}t \; + \delta \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) : (\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) \;\; \mathrm{d}x \; \mathrm{d}t \, . \tag{3.3.10}$$

Thus combining the estimates (3.3.3), (3.3.4), (3.3.9) and (3.3.10) and choosing a small δ suitably, we obtain

$$\mathcal{E}(\tau) + \frac{1}{2} \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) : (\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) \, \mathrm{d}x \; \mathrm{d}t \leq \mathcal{E}_0 + C(r, \mathbf{U}, q) \int_0^\tau \mathcal{E}(t) \; \mathrm{d}t. \tag{3.3.11}$$

The hypothesis on the initial condition helps us to infer $\mathcal{E}_0 = 0$. Since $C(r, \mathbf{U}, q)$ in (3.3.11) is uniformly bounded in $[0, T]$. Therefore, we apply Grönwall's inequality to conclude

$$\mathcal{E} = 0 \text{ a.e. in } (0, T).$$

This ends the proof of the Theorem 3.3.4.

Navier slip boundary condition: In the case of the Navier slip boundary condition, a similar result holds. The main difference in considering the Navier slip boundary condition is the unavailability of the Poincaré inequality for the velocity $\mathbf{u}$, since $\mathbf{u} \in L^2(0, T; W^{1,2}(\Omega))$ and the standard Poincaré inequality (1.1.8) is not true in $L^2(0, T; W^{1,2}(\Omega))$. Therefore, we need to consider the generalized Korn-Poincaré Inequality(1.1.13). Here we state the result in this case.

Proposition 3.3.5. *Let $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$, be a bounded Lipschitz domain. Let the pressure be given by* (2.1.36) *or* (2.1.38), *with $\gamma \geq 2$ and $q \in C^{0,1}[0, \infty)$. We further assume that*

$$0 < r_0 = r(0, \cdot) \in W^{k,2}(\Omega) \text{ and } \mathbf{U}_0 = \mathbf{U}(0, \cdot) \in W^{k,2}(\Omega; \mathbb{R}^d) \text{ with } k > \left[\frac{d}{2}\right] + 2.$$

Suppose that $(\varrho, \mathbf{u})$ is a dissipative weak solution following the Definition 2.2.8 and $(r, \mathbf{U})$ is a strong solution of the problem (2.2.1)-(2.2.3) *with Navier slip boundary condition* (2.2.6) *on the time interval $[0, T)$ such that initial data satisfies*

$$\varrho(0, \cdot) = r(0, \cdot) > 0, \ \varrho\mathbf{u}(0, \cdot) = r(0, \cdot)\mathbf{U}(0, \cdot).$$

Then

$$\varrho = r, \ \mathbf{u} = \mathbf{U} \text{ in } (0, T) \times \Omega.$$

First we rewrite an analogous lemma of (3.3.1) by using the following structural property of the Newton rheological law (2.2.3):

$$\mathbb{S}(\mathbb{A})\colon \mathbb{A} = \mu\mathbb{D}_0(\mathbb{A})\colon \mathbb{D}_0(\mathbb{A}) + \lambda|\mathrm{trace}(\mathbb{A})|^2.$$

where $\mathbb{A} \in \mathbb{R}^{d\times d}$.

Lemma 3.3.6. *Suppose $(\varrho, \mathbf{u})$ is a weak solution of the compressible Navier–Stokes system following the Definition 2.2.8 with pressure law* (2.1.36) *or* (2.1.38) *and $[r, \mathbf{U}]$ is a strong solution of the same system with $r > 0$ satisfying Navier-slip boundary condition. Then the following inequality holds*

$$\begin{aligned}
\mathcal{E}(\tau)+\mu &\int_0^\tau \int_\Omega \mathbb{D}_0(\nabla_x\mathbf{u} - \nabla_x\mathbf{U}) : \mathbb{D}_0(\nabla_x\mathbf{u} - \nabla_x\mathbf{U}) \ \ dx \ dt \\
&+ \lambda \int_0^\tau \int_\Omega |div_x\mathbf{u} - div_x\mathbf{U}|^2 \ dx \ dt \\
&\leq \mathcal{E}_0 + \int_0^\tau \int_\Omega \left(\frac{\varrho}{r} - 1\right) (\mathbf{U} - \mathbf{u}) \ (div_x\mathbb{S}(\nabla_x\mathbf{U}) - \nabla_x q(r)) \ \ dx \ dt \\
&+ \int_0^\tau \int_\Omega \varrho(\mathbf{u} - \mathbf{U}) \cdot ((\mathbf{U} - \mathbf{u}) \cdot \nabla_x)\mathbf{U} \ \ dx \ dt \\
&+ \int_0^\tau \int_\Omega (-h(\varrho) + h(r) + h'(r)(\varrho - r)) \ div_x\mathbf{U} \ \ dx \ dt \\
&+ \int_0^\tau \int_\Omega (div_x\mathbf{u} - div_x\mathbf{U})(q(\varrho) - q(r)) \ \ dx \ dt \ ,
\end{aligned} \tag{3.3.12}$$

for a.e. $\tau \in (0, T)$.

In order to estimate the terms on the right hand side of (3.3.12), we notice that except for the remainder term

$$\int_0^\tau \int_\Omega \left(\frac{\varrho}{r} - 1\right) (\mathbf{U} - \mathbf{u}) \ (\mathrm{div}_x\mathbb{S}(\nabla_x\mathbf{U}) - \nabla_x q(r)) \ \ \mathrm{d}x \ \mathrm{d}t \ ,$$

the estimates are similar to those in the case of no-slip boundary condition (Theorem 3.3.4), since we have used the Poincaré inequality only in this term.

For this term first we consider ψ as in (3.3.5). Then we obtain

$$\begin{aligned}&\int_0^\tau\int_\Omega \psi(\varrho)(\varrho-r)(\mathbf{U}-\mathbf{u})\cdot\frac{1}{r}\big(\mathrm{div}_x\mathbb{S}(\nabla_x\mathbf{U})-\nabla_x q(r)\big)\ \mathrm{d}x\ \mathrm{d}t\\&\leq C(r,\mathbf{U},q)\int_0^\tau \mathcal{E}(t)\ \mathrm{d}t\ .\end{aligned}\tag{3.3.13}$$

Again, we consider

$$(1-\psi(\varrho))(\varrho-r)(\mathbf{U}-\mathbf{u})=(w_1(\varrho)+w_2(\varrho))(\varrho-r)(\mathbf{U}-\mathbf{u}),.$$

where $\mathrm{supp}(w_1)\subset[0,r_1)$ and $\mathrm{supp}(w_2)\subset(r_2,\infty)$.
The following observation

$$w_2(\varrho)(\varrho-r)(\mathbf{U}-\mathbf{u})\leq C(r)(\varrho+\varrho|\mathbf{U}-\mathbf{u}|^2),$$

helps us to deduce

$$\begin{aligned}&\int_0^\tau\int_\Omega w_2(\varrho)(\varrho-r)(\mathbf{U}-\mathbf{u})\cdot\frac{1}{r}\big(\mathrm{div}_x\mathbb{S}(\nabla_x\mathbf{U})-\nabla_x q(r)\big)\ \mathrm{d}x\ \mathrm{d}t\\&\leq C(r,\mathbf{U},q)\int_0^\tau \mathcal{E}(t)\ \mathrm{d}t\ .\end{aligned}\tag{3.3.14}$$

Also, for any $\delta>0$, using Young's inequality (1.1.1) again, we deduce that

$$w_1(\varrho)(\varrho-r)(\mathbf{U}-\mathbf{u})\leq C(\delta)w_1^2(\varrho)(\varrho-r)^2+\delta|\mathbf{U}-\mathbf{u}|^2.$$

Now, we notice the presence of the term $\delta|\mathbf{U}-\mathbf{u}|^2$ in the right hand side of the above inequality. From Generalized Korn-Poincaré inequality (1.1.13) we infer

$$\begin{aligned}\int_\Omega|\mathbf{U}-\mathbf{u}|^2\ \mathrm{d}x\leq C(d,\Omega)\bigg(\int_\Omega&\mathbb{D}_0(\nabla_x\mathbf{u}-\nabla_x\mathbf{U})\colon\mathbb{D}_0(\nabla_x\mathbf{u}-\nabla_x\mathbf{U})\ \mathrm{d}x\\&+\int_\Omega\varrho|\mathbf{U}-\mathbf{u}|^2\ \mathrm{d}x\bigg).\end{aligned}$$

It yields

$$
\begin{aligned}
&\int_0^\tau \int_\Omega w_1(\varrho)(\varrho - r)(\mathbf{U} - \mathbf{u}) \cdot \frac{1}{r}\big(\mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{U}) - \nabla_x q(r)\big) \ \mathrm{d}x \ \mathrm{d}t \\
&\le C \left(\left\| \frac{1}{r}\big(\mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{U})\big) \right\|_{C([0,T]\times\overline{\Omega};\mathbb{R}^d)} + \frac{L_q}{\inf r} \|r\|_{C^1([0,T]\times\overline{\Omega};\mathbb{R}^d)}, \delta \right) \int_0^\tau \mathcal{E}(t) \ \mathrm{d}t \\
&\quad + \delta C(d,\Omega) \bigg(\int_0^\tau \int_\Omega \mathbb{D}_0(\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) \colon \mathbb{D}_0(\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) \ \mathrm{d}x \ \mathrm{d}t \\
&\qquad\qquad + \int_0^\tau \int_\Omega \varrho |\mathbf{U} - \mathbf{u}|^2 \ \mathrm{d}x \ \mathrm{d}t \bigg) \\
&\le C(r, \mathbf{U}, q, \delta) \int_0^\tau \mathcal{E}(t) \ \mathrm{d}t \\
&\quad\ + \delta C(d,\Omega) \int_0^\tau \int_\Omega \mathbb{D}_0(\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) \colon \mathbb{D}_0(\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) \ \mathrm{d}x \ \mathrm{d}t \ .
\end{aligned}
\tag{3.3.15}
$$

Arguing similarly as we have done in the proof of the Theorem 3.3.4 we conclude the claim that the Proposition 3.3.5 is true.

3.3.2 Weak–strong uniqueness for a non-monotone hard-sphere type pressure law

Suppose $(\varrho, \mathbf{u})$ is a dissipative weak solution of the system (2.2.1)-(2.2.3) with pressure law (2.1.40) in $(0,T) \times \mathbb{T}^d$. As mentioned in the Remark 2.2.13, the assumption (2.2.29) is essential for the existence of a weak solution. Then we have the relative energy inequality (3.2.4) which holds for a suitable class of test functions, in particular, for $(r, \mathbf{U}) \in C^1([0,T] \times \mathbb{T}^d) \times C^1([0,T]; C^2(\mathbb{T}^d))$ with $r > 0$.

Furthermore, if we assume that $(r, \mathbf{U}) \in C^1([0,T] \times \mathbb{T}^d) \times C^1([0,T]; C^2(\mathbb{T}^d))$ is a strong solution of the same system with $r > 0$ in $(0,T) \times \mathbb{T}^d$, then the following form of relative energy holds for a.e. $\tau \in (0,T)$:

$$
\mathcal{E}(\tau) + \int_0^\tau \int_{\mathbb{T}^d} \mathbb{S}(\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) : (\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) \ \ \mathrm{d}x \ \mathrm{d}t \ \le \mathcal{E}_0 + \int_0^\tau \mathcal{R}_1(t) \ \mathrm{d}t \ , \tag{3.3.16}
$$

where $\mathcal{R}_1(\cdot)$ is given by

$$
\begin{aligned}
\mathcal{R}_1(t) = &\int_{\mathbb{T}^d} \left(\frac{\varrho}{r} - 1\right) (\mathbf{U} - \mathbf{u}) \left(\mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{U}) - \nabla_x q(r)\right) \ \mathrm{d}x \ \mathrm{d}t \\
&+ \int_{\mathbb{T}^d} \varrho(\mathbf{u} - \mathbf{U}) \cdot ((\mathbf{U} - \mathbf{u}) \cdot \nabla_x) \mathbf{U} \ \ \mathrm{d}x \ \mathrm{d}t \\
&+ \int_{\mathbb{T}^d} (-h(\varrho) + h(r) + h'(r)(\varrho - r)) \, \mathrm{div}_x \mathbf{U} \ \ \mathrm{d}x \ \mathrm{d}t \\
&+ \int_{\mathbb{T}^d} (\mathrm{div}_x \mathbf{u} - \mathrm{div}_x \mathbf{U})(q(\varrho) - q(r)) \ \ \mathrm{d}x \ \mathrm{d}t \ .
\end{aligned}
\tag{3.3.17}
$$

In the context of isentropic pressure law (2.1.35) and (2.1.37), we have Lemma 3.3.3, which leads us to the conclusion that the term

$$\int_0^\tau \int_{\mathbb{T}^d} (-h(\varrho) + h(r) + h'(r)(\varrho - r)) \operatorname{div}_x \mathbf{U} \ \mathrm{d}x \ \mathrm{d}t \ .$$

is controlled by the relative energy $\int_0^\tau \mathcal{E}(t) \ \mathrm{d}t$. Unfortunately, we are unable to obtain a similar lemma for the monotone part of the pressure law (2.1.40). Instead, we have the following lemma from Feireisl, Lu and Novotný [64, Lemma 2.1]:

Lemma 3.3.7. *Let h be the monotone part of the pressure law* (2.1.40) *and H the pressure potential associated with h. Let $\varrho \geq 0$ and $0 < \alpha_0 \leq r \leq \bar{\varrho} - \alpha_0 < \bar{\varrho}$. There exists $\alpha_1 \in (0, \alpha_0)$ and a constant $c > 0$, such that*

$$H(\varrho) - H(r) - H'(r)(\varrho - r) \geq \begin{cases} c(\varrho - r)^2, & \text{if } \alpha_1 \leq \varrho \leq \bar{\varrho} - \alpha_1, \\ \frac{h(r)}{2}, & \text{if } 0 \leq \varrho \leq \alpha_1, \\ \frac{H(\varrho)}{2}, & \text{if } \bar{\varrho} - \alpha_1 \leq \varrho < \bar{\varrho}. \end{cases}$$

We also have,

$$h(\varrho) - h(r) - h'(r)(\varrho - r) \leq \begin{cases} c(\varrho - r)^2, & \text{if } \alpha_1 \leq \varrho \leq \bar{\varrho} - \alpha_1, \\ 1 + h'(r)r - h(r), & \text{if } 0 \leq \varrho \leq \alpha_1, \\ 2h(\varrho), & \text{if } \bar{\varrho} - \alpha_1 \leq \varrho < \bar{\varrho}. \end{cases}$$

Remark 3.3.8. Without loss of generality we can assume on $[\bar{\varrho} - \alpha_1, \bar{\varrho})$, $H(\varrho) > 2$. Furthermore, we consider α_1 such that $supp(q) \subset (\alpha_1, \bar{\varrho} - \alpha_1)$, where q is the non-monotone part in the pressure law (2.1.40).

Remark 3.3.9. In Lemma 3.3.7, the constant c depends on r such that $c(r)$ is uniformly bounded on $(\alpha_0, \bar{\varrho} - \alpha_0)$. Also, $0 \leq \varrho \leq \bar{\varrho} - \alpha_1$ we obtain

$$|h(\varrho) - h(r) - h'(r)(\varrho - r)| \leq C(H(\varrho) - H(r) - H'(r)(\varrho - r)).$$

Moreover, for $\bar{\varrho} - \alpha_1 \leq \varrho < \bar{\varrho}$ we have no control on $h(\varrho) - h(r) - h'(r)(\varrho - r)$ by $H(\varrho) - H(r) - H'(r)(\varrho - r)$.

Relative energy inequality with extra term

First we introduce a few notations and important results.

- In $\mathbb{T}^d$, we denote Δ_x the Laplace operator defined on spatially periodic functions with zero mean.
- For $1 < q < \infty$, we denote $L^q_0(\mathbb{T}^d) := \{f \in L^q(\mathbb{T}^d) | \int_{\mathbb{T}^d} f \ \mathrm{d}x = 0.\}$, then by classical elliptic theory, Δ_x^{-1} is a bounded linear mapping from $L^q_0(\mathbb{T}^d)$ to $W^{2,q}(\mathbb{T}^d) \cap L^q_0(\mathbb{T}^d)$.

To overcome the difficulty stated in Remark 3.3.9, we add one extra term

$$\int_0^\tau \int_{\mathbb{T}^d} b(\varrho)h(\varrho)\,\mathrm{d}x\,\mathrm{d}t$$

on the relative energy inequality, where b is a function that satisfies the hypothesis of renormalized equation. To include this term, we have to use $\nabla_x\Delta_x^{-1}(b(\varrho)-\langle b(\varrho)\rangle)$ as a test function in the momentum equation (2.2.27) and then the renormalized continuity equation (2.2.25) suitably. Then we obtain

$$\int_0^\tau \int_{\mathbb{T}^d} b(\varrho)h(\varrho)\,\mathrm{d}x\,\mathrm{d}t\; = \int_0^\tau \int_{\mathbb{T}^d} \mathcal{R}_2(t)\,\mathrm{d}t\; + \int_{\mathbb{T}^d} \mathcal{R}_3(\tau), \tag{3.3.18}$$

where $\mathcal{R}_2(\cdot)$ is given by

$$\begin{aligned}
\mathcal{R}_2(t) = &\int_{\mathbb{T}^d} h(\varrho)\langle b(\varrho)\rangle\,\mathrm{d}x - \int_{\mathbb{T}^d} (q(\varrho)-q(r))\,(b(\varrho)-\langle b(\varrho)\rangle)\,\mathrm{d}x + \int_{\mathbb{T}^d} q(r)b(\varrho)\,\mathrm{d}x\\
&- \int_{\mathbb{T}^d} \varrho\mathbf{u}\otimes\mathbf{u} : \nabla_x(\nabla_x\Delta_x^{-1}(b(\varrho)-\langle b(\varrho)\rangle))\,\mathrm{d}x\\
&+ \int_{\mathbb{T}^d} \mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x(\nabla_x\Delta_x^{-1}(b(\varrho)-\langle b(\varrho)\rangle))\,\mathrm{d}x\\
&+ \int_{\mathbb{T}^d} \varrho\mathbf{u}\cdot\nabla_x\Delta_x^{-1}\mathrm{div}_x(b(\varrho)\mathbf{u})\,\mathrm{d}x\\
&+ \int_{\mathbb{T}^d} \varrho\mathbf{u}\cdot\nabla_x\Delta_x^{-1}\big((b'(\varrho)\varrho-b(\varrho))\mathrm{div}_x\mathbf{u} - \langle(b'(\varrho)\varrho-b(\varrho)\mathrm{div}_x\mathbf{u}\rangle\big)\,\mathrm{d}x,
\end{aligned} \tag{3.3.19}$$

and $\mathcal{R}_3(\cdot)$ is given by

$$\begin{aligned}
\mathcal{R}_3(\tau) = &\int_{\mathbb{T}^d} \varrho\mathbf{u}\cdot\nabla_x\Delta_x^{-1}(b(\varrho)-\langle b(\varrho)\rangle)(\tau,\cdot)\,\mathrm{d}x\\
&- \int_{\mathbb{T}^d} \varrho_0\mathbf{u}_0\cdot\nabla_x\Delta_x^{-1}(b(\varrho_0)-\langle b(\varrho_0)\rangle)\,\mathrm{d}x
\end{aligned} \tag{3.3.20}$$

Indeed, it is important to verify that the term $\int_0^\tau \int_{\mathbb{T}^d} b(\varrho)h(\varrho)\,\mathrm{d}x\,\mathrm{d}t$ is well defined for suitable b, then the identity (3.3.18) makes sense. We give the following lemma which ensures this.

Lemma 3.3.10. *Suppose the pressure constraint* (2.2.29) *is satisfied, i.e.,*

$$\lim_{\varrho\to\bar{\varrho}} h(\varrho)(\bar{\varrho}-\varrho)^\beta > 0, \qquad \textit{for some } \beta \geq \frac{5}{2}.$$

Let $\{\varrho,\mathbf{u}\}$ *be a dissipative weak solution in* $(0,T)\times\Omega$ *in the sense of definition* (2.2.12). *Let* $(r,\mathbf{U}) \in C^1([0,T]\times\mathbb{T}^d)\times C^1([0,T];C^2(\mathbb{T}^d;\mathbb{R}^d))$ *be a strong solution of the same system with* $0<r<\bar{\varrho}$. *Let* $b\in C^1[0,\varrho)$ *satisfy the condition*

$$|b'(s)|^{\frac{5}{2}} + |b(s)|^{\frac{5}{2}} \leq C(1+h(s)) \textit{ for some constant } C \textit{ and any } s\in[0,\bar{\varrho}). \tag{3.3.21}$$

Then the following relative energy is true for a.e. $\tau \in (0,T)$*,*

$$[\mathcal{E}(t)]_{t=0}^{t=\tau}+\int_0^\tau \int_{\mathbb{T}^d} \mathbb{S}(\nabla_x \mathbf{u}-\nabla_x \mathbf{U}):(\nabla_x \mathbf{u}-\nabla_x \mathbf{U}) \ dx \ dt \ +\int_0^\tau \int_{\mathbb{T}^d} b(\varrho)h(\varrho) \ dx \ dt$$
$$\leq \int_0^\tau \mathcal{R}_1(t) \ dt \ + \int_0^\tau \mathcal{R}_2(t) \ dt \ + \mathcal{R}_3(\tau), \tag{3.3.22}$$

with $\mathcal{R}_1$*,* $\mathcal{R}_2$ *and* $\mathcal{R}_3$ *are given by* (3.3.17),(3.3.19)*, and* (3.3.20)*,respectively.*

Remark 3.3.11. The Lemma 3.3.10 remains true even if we do not assume that $(\varrho, \mathbf{U})$ is a strong solution. It suffices to assume $(r,\mathbf{U}) \in C^1([0,T]\times\mathbb{T}^d)\times C^1([0,T];C^2(\mathbb{T}^d;\mathbb{R}^d))$ with $0<r<\bar{\varrho}$. In this case we need to replace (3.3.16) by (3.2.4).

Remark 3.3.12. The condition for b in (3.3.21) is slightly changed from the similar assumption in the Definition 2.2.12. But this is related to the technical assumption(2.2.29).

Here we give an extended outline of the proof of the Lemma 3.3.10. By (2.2.29), we first observe

$$H(s) \leq C + (\bar{\varrho}-s)h(s) \leq 2C + H(s), \text{ for all } s\in[0,\bar{\varrho}),$$

where C is a constant that depends on $\bar{\varrho}$. Together with (3.3.21), this gives

$$|b(\varrho)|+|b'(\varrho)| \leq C(1+h(s))^{\frac{2}{5}} \leq C(1+H(s))^{\frac{2}{3}}, \text{ for } s\in[0,\bar{\varrho}).$$

Since $q\in C_c^1[0,\bar{\varrho})$, we have $Q\in C^1[0,\bar{\varrho})$. Hence from the energy inequality (2.2.28) in the Definition 2.2.12, we obtain

$$b(\varrho),\ b'(\varrho) \in L^{\frac{5}{2}}((0,T)\times\mathbb{T}^d)\cap L^\infty(0,T;L^{\frac{3}{2}}(\mathbb{T}^d)) \tag{3.3.23}$$

The function $b(\varrho)$ satisfies the renormalized equation of continuity (2.2.25). Therefore, we are able to get the following information on $\partial_t b(\varrho)$:

$$\partial_t b(\varrho) \in \left(L^2(0,T;W^{-1,\frac{6}{5}}(\mathbb{T}^d))\cap L^{\frac{10}{9}}(0,T;W^{-1,\frac{30}{17}}(\mathbb{T}^d)) + L^{\frac{10}{9}}((0,T)\times\mathbb{T}^d)\right).$$

Due to the fact that $b(\varrho)$ satisfies (3.3.23), we choose $\nabla_x\Delta_x^{-1}(b(\varrho)-\langle b(\varrho)\rangle)$ as the test function in the momentum equation (2.2.27) by suitably modifying the test function class

$$\left\{\boldsymbol{\varphi}\in L^{\frac{5}{2}}(0,T;W^{1,\frac{5}{2}}(\mathbb{T}^d;\mathbb{R}^d))\cap L^\infty(0,T;W^{1,\frac{3}{2}}(\mathbb{T}^d;\mathbb{R}^d))\right\}. \tag{3.3.24}$$

It is possible, since the test function class $\{\boldsymbol{\varphi}\in C^1([0,T];C^2(\mathbb{T}^d;\mathbb{R}^d))\}$ is dense in (3.3.24). Hence, we obtain

$$\int_0^\tau \int_{\mathbb{T}^d} b(\varrho)h(\varrho) \ \mathrm{d}x \ \mathrm{d}t \ = \int_0^\tau \int_{\mathbb{T}^d} \mathcal{R}_2(t) \ \mathrm{d}t \ + \int_{\mathbb{T}^d} \mathcal{R}_3(\tau), \tag{3.3.25}$$

where $\mathcal{R}_2$ and $\mathcal{R}_3$ are described as in (3.3.19) and (3.3.20), respectively. Next, we show that the integrals in the right hand side of (3.3.25) make sense from the following bounds for a.e. $\tau \in (0, T)$:

At first, we have

$$\begin{aligned}\int_0^\tau \int_{\mathbb{T}^d} h(\varrho)\langle b(\varrho)\rangle \, \mathrm{d}x \, \mathrm{d}t &\leq \|h(\varrho)\|_{L^\infty(0,T;L^1(\mathbb{T}^d))} \|\langle b(\varrho)\rangle\|_{L^1(0,T)} \\ &\leq \|h(\varrho)\|_{L^\infty(0,T;L^1(\mathbb{T}^d))} \|b(\varrho)\|_{L^1(0,T;L^1(\mathbb{T}^d))}.\end{aligned}$$

As q is compactly supported, we obtain

$$\begin{aligned}&\int_0^\tau \int_{\mathbb{T}^d} \left((q(\varrho) - q(r)) \left(b(\varrho) - \langle b(\varrho)\rangle \right) - q(r) b(\varrho) \right) \, \mathrm{d}x \, \mathrm{d}t \\ &\leq C \left(\|q(\varrho)\|_{L^\infty((0,T)\times\mathbb{T}^d)} + \|q(r)\|_{L^\infty((0,T)\times\mathbb{T}^d)} \right) \|b(\varrho)\|_{L^1(0,T;L^1(\mathbb{T}^d))}\end{aligned}$$

From Definition (2.2.12) and Sobolev embedding $W^{1,2}(\mathbb{T}^d) \subset L^6$, for $d = 3$, we have

$$\varrho\mathbf{u} \otimes \mathbf{u} \in L^\infty(0, T; L^1(\mathbb{T}^d; \mathbb{R}^{d\times d})) \cap L^1(0, T; L^3(\mathbb{T}^d; \mathbb{R}^{d\times d})) \cap L^{\frac{5}{3}}((0, T) \times \mathbb{T}^d; \mathbb{R}^{d\times d}).$$

Therefore, we deduce that

$$\begin{aligned}&\int_0^\tau \int_{\mathbb{T}^d} \varrho\mathbf{u} \otimes \mathbf{u} : \nabla_x (\nabla_x \Delta_x^{-1} (b(\varrho) - \langle b(\varrho)\rangle)) \, \mathrm{d}x \, \mathrm{d}t \\ &\leq C \|\varrho\mathbf{u} \otimes \mathbf{u}\|_{L^{\frac{5}{3}}((0,T)\times\mathbb{T}^d;\mathbb{R}^{d\times d})} \|(b(\varrho) - \langle b(\varrho)\rangle)\|_{L^{\frac{5}{2}}((0,T)\times\mathbb{T}^d)}.\end{aligned}$$

Similarly, we have the following estimate:

$$\begin{aligned}&\int_0^\tau \int_{\mathbb{T}^d} \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x (\nabla_x \Delta_x^{-1} (b(\varrho) - \langle b(\varrho)\rangle)) \, \mathrm{d}x \, \mathrm{d}t \\ &\leq C \|\nabla_x \mathbf{u}\|_{L^2((0,T)\times\mathbb{T}^d;\mathbb{R}^{d\times d})} \|(b(\varrho) - \langle b(\varrho)\rangle)\|_{L^2((0,T)\times\mathbb{T}^d)}.\end{aligned}$$

Again using the Sobolev embedding, we get $\varrho\mathbf{u} \in L^2(0, T; L^6(\mathbb{T}^d; \mathbb{R}^d))$. This implies

$$\begin{aligned}&\int_0^\tau \int_{\mathbb{T}^d} \varrho\mathbf{u} \cdot \nabla_x \Delta_x^{-1} \mathrm{div}_x (b(\varrho)\mathbf{u}) \, \mathrm{d}x \, \mathrm{d}t \\ &\leq C \|\varrho\mathbf{u}\|_{L^2(0,T;L^6(\mathbb{T}^d;\mathbb{R}^d))} \|b(\varrho)\mathbf{u}\|_{L^2(0,T;L^{\frac{6}{5}}(\mathbb{T}^d;\mathbb{R}^d))}.\end{aligned}$$

Analogously, it is easy to verify that

$$\begin{aligned}&\int_0^\tau \int_{\mathbb{T}^d} \varrho\mathbf{u} \cdot \nabla_x \Delta_x^{-1} \big((b'(\varrho)\varrho - b(\varrho)) \mathrm{div}_x \mathbf{u} - \langle (b'(\varrho)\varrho - b(\varrho) \mathrm{div}_x \mathbf{u} \rangle \big) \, \mathrm{d}x \, \mathrm{d}t \\ &\leq \|\varrho\mathbf{u}\|_X \|b(\varrho) - \varrho b'(\varrho)\|_{L^{\frac{5}{2}}((0,T)\times\mathbb{T}^d)} \|\mathrm{div}_x \mathbf{u}\|_{L^2((0,T)\times\mathbb{T}^d)},\end{aligned}$$

where $X = L^2(0,T;L^6(\mathbb{T}^d;\mathbb{R}^d)) \cap L^\infty(0,T;L^2(\mathbb{T}^d;\mathbb{R}^d))$.
Also, we deduce that

$$\int_{\mathbb{T}^d} \varrho\mathbf{u}\cdot\nabla_x\Delta_x^{-1}(b(\varrho)-\langle b(\varrho)\rangle(\tau,\cdot)\,\mathrm{d}x - \int_{\mathbb{T}^d} \varrho_0\mathbf{u}_0\cdot\nabla_x\Delta_x^{-1}(b(\varrho_0)-\langle b(\varrho_0)\rangle\,\mathrm{d}x$$
$$\leq C\left(\|\varrho\mathbf{u}\|_{L^\infty(0,T;L^2(\mathbb{T}^d;\mathbb{R}^d))}\|b(\varrho)\|_{L^\infty(0,T;L^{\frac{3}{2}}(\mathbb{T}^d))} + \|\varrho_0\mathbf{u}_0\|_{L^2(\mathbb{T}^d;\mathbb{R}^d))}\|b(\varrho_0)\|_{L^{\frac{3}{2}}(\mathbb{T}^d)}\right).$$

Thus, we have all the integrals of $\mathcal{R}_2$ and $\mathcal{R}_3$ is bounded. This ends the proof of the Lemma 3.3.10.

Main theorem

Now we will provide the weak-strong uniqueness result for this problem.

Theorem 3.3.13. *Suppose* $(\varrho,\mathbf{u})$ *is a weak solution of the system* (2.2.1)-(2.2.3) *following the Definition 2.2.12 with the pressure law* (2.1.40) *in* $(0,T)\times\mathbb{T}^d$. *Suppose, the monotone part* h *of the pressure satisfies*

$$\lim_{\varrho\to\bar{\varrho}} h(\varrho)(\bar{\varrho}-\varrho)^\beta > 0, \qquad \textit{for some } \beta\geq 3, \tag{3.3.26}$$

and, the non-monotone part q *is compactly supported in* $[0,\bar{\varrho})$. *Let* $(r,\mathbf{U})\in C^1([0,T]\times\mathbb{T}^d)\times C^1([0,T];C^2(\mathbb{T}^d))$ *be a strong solution of the same system with* $0<r<\bar{\varrho}$, *and with the same initial data* $(\varrho_0,(\varrho\mathbf{u})_0)$. *Then there holds,*

$$(\varrho,\mathbf{u}) = (r,\mathbf{U}) \ \textit{in}\ (0,T)\times\mathbb{T}^d$$

Remark 3.3.14. The assumption $\beta\geq 3$ in (3.3.26) is purely technical to prove the weak–strong uniqueness.

The main idea to obtain the weak-strong uniqueness is to estimate the remainder terms $\mathcal{R}_1$, $\mathcal{R}_1$ and $\mathcal{R}_3$ in (3.3.22). We will try to show that either these estimates are bounded by $\eta(\cdot)\mathcal{E}(\cdot)$ for some positive function η, or absorbed it left hand side of the inequality.

Instead of considering b from certain class we can choose a particular b that satisfies (3.3.21).

Choice of b and its properties

Consider $b\in C^\infty[0,\bar{\varrho})$, $b'(s)\geq 0$ as follows:

$$b(s) = \begin{cases} 0 & \text{if } s\leq\bar{\varrho}-\alpha_1, \\ -\log(\bar{\varrho}-s), & \text{if } \bar{\varrho}-\alpha_2\leq s<\bar{\varrho}, \end{cases} \qquad b'(s)>0 \text{ if } \bar{\varrho}-\alpha_1<s<\bar{\varrho}-\alpha_2. \tag{3.3.27}$$

The choice of α_2 is in such a way that

$$-\log(\bar{\varrho}-s)\geq 16\|\mathrm{div}_x\mathbf{U}\|_{C([0,T]\times\mathbb{T}^d)}, \text{ if } \bar{\varrho}-\alpha_2\leq s<\bar{\varrho}. \tag{3.3.28}$$

Considering the assumption (3.3.26), we have

$$\text{For } \gamma > 0, \lim_{s\to\bar{\varrho}-} \frac{h(s)}{(b(s))^{\gamma}} = \lim_{s\to\bar{\varrho}-} \frac{H(s)}{(b(s))^{\gamma}} = \lim_{s\to\bar{\varrho}-} \frac{h(s)}{(b'(s))^{\beta}} = \lim_{s\to\bar{\varrho}-} \frac{H(s)}{(b(s))^{\beta-1}} = +\infty. \tag{3.3.29}$$

This helps to obtain the following results:

- From (3.3.29) and (3.3.7), we have

$$\begin{aligned} \int_{\mathbb{T}^d} |b(\varrho)|^{\gamma} \, \mathrm{d}x &= \int_{\varrho\geq\bar{\varrho}-\alpha_1} |b(\varrho)|^{\gamma} \, \mathrm{d}x \\ &\leq C \int_{\varrho\geq\bar{\varrho}-\alpha_1} H(\varrho) \, \mathrm{d}x \\ &\leq C \int_{\varrho\geq\bar{\varrho}-\alpha_1} (H(\varrho) - H(r) - H'(r)(\varrho - r)) \, \mathrm{d}x, \end{aligned} \tag{3.3.30}$$

  for any $\gamma \geq 1$.

- Also for any $2 \leq \beta_0 \leq \beta$, we have

$$\begin{aligned} \int_{\mathbb{T}^d} |b'(\varrho)|^{\beta_0 - 1} \, \mathrm{d}x &\leq C \int_{\varrho\geq\bar{\varrho}-\alpha_1} H(\varrho) \, \mathrm{d}x \\ &\leq C \int_{\varrho\geq\bar{\varrho}-\alpha_1} (H(\varrho) - H(r) - H'(r)(\varrho - r)) \, \mathrm{d}x, \\ \int_{\mathbb{T}^d} |b'(\varrho)|^{\beta_0} \, \mathrm{d}x &\leq C \int_{\mathbb{T}^d} h(\varrho) \, \mathrm{d}x. \end{aligned} \tag{3.3.31}$$

This choice of b ensures that the Lemma 3.3.10 remains true.

Estimates for the remainder terms $\mathcal{R}_1$, $\mathcal{R}_2$ and $\mathcal{R}_3$

We consider the following subsets of $\mathbb{T}^d$ at any instantaneous time t:

$$\begin{aligned} \Omega_1 &= \{x \in \mathbb{T}^d | 0 \leq \varrho \leq \bar{\varrho} - \alpha_1\}, \\ \Omega_2 &= \{x \in \mathbb{T}^d | \bar{\varrho} - \alpha_1 \leq \varrho \leq \bar{\varrho} - \alpha_2\}, \\ \Omega_3 &= \{x \in \mathbb{T}^d | \bar{\varrho} - \alpha_2 \leq \varrho < \bar{\varrho}\}. \end{aligned}$$

For any $\psi \in L^2(0, T; L^2(\mathbb{T}^d))$ we have

$$\int_{\mathbb{T}^d} (q(\varrho) - q(r))\psi \, \mathrm{d}x \leq C_\delta \left(\int_{\mathbb{T}^d} \mathbf{1}_{\Omega_1} (\varrho - r)^2 \, \mathrm{d}x + \int_{\mathbb{T}^d} \mathbf{1}_{\Omega_2\cup\Omega_3} \, \mathrm{d}x \right) + \delta \int_{\mathbb{T}^d} \psi^2 \, \mathrm{d}x,$$

since $\mathrm{supp} q \subset (0, \bar{\varrho} - \alpha_1)$. From the Lemma 3.3.7 we note that the first term on the right hand side of the above inequality is controlled by the relative energy.

First, we focus on the term $\mathcal{R}_1$ and consider

$$
\begin{aligned}
\mathcal{R}_{1,1}(t) = & \int_{\mathbb{T}^d} \left(\frac{\varrho}{r} - 1\right) (\mathbf{U} - \mathbf{u}) \left(\mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{U}) - \nabla_x q(r)\right) \mathrm{d}x \\
& + \int_{\mathbb{T}^d} \varrho(\mathbf{u} - \mathbf{U}) \cdot ((\mathbf{U} - \mathbf{u}) \cdot \nabla_x)\mathbf{U} \ \mathrm{d}x \\
& + \int_{\mathbb{T}^d} \mathbf{1}_{\Omega_1} (-h(\varrho) + h(r) + h'(r)(\varrho - r)) \mathrm{div}_x \mathbf{U} \ \mathrm{d}x \\
& + \int_{\mathbb{T}^d} (\mathrm{div}_x \mathbf{u} - \mathrm{div}_x \mathbf{U})(q(\varrho) - q(r)) \ \mathrm{d}x.
\end{aligned}
$$

We have calculated these terms in the Subsection 3.3.1. However, we need to be a little careful as we consider domain $\mathbb{T}^d$. Instead of the Poincaré inequality(1.1.8), we need to use the generalized Korn–Poincaré inequality (1.1.13). Thus we obtain

$$
\begin{aligned}
\int_0^\tau \mathcal{R}_{1,1}(t) \ \mathrm{d}t \ \leq & C(\delta, q, r\mathbf{U}) \int_0^\tau \int_{\mathbb{T}^d} \mathcal{E}(t) \ \mathrm{d}t \\
& + \delta C_p \Bigg(\int_0^\tau \int_{\mathbb{T}^d} \mathbb{D}_0(\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) \colon \mathbb{D}_0(\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) \ \mathrm{d}x \ \mathrm{d}t \\
& \qquad + \int_0^\tau \int_{\mathbb{T}^d} |\mathrm{div}_x \mathbf{u} - \mathrm{div}_x \mathbf{U}|^2 \ \mathrm{d}x \ \mathrm{d}t \Bigg),
\end{aligned}
$$

for any $\delta > 0$ and C_p is the constant coming from the Korn–Poincaré inequality. On the other hand, the choice of b in (3.3.27) yields

$$
\int_0^\tau \int_{\mathbb{T}^d} \mathbf{1}_{\Omega_2 \cup \Omega_3} (-h(\varrho) + h(r) + h'(r)(\varrho - r)) \mathrm{div}_x \mathbf{U} \ \mathrm{d}x \ \mathrm{d}t \leq \frac{1}{8} \int_0^\tau \int_{\mathbb{T}^d} b(\varrho) h(\varrho) \, \mathrm{d}x \, \mathrm{d}t.
$$

From this we deduce that

$$
\begin{aligned}
\int_0^\tau \mathcal{R}_1(t) \ \mathrm{d}t \ \leq & \ C(\delta, \bar{\varrho}, r, \mathbf{U}, q) \int_0^\tau \mathcal{E}(t) \ \mathrm{d}t \ + \frac{1}{8} \int_0^\tau \int_{\mathbb{T}^d} b(\varrho) h(\varrho) \ \mathrm{d}x \ \mathrm{d}t \\
& + \delta C_p \Bigg(\int_0^\tau \int_{\mathbb{T}^d} \mathbb{D}_0(\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) \colon \mathbb{D}_0(\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) \ \mathrm{d}x \ \mathrm{d}t \\
& \qquad + \int_0^\tau \int_{\Omega} |\mathrm{div}_x \mathbf{u} - \mathrm{div}_x \mathbf{U}|^2 \ \mathrm{d}x \ \mathrm{d}t \Bigg).
\end{aligned}
\tag{3.3.32}
$$

Now we recall the remainder term $\mathcal{R}_2$,

$$
\begin{aligned}
\mathcal{R}_2(t) = &\int_{\mathbb{T}^d} h(\varrho)\langle b(\varrho)\rangle \, \mathrm{d}x - \int_\Omega (q(\varrho) - q(r))\,(b(\varrho) - \langle b(\varrho)\rangle) \, \mathrm{d}x + \int_\Omega q(r) b(\varrho) \, \mathrm{d}x \\
&- \int_{\mathbb{T}^d} \varrho\mathbf{u}\otimes\mathbf{u} : \nabla_x(\nabla_x\Delta_x^{-1}(b(\varrho) - \langle b(\varrho)\rangle)) \, \mathrm{d}x \\
&+ \int_{\mathbb{T}^d} \mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x(\nabla_x\Delta_x^{-1}(b(\varrho) - \langle b(\varrho)\rangle)) \, \mathrm{d}x \\
&+ \int_{\mathbb{T}^d} \varrho\mathbf{u}\cdot\nabla_x\Delta_x^{-1}\mathrm{div}_x(b(\varrho)\mathbf{u}) \, \mathrm{d}x \\
&+ \int_{\mathbb{T}^d} \varrho\mathbf{u}\cdot\nabla_x\Delta_x^{-1}\big((b'(\varrho)\varrho - b(\varrho))\mathrm{div}_x\mathbf{u} - \langle (b'(\varrho)\varrho - b(\varrho)\mathrm{div}_x\mathbf{u}\rangle\big) \, \mathrm{d}x \\
= &\sum_{j=1}^{7} \mathcal{T}_j.
\end{aligned}
$$

We quickly recall the regularity class of the variables:

$$
\begin{aligned}
&\varrho \in C_w([0,T]; L^\gamma(\mathbb{T}^d)) \text{ for any } \gamma > 1, \ p(\varrho) \in L^1((0,T)\times\mathbb{T}^d) \\
&\mathbf{u} \in L^2(0,T; W^{1,2}(\mathbb{T}^d;\mathbb{R}^d)), \ \varrho\mathbf{u} \in C_w([0,T]; L^2(\mathbb{T}^d;\mathbb{R}^d)).
\end{aligned}
$$

Now using the Sobolev embedding we have $\varrho\mathbf{u} \in L^2(0,T; L^6(\mathbb{T}^d;\mathbb{R}^d))$, and

$$
\varrho\mathbf{u}\otimes\mathbf{u} \in L^\infty(0,T; L^1(\mathbb{T}^d;\mathbb{R}^{d\times d})) \cap L^1(0,T; L^3(\mathbb{T}^d;\mathbb{R}^{d\times d})) \cap L^{\frac{5}{3}}((0,T)\times\mathbb{T}^d;\mathbb{R}^{d\times d}).
$$

Further we note that $b(\varrho)$ and $b'(\varrho)$ satisfies (3.3.23). Moreover, from (3.3.30) and (3.3.31) we have

$$
b(\varrho) \in L^\infty(0,T; L^\gamma(\mathbb{T}^d)) \text{ for any } \gamma \geq 1.
$$

From (3.3.30) we get

$$
\langle b(\varrho)\rangle = \frac{1}{\mathcal{L}(\mathbb{T}^d)} \int_{\mathbb{T}^d} b(\varrho) \leq \mathcal{E}(t),
$$

for a.e. $t \in (0,T)$. Also, from the Definition 2.2.12, we have $h(\varrho) \in L^\infty(0,T; L^1(\mathbb{T}^d))$. This implies

$$
\int_{\mathbb{T}^d} h(\varrho)\langle b(\varrho)\rangle \, \mathrm{d}x \leq \eta(t)\mathcal{E}(t),
$$

where $\eta \in L^1(0,T)$.

Henceforth, we use η as a generic function in $L^1(0,T)$ which depends on $q, \bar{\varrho}$, the initial data $(\varrho_0, (\varrho\mathbf{u})_0)$, the initial energy E_0 and the strong solution $(r, \mathbf{U})$.

We have q is compactly supported and $b(\varrho)$ satisfies (3.3.23), these yield

$$
\int_{\mathbb{T}^d} h(\varrho)\langle b(\varrho)\rangle \, \mathrm{d}x - \int_{\mathbb{T}^d} (q(\varrho) - q(r))\,(b(\varrho) - \langle b(\varrho)\rangle) \, \mathrm{d}x \leq \eta(t)\mathcal{E}(t).
$$

This helps us to conclude

$$\mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3 \leq \eta(t)\mathcal{E}(t). \tag{3.3.33}$$

Let us consider the term $\mathcal{T}_4$, i.e.,

$$\int_{\mathbb{T}^d} \varrho \mathbf{u} \otimes \mathbf{u} : \nabla_x(\nabla_x \Delta_x^{-1}(b(\varrho) - \langle b(\varrho)\rangle))\, \mathrm{d}x.$$

We have the following identity

$$\begin{aligned}\varrho \mathbf{u} \otimes \mathbf{u} =& \varrho(\mathbf{u} - \mathbf{U}) \otimes (\mathbf{u} - \mathbf{U}) + \varrho \mathbf{U} \otimes (\mathbf{u} - \mathbf{U}) \\ &+ \varrho(\mathbf{u} - \mathbf{U}) \otimes \mathbf{U} + (\varrho - r)\mathbf{U} \otimes \mathbf{U} + r\mathbf{U} \otimes \mathbf{U}.\end{aligned}$$

From our choice of b, we note that

$$\|\nabla_x(\nabla_x \Delta_x^{-1}(b(\varrho) - \langle b(\varrho)\rangle))\|_{L^\gamma(\mathbb{T}^d;\mathbb{R}^{d\times d})} \leq \|b(\varrho)\|_{L^\gamma(\mathbb{T}^d)}, \text{ for any } \gamma \geq 1.$$

We employ the Sobolev embedding theorem to obtain

$$\begin{aligned}&\int_{\mathbb{T}^d} \varrho(\mathbf{u} - \mathbf{U}) \otimes (\mathbf{u} - \mathbf{U}) \colon \nabla_x(\nabla_x \Delta_x^{-1}(b(\varrho) - \langle b(\varrho)\rangle)\, \mathrm{d}x \\ &\leq C(\bar{\varrho})\|\sqrt{\varrho}(\mathbf{u} - \mathbf{U})\|_{L^2(\mathbb{T}^d)}\|\mathbf{u} - \mathbf{U}\|_{L^6(\mathbb{T}^d)}\|b(\varrho)\|_{L^3(\mathbb{T}^d)} \\ &\leq C(\bar{\varrho}, \delta)\|\sqrt{\varrho}|\mathbf{u} - \mathbf{U}\|^2_{L^2(\mathbb{T}^d)}\|b(\varrho)\|^2_{L^3(\mathbb{T}^d)} + \delta\|\mathbf{u} - \mathbf{U}\|^2_{W^{1,2}(\mathbb{T}^d)}.\end{aligned}$$

Eventually, we use the Generalized Korn-Poincaré inequality (1.1.13) to deduce

$$\begin{aligned}&\int_{\mathbb{T}^d} \varrho(\mathbf{u} - \mathbf{U}) \otimes (\mathbf{u} - \mathbf{U}) \colon \nabla_x(\nabla_x \Delta_x^{-1}(b(\varrho) - \langle b(\varrho)\rangle)\, \mathrm{d}x \\ &\leq C(\delta, \bar{\varrho})\eta(t)\mathcal{E}(t) + \delta\left(C_p \int_{\mathbb{T}^d} \mathbb{D}_0(\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) \colon \mathbb{D}_0(\nabla_x \mathbf{u} - \nabla_x \mathbf{U})\, \mathrm{d}x\, \mathrm{d}t\right).\end{aligned}$$

Analogously, we have

$$\begin{aligned}&\int_{\mathbb{T}^d} \varrho \mathbf{U} \otimes (\mathbf{u} - \mathbf{U}) \colon \nabla_x(\nabla_x \Delta_x^{-1}(b(\varrho) - \langle b(\varrho)\rangle)\, \mathrm{d}x \\ &\leq C(\bar{\varrho}, \mathbf{U})\left(\int_{\mathbb{T}^d} \varrho|\mathbf{u} - \mathbf{U}|^2\, \mathrm{d}x + \int_{\mathbb{T}^d} |b(\varrho)|^2\, \mathrm{d}x\right).\end{aligned}$$

We notice that both the terms in the right hand side of the last inequality are dominated by the relative energy.

By a similar argument, we get

$$\int_{\mathbb{T}^d} (\varrho - r)\mathbf{U} \otimes \mathbf{U} \colon \nabla_x(\nabla_x \Delta_x^{-1}(b(\varrho) - \langle b(\varrho)\rangle)\, \mathrm{d}x \leq \eta(t)\mathcal{E}(t) + \int_{\mathbb{T}^d} |b(\varrho)|^2\, \mathrm{d}x$$

and

$$\int_{\mathbb{T}^d} r\mathbf{U} \otimes \mathbf{U} \colon \nabla_x(\nabla_x \Delta_x^{-1}(b(\varrho) - \langle b(\varrho)\rangle)\, \mathrm{d}x \leq \eta(t)\mathcal{E}(t).$$

Thus, collecting all estimates of the term $\mathcal{T}_4$, we obtain

$$\begin{aligned}\mathcal{T}_4 &= \int_{\mathbb{T}^d} \varrho \mathbf{u} \otimes \mathbf{u} : \nabla_x (\nabla_x \Delta_x^{-1} (b(\varrho) - \langle b(\varrho) \rangle)) \, \mathrm{d}x \\ &\leq C(\delta, \bar{\varrho}, \mathbf{U}, E_0) \, \eta(t) \mathcal{E}(t) \\ &\quad + \delta \left(C_p \int_{\mathbb{T}^d} \mathbb{D}_0(\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) \colon \mathbb{D}_0(\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) \, \mathrm{d}x \, \mathrm{d}t \right).\end{aligned}$$

For the term $\mathcal{T}_5$, at first we rewrite

$$\mathbb{S}(\nabla_x \mathbf{u}) = (\mathbb{S}(\nabla_x \mathbf{u}) - \mathbb{S}(\nabla_x \mathbf{U})) + (\mathbb{S}(\nabla_x \mathbf{U})).$$

Then, with the help of the Young's inequality and the generalized Korn-Poincaré inequality (1.1.13), we get

$$\begin{aligned}&\int_{\mathbb{T}^d} \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x (\nabla_x \Delta_x^{-1} (b(\varrho) - \langle b(\varrho) \rangle)) \, \mathrm{d}x \\ &\leq C(\delta) \eta(t) \mathcal{E}(t) + \delta \left(C_p \int_{\mathbb{T}^d} \mathbb{D}_0(\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) \colon \mathbb{D}_0(\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) \, \mathrm{d}x \, \mathrm{d}t \right).\end{aligned}$$

For the term $\mathcal{T}_6$, we consider

$$\begin{aligned}\varrho \mathbf{u} \cdot \nabla_x \Delta_x^{-1} \mathrm{div}_x (b(\varrho) \mathbf{u}) &= \varrho (\mathbf{u} - \mathbf{U}) \cdot \nabla_x \Delta_x^{-1} \mathrm{div}_x (b(\varrho)(\mathbf{u} - \mathbf{U})) \\ &\quad + \varrho (\mathbf{u} - \mathbf{U}) \cdot \nabla_x \Delta_x^{-1} \mathrm{div}_x (b(\varrho) \mathbf{U}) \\ &\quad + \varrho \mathbf{U} \cdot \nabla_x \Delta_x^{-1} \mathrm{div}_x (b(\varrho)(\mathbf{u} - \mathbf{U})) + \varrho \mathbf{U} \cdot \nabla_x \Delta_x^{-1} \mathrm{div}_x (b(\varrho) \mathbf{U})\end{aligned}$$

Therefore, we deduce that

$$\begin{aligned}&\int_{\mathbb{T}^d} \varrho (\mathbf{u} - \mathbf{U}) \cdot \nabla_x \Delta_x^{-1} \mathrm{div}_x (b(\varrho)(\mathbf{u} - \mathbf{U})) \, \mathrm{d}x \\ &\leq C(\bar{\varrho}) \| \sqrt{\varrho} (\mathbf{u} - \mathbf{U}) \|_{L^2(\mathbb{T}^d)} \| \mathbf{u} - \mathbf{U} \|_{L^6(\mathbb{T}^d)} \| b(\varrho) \|_{L^3(\mathbb{T}^d)} \\ &\leq C(\bar{\varrho}, \delta) \| \sqrt{\varrho} | \mathbf{u} - \mathbf{U} \|_{L^2(\mathbb{T}^d)}^2 \| b(\varrho) \|_{L^3(\mathbb{T}^d)}^2 + \delta \| | \mathbf{u} - \mathbf{U} \|_{W^{1,2}(\mathbb{T}^d)}^2 .\end{aligned}$$

In a similar way, we estimate the other terms in $\mathcal{T}_6$ and obtain

$$\begin{aligned}\mathcal{T}_6 &= \int_{\mathbb{T}^d} \varrho \mathbf{u} \cdot \nabla_x \Delta_x^{-1} \mathrm{div}_x (b(\varrho) \mathbf{u}) \, \mathrm{d}x \\ &\leq C(\delta) \eta(t) \mathcal{E}(t) + \delta \left(C \int_0^\tau \int_{\mathbb{T}^d} \mathbb{D}_0(\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) \colon \mathbb{D}_0(\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) \, \mathrm{d}x \, \mathrm{d}t \right)\end{aligned}$$

Now the only remaining term to be estimated from $\mathcal{R}_2$ is

$$\mathcal{T}_7 = \int_{\mathbb{T}^d} \varrho \mathbf{u} \cdot \nabla_x \Delta_x^{-1} \big((b'(\varrho) \varrho - b(\varrho)) \mathrm{div}_x \mathbf{u} - \langle (b'(\varrho) \varrho - b(\varrho) \mathrm{div}_x \mathbf{u} \rangle \big) \, \mathrm{d}x.$$

The first observation is

$$\int_{\mathbb{T}^d} \varrho\mathbf{u}\cdot\nabla_x\Delta_x^{-1}\big((b'(\varrho)\varrho-b(\varrho))\mathrm{div}_x\mathbf{u}-\langle(b'(\varrho)\varrho-b(\varrho)\mathrm{div}_x\mathbf{u}\rangle\big)\;\mathrm{d}x$$
$$=-\int_{\mathbb{T}^d}\mathrm{div}_x\Delta_x^{-1}(\varrho\mathbf{u}-\langle\varrho\mathbf{u}\rangle)(b'(\varrho)\varrho-b(\varrho))\mathrm{div}_x\mathbf{u}.$$

Again, we split the integral into several in the following way:

$$\begin{aligned}&\int_{\mathbb{T}^d}\mathrm{div}_x\Delta_x^{-1}(\varrho\mathbf{u}-\langle\varrho\mathbf{u}\rangle)(b'(\varrho)\varrho-b(\varrho))\mathrm{div}_x\mathbf{u}\\
&=\int_{\mathbb{T}^d}\mathrm{div}_x\Delta_x^{-1}(\varrho\mathbf{u}-\langle\varrho\mathbf{u}\rangle)(b'(\varrho)\varrho-b(\varrho))\mathrm{div}_x\mathbf{U}\\
&+\int_{\mathbb{T}^d}\mathrm{div}_x\Delta_x^{-1}\left((\varrho\mathbf{u}-\langle\varrho\mathbf{u}\rangle)-(\varrho\mathbf{U}-\langle\varrho\mathbf{U}\rangle)\right)(b'(\varrho)\varrho-b(\varrho))(\mathrm{div}_x\mathbf{u}-\mathrm{div}_x\mathbf{U})\\
&+\int_{\mathbb{T}^d}\mathrm{div}_x\Delta_x^{-1}(\varrho\mathbf{U}-\langle\varrho\mathbf{U}\rangle)(b'(\varrho)\varrho-b(\varrho))(\mathrm{div}_x\mathbf{u}-\mathrm{div}_x\mathbf{U}).\end{aligned}$$

We notice the importance of considering $\beta\geq 3$ in (3.3.26) in the following expression:

$$\begin{aligned}&\int_{\mathbb{T}^d}\mathrm{div}_x\Delta_x^{-1}(\varrho(\mathbf{u}-\mathbf{U})-\langle\varrho(\mathbf{u}-\mathbf{U})\rangle)(b'(\varrho)\varrho-b(\varrho))(\mathrm{div}_x\mathbf{u}-\mathrm{div}_x\mathbf{U})\\
&\leq\|\mathrm{div}_x\Delta_x^{-1}(\varrho(\mathbf{u}-\mathbf{U})-\langle\varrho(\mathbf{u}-\mathbf{U})\rangle)\|_{L^6(\mathbb{T}^d)}\\
&\qquad\times\Big(\|b'(\varrho)\varrho-b(\varrho)\|_{L^3(\mathbb{T}^d)}\|\mathrm{div}_x\mathbf{u}-\mathrm{div}_x\mathbf{U}\|_{L^2(\mathbb{T}^d)}\Big)\\
&\leq C(\delta)\|\sqrt{\varrho}(\mathbf{u}-\mathbf{U})\|^2_{L^2(\mathbb{T}^d)}\|b'(\varrho)\varrho-b(\varrho)\|^2_{L^3(\mathbb{T}^d)}+\delta\|\mathrm{div}_x\mathbf{u}-\mathrm{div}_x\mathbf{U}\|^2_{L^2(\mathbb{T}^d)}.\end{aligned}$$

It is necessary because we need a uniform bound of the term $\|b'(\varrho)\varrho-b(\varrho)\|^2_{L^3(\Omega)}$. Finally, we can estimate the other remaining terms and infer that

$$\begin{aligned}\mathcal{T}_7&=\int_{\mathbb{T}^d}\varrho\mathbf{u}\cdot\nabla_x\Delta_x^{-1}\big((b'(\varrho)\varrho-b(\varrho))\mathrm{div}_x\mathbf{u}-\langle(b'(\varrho)\varrho-b(\varrho)\mathrm{div}_x\mathbf{u}\rangle\big)\;\mathrm{d}x\\
&\leq\eta(t)\mathcal{E}(t)+\delta C_p\Bigg(\int_{\mathbb{T}^d}\mathbb{D}_0(\nabla_x\mathbf{u}-\nabla_x\mathbf{U})\colon\mathbb{D}_0(\nabla_x\mathbf{u}-\nabla_x\mathbf{U})\;\mathrm{d}t\\
&+\int_{\mathbb{T}^d}|\mathrm{div}_x\mathbf{u}-\mathrm{div}_x\mathbf{U}|^2\;\mathrm{d}x\Bigg).\end{aligned}$$

Thus, by combining all the estimates, we get

$$\begin{aligned}\int_0^\tau\mathcal{R}_2(t)\;\mathrm{d}t\;&\leq\int_0^\tau\eta(t)\mathcal{E}(t)\;\mathrm{d}t\\
&+\delta C_p\Big(C\int_0^\tau\int_{\mathbb{T}^d}\mathbb{D}_0(\nabla_x\mathbf{u}-\nabla_x\mathbf{U})\colon\mathbb{D}_0(\nabla_x\mathbf{u}-\nabla_x\mathbf{U})\;\mathrm{d}x\;\mathrm{d}t\\
&\quad+\int_0^\tau\int_{\mathbb{T}^d}|\mathrm{div}_x\mathbf{u}-\mathrm{div}_x\mathbf{U}|^2\;\mathrm{d}x\;\mathrm{d}t\;\Big)\end{aligned}\tag{3.3.34}$$

Now for initial data $\varrho_0 = r_0 \in [\alpha_0, \bar{\varrho} - \alpha_0]$, we have $b(\varrho_0) \equiv 0$. Hence, for $\mathcal{R}_3$, we get

$$\mathcal{R}_3(\tau) \leq \frac{1}{4}\int_{\mathbb{T}^d} \varrho|\mathbf{u}-\mathbf{U}|^2(\tau,\cdot)\ \mathrm{d}x + \frac{1}{2}\int_{\varrho \geq \bar{\varrho}-\alpha_1} (H(\varrho) - H(r) - H'(r)(\varrho - r))\ \mathrm{d}x. \tag{3.3.35}$$

We collect all the estimates of $\mathcal{R}_1$, $\mathcal{R}_3$ and $\mathcal{R}_3$ from (3.3.32), (3.3.34) and (3.3.35), respectively and obtain

$$\begin{aligned}
\mathcal{E}(\tau) + &\int_0^\tau \int_{\mathbb{T}^d} \mathbb{S}(\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) : (\nabla_x \mathbf{u} - \nabla_x \mathbf{U})\ \mathrm{d}x\ \mathrm{d}t\ + \int_0^\tau \int_{\mathbb{T}^d} b(\varrho)h(\varrho) \\
&\leq \mathcal{E}_0 + \int_0^\tau \eta(t)\mathcal{E}(t)\ \mathrm{d}t\ + \frac{1}{8}\int_0^\tau \int_{\mathbb{T}^d} b(\varrho)h(\varrho)\ \mathrm{d}x\ \mathrm{d}t \\
&+ \delta C_p \Bigg(\int_0^\tau \int_{\mathbb{T}^d} \mathbb{D}_0(\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) \colon \mathbb{D}_0(\nabla_x \mathbf{u} - \nabla_x \mathbf{U})\ \mathrm{d}x\ \mathrm{d}t \\
&\qquad + \int_0^\tau \int_{\mathbb{T}^d} |\mathrm{div}_x \mathbf{u} - \mathrm{div}_x \mathbf{U}|^2\ \mathrm{d}x\ \mathrm{d}t \Bigg),
\end{aligned}$$

for a $\eta \in L^1(0,T)$.

We use the following identity:

$$\mathbb{S}(\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) : (\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) = \mu \mathbb{D}_0(\mathbf{u} - \mathbf{U}) \colon \mathbb{D}_0(\mathbf{u} - \mathbf{U}) + \lambda |\mathrm{div}_x \mathbf{u} - \mathrm{div}_x \mathbf{U}|^2,$$

and, choose δ small, depending on μ, λ such that we have

$$\begin{aligned}
\mathcal{E}(\tau) &+ \frac{\mu}{4}\int_0^\tau \int_{\mathbb{T}^d} \mathbb{D}_0(\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) : \mathbb{D}_0(\nabla_x \mathbf{u} - \nabla_x \mathbf{U})\ \mathrm{d}x\ \mathrm{d}t \\
&+ \frac{\lambda}{4}\int_0^\tau \int_{\mathbb{T}^d} |\mathrm{div}_x \mathbf{u} - \mathrm{div}_x \mathbf{U}|^2\ \mathrm{d}x\ \mathrm{d}t\ + \frac{1}{4}\int_0^\tau \int_{\mathbb{T}^d} b(\varrho)h(\varrho)\ \mathrm{d}x\ \mathrm{d}t \\
&\leq \mathcal{E}_0 + \int_0^\tau \eta(t)\mathcal{E}(t)\ \mathrm{d}t\ ,
\end{aligned} \tag{3.3.36}$$

where $\eta \in L^1(0,T)$.

Proof of the Theorem 3.3.13: From the hypothesis of the theorem, we have

$$(r_0, r_0\mathbf{U}_0) = (\varrho_0, (\varrho\mathbf{u})_0).$$

This concludes $\mathcal{E}_0 = 0$. Since $b \geq 0$, as a consequence of Grönwall's lemma, we conclude that

$$\mathcal{E}(t) = 0 \text{ for a.e. } t \in (0,T).$$

This ends the proof of the theorem 3.3.13.

3.3.3 Generalized weak–strong uniqueness for a non-monotone isentropic pressure

Now we are in the last part of this chapter. Here we consider the *renormalized dissipative measure-valued*(rDMV) solutions of the compressible Navier–Stokes system

(2.2.1)-(2.2.3), following the Definition 2.5.3 with no-slip boundary condition (2.2.5). We consider a non-monotone barotropic pressure law, that follows (2.1.36) or (2.1.38).

Now we state the main result that describes a generalized weak-strong uniqueness result for the Navier–Stokes system.

Theorem 3.3.15. *Let $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$ be a Lipschitz bounded domain. Suppose the pressure p satisfies* (2.1.36) *or* (2.1.38)*, with $q \in C_c^1(0, \infty)$ and $\gamma \geq 1$. Let $\{\mathcal{V}_{t,x}, \mathcal{D}\}$ be a rDMV solution to the Navier–Stokes system* (2.2.1)-(2.2.3) *in $(0,T)\times\Omega$, with initial state represented by $\mathcal{V}_0$ and no-slip boundary condition, as defined in Definition* (2.5.3)*. Let (r, U) be a strong solution to the same system in $(0,T)\times\Omega$ with initial data $(r_0, \mathbf{U}_0)$ satisfying $r_0 > 0$ in Ω. We assume that the strong solution belongs to the class*

$$r,\ \nabla_x r,\ \mathbf{U},\ \nabla_x \mathbf{U} \in C([0,T] \times \overline{\Omega}),\ \partial_t \mathbf{U} \in L^2(0,T; C(\overline{\Omega}; \mathbb{R}^d)),\ r > 0,\ \mathbf{U}|_{\partial\Omega} = 0. \tag{3.3.37}$$

Then there exists a constant $\Lambda = \Lambda(T)$, depending only on the norms of r, r^{-1}, $\mathbf{U}$, and the initial data $(r_0, \mathbf{U}_0)$ in the aforementioned spaces, such that

$$\begin{aligned}
&\int_\Omega \left[\left\langle \mathcal{V}_{\tau,x}; \frac{1}{2}\tilde{\varrho}|\tilde{\mathbf{u}} - \mathbf{U}|^2 + H(\tilde{\varrho}) - H(r) - H'(r)(\tilde{\varrho} - r) \right\rangle \right] dx + \mathcal{D}(\tau) \\
&\leq \Lambda(T) \int_\Omega \left[\left\langle \mathcal{V}_{0,x}; \frac{1}{2}\tilde{\varrho}|\tilde{\mathbf{u}} - \mathbf{U}_0(x)|^2 + H(\tilde{\varrho}) - H(r_0(x)) - H'(r_0(x))(\tilde{\varrho} - r_0(x)) \right\rangle \right] dx,
\end{aligned}$$

for a.e. $\tau \in (0,T)$, $\mathbf{U}_0(x) = \mathbf{U}(0,x)$ and $r_0(x) = r(0,x)$ for $x \in \Omega$. In particular, if the initial states coincide, i.e.

$$\mathcal{V}_{0,x} = \delta_{\{r_0(x), \mathbf{U}_0(x)\}},\ \textit{for a.e. } x \in \Omega \tag{3.3.38}$$

then $\mathcal{D} = 0$, and

$$\mathcal{V}_{\tau,x} = \delta_{\{r(\tau,x), \mathbf{U}(\tau,x), \nabla_x \mathbf{U}(\tau,x)\}}\ \textit{for a.e. } (\tau, x) \in (0,T) \times \Omega.$$

Since, In initial energy is dependent on the density and the velocity it enough to consider $\mathcal{V}_{0,x}$ as described in (3.3.38).

We recall that considering suitable test functions we already have the relative energy inequality, see Lemma 3.2.6. From the hypothesis of the Theorem 3.3.15 we have assumption $r_0(x) > 0$ in Ω. Using the above observation on the strong solution $(r, \mathbf{U})$ we rewrite the Lemma 3.2.6 in the following way:

Lemma 3.3.16. *Let $(\mathcal{V}, \mathcal{D})$ be a rDMV solution that follows the Definition 2.5.3. Suppose$(r, \mathbf{U})$ is a strong solution in the class* (3.3.37) *with initial data $(r_0, \mathbf{U}_0)$ such* *that $r_0 > 0$ in Ω. Then the following inequality is true for a.e. $\tau \in (0,T)$:*

$$
\begin{aligned}
&\mathcal{E}_{mv}(\tau) + \mu \int_0^\tau \int_\Omega \langle \mathcal{V}_{t,x}; \mathbb{D}_0(\widetilde{\mathbb{D}_{\mathbf{v}}} - \nabla_x \mathbf{U}) : \mathbb{D}_0(\widetilde{\mathbb{D}_{\mathbf{v}}} - \nabla_x \mathbf{U}) \rangle \ dx \ dt \\
&\quad + \lambda \int_0^\tau \int_\Omega \langle \mathcal{V}_{t,x}; |\, Tr(\widetilde{\mathbb{D}_{\mathbf{v}}}) - div_x \mathbf{U}|^2 \rangle \ dx \ dt \ + \mathcal{D}(\tau) \\
&\quad \leq \int_\Omega \left[\left\langle \mathcal{V}_{0,x}; \frac{1}{2} \tilde{\varrho} |\tilde{\mathbf{u}} - \mathbf{U}_0|^2 + H(\tilde{\varrho}) - H(r_0) - H'(r_0)(\tilde{\varrho} - r_0) \right\rangle \right] dx \\
&\quad + \int_0^\tau \int_\Omega \langle \mathcal{V}_{t,x}; (\tilde{\varrho}(\tilde{\mathbf{u}} - \mathbf{U}) \cdot \nabla_x) \mathbf{U} \cdot (\mathbf{U} - \tilde{\mathbf{u}}) \rangle \ dx \ dt \\
&\quad + \int_0^\tau \int_\Omega \langle \mathcal{V}_{t,x}; (\tilde{\varrho} - r)(\mathbf{U} - \tilde{\mathbf{u}}) \rangle \cdot \frac{1}{r} \big(div_x \mathbb{S}(\nabla_x \mathbf{U}) - \nabla_x q(r) \big) \ dx \ dt \\
&\quad + \int_0^\tau \int_\Omega \langle \mathcal{V}_{t,x}; (-h(\tilde{\varrho}) + h(r) + h'(r)(\tilde{\varrho} - r)) \rangle \ div_x \mathbf{U} \ \ dx \ dt \\
&\quad + \int_0^\tau \int_\Omega \big\langle \mathcal{V}_{t,x}; \big(q(\tilde{\varrho}) - q(r) \big) \big(Tr(\widetilde{\mathbb{D}_{\mathbf{v}}}) - div_x \mathbf{U} \big) \big\rangle \ dx \ dt \\
&\quad + \|\mathbf{U}\|_{C^1([0,T]\times\overline{\Omega};\mathbb{R}^N)} \int_0^\tau \xi(t) \mathcal{D}(t) \ dt \ = \Sigma_{i=1}^6 \mathcal{I}_i.
\end{aligned}
\tag{3.3.39}
$$

Estimates for the remainder terms

To simplify the calculation, we assume

$$\partial_t \mathbf{U} \in C([0,T] \times \overline{\Omega}; R^d). \tag{3.3.40}$$

First, we note that for our assumption of the pressure law, the Lemma 3.3.2 and 3.3.3 remain true in this case. To obtain our desired result, let us estimate the terms $\mathcal{I}_i$, for $i = 1, \cdots, 6$ in (3.3.39).

Remainder term $\mathcal{I}_2$: We have

$$|\mathcal{I}_2| \leq \|\mathbf{U}\|_{C^1([0,T]\times\overline{\Omega};\mathbb{R}^d)} \int_0^\tau \mathcal{E}_{mv}(t) \ \mathrm{d}t \ . \tag{3.3.41}$$

Remainder term $\mathcal{I}_4$: Similarly, using the Lemma 3.3.3 we get

$$|\mathcal{I}_4| \leq C \int_0^\tau \mathcal{E}_{mv}(t) \ \mathrm{d}t \ . \tag{3.3.42}$$

Remainder term $\mathcal{I}_3$: Here we introduce a function $\psi \in C_c^\infty(0,\infty)$ with $0 \leq \psi \leq 1$, such that

$$\psi(\tilde{\varrho}) = 1 \text{ for } \tilde{\varrho} \in (r_1, r_2),$$

where r_1, r_2 is related with the Lemma 3.3.2 and (3.3.3). Without loss of generality we assume $r_1 \leq \frac{1}{2} \inf r$, $r_2 \geq 2 \times \sup r$ and $\mathrm{supp}(q) \subset (r_1, r_2)$. We rewrite

$$
\begin{aligned}
&\langle \mathcal{V}_{t,x}; (\tilde{\varrho} - r)(\mathbf{U} - \tilde{\mathbf{u}}) \rangle \\
&= \langle \mathcal{V}_{t,x}; \psi(\tilde{\varrho})(\tilde{\varrho} - r)(\mathbf{U} - \tilde{\mathbf{u}}) \rangle + \langle \mathcal{V}_{t,x}; (1 - \psi(\tilde{\varrho}))(\tilde{\varrho} - r)(\mathbf{U} - \tilde{\mathbf{u}}) \rangle.
\end{aligned}
$$

Consequently we obtain

$$\langle \mathcal{V}_{t,x}; \psi(\tilde{\varrho})(\tilde{\varrho}-r)(\mathbf{U}-\tilde{\mathbf{u}})\rangle \leq \frac{1}{2}\left\langle \mathcal{V}_{t,x}; \frac{\psi^2(\tilde{\varrho})}{\sqrt{\tilde{\varrho}}}(\tilde{\varrho}-r)^2\right\rangle + \frac{1}{2}\left\langle \mathcal{V}_{t,x}; \frac{\psi^2(\tilde{\varrho})}{\sqrt{\tilde{\varrho}}}\tilde{\varrho}|\mathbf{U}-\tilde{\mathbf{u}}|^2\right\rangle. \tag{3.3.43}$$

Now using that ψ is compactly supported in $(0,\infty)$ and Lemma 3.3.2 we conclude that,

$$\begin{aligned} &\int_0^\tau \int_\Omega \langle \mathcal{V}_{t,x}; \psi(\tilde{\varrho})(\tilde{\varrho}-r)(\mathbf{U}-\tilde{\mathbf{u}})\rangle \cdot \frac{1}{r}\big(\mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{U}) - \nabla_x q(r)\big) \, \mathrm{d}x \, \mathrm{d}t \\ &\leq \left\| \frac{1}{r}\big(\mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{U}) - \nabla_x q(r)\big)\right\|_{C([0,T]\times\overline{\Omega};\mathbb{R}^d)} \int_0^\tau \mathcal{E}_{mv}(t) \, \mathrm{d}t \, . \end{aligned} \tag{3.3.44}$$

We rewrite $1-\psi(\tilde{\varrho}) = w_1(\tilde{\varrho}) + w_2(\tilde{\varrho})$, where $\mathrm{supp}(w_1) \subset [0, r_1)$ and $\mathrm{supp}(w_2) \subset (r_2, \infty)$,

$$\langle \mathcal{V}_{t,x}; (1-\psi(\tilde{\varrho}))(\tilde{\varrho}-r)(\mathbf{U}-\tilde{\mathbf{u}})\rangle = \langle \mathcal{V}_{t,x}; (w_1(\tilde{\varrho})+w_2(\tilde{\varrho}))(\tilde{\varrho}-r)(\mathbf{U}-\tilde{\mathbf{u}})\rangle.$$

For $\delta > 0$ we obtain,

$$\langle \mathcal{V}_{t,x}; w_1(\tilde{\varrho})(\tilde{\varrho}-r)(\mathbf{U}-\tilde{\mathbf{u}})\rangle \leq C(\delta)\langle \mathcal{V}_{t,x}; w_1^2(\tilde{\varrho})(\tilde{\varrho}-r)^2\rangle + \delta\langle \mathcal{V}_{t,x}; |\mathbf{U}-\tilde{\mathbf{u}}|^2\rangle.$$

Hence, the generalized Korn–Poincaré inequality (2.5.15) implies

$$\begin{aligned} &\int_0^\tau \int_\Omega \langle \mathcal{V}_{t,x}; w_1(\tilde{\varrho})(\tilde{\varrho}-r)(\mathbf{U}-\tilde{\mathbf{u}})\rangle \, \mathrm{d}x \, \mathrm{d}t \\ &\qquad \leq C \int_0^\tau \mathcal{E}_{mv}(t) \, \mathrm{d}t \; + \delta \int_0^\tau \int_\Omega \langle \mathcal{V}_{t,x}; |\mathbb{D}_0(\widetilde{\mathbb{D}_{\mathbf{v}}}) - \mathbb{D}_0(\nabla_x \mathbf{U})|^2\rangle \, \mathrm{d}x \, \mathrm{d}t \, . \end{aligned}$$

We know $w_2(\tilde{\varrho}) > 0$ and, $|\tilde{\varrho} - r| \leq 2\tilde{\varrho}$ if $\tilde{\varrho} > 2r_2$. Now using Youngs inequality (1.1.1), we obtain

$$\langle \mathcal{V}_{t,x}; w_2(\tilde{\varrho})(\tilde{\varrho}-r)(\mathbf{U}-\tilde{\mathbf{u}})\rangle \leq C\langle \mathcal{V}_{t,x}; w_2(\tilde{\varrho})(\tilde{\varrho} + \tilde{\varrho}|\mathbf{U}-\tilde{\mathbf{u}}|^2)\rangle.$$

It yields

$$\int_0^\tau \int_\Omega \langle \mathcal{V}_{t,x}; w_2(\tilde{\varrho})(\tilde{\varrho}-r)(\mathbf{U}-\tilde{\mathbf{u}})\rangle \, \mathrm{d}x \, \mathrm{d}t \; \leq C(r) \int_0^\tau \mathcal{E}_{mv}(t) \, \mathrm{d}t \, .$$

We take δ small enough and combine all the above terms to obtain

$$|\mathcal{I}_3| \leq C \int_0^\tau \mathcal{E}_{mv}(t) \; \mathrm{d}t \; + \frac{\mu}{4} \int_0^\tau \int_\Omega \langle \mathcal{V}_{t,x}; |\mathbb{D}_0(\widetilde{\mathbb{D}_{\mathbf{v}}}) - \mathbb{D}_0(\nabla_x \mathbf{U})|^2\rangle \, \mathrm{d}x \, \mathrm{d}t \, . \tag{3.3.45}$$

Remainder term $\mathcal{I}_5$: From our choice of q and ψ we get

$$q(\tilde{\varrho}) - q(r) = \psi(\tilde{\varrho})(q(\tilde{\varrho}) - q(r)) - (1-\psi(\tilde{\varrho}))q(r).$$

Since q is compactly supported C^1 function, we have

$$|q(\tilde{\varrho}) - q(r)| \le C\big(\psi(\tilde{\varrho})|\tilde{\varrho} - r| + (1 - \psi(\tilde{\varrho}))\big).$$

As a direct consequence of Young's inequality (1.1.1) with $\delta > 0$ we deduce that

$$\begin{aligned}
&\big\langle \mathcal{V}_{t,x}; \big(q(\tilde{\varrho}) - q(r)\big)(\mathrm{Tr}(\widetilde{\mathbb{D}_{\mathbf{v}}}) - \mathrm{div}_x \mathbf{U})\big\rangle \\
&\le \frac{C}{4\delta}\big\langle \mathcal{V}_{t,x}; \big(\psi(\tilde{\varrho})(\tilde{\varrho} - r)^2 + (1 - \psi(\tilde{\varrho}))\big)\big\rangle + \delta\big\langle \mathcal{V}_{t,x}; |\mathrm{Tr}(\widetilde{\mathbb{D}_{\mathbf{v}}}) - \mathrm{div}_x \mathbf{U}|^2\big\rangle.
\end{aligned}$$

Further using Lemma 3.3.2 and choosing an appropriate δ, we infer

$$|\mathcal{I}_5| \le C \int_0^\tau \mathcal{E}_{mv}(t) \ \mathrm{d}t \ + \frac{\lambda}{2}\int_0^\tau \int_\Omega \langle \mathcal{V}_{t,x}; |\mathrm{Tr}(\widetilde{\mathbb{D}_{\mathbf{v}}}) - \mathrm{div}_x \mathbf{U}|^2\rangle \ \mathrm{d}x \ \mathrm{d}t \, . \tag{3.3.46}$$

Remainder term $\mathcal{I}_6$: From the definition of defect measure we know

$$|\langle r^M(\tau); \nabla_x \boldsymbol{\varphi}\rangle_{\{\mathcal{M}(\overline{\Omega};\mathbb{R}^{d\times d}), C(\overline{\Omega};\mathbb{R}^{d\times d})\}}| \le \xi(\tau)\mathcal{D}(\tau)\|\boldsymbol{\varphi}\|_{C^1(\overline{\Omega})}.$$

This implies

$$|\mathcal{I}_6| \le C \int_0^\tau \xi(t)\mathcal{D}(t) \ \mathrm{d}t \, . \tag{3.3.47}$$

Proof of the Theorem 3.3.15:

Considering the above discussion, additional assumption (3.3.40) and combining all estimates of $\mathcal{I}_i$ for $i = 2, 3, 4, 5, 6$, we have

$$\begin{aligned}
&\mathcal{E}_{mv}(\tau) + \frac{\mu}{4}\int_0^\tau \int_\Omega \langle \mathcal{V}_{t,x}; \mathbb{D}_0(\widetilde{\mathbb{D}_{\mathbf{v}}} - \nabla_x \mathbf{U}) : \mathbb{D}_0(\widetilde{\mathbb{D}_{\mathbf{v}}} - \nabla_x \mathbf{U})\rangle \ \mathrm{d}x \ \mathrm{d}t \\
&\quad + \frac{\lambda}{2}\int_0^\tau \int_\Omega \langle \mathcal{V}_{t,x}; |\mathrm{Tr}(\widetilde{\mathbb{D}_{\mathbf{v}}}) - \mathrm{div}_x \mathbf{U}|^2\rangle \ \mathrm{d}x \ \mathrm{d}t \ + \mathcal{D}(\tau) \\
&\quad \le \int_\Omega \Big[\big\langle \mathcal{V}_{0,x}; \frac{1}{2}\tilde{\varrho}|\tilde{\mathbf{u}} - \mathbf{U}_0|^2 + H(\tilde{\varrho}) - H(r_0) - H'(r_0)(\tilde{\varrho} - r_0)\big\rangle\Big] \ \mathrm{d}x \\
&\quad + C(r, \mathbf{U}, q)\int_0^\tau \mathcal{E}_{mv}(t) \ \mathrm{d}t \ + \int_0^\tau \xi(t)\mathcal{D}(t) \ \mathrm{d}t \, .
\end{aligned} \tag{3.3.48}$$

Now applying Grönwall's lemma, we conclude

$$\begin{aligned}
&\int_\Omega \Big[\Big\langle \mathcal{V}_{\tau,x}; \frac{1}{2}\tilde{\varrho}|\tilde{\mathbf{u}} - \mathbf{U}|^2 + H(\tilde{\varrho}) - H(r) - H'(r)(\tilde{\varrho} - r)\Big\rangle\Big] \ \mathrm{d}x + \mathcal{D}(\tau) \\
&\le \Lambda(T)\int_\Omega \Big[\Big\langle \mathcal{V}_{0,x}; \frac{1}{2}\tilde{\varrho}|\tilde{\mathbf{u}} - \mathbf{U}_0(x)|^2 + H(\tilde{\varrho}) - H(r_0(x)) - H'(r_0(x))(\tilde{\varrho} - r_0(x))\Big\rangle\Big] \mathrm{d}x,
\end{aligned}$$

for a.e. $\tau \in [0, T]$.

Remark 3.3.17. For simplicity of the proof we assume (3.3.40). If we stick to only (3.3.37), then we have $\int_0^\tau \eta_{(r,\mathbf{U},q)}(t)\mathcal{E}_{mv}(t) \ \mathrm{d}t$, where $\eta_{(r,\mathbf{U},q)} \in L^1(0, T)$ instead of the term $c(r, \mathbf{U}, q)\int_0^\tau \mathcal{E}_{mv}(t) \ \mathrm{d}t$ in (3.3.48).

Navier Slip boundary condition

A possible adaptation in the context of Navier slip boundary condition(2.2.6) is possible. Here we have the following proposition:

Proposition 3.3.18. *Let $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$ be a Lipschitz bounded domain. Suppose the pressure p satisfies* (2.1.36) *or* (2.1.38)*, with $q \in C_c^1(0, \infty)$. Let $\{\mathcal{V}_{t,x}, \mathcal{D}\}$ be a rDMV solution to the Navier–Stokes system* (2.2.1)-(2.2.3) *in $(0, T) \times \Omega$, with initial state represented by $\mathcal{V}_0$ and Navier-slip boundary condition, as defined in Definition* (2.5.7)*. Let (r, U) be a strong solution to the same system in $(0, T) \times \Omega$ with initial data $(r_0, \mathbf{U}_0)$ satisfying $r_0 > 0$ in Ω. We assume that the strong solution belongs to the class*

$$r,\ \nabla_x r,\ \mathbf{U},\ \nabla_x \mathbf{U} \in C([0,T] \times \overline{\Omega}),\ \partial_t \mathbf{U} \in L^2(0,T; C(\overline{\Omega}; \mathbb{R}^d)),\ r > 0,\ \mathbf{U}|_{\partial\Omega} = 0. \tag{3.3.49}$$

Then there exists a constant $\Lambda = \Lambda(T)$, depending only on the norms of r, r^{-1}, $\mathbf{U}$, and the initial data $(r_0, \mathbf{U}_0)$ in the aforementioned spaces, such that

$$\int_\Omega \left[\left\langle \mathcal{V}_{\tau,x}; \frac{1}{2}\tilde{\varrho}|\tilde{\mathbf{u}} - \mathbf{U}|^2 + H(\tilde{\varrho}) - H(r) - H'(r)(\tilde{\varrho} - r) \right\rangle \right] dx + \mathcal{D}(\tau)$$
$$\leq \Lambda(T) \int_\Omega \left[\left\langle \mathcal{V}_{0,x}; \frac{1}{2}\tilde{\varrho}|\tilde{\mathbf{u}} - \mathbf{U}_0(x)|^2 + H(\tilde{\varrho}) - H(r_0(x)) - H'(r_0(x))(\tilde{\varrho} - r_0(x)) \right\rangle \right] dx,$$

for a.e. $\tau \in (0, T)$, $\mathbf{U}_0(x) = \mathbf{U}(0, x)$ and $r_0(x) = r(0, x)$ for $x \in \Omega$. In particular, if the initial states coincide, i.e.

$$\mathcal{V}_{0,x} = \delta_{\{r_0(x), \mathbf{U}_0(x)\}},\ \textit{for a.e. } x \in \Omega$$

then $\mathcal{D} = 0$, and

$$\mathcal{V}_{\tau,x} = \delta_{\{r(\tau,x), \mathbf{U}(\tau,x), \nabla_x \mathbf{U}(\tau,x)\}}\ \textit{for a.e. } (\tau, x) \in (0, T) \times \Omega.$$

The proof is almost similar to the proof of the Theorem 3.3.15. We have the relative energy inequality from the Lemma 3.2.7. We need to obtain a lemma similar to Lemma 3.3.16.

Now, if we compare the Definition 2.5.3 for the no-slip boundary condition with the Definition 2.5.7 for the Navier slip boundary condition, we notice two different variants of Generalized Korn-Poincaré inequalty (2.5.15) and (2.5.21). In the context of Navier-slip boundary condition, we need to use (2.5.21) appropriately for the term $\mathcal{I}_3$ in (3.3.39).

3.4 Concluding remark

Hypothesis on the adiabatic constant γ and q in (3.2.5) is related to the growth of the perturbation q when $\varrho \to \infty$. The weak-strong uniqueness result (Theorem 3.3.4) remains valid as soon as

$$q'(\varrho) \approx \varrho^\alpha \text{ for } \varrho \to \infty, \text{ where } \alpha + 1 \leq \frac{\gamma}{2}, \gamma \geq 1.$$

The weak- strong uniqueness results are available for different systems of fluid dynamics by Berthelin and Vasseur [13], Mellet and Vasseur [101], or Saint–Raymond [109] to name a few examples. In the context of hyperbolic conservation law, there is a generalized weak-strong uniqueness result by Brenier, De Lellis and Székelyhidi [18]

For a monotone pressure law and a bounded domain Ω, the weak-strong uniqueness property for the compressible Navier–Stokes system with inflow-outflow boundary condition is proved by Kwon, Novotný and Satko [93]. We expect a similar result for a non-monotone pressure law.

For Navier–Stokes-Fourier system weak strong uniqueness result was proved by Feireisl and Novotný [73] and later extended by Jesslé, Jin and Novotný [87].

We have introduced a dissipative solution of the Navier–Stokes system for a monotone pressure law. We can observe a similar generalized weak–strong uniqueness for this class of solutions. It is fairly straightforward, but outside our scope in this thesis.

In the context of weak-strong uniqueness for the isentropic Euler system with a non-monotone pressure law, a limitation arises in the adaptation of the similar argument, since the viscous term plays a central role.

For a hard-sphere type pressure law, the proof of the weak-strong uniqueness property for a bounded domain without slip boundary condition is still open. We use $\nabla_x \Delta_x^{-1}(b(\varrho) - \langle b(\varrho) \rangle)$ as a test function in the momentum equation and adjoint of operator $\nabla_x \Delta_x^{-1}$ plays a crucial role later. If we consider some different boundary condition like no slip, we have to replace $\nabla_x \Delta_x^{-1}$ operator by Bogovskii operator $\mathcal{B}$, see Galdi [84, Chapter III]. In that case adjoint of $\mathcal{B}$ is quite different from adjoint of $\nabla_x \Delta_x^{-1}$.

Chapter 4

Singular limit and multiple scale analysis for a perfect fluid

4.1 Introduction

In this chapter we are interested in singular limit problems. We consider a system with characteristic numbers. As mentioned earlier, this system is called the *primitive system*. Then we will identify the *target system* when the characteristic numbers are small or large. The classical approach is to consider *classical (strong)* solutions of the *primitive system* and expect them to converge to the classical solutions of the *target system*. We have already pointed out in the introduction that the main limitation of this approach is that in most cases there is no global existence of a strong solution of the primitive system. The second approach is based on the theory of generalized solution. As mentioned earlier, the Navier–Stokes and the Euler system admit a global in time generalized solution. Two methods may be adopted here to solve the problem:

I. The first method is to consider generalized solutions of the primitive system and expect them to converge to a generalized solution of the target system. Then the generalized weak-strong uniqueness of the target system provides convergence in the life span of the strong solution of the target system.

II. Another method is based on the use of the *relative energy*. We consider the generalized solution of the primitive system. Assuming that the existence of a strong solution of the *target system* is known a priori, we use it as a test function in the relative energy and obtain convergence.

We have the following advantages:

- Generalized solutions of the primitive system exist globally in time. Therefore, the result depends only on the life span of the target problem, which may be finite.
- Convergence holds for a large class of generalized solutions, indicating some stability of the limit solution of the target system.

Here we describe the two problems we are interested in

- In the first problem, we consider the compressible Euler system with *low Mach number limit*. We assume that the spatial domain is $\mathbb{R}^d$. Here we consider a *dissipative solution* of the compressible Euler system and find that it converges to a dissipative solution of the incompressible Euler system. In $\mathbb{R}^2$, the incompressible Euler system has a global in time strong solution, so we obtain the desired result using weak-strong uniqueness for suitable initial data. On the other hand, in $\mathbb{R}^3$, we have only local existence of strong solution for the incompressible Euler system, so we can prove convergence only locally in time.

- In the second case, we consider a rotating compressible Euler system in $\mathbb{R}^2 \times (0,1)$. We study the effect of the *low Mach number limit* (also called *incompressible limit*), *low Rossby number limit* and *low Froude number limit* acting *simultaneously* on the system and with different scale interactions. Since a low Rossby number indicates fast rotation, the resulting flow is expected to be planar. Depending on the multiple scaling, we obtain different target systems, although all describe incompressible fluids.

There are a number of articles dealing with the low Mach number limit in the context of measure–valued solutions. In Feireisl, Klingenberg and Markfelder [62], Bruell and Feireisl [23], Březina and Mácha [25], it is shown that *measure–valued solution* of *primitive system*, describing a compressible inviscid fluid, converges to a *strong solution* of the incompressible *target system* given suitable initial data. The 'single–scale' limit of the rotating Euler system was studied by Nečasová and Tong in [103] using the measure–valued solution.

4.2 Low Mach number limit for the Compressible Euler system

We consider $T > 0$ and $\Omega = \mathbb{R}^d$ with $d = 2, 3$. We quickly revisit the scaled compressible Euler system in the time-space cylinder $Q_T = (0,T) \times \mathbb{R}^d$, which describes the time evolution of the mass density $\varrho = \varrho(t,x)$ and the momentum field $\mathbf{m} = \mathbf{m}(t,x)$ of the fluid :

$$\begin{aligned} &\mathrm{Sr}\,\partial_t \varrho + \mathrm{div}_x \mathbf{m} = 0, \\ &\mathrm{Sr}\,\partial_t \mathbf{m} + \mathrm{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) + \frac{1}{\mathrm{Ma}^2} \nabla_x p(\varrho) = 0. \end{aligned} \tag{4.2.1}$$

- **Pressure Law:** In an isentropic setting, the pressure p and the density ϱ of the fluid are interrelated by

$$p(\varrho) = a\varrho^\gamma, \; a > 0, \; \gamma > 1. \tag{4.2.2}$$

- **Scaling:** The scaled system contains these characteristic numbers:

Sr– Strouhal number,

Ma– Mach number.

Here we consider,

$$\mathrm{Sr} \approx 1 \text{ and } \mathrm{Ma} \approx \epsilon \text{ for } \epsilon > 0. \tag{4.2.3}$$

For each $\epsilon > 0$, we denote $(\varrho_\epsilon, \mathbf{m}_\epsilon)$ the solution of the system. We provide a suitable initial data condition and a far field condition as follows:

- **Far field condition:** Let $(\bar{\varrho}, \mathbf{0})$ be a static solution of the above system with $\bar{\varrho} > 0$. We assume

$$\varrho_\epsilon \to \bar{\varrho},\ \mathbf{m} \to 0 \text{ as } |x| \to \infty. \tag{4.2.4}$$

- **Initial data:** We supplement the initial data as

$$\varrho_\epsilon(0, \cdot) = \varrho_{\epsilon,0},\ \mathbf{m}_\epsilon(0, \cdot) = \mathbf{m}_{\epsilon,0}. \tag{4.2.5}$$

The main goal is to study the low Mach number limit of the system, i.e., as $\epsilon \to 0$, $(\varrho_\epsilon, \mathbf{m}_\epsilon)$ converges to certain functions satisfying a system of equations. We are interested in the dissipative solution of the primitive system.

Remark 4.2.1. Here we consider a dissipative solution of the system following the Definition 2.6.6. Although the definition is given for $\epsilon = 1$. We can modify it for scaling (4.2.3). The term $\int_0^T \int_{\mathbb{R}^d} p(\varrho) \mathrm{div}_x \boldsymbol{\varphi} \, \mathrm{d}x \, \mathrm{d}t$ in the equation (2.6.17) is replaced by $\frac{1}{\epsilon^2} \int_0^T \int_{\mathbb{R}^d} p(\varrho_\epsilon) \mathrm{div}_x \boldsymbol{\varphi} \, \mathrm{d}x \, \mathrm{d}t$ and the energy inequality is given by

$$\int_{\mathbb{R}^d} \left(\frac{|\mathbf{m}_\epsilon|^2}{\varrho_\epsilon} + \frac{1}{\epsilon^2} (P(\varrho_\epsilon) - (\varrho_\epsilon - \bar{\varrho}) P'(\bar{\varrho}) - P(\bar{\varrho})) \right) (\tau, \cdot) \, \mathrm{d}x + \int_{\mathbb{R}^d} \mathrm{d}\mathfrak{C}_e(\tau)$$
$$\leq \int_{\mathbb{R}^d} \left(\frac{|\mathbf{m}_{0,\epsilon}|^2}{\varrho_\epsilon} + \frac{1}{\epsilon^2} (P(\varrho_{0,\epsilon}) - (\varrho_{0,\epsilon} - \bar{\varrho}) P'(\bar{\varrho}) - P(\bar{\varrho})) \right) \mathrm{d}x.$$

Here we fix some notation and recall important results for this chapter:

- We denote

$$E_\epsilon(\tau) = \int_{\mathbb{R}^d} \left(\frac{|\mathbf{m}_\epsilon|^2}{\varrho_\epsilon} + \frac{1}{\epsilon^2} (P(\varrho_\epsilon) - (\varrho_\epsilon - \bar{\varrho}) P'(\bar{\varrho}) - P(\bar{\varrho}) \right) (\tau, \cdot)) \, \mathrm{d}x.$$

- Here we consider $P(\varrho) = \frac{a}{\gamma - 1} \varrho^\gamma$ for the pressure law (4.2.2).
- Let r be a positive real valued function and let its range lie in a compact subset of $(0, \infty)$. Then for any $\varrho \geq 0$, there exists $r_1, r_2 > 0$, that depends on r such that

$$P(\varrho) - (\varrho - r) P'(r) - P(r) \geq c(r) \begin{cases} (\varrho - r)^2 \text{ for } r_1 \leq \varrho < r_2 \\ 1 + \varrho^\gamma, \text{ otherwise} \end{cases}, \tag{4.2.6}$$

  where $c(r)$ is a constant dependent on r. We note that if $\gamma > 2$, one can consider $1 + \varrho^2$ instead of $1 + \varrho^\gamma$ in (4.2.6). Taking the last observation into account, we can replace γ by $\gamma' = \min\{2, \gamma\}$ in (4.2.6).

- **Essential and residual part of a function:** We introduce a function $\chi = \chi(\varrho)$ such that

$$\chi(\cdot) \in C_c^\infty(0,\infty),\ 0 \le \chi \le 1,\ \chi(\varrho) = 1 \text{ if } \varrho_1 \le \varrho \le \varrho_2,$$

where $\varrho_1, \varrho_2 > 0$. For a function, $H = H(\varrho, \mathbf{u})$ we set

$$[H]_{\text{ess}} = \chi(\varrho)H(\varrho,\mathbf{u}),\ [H]_{\text{res}} = (1-\chi(\varrho))H(\varrho,\mathbf{u}). \tag{4.2.7}$$

4.2.1 Derivation of the target system

Here is an informal justification of how to obtain the *target system*. First, we note that $(\bar\varrho, 0)$ is a *static* solution of the scaled Euler system (4.2.1)-(4.2.3) with far field condition (4.2.4). Consider

$$\begin{aligned}\varrho_\epsilon &= \bar\varrho + \epsilon\varrho_\epsilon^{(1)} + \epsilon^2\varrho_\epsilon^{(2)} + \cdots,\\ \mathbf{m}_\epsilon &= \bar\varrho\mathbf{v} + \epsilon\mathbf{m}_\epsilon^{(1)} + \epsilon^2\mathbf{m}_\epsilon^{(2)} + \cdots.\end{aligned}$$

As a consequence of the above we obtain

$$p(\varrho_\epsilon) = p(\bar\varrho) + \epsilon p'(\bar\varrho)\varrho_\epsilon^{(1)} + \epsilon^2(p'(\bar\varrho)\varrho_\epsilon^{(2)} + \frac{1}{2}p''(\bar\varrho)(\varrho_\epsilon^{(1)})^2) + o(\epsilon^3).$$

From the continuity equation and the momentum equation we get

$$\bar\varrho\mathrm{div}_x\mathbf{v} + \epsilon(\partial_t\varrho_\epsilon^{(1)} + \mathrm{div}_x\mathbf{m}_\epsilon^{(1)}) + o(\epsilon^2) = 0$$

and

$$\begin{aligned}&\bar\varrho(\partial_t\mathbf{v} + (\mathbf{v}\cdot\nabla_x)\mathbf{v}) + \nabla_x(p'(\bar\varrho)\varrho_\epsilon^{(2)} + \frac{1}{2}p''(\bar\varrho)(\varrho_\epsilon^{(1)})^2)\\ &+ \frac{1}{\epsilon}(p'(\bar\varrho)\nabla_x\varrho_\epsilon^{(1)}) + o(\epsilon) = 0.\end{aligned}$$

Further, we assume $\left(\frac{\varrho_\epsilon - \bar\varrho}{\epsilon}\right) \to q$ and $\mathbf{m}_\epsilon \to \bar\varrho\mathbf{v}$ in a strong sense. This implies

$$\mathrm{div}_x\mathbf{v} = 0.$$

Let $\mathbb{H}$ be the Helmholtz projection. Now $\mathrm{div}_x\mathbf{v} = 0$ implies $\mathbb{H}[\mathbf{v}] = \mathbf{v}$. Let $\phi : \mathbb{R}^d \to \mathbb{R}$ be a smooth scalar field. Then we have $\mathbb{H}(\nabla_x\phi) = 0$. It yields

$$\begin{aligned}&\mathrm{div}_x\mathbf{v} = 0,\\ &\mathbb{H}\big[\partial_t\mathbf{v} + (\mathbf{v}\cdot\nabla_x)\mathbf{v}\big] = 0.\end{aligned} \tag{4.2.8}$$

In other words, the system (4.2.8) is also described as

$$\begin{aligned}&\mathrm{div}_x\mathbf{v} = 0,\\ &\partial_t\mathbf{v} + (\mathbf{v}\cdot\nabla_x)\mathbf{v} + \nabla_x\Pi = 0,\end{aligned} \tag{4.2.9}$$

where Π is the *pressure*, which is unknown.

Dissipative solution of target system

For the incompressible Navier–Stokes the idea of dissipative solutions was introduced by Abbatiello and Feireisl [1], which can be extended analogously for the incompressible Euler system in $\mathbb{R}^d$. First, we prescribe an initial data for the system (4.2.9)as

$$\mathbf{v}(0,\cdot) = \mathbf{v}_0 \text{ with } \mathbf{v}_0 \in L^2(\mathbb{R}^d;\mathbb{R}^d). \tag{4.2.10}$$

The definition is as follows:

Definition 4.2.2. We say that $\mathbf{v} \in C_{\text{weak}}([0,T];L^2(\mathbb{R}^d;\mathbb{R}^d))$ is a *dissipative solution* of the problem (4.2.9) with initial data (4.2.10) if there exist *turbulent defect measures* $(\mathfrak{C}_m, \mathfrak{C}_e)$ with

$$\mathfrak{C}_m \in L^\infty_{\text{weak-}(*)}(0,T;\mathcal{M}^+(\mathbb{R}^d;\mathbb{R}^{d\times d}_{\text{sym}})),\ \mathfrak{C}_e \in L^\infty_{\text{weak-}(*)}(0,T;\mathcal{M}^+(\mathbb{R}^d)),$$

such that the following holds:

- **Compatibility of turbulent defect measures:** There exists $\Lambda_1, \Lambda_2 > 0$ such that

$$\Lambda_1 \mathrm{Tr}(\mathfrak{C}_m) \le \mathfrak{C}_e \le \Lambda_2 \mathrm{Tr}(\mathfrak{C}_m). \tag{4.2.11}$$

- **Incompressiblity:** For any $\tau \in (0,T)$ and any $\varphi \in C^1_c([0,T)\times\mathbb{R}^d)$ it holds

$$\int_0^\tau \int_{\mathbb{R}^d} \mathbf{v}\cdot\nabla_x\varphi \, \mathrm{d}x \, \mathrm{d}t \ = 0. \tag{4.2.12}$$

- **Momentum equation:** For any $\tau \in (0,T)$ and any $\boldsymbol{\varphi} \in C^1_c([0,T)\times\mathbb{R}^d;\mathbb{R}^d)$ with $\mathrm{div}_x\boldsymbol{\varphi} = 0$, it holds

$$\begin{aligned} &\left[\int_{\mathbb{R}^d} \mathbf{v}(\tau,\cdot)\cdot\boldsymbol{\varphi}(\tau,\cdot)\, \mathrm{d}x\right]_{t=0}^{t=\tau} \\ &= \int_0^\tau \int_{\mathbb{R}^d} [\mathbf{v}\cdot\partial_t\boldsymbol{\varphi} + \mathbf{v}\otimes\mathbf{v} : \nabla_x\boldsymbol{\varphi}] \, \mathrm{d}x \, \mathrm{d}t \ + \int_0^\tau \int_{\mathbb{R}^d} \nabla_x\boldsymbol{\varphi} : \mathrm{d}\mathfrak{C}_m \, \mathrm{d}t \ . \end{aligned} \tag{4.2.13}$$

- **Energy inequality:** For a.e. $\tau \in [0,T)$, we have

$$\int_{\mathbb{R}^d} \frac{1}{2}|\mathbf{v}(\tau,\cdot)|^2 \, \mathrm{d}x + \int_{\mathbb{R}^d} \mathrm{d}\ \mathfrak{C}_e(\tau,\cdot) \le \int_{\mathbb{R}^d} \frac{1}{2}|\mathbf{v}_0|^2 \, \mathrm{d}x. \tag{4.2.14}$$

4.2.2 Main Result

We say that the set of initial data $\{(\varrho_{0,\epsilon}, \mathbf{m}_{0,\epsilon})\}_{\epsilon>0}$ is *well-prepared* if

$$0 < \varrho_{0,\epsilon} \in L^2 \cap L^\infty(\mathbb{R}^d) \text{ and } \frac{\mathbf{m}_{0,\epsilon}}{\varrho_{0,\epsilon}} \in L^2(\mathbb{R}^d;\mathbb{R}^d),$$

for each $\epsilon > 0$, and they have the following property

$$\begin{aligned} &\varrho_{0,\epsilon} = \bar{\varrho} + \epsilon\tilde{\varrho}_\epsilon^{(1)} \text{ with } \tilde{\varrho}_\epsilon^{(1)} \to 0 \text{ in } L^2(\mathbb{R}^d), \\ &\mathbf{u}_{0,\epsilon} = \frac{\mathbf{m}_{0,\epsilon}}{\varrho_{0,\epsilon}} \to \mathbf{v}_0 \text{ in } L^2(\mathbb{R}^d;\mathbb{R}^3) \text{ with } \mathrm{div}_x\mathbf{v}_0 = 0. \end{aligned} \tag{4.2.15}$$

Theorem 4.2.3. *Let pressure p follows* (4.2.2)*. Let $\{(\varrho_{0,\epsilon}, \mathbf{m}_{0,\epsilon})\}_{\epsilon>0}$ be a family of well prepared initial data, i.e., it satisfies* (4.2.15)*. For each $\epsilon > 0$, $(\varrho_\epsilon, \mathbf{m}_\epsilon, \mathfrak{R}_{m_\epsilon}, \mathfrak{R}_{e_\epsilon})$ is a dissipative solution of the compressible Euler system for the initial data $(\varrho_{0,\epsilon}, \mathbf{m}_{0,\epsilon})$. As $\epsilon \to 0$, $(\varrho_\epsilon, \mathbf{m}_\epsilon, \mathfrak{R}_{m_\epsilon}, \mathfrak{R}_{e_\epsilon})$ converges to a* dissipative solution $(\mathbf{v}, \mathfrak{C}_m, \mathfrak{C}_e)$ *of the incompressible Euler system with initial data $\mathbf{v}_0$.*

Moreover, an additional assumption

$$\mathbf{v}_0 \in W^{k,2}(\mathbb{R}^d) \textit{ with } div_x \mathbf{v}_0 = 0, \textit{ for } k \geq d+1 \tag{4.2.16}$$

ensures a local in time strong solution $\mathbf{V}$ of the incompressible Euler system, i.e., there exists $T_{max} > 0$ such that $\mathbf{V} \in C^1([0, T_{max}); W^{k,2}(\mathbb{R}^d; \mathbb{R}^d))$ and it solves (4.2.9) *in point wise sense. The condition* (4.2.16) *implies $\mathbf{v} = \mathbf{V}$ in $C([0, T_{max}); C^1(\mathbb{R}^d; \mathbb{R}^d))$, $\mathfrak{C}_m, \mathfrak{C}_e = 0$ and*

$$\frac{\mathbf{m}_\epsilon}{\sqrt{\varrho_\epsilon}} \to \mathbf{v} \textit{ in } L^2_{loc}((0, T_{max}) \times \mathbb{R}^d; \mathbb{R}^d).$$

Remark 4.2.4. If $d = 2$ and $\mathbf{v}_0$ satisfies (4.2.16), the incompressible Euler system has global in time strong solution, see [89]. In this case Theorem 4.2.3 remains true for any $T > 0$.

Here we provide the proof of the theorem4.2.3. For the sake of simplicity, we assume $\bar{\varrho} = 1$.

Uniform bounds and convergence

First, we want to obtain certain uniform bounds on the variables $(\varrho_\epsilon, \mathbf{m}_\epsilon)$. We assume that $(\varrho_\epsilon, \mathbf{m}_\epsilon)$ is a dissipative solution. We recall the energy inequality

$$E_\epsilon(\tau) + \int_{\mathbb{R}^d} \mathrm{d}\, \mathfrak{R}_{e_\epsilon}(\tau, \cdot) \leq E_{0,\epsilon} \tag{4.2.17}$$

for a.e. $\tau > 0$, with

$$E_\epsilon(\tau) = \int_{\mathbb{R}^d} \left(\frac{1}{2} \frac{|\mathbf{m}_\epsilon|^2}{\varrho_\epsilon} + P(\varrho_\epsilon) - P(1) - P'(1)(\varrho_\epsilon - 1) \right) \mathrm{d}x.$$

The choice of *well-prepared* initial data (4.2.15) gives an uniform estimate for the initial energy, i.e.,

$$E_{0,\epsilon} \leq C,$$

where C independent of ϵ. This along with (4.2.6) gives the following uniform estimates:

$$\begin{aligned}
&\operatorname*{ess\ sup}_{t \in (0,T)} \left\| \frac{\mathbf{m}_\epsilon}{\sqrt{\varrho_\epsilon}} \right\|_{L^2(\mathbb{R}^d; \mathbb{R}^3)} \leq C, \\
&\operatorname*{ess\ sup}_{t \in (0,T)} \left\| \left[\frac{\varrho_\epsilon - 1}{\epsilon} \right]_{\text{ess}} \right\|_{L^2(\mathbb{R}^d)} \leq C, \\
&\operatorname*{ess\ sup}_{t \in (0,T)} \|[\varrho_\epsilon]_{\text{res}}\|^\gamma_{L^\gamma(\mathbb{R}^d)} + \operatorname*{ess\ sup}_{t \in (0,T)} \|[1]_{\text{res}}\|^\gamma_{L^\gamma(\mathbb{R}^d)} \leq \epsilon^2 C.
\end{aligned}$$

In the last expression, the essential part $[\,\cdot\,]_{\mathrm{ess}}$ and the residual part $[\,\cdot\,]_{\mathrm{res}}$ is considered in the similar line of (4.2.7), particularly choosing $\varrho_1 = \frac{1}{2}$ and $\varrho_2 = 2$. Hence we have

$$\operatorname*{ess\ sup}_{t\in(0,T)} \|[\varrho_\epsilon - \bar{\varrho}]_{\mathrm{res}}\|_{L^\gamma(\mathbb{R}^d)} \le \epsilon^{\frac{2}{\gamma}} C$$

From the last equation we get

$$\left\| \left(\frac{\varrho_\epsilon - 1}{\epsilon} \right) \right\|_{L^\infty(0,T;L^2+L^\gamma(\mathbb{R}^d))} \le (1 + \epsilon^{\frac{2}{\gamma}-1}) C$$

First we observe that for $1 < \gamma \le 2$,

$$\left\| \left(\frac{\varrho_\epsilon - 1}{\epsilon} \right) \right\|_{L^\infty(0,T;L^2+L^\gamma(\mathbb{R}^d))} \le C,$$

with C is independent of ϵ.
For $\gamma > 2$ is a bit simpler. From the Remark 4.2.1 we note that

$$\left\| \left(\frac{\varrho_\epsilon - 1}{\epsilon} \right) \right\|_{L^\infty(0,T;L^2(\mathbb{R}^d)} \le C,$$

where C is independent of ϵ. In other words, we infer that

$$\left\| \left(\frac{\varrho_\epsilon - 1}{\epsilon} \right) \right\|_{L^\infty(0,T;L^2+L^{\gamma'}(\mathbb{R}^d))} \le C,$$

with $\gamma' = \min\{\gamma, 2\}$.
Thus, we have the following convergence:

$$\begin{aligned} &\varrho_\epsilon^{(1)} \equiv \left(\frac{\varrho_\epsilon - 1}{\epsilon} \right) \to \varrho^{(1)} \begin{cases} \text{weak-(*)ly in } L^\infty(0,T;L^2 + L^\gamma(\mathbb{R}^d)), \text{ for } 1 < \gamma \le 2, \\ \text{weak-(*)ly in } L^\infty(0,T;L^2(\mathbb{R}^d)) \text{ for } \gamma > 2, \end{cases} \\ &\frac{\mathbf{m}_\epsilon}{\sqrt{\varrho_\epsilon}} \to \mathbf{v} \text{ weak-(*)ly in } L^\infty(0,T;L^2(\mathbb{R}^d;\mathbb{R}^d)), \end{aligned} \tag{4.2.18}$$

at least for a suitable subsequence. For $1 < \gamma \le 2$, we obtain

$$\varrho_\epsilon \to 1 \text{ in } L^\infty(0,T;L^2 + L^\gamma(\mathbb{R}^d)).$$

and

$$\|\mathbf{m}_\epsilon\|_{L^\infty(0,T;L^2+L^{2\gamma/\gamma\,+\,1}(\mathbb{R}^d;\mathbb{R}^d))} \le C.$$

Thus passing to a suitable subsequence, we have

$$\mathbf{m}_\epsilon \to \mathbf{m} \text{ weak-(*)ly in } L^\infty(0,T;L^2 + L^{2\gamma/\gamma\,+\,1}(\mathbb{R}^d;\mathbb{R}^d)).$$

The strong convergence of ϱ_ϵ together with (4.2.18) implies $\mathbf{m} = \mathbf{v}$, in the sense of distribution.

Arguing similarly for $\gamma > 2$, it holds

$$\varrho_\epsilon \to 1 \text{ in } L^\infty(0,T;L^2(\mathbb{R}^d))$$

and

$$\|\mathbf{m}_\epsilon\|_{L^\infty(0,TL^2+L^{4/3}(\mathbb{R}^d;\mathbb{R}^d))} \leq C.$$

Therefore, we again deduce that

$$\mathbf{m}_\epsilon \to \mathbf{m} \text{ weak-(*)ly in } L^\infty(0,T;L^2+L^{4/3}(\mathbb{R}^d;\mathbb{R}^d)),$$

for a suitable subsequence as the case may be. Furthermore, we have $\mathbf{m} = \mathbf{v}$, in the sense of distribution. Thus we have,

$$\frac{\mathbf{m}_\epsilon}{\sqrt{\varrho_\epsilon}} \to \mathbf{v} \text{ weak-(*)ly in } L^\infty(0,T;L^2(\mathbb{R}^d;\mathbb{R}^d)).$$

Defect measure and limit passage

Letting $\epsilon \to 0$ in the continuity equation we have,

$$\int_0^\tau \int_{\mathbb{R}^d} \mathbf{v} \cdot \nabla_x \varphi \, \mathrm{d}x \, \mathrm{d}t \; = 0,$$

for any $\tau \in (0,T)$ and any $\varphi \in C_c([0,T) \times \mathbb{R}^d)$. We observe the following uniform estimate

$$\left\| \frac{\mathbf{m}_\epsilon \otimes \mathbf{m}_\epsilon}{\varrho_\epsilon} \right\|_{L^\infty(0,T;L^1(\mathbb{R}^d;\mathbb{R}^{d\times d})} \leq C$$

Also energy equation gives us uniform bound for the convective term $\frac{\mathbf{m}_\epsilon \otimes \mathbf{m}_\epsilon}{\varrho_\epsilon}$ in $L^\infty(0,T;L^1(\mathbb{R}^d;\mathbb{R}^d))$ norm. Hence we obtain

$$\frac{\mathbf{m}_\epsilon \otimes \mathbf{m}_\epsilon}{\varrho_\epsilon} \to \mathcal{C}_1 \text{ weak-(*)ly in } L^\infty_{\text{weak-(*)}}(0,T;\mathcal{M}(\mathbb{R}^d;\mathbb{R}^{d\times d}_{\text{sym}})).$$

Analogously for the kinetic energy we observe

$$\frac{1}{2}\frac{|\mathbf{m}_\epsilon|^2}{\varrho_\epsilon} \to \mathcal{C}_2 \text{ weak-(*)ly in } L^\infty_{\text{weak-(*)}}(0,T;\mathcal{M}(\mathbb{R}^d)).$$

Furthermore, we already have

$$\mathbf{w}_\epsilon = \frac{\mathbf{m}_\epsilon}{\sqrt{\varrho_\epsilon}} \to \mathbf{v} \text{ weak-(*)ly } \text{ in } L^\infty_{\text{weak-(*)}}(0,T;L^2(\mathbb{R}^d;\mathbb{R}^d)),$$

Consider $b(\lambda) = \frac{1}{2}|\lambda|^2$ for $\lambda \in \mathbb{R}^d$. It is a convex function. We note that $\mathcal{C}_2 = \overline{b(\mathbf{w})}$, where $\overline{b(\mathbf{w})}$ is the weak-(*) limit of $b(\mathbf{w}_\epsilon)$ in $L^\infty_{\text{weak-(*)}}(0,T\mathcal{M}(\mathbb{R}^d))$. Proposition 1.3.17 implies that

$$\mathfrak{C}_e^2 \equiv \mathcal{C}_2 - \frac{|\mathbf{v}|^2}{2} \in L^\infty_{\text{weak-(*)}}(0,T\mathcal{M}^+(\mathbb{R}^d)).$$

Next consider

$$\mathfrak{C}_m^2 = \mathcal{C}_1 - \mathbf{v} \otimes \mathbf{v}.$$

It is easy to verify that for any $\xi \in \mathbb{R}^d$ and $\mathbf{V} \in \mathbb{R}^{d\times d}$,

$$\mathbf{V} \otimes \mathbf{V} \colon \xi \otimes \xi = |\mathbf{V} \cdot \xi|^2.$$

and the mapping $\mathbf{V} \mapsto |\mathbf{V} \cdot \xi|^2$ is a convex function for any $\xi \in \mathbb{R}^d$. It yields

$$\mathfrak{C}_m^2 = \mathcal{C}_1 - \mathbf{v} \otimes \mathbf{v} \in L^\infty_{\text{weak-(*)}}(0,T;\mathcal{M}^+(\mathbb{R}^d;\mathbb{R}^{d\times d}_{\text{sym}})).$$

Since

$$\int_{\mathbb{R}^d} \mathrm{d}\mathfrak{R}_{\mathrm{e}_\epsilon}(\tau,\cdot) \leq \mathrm{C},$$

we have the uniform estimate for energy defect measure for a.e. $\tau \in (0,T)$ The compatibility of defect measures $\mathfrak{R}_{m_\epsilon}$ and $\mathfrak{R}_{e_\epsilon}$ implies

$$\begin{aligned}
&\mathfrak{R}_{m_\epsilon} \to \mathfrak{C}_m^1 \text{ weak-(*)ly in } L^\infty_{\text{weak-(*)}}(0,T;\mathcal{M}(\mathbb{R}^d;\mathbb{R}^{d\times d}_{\text{sym}})),\\
&\mathfrak{R}_{e_\epsilon} \to \mathfrak{C}_e^1 \text{ weak-(*)ly in } L^\infty_{\text{weak-(*)}}(0,T;\mathcal{M}(\mathbb{R}^d)).
\end{aligned}$$

We define the total turbulent defect measure as

$$\mathfrak{C}_m = \mathfrak{C}_m^1 + \mathfrak{C}_m^2 \text{ and } \mathfrak{C}_e = \mathfrak{C}_e^1 + \mathfrak{C}_e^2.$$

Using the fact

$$\mathrm{Tr}\left(\frac{\mathbf{m}_\epsilon \otimes \mathbf{m}_\epsilon}{\varrho_\epsilon}\right) = \frac{|\mathbf{m}_\epsilon|^2}{\varrho_\epsilon}$$

and the relation between pressure and pressure potential we infer the compatibility of the defect measure $\mathfrak{C}_m$ and $\mathfrak{C}_e$ as in (4.2.11).

We notice that the class of test function for incompressible Euler system is $\boldsymbol{\varphi} \in C_c^1([0,T) \times \mathbb{R}^d;\mathbb{R}^d)$ such that $\mathrm{div}_x\boldsymbol{\varphi} = 0$. For such $\boldsymbol{\varphi}$ in the scaled momentum equation we have,

$$\begin{aligned}
&\left[\int_{\mathbb{R}^d} \mathbf{m}_\epsilon(\tau,\cdot)\cdot\boldsymbol{\varphi}(\tau,\cdot)\ \mathrm{d}x\right]_{t=0}^{t=\tau}\\
&= \int_0^\tau \int_{\mathbb{R}^d} \left[\mathbf{m}_\epsilon \cdot \partial_t\boldsymbol{\varphi} + \left(\frac{\mathbf{m}_\epsilon \otimes \mathbf{m}_\epsilon}{\varrho_\epsilon}\right) : \nabla_x\boldsymbol{\varphi}\right] \mathrm{d}x\ \mathrm{d}t\ + \int_0^\tau \int_{\mathbb{R}^d} \nabla_x\boldsymbol{\varphi} : \mathrm{d}\mathfrak{R}_{m_\epsilon}\ \mathrm{d}t\ .
\end{aligned} \tag{4.2.19}$$

Therefore, using the convergence of state variables and the characterization of nonlinear term by a defect measure, we conclude

$$\begin{aligned}
&\left[\int_{\mathbb{R}^d} \mathbf{v}(\tau,\cdot)\cdot\boldsymbol{\varphi}(\tau,\cdot)\ \mathrm{d}x\right]_{t=0}^{t=\tau}\\
&= \int_0^\tau \int_{\mathbb{R}^d} [\mathbf{v}\cdot\partial_t\boldsymbol{\varphi} + \mathbf{v}\otimes\mathbf{v} : \nabla_x\boldsymbol{\varphi}]\ \mathrm{d}x\ \mathrm{d}t\ + \int_0^\tau \int_{\mathbb{R}^d} \nabla_x\boldsymbol{\varphi} : \mathrm{d}\mathfrak{C}_m\ \mathrm{d}t\ .
\end{aligned}$$

From the lower semicontinuity of the norm, we obtain the following form of energy inequality:

$$\int_{\mathbb{R}^d} \frac{1}{2}|\mathbf{v}(\tau,\cdot)|^2 \,\mathrm{d}x + \int_{\mathbb{R}^d} \mathrm{d}\ \mathfrak{C}_e(\tau,\cdot) \leq \int_{\mathbb{R}^d} \frac{1}{2}|\mathbf{v}_0|^2 \,\mathrm{d}x.$$

We conclude $(\mathbf{v}, \mathfrak{C}_m, \mathfrak{C}_e)$ is a *dissipative solution* for incompressible Euler equation.

Weak(dissipative)–strong uniqueness for incompressible Euler system

In the review article Wiedemann [121, Theorem 3.4] proves generalized weak–strong uniqueness for the incompressible Euler system. Drawing inspiration from that, we consider the relative energy as

$$e(\mathbf{v}\,|\,\mathbf{V})(\tau) = \int_{\mathbb{R}^d} |\mathbf{v} - \mathbf{V}|^2(\tau)\,\mathrm{d}x, \tag{4.2.20}$$

for $\tau \in (0, T_{\max})$, $\mathbf{v}$ is an dissipative solution and $\mathbf{V}$ is the strong solution. It is easy to verify that

$$\begin{aligned} &e(\mathbf{v}\,|\,\mathbf{V})(\tau) + \int_{\mathbb{R}^d} \mathrm{d}\ \mathfrak{C}_e(\tau,\cdot) \\ &\leq e(\mathbf{v}_0, \mathbf{V}(0)) + C\left(\int_0^\tau e(\mathbf{v}\,|\,\mathbf{V})(t)\,\mathrm{d}t + \int_0^\tau \int_{\mathbb{R}^d} \mathrm{Tr}\,(\mathrm{d}\mathfrak{C}_m(\cdot))\,\mathrm{d}t\right), \end{aligned}$$

for a.e. $\tau \in (0, T_{\max})$. Furthermore, the choice of initial data $\mathbf{v}_0$ in (4.2.16) and the assumption $\mathbf{v}_0 = \mathbf{V}(0,\cdot)$ imply the desired weak-strong uniqueness in $(0, T_{\max}) \times \mathbb{R}^d$. This infer that the defect measures vanish, i.e.,

$$\mathfrak{C}_m = 0 \text{ and } \mathfrak{C}_e = 0. \tag{4.2.21}$$

Local strong convergence

The equation (4.2.21) shows that

$$\overline{b(\mathbf{w})} = \frac{1}{2}|\mathbf{v}|^2.$$

Let B be a bounded subset of $\mathbb{R}^d$ and $T < T_{\max}$. We notice that b and $\mathbf{w}_\epsilon$ satisfy the hypothesis of the Lemma 1.3.26, and as an immediate consequence, we have

$$b(\mathbf{w}_\epsilon) \to \frac{1}{2}|\mathbf{v}|^2 \text{ weakly in } L^1((0,T)\times B).$$

It implies

$$\int_0^T \int_B b(\mathbf{w}_\epsilon)\,\mathrm{d}x\,\mathrm{d}t \to \int_0^T \int_B |\mathbf{v}|^2\,\mathrm{d}x\,\mathrm{d}t\,,$$

the L^2-norm convergence of $\mathbf{w}_\epsilon$.

Moreover, we have

$$\mathbf{w}_\epsilon \to \mathbf{v} \text{ weakly in } L^2((0,T)\times B).$$

Weak convergence together with norm convergence in $(0,T)\times B$ asserts that

$$\mathbf{w}_\epsilon \to \mathbf{v} \text{ in } L^2((0,T)\times B).$$

This ends the proof of the Theorem 4.2.3.

4.3 Multi-scale analysis of a compressible rotating inviscid fluid

In this section we are interested in studying the singular limit problems of a rotating compressible inviscid fluid. We consider the model of a rotating fluid as described in Chemin et al. [32]. Let $T>0$ and $\Omega(\subset\mathbb{R}^3)=\mathbb{R}^2\times(0,1)$ be an infinite slab. The rotating Euler system is the barotropic Euler system (2.3.1)-(2.3.2) with forcing term

$$\varrho\mathbf{f} = -\mathbf{b}\times\mathbf{m} + \varrho\nabla_x G. \tag{4.3.1}$$

The term $\mathbf{b}\times\mathbf{m}$ represents the effect of *rotation (Coriolis force)* with the axis of rotation $\mathbf{b}$ and the effect of *gravitational force* is given by $\nabla_x G$. We have neglected the effect of the centrifugal force.

We consider the scaled compressible Euler equation in the time-space cylinder $Q_T=(0,T)\times\Omega$ which describes the time evolution of the mass density $\varrho=\varrho(t,x)$ and the momentum field $\mathbf{m}=\mathbf{m}(t,x)$ of a rotating inviscid fluid with axis of rotation $\mathbf{b}=(0,0,1)$:

- **Conservation of mass:**

$$\mathrm{Sr}\;\partial_t\varrho + \mathrm{div}_x\mathbf{m} = 0. \tag{4.3.2}$$

- **Conservation of momentum:**

$$\mathrm{Sr}\;\partial_t\mathbf{m} + \mathrm{div}_x\left(\frac{\mathbf{m}\otimes\mathbf{m}}{\varrho}\right) + \frac{1}{\mathrm{Ma}^2}\nabla_x p(\varrho) + \frac{1}{\mathrm{Ro}}\mathbf{b}\times\mathbf{m} = \frac{1}{\mathrm{Fr}^2}\varrho\nabla_x G. \tag{4.3.3}$$

- **Multiple scaling:** The scaled system contains characteristic numbers: Strouhal number(Sr), Mach number(Ma), Rossby number(Ro) and Froude number(Fr).
  Here we consider a multiple scaling as

$$\mathrm{Sr}\ \approx 1,\ \mathrm{Ma}\approx\epsilon^m,\ \mathrm{Ro}\approx\epsilon,\ \mathrm{Fr}\approx\epsilon^n \text{ for } \epsilon>0,\ m,n>0. \tag{4.3.4}$$

- **Pressure law:** The pressure p and the density ϱ of the fluid are interrelated by the standard isentropic law

$$p(\varrho) = a\varrho^{\gamma},\ a > 0,\ \gamma > 1. \tag{4.3.5}$$

Thus for each $\epsilon > 0$, a solution of the system is denoted by $(\varrho_\epsilon, \mathbf{m}_\epsilon)$, just to indicate the dependence of solutions on scaling parameter. Now to complete the formulation we introduce the initial, boundary, far field and other important conditions.

- **Boundary condition:** Here we consider impermeability or slip condition on the horizontal boundary, i.e.

$$\mathbf{m}_\epsilon \cdot \mathbf{n} = 0 \text{ on } \partial\Omega,\ \text{ where }\ \mathbf{n} = (0, 0, \pm 1). \tag{4.3.6}$$

- **Far field condition:** Let us introduce the notation $x = (x_h, x_3) \in \Omega$ and the projection of x in $\mathbb{R}^2$ as $P_h(x) = x_h$.

  For each $\epsilon > 0$, we identify a static solution $(\tilde{\varrho}_\epsilon, \mathbf{0})$ that satisfies (4.3.2)-(4.3.3) with (4.3.4). More specifically, a static solutions is a pair $(\tilde{\varrho}_\epsilon, \mathbf{0})$, where the density profile $\tilde{\varrho}_\epsilon$ satisfies

$$\nabla_x p(\tilde{\varrho}_\epsilon) = \epsilon^{2(m-n)} \tilde{\varrho}_\epsilon \nabla_x G. \tag{4.3.7}$$

  We assume the *far field* condition as,

$$|\varrho_\epsilon - \tilde{\varrho}_\epsilon| \to 0,\ \mathbf{m}_\epsilon \to \mathbf{0} \text{ as } |x_h| \to \infty. \tag{4.3.8}$$

- **Initial data:** For each $\epsilon > 0$, we supplement the initial data as

$$\varrho_\epsilon(0, \cdot) = \varrho_{\epsilon,0},\ \mathbf{m}_\epsilon(0, \cdot) = \mathbf{m}_{\epsilon,0}. \tag{4.3.9}$$

- **Choice of G:** We consider

$$G(x) = -\Lambda x_3 \text{ in } \Omega,\ \text{with } \Lambda = 0 \text{ or } 1. \tag{4.3.10}$$

  In certain cases we can ignore the gravitational effect by considering $\Lambda = 0$.
  For $\Lambda = 1$, G corresponds to the gravitational force acting in the vertical direction.

Remark 4.3.1. As a matter of fact, the driving potential G can be seen as a sum of the centrifugal force proportional to the norm of the horizontal component of the spatial variable i.e. $(x_1^2 + x_2^2)$ and the gravitational force acting in the vertical direction x_3. We omit the effect of the centrifugal force in the present section motivated by certain meteorological models.

Remark 4.3.2. We are interested in *multiple scaling* of (4.3.4), i.e., we choose different $m, n \in \mathbb{N}$.

Dissipative solution for a rotating Euler system

We are interested in the dissipative solutions of the primitive system (4.3.2)-(4.3.3). For the domain $\Omega = \mathbb{R}^2 \times (0,1)$, we introduced the definition of a dissipative solution for the Euler system in Chapter 2, see Definition 2.6.12. For the definition of a dissipative solution for a rotating fluid, we now substitute $\mathbf{f}$ following (4.3.1) in Definition 2.6.12. However, we note again that the definition is given for $\epsilon = 1$. For any $\epsilon > 0$, we can modify it by observing the Remark 4.2.1 from the last section.

Thus, for $\epsilon > 0$, we denote $(\varrho_\epsilon, \mathbf{m}_\epsilon)$ as a dissipative solution with *turbulent defect measures* $(\mathfrak{R}_{m_\epsilon}, \mathfrak{R}_{e_\epsilon})$ of the system (4.3.2)-(4.3.3) with constraints (4.3.4)-(4.3.10) following the Definition 2.6.12.

Remark 4.3.3. It is worth to noting that the term $(-\mathbf{b} \times \mathbf{m})$ does not contribute in the energy, since the following identity is true for any vector $\mathbf{v} \in \mathbb{R}^3$:

$$\mathbf{b} \times \mathbf{v} \cdot \mathbf{v} = 0, \text{ for } \mathbf{b} = (0,0,1).$$

4.3.1 Relative energy inequality

In our approach, relative energy functional plays an important role. We consider a scaled version of relative energy as

$$\begin{aligned}\mathcal{E}_\epsilon(t) &= \mathcal{E}(\varrho_\epsilon, \mathbf{m}_\epsilon | \tilde{\varrho}, \tilde{\mathbf{u}})(t) \\ &:= \int_\Omega \left[\frac{1}{2} \varrho_\epsilon \left| \frac{\mathbf{m}_\epsilon}{\varrho_\epsilon} - \tilde{\mathbf{u}} \right|^2 + \frac{1}{\epsilon^{2m}} (P(\varrho_\epsilon) - P(\tilde{\varrho}) - P'(\tilde{\varrho})(\varrho_\epsilon - \tilde{\varrho})) \right] (t, \cdot) \, \mathrm{d}x,\end{aligned} \tag{4.3.11}$$

where $\tilde{\varrho}, \tilde{\mathbf{u}}$ satisfies

$$\begin{aligned}&0 < \tilde{\varrho} \in C^\infty([0,T]) \times \overline{\Omega}) \text{ with } \tilde{\varrho} - \tilde{\varrho}_\epsilon \text{ having compact support in } [0,T] \times \overline{\Omega}, \\ &\tilde{\mathbf{u}} \in C_c^\infty([0,T] \times \overline{\Omega}; \mathbb{R}^d) \text{ with } \tilde{\mathbf{u}} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega,\end{aligned} \tag{4.3.12}$$

and, $(\varrho_\epsilon, \mathbf{m}_\epsilon)$ is a dissipative solution of the system.

Remark 4.3.4. Let $\tilde{\varrho}, \tilde{\mathbf{u}}$ satisfies (4.3.12) and $0 < \varrho_1 < \varrho_2$. We consider a function $\chi = \chi(\varrho)$ such that

$$\chi(\cdot) \in C_c^\infty(0,\infty), \; 0 \leq \chi \leq 1, \; \chi(\varrho) = 1 \text{ if } \varrho_1 \leq \varrho \leq \varrho_2,$$

For a function, $H = H(\varrho, \mathbf{u})$ we set

$$[H]_{\text{ess}} = \chi(\varrho) H(\varrho, \mathbf{m}), \; [H]_{\text{res}} = (1 - \chi(\varrho)) H(\varrho, \mathbf{m}). \tag{4.3.13}$$

The relative energy is a coercive functional (see. Bruell and Feireisl [23, Section 2]), that satisfies the following estimate:

$$\begin{aligned}\mathcal{E}(\varrho_\epsilon, \mathbf{u}_\epsilon|\tilde{\varrho}, \tilde{\mathbf{u}})(t) \geq &\int_\Omega \left[\left|\frac{\mathbf{m}_\epsilon}{\varrho_\epsilon} - \tilde{\mathbf{m}}\right|^2\right]_{\text{ess}} \mathrm{d}x + \int_\Omega \left[\frac{|\mathbf{m}_\epsilon|^2}{\varrho_\epsilon}\right]_{\text{res}} \mathrm{d}x \\ &+ \frac{1}{\epsilon^{2m}} \int_\Omega [(\varrho_\epsilon - \tilde{\varrho})^2]_{\text{ess}} \mathrm{d}x + \frac{1}{\epsilon^{2m}} \int_\Omega [1]_{\text{res}} + [\varrho_\epsilon^\gamma]_{\text{res}} \mathrm{d}x,\end{aligned} \tag{4.3.14}$$

for $t \in (0, T)$.

In Section 3.2, we derive a relative energy inequality for viscous fluids. In this case, a similar scaled version of the relative energy inequality is possible and given in the next lemma.

Lemma 4.3.5. *Let $(\tilde{\varrho}_\epsilon, 0)$ be a static solution and $(\varrho_\epsilon, \mathbf{m}_\epsilon)$ be a dissipative solution of system (4.3.2)-(4.3.10) with defect measure $(\mathfrak{R}_{e_\epsilon}, \mathfrak{R}_{m_\epsilon})$ for finite enrgy initial data $(\varrho_{0,\epsilon}, \mathbf{m}_{0,\epsilon})$ that follows the Definition (2.6.12). Suppose $(\tilde{\varrho}, \tilde{\mathbf{u}})$ satisfies (4.3.12). Then we have the following inequality*

$$\begin{aligned}&\mathcal{E}_\epsilon(\tau) + \int_{\overline{\Omega}} \mathrm{d}\, \mathfrak{R}_{e_\epsilon}(\tau, \cdot) \\ &\leq \mathcal{E}_{0,\epsilon} - \int_0^\tau \int_\Omega (\mathbf{m}_\epsilon - \varrho_\epsilon \tilde{\mathbf{u}}) \cdot \partial_t \tilde{\mathbf{u}} \ dx \ dt \ - \int_0^\tau \int_\Omega \left(\frac{(\mathbf{m}_\epsilon - \varrho_\epsilon \tilde{\mathbf{u}}) \otimes \mathbf{m}_\epsilon}{\varrho_\epsilon} \right) : \nabla_x \tilde{\mathbf{u}} \ dx \ dt \\ &- \frac{1}{\epsilon^{2m}} \int_0^\tau \int_\Omega (p(\varrho_\epsilon) - p(\tilde{\varrho})) div_x \tilde{\mathbf{u}} \ dx \ dt \ + \frac{1}{\epsilon^{2m}} \int_0^\tau \int_\Omega (\tilde{\varrho} - \varrho_\epsilon) \partial_t P'(\tilde{\varrho}) \ dx \ dt \\ &+ \frac{1}{\epsilon} \int_0^\tau \int_\Omega \mathbf{b} \times \mathbf{m}_\epsilon \cdot \tilde{\mathbf{u}} \ dx \ dt \\ &+ \frac{1}{\epsilon^{2m}} \int_0^\tau \int_\Omega (\tilde{\varrho} \tilde{\mathbf{u}} - \mathbf{m}_\epsilon) \cdot (\nabla_x P'(\tilde{\varrho}) - \nabla_x P'(\tilde{\varrho}_\epsilon)) \ dx \ dt \\ &- \frac{1}{\epsilon^{2n}} \int_0^\tau \int_\Omega (\varrho_\epsilon - \tilde{\varrho}) \nabla_x G \cdot \tilde{\mathbf{u}} \ dx \ dt \ - \int_0^\tau \int_{\overline{\Omega}} \nabla_x \tilde{\mathbf{u}} : d\mathfrak{R}_{m_\epsilon}(t, \cdot) \ dt \ .\end{aligned} \tag{4.3.15}$$

where $\mathcal{E}_{0,\epsilon}$ is the following expression

$$\int_\Omega \left[\frac{1}{2} \varrho_{0,\epsilon} \left| \frac{\mathbf{m}_{0,\epsilon}}{\varrho_{0,\epsilon}} - \tilde{\mathbf{u}}(0, \cdot) \right|^2 + \frac{1}{\epsilon^{2m}} \left(P(\varrho_{0,\epsilon}) - P(\tilde{\varrho}(0, \cdot)) - P'(\tilde{\varrho}(0, \cdot))(\varrho_{0,\epsilon} - \tilde{\varrho}(0, \cdot)) \right) \right] dx.$$

Remark 4.3.6. The proof of this lemma is similar to the one we performed in the last chapter (see, Section 3.2). We only need to deal with the scaled system by considering the Remark 4.2.1.

Remark 4.3.7. Instead of considering $(\tilde{\varrho}, \tilde{\mathbf{u}})$ as a smooth function, we can extend this to a suitable class of Sobolev functions as we have described in Proposition 3.2.8.

4.3.2 Derivation of target systems: Multiple scales

Here is an informal justification of how we obtain the limiting system, which we call the *target system.* First, we note that $(\tilde{\varrho}_\epsilon, 0)$ is a *static* solution of (4.3.2)-(4.3.3) satisfying

$$\nabla_x p(\tilde{\varrho}_\epsilon) = \Lambda \epsilon^{2(m-n)} \tilde{\varrho}_\epsilon \nabla_x G. \tag{4.3.16}$$

Let us consider an asymptotic expansion around the $(\tilde{\varrho}_\epsilon, \mathbf{v})$ as

$$\begin{aligned} \varrho_\epsilon &= \tilde{\varrho}_\epsilon + \epsilon^m \varrho_\epsilon^{(1)} + \epsilon^{2m} \varrho_\epsilon^{(2)} + \cdots , \\ \mathbf{u}_\epsilon &= \mathbf{v} + \epsilon^m \mathbf{v}_\epsilon^{(1)} + \epsilon^{2m} \mathbf{v}_\epsilon^{(2)} + \cdots . \end{aligned}$$

Using the fact $\mathbf{m}_\epsilon = \varrho_\epsilon \mathbf{u}_\epsilon$, we have

$$\mathbf{m}_\epsilon = \tilde{\varrho}_\epsilon \mathbf{v} + \epsilon^m \mathbf{m}_\epsilon^{(1)} + \epsilon^{2m} \mathbf{m}_\epsilon^{(2)} + \cdots ,$$

where $\mathbf{m}_\epsilon^{(1)} = \varrho_\epsilon \mathbf{v}_\epsilon^{(1)} + \varrho_\epsilon^{(1)} \mathbf{v}$. As a consequence of the above, we obtain

$$p(\varrho_\epsilon) = p(\tilde{\varrho}_\epsilon) + \epsilon^m p'(\tilde{\varrho}_\epsilon) \varrho_\epsilon^{(1)} + \epsilon^{2m} (p'(\tilde{\varrho}_\epsilon) \varrho_\epsilon^{(2)} + \frac{1}{2} p''(\tilde{\varrho}_\epsilon)(\varrho_\epsilon^{(1)})^2) + o(\epsilon^{3m}).$$

Substituting this into the continuity and momentum equation, we get

$$\operatorname{div}_x(\tilde{\varrho}_\epsilon \mathbf{v}) + \epsilon^m (\partial_t \varrho_\epsilon^{(1)} + \operatorname{div}_x(\mathbf{v}_\epsilon^{(1)} \bar{\varrho} + \mathbf{v} \varrho_\epsilon^{(1)}) + o(\epsilon^{2m}) = 0 \tag{4.3.17}$$

and

$$\begin{aligned} &\partial_t(\tilde{\varrho}_\epsilon \mathbf{v}) + (\tilde{\varrho}_\epsilon \mathbf{v} \cdot \nabla_x) \mathbf{v} + \nabla_x \left(p'(\tilde{\varrho}_\epsilon) \varrho_\epsilon^{(2)} + \frac{1}{2} p''(\tilde{\varrho}_\epsilon)(\varrho_\epsilon^{(1)})^2 \right) \\ &+ \epsilon^{m-1} \mathbf{b} \times \mathbf{m}_\epsilon^{(1)} + \frac{1}{\epsilon^{2m}} \nabla_x p(\tilde{\varrho}_\epsilon) + \frac{1}{\epsilon^m} \nabla_x (p'(\tilde{\varrho}_\epsilon) \varrho_\epsilon^{(1)}) + \frac{1}{\epsilon} (\bar{\varrho} \mathbf{b} \times \mathbf{v}) \\ &- \Lambda \frac{1}{\epsilon^{2n}} \tilde{\varrho}_\epsilon \nabla_x G + o(\epsilon^{m-1}, \epsilon^m) = 0. \end{aligned} \tag{4.3.18}$$

In order to take into account 'multiple scaling', we consider three different cases as

- **(Case I:)** the gravitational force is absent, i.e., $\Lambda = 0$, $m = 1$,
- **(Case II:)** the gravitational force is present and Mach and Froude number has same scaling, i.e., $\Lambda = 1, \ m = n = 1$,
- **(Case III:)** the gravitational force is present and the effect of Mach number is dominant, i.e., $\Lambda = 1, \ \frac{m}{2} > n \geq 1$.

Case I: $\Lambda = 0$, $m = 1$; Target System: Geophysical Flow

First, from (4.3.7) we observe $\tilde{\varrho}_\epsilon = \bar{\varrho}$, a constant. Without loss of generality, we assume $\bar{\varrho} > 0$. Therefore, we rewrite (4.3.17) and (4.3.18) as

$$\bar{\varrho}\mathrm{div}_x \mathbf{v} + \epsilon(\partial_t \varrho_\epsilon^{(1)} + \mathrm{div}_x(\mathbf{v}_\epsilon^{(1)}\bar{\varrho} + \mathbf{v}\varrho_\epsilon^{(1)}) + o(\epsilon^2) = 0,$$

and

$$\begin{aligned}&\bar{\varrho}(\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla_x)\mathbf{v}) + \nabla_x\left(p'(\bar{\varrho})\varrho_\epsilon^{(2)} + \frac{1}{2}p''(\bar{\varrho})(\varrho_\epsilon^{(1)})^2\right)\\ &+ \mathbf{b}\times\mathbf{m}_\epsilon^{(1)} + \frac{1}{\epsilon}(p'(\bar{\varrho})\nabla_x\varrho_\epsilon^{(1)} + \bar{\varrho}\mathbf{b}\times\mathbf{v}) + o(\epsilon) = 0,\end{aligned}$$

respectively, where $\mathbf{m}_\epsilon^{(1)} = \mathbf{v}_\epsilon^{(1)}\bar{\varrho} + \mathbf{v}\varrho_\epsilon^{(1)}$.

Furthermore, we assume $\left(\frac{\varrho_\epsilon - \bar{\varrho}}{\epsilon}\right) \to q$ and $\mathbf{u}_\epsilon \to \mathbf{v}$ in some strong sense. Then, as a consequence, we have

$$\begin{aligned}&p'(\bar{\varrho})\nabla_x q + \bar{\varrho}\mathbf{b}\times\mathbf{v} = 0,\\ &\mathrm{div}_x \mathbf{v} = 0,\\ &\bar{\varrho}(\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla_x)\mathbf{v}) + \nabla_x \Pi_1 + \mathbf{b}\times\mathbf{m}_1 = 0,\\ &\partial_t q + \mathrm{div}_x \mathbf{m}_1 = 0.\end{aligned}$$

From the above equations, it yields

$$\begin{aligned}&q_{x_3} = 0,\ q(x) = q(x_h), \nabla_{x_h}^{\perp} q = \frac{\bar{\varrho}}{p'(\bar{\varrho})}\mathbf{v}_h,\ \mathbf{v}_h = (v_1, v_2),\\ &\mathrm{div}_{x_h}\mathbf{v}_h = 0, v_{3_{x_3}} = 0,\ \text{where } \nabla_{x_h}^{\perp} \equiv \left(-\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1}\right).\end{aligned}$$

If we additionally assume smoothness of the variables, we derive

$$v_{1_{x_3}} = 0,\ v_{2_{x_3}} = 0.$$

We conclude that the quantity $\mathbf{v}$ depends only on $x_h = (x_1, x_2)$, i.e.,

$$\mathbf{v}(x) = \mathbf{v}(x_h).$$

Moreover, the slip boundary condition (4.3.6) gives

$$v_3(x_h, x_3) = 0.$$

Thus, we infer $\mathbf{v} = (\mathbf{v}_h(x_h), 0)$ and

$$\begin{aligned}&\frac{p'(\bar{\varrho})}{\bar{\varrho}}\nabla_{x_h} q + \mathbf{b}\times\mathbf{v} = 0,\\ &\partial_t\left(\Delta_{x_h} q - \frac{1}{p'(\bar{\varrho})}q\right) + \nabla_{x_h}^{\perp} q\cdot\nabla_{x_h}\left(\Delta_{x_h} q - \frac{1}{p'(\bar{\varrho})}q\right) = 0.\end{aligned} \tag{4.3.19}$$

In (4.3.19), q can be regarded as a kind of *stream function* and the relation between q and $\mathbf{u}$ gives the *incompressibility condition*, i.e., $\mathrm{div}_{x_h}\mathbf{v}_h = 0$.

Case II: $\Lambda = 1,\ m = n = 1$**; Target System: Stratified fluid, Geophysical flow**

In this case, a static solution $\tilde{\varrho}_\epsilon$ satisfying

$$\nabla_x p(\tilde{\varrho}_\epsilon) = \tilde{\varrho}_\epsilon \nabla_x G. \tag{4.3.20}$$

is independent of ϵ. We denote it by $\hat{\varrho}$. From our choice of $G(x) = -x_3$, we have $\hat{\varrho}(x) = \hat{\varrho}(x_3)$. Furthermore, we can choose $0 < \hat{\varrho} \in C^\infty([0,1])$.

We rewrite (4.3.17) and (4.3.18) as

$$\operatorname{div}_x(\hat{\varrho}\mathbf{v}) + \epsilon(\partial_t \varrho_\epsilon^{(1)} + \operatorname{div}_x(\mathbf{v}_\epsilon^{(1)}\hat{\varrho} + \mathbf{v}\varrho_\epsilon^{(1)}) + o(\epsilon^2) = 0$$

and

$$\begin{aligned}&\partial_t(\hat{\varrho}\mathbf{v}) + \operatorname{div}_x(\hat{\varrho}\mathbf{v}\otimes\mathbf{v}) + \nabla_x\left(p'(\hat{\varrho})\varrho_\epsilon^{(2)} + \frac{1}{2}p''(\hat{\varrho})(\varrho_\epsilon^{(1)})^2\right)\\ &+ \mathbf{b}\times\mathbf{m}_\epsilon^{(1)} + \frac{1}{\epsilon}(\nabla_x(p'(\hat{\varrho})\varrho_\epsilon^{(1)}) + \mathbf{b}\times\hat{\varrho}\mathbf{v}) = \frac{1}{\epsilon}\varrho_\epsilon^{(1)}\nabla_x G + o(\epsilon),\end{aligned}$$

respectively, where $\mathbf{m}_\epsilon^{(1)} = \mathbf{v}_\epsilon^{(1)}\hat{\varrho} + \mathbf{v}\varrho_\epsilon^{(1)}$.

We assume $\left(\frac{\varrho_\epsilon - \hat{\varrho}}{\epsilon}\right) \to q$ and $\mathbf{u}_\epsilon \to \mathbf{v}$ in some strong sense. Consequently, we get

$$\begin{aligned}&\nabla_x(p'(\hat{\varrho})q) + \mathbf{b}\times\hat{\varrho}\mathbf{v} = q\nabla_x G,\\ &\operatorname{div}_x(\hat{\varrho}\mathbf{v}) = 0,\\ &\partial_t(\hat{\varrho}\mathbf{v}) + \operatorname{div}_x(\hat{\varrho}\mathbf{v}\otimes\mathbf{v}) + \nabla_x\Pi_1 + \mathbf{b}\times\mathbf{m}_1 = 0,\\ &\partial_t q + \operatorname{div}_x\mathbf{m}_1 = 0.\end{aligned}$$

From the above equations, we deduce that

$$\nabla_x(P'(\hat{\varrho})q) + \mathbf{b}\times\hat{\varrho}\mathbf{v} = 0.$$

The choice of $\mathbf{b}$ implies

$$(P''(\hat{\varrho})q)_{x_3} = 0.$$

Further, an additional smoothness assumption gives

$$(P''(\hat{\varrho})q)(x_h, x_3) = (P''(\hat{\varrho})q)(x_h, 0) \text{ for } x_3 \in (0,1).$$

Also using the boundary condition, we obtain $\mathbf{v}(x_h, x_3) = (\mathbf{v}_h(x_h), 0)$.
From our choice of G, we consider $C = P''(\hat{\varrho}(0)) > 0$ and $P''(\hat{\varrho}(x_3)) \neq 0$ for $x_3 \in [0,1]$. Thus, we have a definite structure of q and it is given by

$$q(x_h, x_3) = C\frac{q(x_h, 0)}{P''(\hat{\varrho}(x_3))}.$$

Finally, we deduce that for each $x_3 \in (0,1)$, $(q, \mathbf{v})$ satisfies

$$\begin{aligned}&\nabla_x(P''(\hat{\varrho})q) + \mathbf{b}\times\mathbf{v} = 0,\\ &\partial_t(\Delta_{x_h}(p'(\hat{\varrho})q) - q) + \nabla_{x_h}^\perp(p'(\hat{\varrho})q)\cdot\nabla_{x_h}(\Delta_{x_h}(P''(\hat{\varrho})q)) = 0,\end{aligned} \tag{4.3.21}$$

and $\mathbf{v}(x_h, x_3) = (\mathbf{v}_h(x_h), 0)$.

Remark 4.3.8. The system (4.3.21) has similarity with (4.3.19). Unfortunately, q in (4.3.21) is dependent on x_3, contrary to (4.3.19). The system (4.3.21) is a damped variant of the incompressible Euler system, see Zeitlin [124].

Case III: $\Lambda = 1$, $\frac{m}{2} > n \geq 1$; Target System: 2D Euler Fluid

We observe that a static solution $(\tilde{\varrho}_\epsilon, \mathbf{0})$ satisfying

$$\nabla_x p(\tilde{\varrho}_\epsilon) = \epsilon^{2(m-n)} \tilde{\varrho}_\epsilon \nabla_x G,$$

has the following property:

$$\lim_{\epsilon \to 0} \nabla_x P'(\tilde{\varrho}_\epsilon) = 0,$$

as soon as $G(x) = -x_3$ in Ω and $\frac{m}{2} > n$. Without loss of generality we assume

$$\tilde{\varrho}_\epsilon \approx \bar{\varrho} + \epsilon^{2(m-n)}.$$

From (4.3.17) and (4.3.18), we obtain

$$\bar{\varrho}\mathrm{div}_x \mathbf{v} + \epsilon^m (\partial_t \varrho_\epsilon^{(1)} + \mathrm{div}_x(\mathbf{v}_\epsilon^{(1)} \bar{\varrho} + \mathbf{v}\varrho_\epsilon^{(1)}) + o(\epsilon^{2m}) = 0$$

and

$$\begin{aligned} &\bar{\varrho}(\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla_x)\mathbf{v}) + \nabla_x \left(p'(\bar{\varrho})\varrho_\epsilon^{(2)} + \frac{1}{2} p''(\bar{\varrho})(\varrho_\epsilon^{(1)})^2 \right) \\ &+ \epsilon^{m-1} \mathbf{b} \times \mathbf{m}_\epsilon^{(1)} + \frac{1}{\epsilon^m} p'(\bar{\varrho}) \nabla_x \varrho_\epsilon^{(1)} + \frac{1}{\epsilon}(\bar{\varrho}\mathbf{b} \times \mathbf{v}) - \frac{1}{\epsilon^{2n}} \bar{\varrho} \nabla_x G + o(\epsilon^m) = 0, \end{aligned} \tag{4.3.22}$$

respectively, where $\mathbf{m}_\epsilon^{(1)} = \mathbf{v}_\epsilon^{(1)} \bar{\varrho} + \mathbf{v}\varrho_\epsilon^{(1)}$. In addition, we assume that $\varrho_\epsilon \mathbf{u}_\epsilon \to \bar{\varrho}\mathbf{v}$ and $\mathbf{u}_\epsilon \to \mathbf{v}$ in some strong sense.

Let $\mathbb{H}$ be the Helmholtz projection, then we have

$$\mathbb{H}\left(\partial_t(\varrho_\epsilon \mathbf{u}_\epsilon) + \mathrm{div}_x(\varrho_\epsilon \mathbf{u}_\epsilon \otimes \mathbf{u}_\epsilon) + \frac{1}{\epsilon} \mathbf{b} \times \varrho_\epsilon \mathbf{u}_\epsilon \right) = \mathbb{H}(\epsilon^\alpha \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}_\epsilon) + \frac{1}{\epsilon^{2n}} \varrho_\epsilon \nabla_x G.$$

Multiplying the above equation by ϵ, we get

$$\mathbb{H}[\mathbf{b} \times \bar{\varrho}\mathbf{v}] = 0.$$

This implies that there exists a scalar field ψ such that

$$\mathbf{b} \times \bar{\varrho}\mathbf{v} = \nabla_x \psi.$$

Now using the slip boundary(4.3.6) as in 'Case I', we have $\mathbf{v}(x) = (\mathbf{v}_h(x_h), 0)$. The limit system is identified as *incompressible Euler system* in $\mathbb{R}^2$, i.e.,

$$\begin{aligned} &\mathrm{div}_{x_h} \mathbf{v}_h = 0, \\ &\partial_t \mathbf{v}_h + (\mathbf{v}_h \cdot \nabla_{x_h})\mathbf{v}_h + \nabla_{x_h} \Pi = 0. \end{aligned} \tag{4.3.23}$$

Here we summarize the above discussion.

Multiple Scales		
Case	Relation between m, n and Λ	Target System
I	$\Lambda = 0$, $m = 1$	Quasi-geophysical flow, see (4.3.19).
II	$\Lambda = 1$, $m = 1$, $n = 1$	Stratified quasi-geophysical flow, see (4.3.21).
III	$\Lambda = 1$, $\frac{m}{2} > n \geq 1$	2D Euler Equation, see, (4.3.23)

4.3.3 Case I: Low mach and Rossby number limit in the absence of gravitational potential

First, we recall the consideration of **Case I**, i.e., $G = 0,\ m = 1$. The choice of initial data plays an important role in our analysis. Hence, we give an appropriate notion of *well-prepared* data for this case.

Definition 4.3.9. We say that the set of initial data $\{(\varrho_{0,\epsilon}, \mathbf{m}_{0,\epsilon})\}_{\{\epsilon>0\}}$ is *well-prepared* if,

$$
\begin{aligned}
&\varrho_{0,\epsilon} = \bar{\varrho} + \epsilon \varrho_{0,\epsilon}^{(1)},\ \{\varrho_{0,\epsilon}\}_{\{\epsilon>0\}} \text{ is bounded in } L^2 \cap L^\infty(\Omega),\ \varrho_{0,\epsilon}^{(1)} \to q_0 \text{ in } L^2(\Omega),\\
&\frac{\mathbf{m}_{0,\epsilon}}{\varrho_{0,\epsilon}} \to \mathbf{v}_0 = \left(\mathbf{v}_0^{(1)}, \mathbf{v}_0^{(2)}, 0\right) \text{ in } L^2(\Omega; \mathbb{R}^3\right) \text{ with the following relation}\\
&-\Delta_{x_h} q_0 = \bar{\varrho} \mathrm{Curl}_{x_h} P_h(\mathbf{v}_0).
\end{aligned}
\tag{4.3.24}
$$

Existence result for target system

In the last section we informally identify the limit system (4.3.19) and it has a similar structure to the 2D Euler equations. We expect the existence of a global in time strong solution for regular initial data. In particular, we may use the abstract theory of Oliver [106, Theorem 3], to obtain the result:

Proposition 4.3.10. *Suppose that*

$$q_0 \in W^{m,2}(\mathbb{R}^2) \text{ for } m \geq 4.$$

Then, the problem (4.3.19) *admits a solution q, unique in the class*

$$q \in C([0,T]; W^{m,2}(\mathbb{R}^2)) \cap C^1([0,T]; W^{m-1,2}(\mathbb{R}^2)).$$

Main Theorem

Here we state the main theorem that we want to prove.

Theorem 4.3.11. *Let $(\varrho_\epsilon, \mathbf{m}_\epsilon)$ be a dissipative solution of the system* (4.3.2)-(4.3.10) *with $\Lambda = 0$ and $m = 1$. Moreover, we assume that the initial data is well-prepared, i.e.*

it satisfies (4.3.24) *and* $q_0 \in W^{k,2}$ *with* $k \geq 4$. *Let* $(q, \mathbf{v})$ *solves* (4.3.19) *for initial data* q_0. *Then after taking a subsequence, the following holds,*

$$\begin{aligned} &\varrho_\epsilon^{(1)} \equiv \frac{\varrho_\epsilon - \bar{\varrho}}{\epsilon} \to q \text{ weak-(*)ly in } L^\infty(0,T; L^2 + L^{\gamma'}(\Omega)), \\ &\mathbf{m}_\epsilon \to \bar{\varrho}\mathbf{v} \text{ weak-(*)ly in } L^\infty(0,T; L^2 + L^{\frac{2\gamma}{\gamma+1}}(\Omega)), \end{aligned} \tag{4.3.25}$$

where $\gamma' = \min\{2, \gamma\}$. *Furthermore, we have*

$$\frac{\mathbf{m}_\epsilon}{\sqrt{\varrho_\epsilon}} \to \mathbf{v} \text{ strongly in } L^1_{loc}((0,T) \times \Omega; \mathbb{R}^3).$$

To prove the theorem, we will first try to obtain some bounds on the state variables.

Uniform bounds and weak convergence of variables

We consider $(\tilde{\mathbf{u}} = 0,\ \tilde{\varrho} = \bar{\varrho})$ as a test function in (4.3.15). Obviously, it belongs to the test function class (4.3.12). For this test function the relative energy inequality reduces to the energy inequality. From the consideration of the well-prepared data, we obtain

$$\int_\Omega \left(\frac{|\mathbf{m}_{0,\epsilon}|^2}{\varrho_\epsilon} + \frac{1}{\epsilon^2} P(\varrho_{0,\epsilon}) - (\varrho_{0,\epsilon} - \bar{\varrho})P'(\bar{\varrho}) - P(\bar{\varrho}) \right) \,\mathrm{d}x < E,$$

where E is independent of ϵ. Using (4.3.14), we have the following uniform bounds

$$\begin{aligned} &\operatorname*{ess\ sup}_{t\in(0,T)} \left\| \frac{\mathbf{m}_\epsilon}{\sqrt{\varrho_\epsilon}} \right\|_{L^2(\Omega;\mathbb{R}^3)} \leq C, \\ &\operatorname*{ess\ sup}_{t\in(0,T)} \left\| \left[\frac{\varrho_\epsilon - \bar{\varrho}}{\epsilon} \right]_{\text{ess}} \right\|_{L^2(\Omega)} \leq C, \\ &\operatorname*{ess\ sup}_{t\in(0,T)} \|[\varrho_\epsilon]_{\text{res}}\|^\gamma_{L^\gamma(\Omega)} + \operatorname*{ess\ sup}_{t\in(0,T)} \|[1]_{\text{res}}\|^\gamma_{L^\gamma(\Omega)} \leq \epsilon^2 C, \end{aligned}$$

where $[\,\cdot\,]_{\text{ess}}$ and $[\,\cdot\,]_{\text{res}}$ are following (4.3.13) with $\varrho_1 = \frac{1}{2}$ and $\varrho_2 = 2$.
Passing to a subsequence(not relabeled), we obtain

$$\begin{aligned} &\varrho_\epsilon^{(1)} \equiv \left(\frac{\varrho_\epsilon - \bar{\varrho}}{\epsilon} \right) \to \varrho^{(1)} \text{ weak-(*)ly in } L^\infty(0,T; L^2 + L^{\gamma'}(\Omega)), \\ &\frac{\mathbf{m}_\epsilon}{\sqrt{\varrho_\epsilon}} \to \hat{\mathbf{m}} \text{ weak-(*)ly in } L^\infty(0,T; L^2(\Omega;\mathbb{R}^3)) \end{aligned}$$

where $\gamma' = \min\{2, \gamma\}$. We define $\mathbf{u} = \frac{\hat{\mathbf{m}}}{\sqrt{\bar{\varrho}}}$. Furthermore, we deduce that

$$\varrho_\epsilon \to \bar{\varrho} \text{ in } L^\infty(0,T; L^2 + L^\gamma(\Omega)).$$

From the observation $\mathbf{m}_\epsilon = \sqrt{\varrho_\epsilon}\frac{\mathbf{m}_\epsilon}{\sqrt{\varrho_\epsilon}}$, it yields

$$\|\mathbf{m}_\epsilon\|_{L^\infty(0,T;L^2+L^{\grave{\gamma}}(\Omega;\mathbb{R}^3))} \leq C,$$

where $\grave{\gamma} = \min\left\{\frac{4}{3}, \frac{2\gamma}{\gamma+1}\right\}$. Again, passing to a subsequence(not relabeled), we get

$$\mathbf{m}_\epsilon \to \mathbf{m} \text{ weak-(*)ly in } L^\infty(0,T;L^2 + L^{2\gamma/\gamma+1}(\Omega;\mathbb{R}^3)).$$

From, the strong convergence of ϱ_ϵ, we infer that

$$\mathbf{m} = \bar{\varrho}\mathbf{u}$$

in the sense of distribution. Now, letting $\epsilon \to 0$ in the continuity equation, we have

$$\bar{\varrho}\int_0^\tau \int_\Omega \mathbf{u}\cdot\nabla_x\varphi \,\mathrm{d}x\,\mathrm{d}t = 0.$$

This is the weak form of the incompressibility condition. Furthermore multiplying momentum equation by ϵ and letting $\epsilon \to 0$ we get the diagnostic equation,

$$\mathbf{b}\times\mathbf{u} + \frac{p'(\bar{\varrho})}{\bar{\varrho}}\nabla_x\varrho^{(1)} = 0,$$

in the sense of distribution.
Clearly, from last relation we have $\varrho^{(1)}$ is independent of x_3, i.e. $\varrho^{(1)} = \varrho^{(1)}(x_h)$.

Relative energy and convergence to the target system

We recall here the target system:

$$\frac{p'(\bar{\varrho})}{\bar{\varrho}}\nabla_x q + \mathbf{b}\times\mathbf{v} = 0,$$
$$\partial_t\left(\Delta_{x_h} q - \frac{1}{p'(\bar{\varrho})}q\right) + \nabla_{x_h}^\perp q\cdot\nabla_{x_h}\left(\Delta_{x_h} q - \frac{1}{p'(\bar{\varrho})}q\right) = 0.$$

Let $q_0 \in W^{k,2}(\mathbb{R}^2)$ with $k \geq 4$ and $(q_0, \mathbf{v}_0)$ satisfies (4.3.24). Further we assume that $(q, \mathbf{v})$ is the strong solution of the target system (4.3.19) with initial data $(q_0, \mathbf{v}_0)$.

Our goal is to show that $(\varrho^{(1)}, \mathbf{u}) \equiv (q, \mathbf{v})$. Here we choose proper test functions and will show that the relative energy goes to zero as $\epsilon \to 0$, i.e., $\lim_{\epsilon\to 0}\mathcal{E}_\epsilon(t) = 0$.
We consider

$$\tilde{\mathbf{u}} = \mathbf{v},\ \tilde{\varrho} = \bar{\varrho} + \epsilon q.$$

The remark (4.3.7) helps us to consider such test functions. We rewrite the relative energy inequality as

$$\begin{aligned}
\mathcal{E}_\epsilon(\tau)+&\int_{\overline{\Omega}} \mathrm{d}\mathfrak{R}_{e_\epsilon}(\tau,\cdot)\\
&\leq \mathcal{E}_\epsilon(0) - \int_0^\tau \int_\Omega (\mathbf{m}_\epsilon - \varrho_\epsilon \mathbf{v})\cdot(\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla_{x_h})\mathbf{v})\ \mathrm{d}x\ \mathrm{d}t\\
&\quad - \int_0^\tau \int_\Omega \left(\frac{(\mathbf{m}_\epsilon - \varrho_\epsilon\mathbf{v})\otimes(\mathbf{m}_\epsilon - \varrho_\epsilon\mathbf{v})}{\varrho_\epsilon}\right) : \nabla_x \mathbf{v}\ \mathrm{d}x\ \mathrm{d}t\\
&\quad - \frac{1}{\epsilon^2}\int_0^\tau \int_\Omega (p(\varrho_\epsilon) - p(\tilde{\varrho}))\mathrm{div}_x \mathbf{v}\ \mathrm{d}x\ \mathrm{d}t\ + \frac{1}{\epsilon}\int_0^\tau \int_\Omega \mathbf{b}\times\mathbf{m}_\epsilon\cdot\mathbf{v}\ \mathrm{d}x\ \mathrm{d}t\\
&\quad + \frac{1}{\epsilon^2}\int_0^\tau \int_\Omega (\bar{\varrho} + \epsilon q - \varrho_\epsilon)P''(\tilde{\varrho})\ \partial_t(\bar{\varrho} + \epsilon q)\ \mathrm{d}x\ \mathrm{d}t\\
&\quad + \frac{1}{\epsilon^2}\int_0^\tau \int_\Omega ((\bar{\varrho} + \epsilon q)\mathbf{v} - \mathbf{m}_\epsilon)\nabla_x P''(\tilde{\varrho})\nabla_x(\bar{\varrho} + \epsilon q)\ \mathrm{d}x\ \mathrm{d}t\\
&\quad - \int_0^\tau \int_{\overline{\Omega}} \nabla_x \tilde{\mathbf{u}} : \mathrm{d}\mathfrak{R}_{m_\epsilon}(t,\cdot)\ \mathrm{d}t\ .
\end{aligned}$$

Using the fact $\mathrm{div}_{x_h}\mathbf{v} = 0$, we obtain

$$\begin{aligned}
\mathcal{E}_\epsilon(\tau)+&\int_{\overline{\Omega}} \mathrm{d}\mathfrak{R}_{e_\epsilon}(\tau,\cdot)\\
&\leq \mathcal{E}_\epsilon(0) - \int_0^\tau \int_\Omega (\mathbf{m}_\epsilon - \varrho_\epsilon \mathbf{v})\cdot(\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla_{x_h})\mathbf{v})\ \mathrm{d}x\ \mathrm{d}t\\
&\quad - \int_0^\tau \int_\Omega \left(\frac{(\mathbf{m}_\epsilon - \varrho_\epsilon\mathbf{v})\otimes(\mathbf{m}_\epsilon - \varrho_\epsilon\mathbf{v})}{\varrho_\epsilon}\right) : \nabla_x \mathbf{v}\ \mathrm{d}x\ \mathrm{d}t\\
&\quad + \frac{1}{\epsilon}\int_0^\tau \int_\Omega P''(\bar{\varrho})\mathbf{m}_\epsilon\cdot\nabla_{x_h} q\ \mathrm{d}x\ \mathrm{d}t\ + \int_0^\tau \int_\Omega (q - \varrho_\epsilon^1)P''(\bar{\varrho} + \epsilon q)\ \partial_t q\ \mathrm{d}x\ \mathrm{d}t\\
&\quad - \frac{1}{\epsilon}\int_0^\tau \int_\Omega \mathbf{m}_\epsilon\cdot P''(\bar{\varrho} + \epsilon q)\nabla_x q\ \mathrm{d}x\ \mathrm{d}t\ - \int_0^\tau \int_{\overline{\Omega}} \nabla_x \tilde{\mathbf{u}} : \mathrm{d}\mathfrak{R}_{m_\epsilon}(t,\cdot)\ \mathrm{d}t\\
&= \Sigma_{i=1}^7 \mathcal{L}_i.
\end{aligned}$$

Consideration of *well prepared data* yields

$$\mathcal{E}_\epsilon(\varrho_{0,\epsilon}, \mathbf{m}_{0,\epsilon} \mid \bar{\varrho} + \epsilon q_0, \mathbf{v}_0) \leq \left\| \frac{\mathbf{m}_{0,\epsilon}}{\varrho_{0,\epsilon}} - \mathbf{v}_0 \right\|^2_{L^2(\Omega)} + \|\varrho^{(1)}_{0,\epsilon} - q_0\|^2_{L^2(\Omega)}.$$

From this we conclude

$$|\mathcal{L}_1| \leq c(\epsilon). \tag{4.3.26}$$

From now on we use $c(\epsilon)$ as a generic function such that $c(\epsilon) \to 0$ as $\epsilon \to 0$.
We have the following observation

$$\mathbf{m}_\epsilon - \varrho_\epsilon \mathbf{v} = (\mathbf{m}_\epsilon - \bar{\varrho}\mathbf{u}) + ((\bar{\varrho} - \varrho_\epsilon)\mathbf{u}) + (\varrho_\epsilon(\mathbf{u} - \mathbf{v}))$$

and

$$P''(\bar{\varrho}+\epsilon q) = P''(\bar{\varrho}+\epsilon q) - P''(\bar{\varrho}) + P''(\bar{\varrho}).$$

This together with the weak convergence of state variables implies

$$\begin{aligned}|\mathcal{L}_2+\mathcal{L}_5| \leq & c(\epsilon) + \int_0^\tau \int_\Omega \bar{\varrho}(\mathbf{v}-\mathbf{u})\cdot(\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla_{x_h})\mathbf{v}) \ \mathrm{d}x \ \mathrm{d}t \\ & + \int_0^\tau \int_\Omega (q-\varrho^{(1)})P''(\bar{\varrho}) \ \partial_t q \ \mathrm{d}x \ \mathrm{d}t \ .\end{aligned}$$

The fact $(q, \mathbf{v})$ is a strong solution of (4.3.19) implies

$$\begin{aligned}&\int_\Omega \bar{\varrho}(\mathbf{v}-\mathbf{u})\cdot(\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla_{x_h})\mathbf{v}) \ \mathrm{d}x \ \mathrm{d}t \ + \int_0^\tau \int_\Omega (q-\varrho^{(1)})P''(\bar{\varrho}) \ \partial_t q \ \mathrm{d}x \ \mathrm{d}t \\ &= \frac{p'(\bar{\varrho})^2}{\bar{\varrho}} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(|\nabla_{x_h} q|^2 + \frac{1}{p'(\bar{\varrho})}|q|^2\right) \ \mathrm{d}x - \bar{\varrho}\int_\Omega \mathbf{u}\cdot\nabla_{x_h}|\mathbf{v}|^2 \ \mathrm{d}x.\end{aligned}$$

Eventually, using properties of $q, \mathbf{u}$ and $\mathbf{v}$ we obtain

$$\int_0^\tau \left(\int_\Omega (\bar{\varrho}(\mathbf{v}-\mathbf{u})\cdot(\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla_{x_h})\mathbf{v}) \ \mathrm{d}x \ \mathrm{d}t \ + (q-\varrho^{(1)})\partial_t q P''(\bar{\varrho})) \ \mathrm{d}x \right) \ \mathrm{d}t \ = 0.$$

Thus we have

$$|\mathcal{L}_2+\mathcal{L}_5| \leq c(\epsilon). \tag{4.3.27}$$

From the definition of relative energy, we deduce

$$|\mathcal{L}_3| \leq \int_0^\tau \mathcal{E}_\epsilon(t) \ \mathrm{d}t \tag{4.3.28}$$

It is easy to verify that

$$\frac{1}{\epsilon}\big(P''(\bar{\varrho}+\epsilon q) - P''(\bar{\varrho})\big) \to P'''(\bar{\varrho})q \text{ in } L^\infty(0,T;L^\infty \cap L^2(\Omega)), \text{ as } \epsilon \to 0.$$

The above statement helps us to get

$$\mathcal{L}_4+\mathcal{L}_6 \to \int_0^\tau \int_\Omega \bar{\varrho}P'''(\bar{\varrho})q\mathbf{v}\cdot(\nabla_{x_h} q, 0) \ \mathrm{d}x \ \mathrm{d}t \ = 0.$$

Therefore, we obtain

$$|\mathcal{L}_4+\mathcal{L}_6| \leq c(\epsilon). \tag{4.3.29}$$

Using compatibility of defect measures, we have the following estimate:

$$|\mathcal{L}_7| \leq \int_0^\tau \int_{\overline{\Omega}} \mathrm{d}\mathfrak{R}_{e_\epsilon}(t,\cdot) \ \mathrm{d}t \ . \tag{4.3.30}$$

Combining (4.3.26)-(4.3.30), we infer that

$$\mathcal{E}_\epsilon(\tau) + \int_\Omega \mathrm{d}\mathfrak{R}_{e_\epsilon}(\tau,\cdot) \leq c(\epsilon) + \int_0^\tau \mathcal{E}_\epsilon(t)\ \mathrm{d}t\ + C\int_0^\tau \int_\Omega \mathrm{d}\mathfrak{R}_{e_\epsilon}(t,\cdot)\ \mathrm{d}t\ ,$$

for a.e. $\tau \in (0,T)$.

Finally, using Grönwall's lemma 1.1.7, we have

$$\mathcal{E}_\epsilon(\tau) + \int_{\overline{\Omega}} \mathrm{d}\mathfrak{R}_{e_\epsilon}(\tau,\cdot) \leq c(\epsilon)C(T),$$

where $c(\epsilon) \to 0$ as $\epsilon \to 0$. Hence, we obtain our deired result

$$\lim_{\epsilon\to 0} \mathcal{E}_\epsilon(\tau) = 0. \tag{4.3.31}$$

Now we use the coerceivity of relative energy functional(4.3.14) and conclude

$$\begin{aligned} &\frac{\varrho_\epsilon - \bar{\varrho}}{\epsilon} \to q \textit{ strongly} \text{ in } L^1_{\mathrm{loc}}((0,T)\times\Omega),\\ &\frac{\mathbf{m}_\epsilon}{\sqrt{\varrho_\epsilon}} \to \sqrt{\bar{\varrho}}\mathbf{v} \textit{ strongly} \text{ in } L^1_{\mathrm{loc}}((0,T)\times\Omega;\mathbb{R}^3). \end{aligned}$$

It ends the proof of the Theorem 4.3.11.

4.3.4 Case II: Low Mach and Rossby number limit in the presence of strong stratification

In this case $\Lambda = 1$ and $m = n = 1$ implies that a static solution $\tilde{\varrho}_\epsilon$ is independent of ϵ. We denote it by $\hat{\varrho}$ and it satisfies

$$\nabla_x p(\hat{\varrho}) = \hat{\varrho}\nabla_x G.$$

Eventually, we also have

$$P'(\hat{\varrho}) = G + C,$$

where C is a constant and as $G(x) = -x_3$ in Ω we also have $\hat{\varrho}(x_h, x_3) = \hat{\varrho}(x_3)$ in Ω. In particular, we choose a particular static solution ($\hat{\varrho} = \hat{\varrho}(x_3)$) such that

$$0 < \hat{\varrho} \in C^3(\Omega) \cap W^{1,\infty}(\Omega).$$

First we recall the target system. It states that for each$x_3 \in [0,1]$, $(q(x_h,x_3), \mathbf{v}(x_h,x_3) = (\mathbf{v}_h(x_h), 0))$ solves

$$\nabla_x(P''(\hat{\varrho}(x_3))q) + \mathbf{b}\times\mathbf{v} = 0, \tag{4.3.32}$$

$$\partial_t(\Delta_{x_h}(p'(\hat{\varrho}(x_3))q) - q) + \nabla^\perp_{x_h}(p'(\hat{\varrho}(x_3))q)\cdot\nabla_{x_h}(\Delta_{x_h}(P''(\hat{\varrho}(x_3))q)) = 0, \tag{4.3.33}$$

supplemented with initial data $q(0,\cdot) = q_0$ in Ω. From the first equation we have

$$\left(P''(\hat{\varrho})q\right)_{x_3} = 0.$$

This leads to consider $q(x_h, x_3) = f(\hat{\varrho}(x_3))\hat{q}(x_h)$, for some smooth function $f : (0,\infty) \to (0,\infty)$, such that

$$f(\hat{\varrho}) = \frac{C}{P'(\hat{\varrho})},$$

with $C > 0$. The above system is well defined in $\mathbb{R}^2 \times (0,1)$, for each $x_3 \in (0,1)$, it satisfies an equation similar to (4.3.19).

If assume $q_0 \in W^{k,2}(\mathbb{R}^d)$ with $k > 4$. For each $x_3 \in [0,1]$, the equation (4.3.33) admits strong solution. Furthermore, following Oliver [106, Theorem 3], we have for each $x_3 \in [0,1]$, $q(\cdot, x_3) \in C([0,T]; W^{k,2}(\mathbb{R}^2)) \cap C^1([0,T]; W^{k-1,2}(\mathbb{R}^2))$.

Finally from the above discussion we state the regularity of the target system as

Proposition 4.3.12. *Suppose that*

$$q_0 \in W^{k,2}(\mathbb{R}^2) \text{ for } k \geq 4.$$

Then, the problem (4.3.21) *admits a solution q, unique in the class*

$$q \in C([0,T]; W^{k,2}(\mathbb{R}^2) \times C^3([0,1])) \cap C^1([0,T]; W^{k-1,2}(\mathbb{R}^2) \times C^3([0,1])).$$

We define the well prepared data as

Definition 4.3.13. We say that the set of initial data $\{(\varrho_{0,\epsilon}, \mathbf{m}_{0,\epsilon})\}_{\{\epsilon>0\}}$ is *well-prepared* if,

$$\begin{aligned}
&\varrho_{0,\epsilon} = \hat{\varrho} + \epsilon\varrho_{0,\epsilon}^{(1)},\ \{\varrho_{0,\epsilon}\}_{\{\epsilon>0\}} \text{ is bounded in } L^2 \cap L^\infty(\Omega),\ \varrho_{0,\epsilon}^{(1)} \to q_0 \text{ in } L^2(\Omega),\\
&\frac{\mathbf{m}_{0,\epsilon}}{\varrho_{0,\epsilon}} \to \mathbf{v}_0 = \left(\mathbf{v}_0^{(1)}, \mathbf{v}_0^{(2)}, 0\right) \text{ in } L^2(\Omega;\mathbb{R}^3\big) \text{ with the relation}\\
&-\Delta_{x_h} q_0 = \hat{\varrho}\operatorname{Curl}_{x_h} P_h(\mathbf{v}_0),\ \text{ and } q_0 \in L^2(\mathbb{R}^2).
\end{aligned} \tag{4.3.34}$$

Theorem 4.3.14. *Let $(\varrho_\epsilon, \mathbf{m}_\epsilon)$ be a dissipative solution of the system* (4.3.2)-(4.3.10) *with $\Lambda = 1$ and $m = n = 1$. Moreover, we assume that the initial data is well-prepared, i.e., it satisfies* (4.3.34) *and $q_0 \in W^{k,2}$ with $k \geq 4$. Let $(q, \mathbf{v})$ solves* (4.3.21) *for initial data q_0. Then after taking a subsequence, the following holds,*

$$\begin{aligned}
&\varrho_\epsilon^{(1)} \equiv \frac{\varrho_\epsilon - \hat{\varrho}}{\epsilon} \to q \textit{ weak-(*)ly in } L^\infty(0,T; L^2 + L^{\gamma'}(\Omega)),\\
&\mathbf{m}_\epsilon \to \bar{\varrho}\mathbf{v} \textit{ weak-(*)ly in } L^\infty(0,T; L^2 + L^{\frac{2\gamma}{\gamma+1}}(\Omega)),
\end{aligned}$$

where $\gamma' = \min\{2, \gamma\}$. Furthermore, we have

$$\frac{\mathbf{m}_\epsilon}{\sqrt{\varrho_\epsilon}} \to \mathbf{v} \textit{ strongly in } L^1_{loc}((0,T) \times \Omega; \mathbb{R}^3).$$

Before heading towards the proof, we recall the relative energy inequality for this case,

$$
\begin{aligned}
&\mathcal{E}_\epsilon(\tau) + \int_{\overline{\Omega}} \mathrm{d}\, \mathfrak{R}_{e_\epsilon}(\tau,\cdot) \\
&\leq \mathcal{E}_\epsilon(0) - \int_0^\tau \int_\Omega (\mathbf{m}_\epsilon - \varrho_\epsilon \tilde{\mathbf{u}}) \cdot \partial_t \tilde{\mathbf{u}} \,\mathrm{d}x\,\mathrm{d}t - \int_0^\tau \int_\Omega \left(\frac{(\mathbf{m}_\epsilon - \varrho_\epsilon \tilde{\mathbf{u}}) \otimes \mathbf{m}_\epsilon}{\varrho_\epsilon} \right) : \nabla_x \tilde{\mathbf{u}} \,\mathrm{d}x\,\mathrm{d}t \\
&- \frac{1}{\epsilon^2} \int_0^\tau \int_\Omega (p(\varrho_\epsilon) - p(\tilde{\varrho})) \mathrm{div}_x \tilde{\mathbf{u}} \,\mathrm{d}x \,\mathrm{d}t \, + \frac{1}{\epsilon^2} \int_0^\tau \int_\Omega (\tilde{\varrho} - \varrho_\epsilon) \partial_t P'(\tilde{\varrho}) \,\mathrm{d}x \,\mathrm{d}t \\
&+ \frac{1}{\epsilon} \int_0^\tau \int_\Omega \mathbf{b} \times \mathbf{m}_\epsilon \cdot \tilde{\mathbf{u}} \,\mathrm{d}x \,\mathrm{d}t \\
&+ \frac{1}{\epsilon^2} \int_0^\tau \int_\Omega (\tilde{\varrho} \tilde{\mathbf{u}} - \mathbf{m}_\epsilon) \cdot (\nabla_x P'(\tilde{\varrho}) - \nabla_x P'(\tilde{\varrho}_\epsilon)) \,\mathrm{d}x \,\mathrm{d}t \\
&- \frac{1}{\epsilon^2} \int_0^\tau \int_\Omega (\varrho_\epsilon - \tilde{\varrho}) \nabla_x G \cdot \tilde{\mathbf{u}} \,\mathrm{d}x \,\mathrm{d}t \, - \int_0^\tau \int_\Omega \nabla_x \tilde{\mathbf{u}} : \mathrm{d}\mathfrak{R}_{m_\epsilon}(t,\cdot) \,\mathrm{d}t \, ,
\end{aligned}
$$

with $(\tilde{\varrho}, \tilde{\mathbf{u}})$ satisfies (4.3.12) with $\tilde{\varrho}_\epsilon = \hat{\varrho}$.

Uniform bounds and weak convergence

To obtain a uniform bound, we proceed similarly to Section 4.2.3. First, using $\tilde{\mathbf{u}} = 0,\ \tilde{\varrho} = \hat{\varrho}$ as test functions and well-prepared data(4.3.34), we obtain the following uniform bounds:

$$
\begin{aligned}
&\operatorname*{ess\ sup}_{t\in(0,T)} \left\| \frac{\mathbf{m}_\epsilon}{\sqrt{\varrho_\epsilon}} \right\|_{L^2(\Omega;\mathbb{R}^3)} \leq C, \\
&\operatorname*{ess\ sup}_{t\in(0,T)} \left\| \left[\frac{\varrho_\epsilon - \hat{\varrho}}{\epsilon} \right]_{\mathrm{ess}} \right\|_{L^2(\Omega)} \leq C, \\
&\operatorname*{ess\ sup}_{t\in(0,T)} \|[\varrho_\epsilon]_{\mathrm{res}}\|^\gamma_{L^\gamma(\Omega)} + \operatorname*{ess\ sup}_{t\in(0,T)} \|[1]_{\mathrm{res}}\|^\gamma_{L^\gamma(\Omega)} \leq \epsilon^2 C.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
&\varrho_\epsilon^{(1)} \equiv \left(\frac{\varrho_\epsilon - \hat{\varrho}}{\epsilon} \right) \to \varrho^{(1)} \text{ weak-(*)ly in } L^\infty(0,T; L^2 + L^{\gamma'}(\Omega)), \\
&\frac{\mathbf{m}_\epsilon}{\sqrt{\varrho_\epsilon}} \to \hat{\mathbf{m}} \text{ weak-(*)ly in } L^\infty(0,T; L^2(\Omega;\mathbb{R}^3)),
\end{aligned}
$$

passing to a suitable subsequence as the case may be, here $\gamma' = \min\{2,\gamma\}$.
We also deduce that

$$
\varrho_\epsilon \to \hat{\varrho} \text{ in } L^\infty(0,T; L^2 + L^{\gamma'}(\Omega)).
$$

Furthermore, we have

$$
\|\mathbf{m}_\epsilon\|_{L^\infty(0,T;L^2 + L^{\hat{\gamma}}(\Omega;\mathbb{R}^3))} \leq C,
$$

where $\grave{\gamma} = \left\{ \frac{4}{3}, \frac{2\gamma}{\gamma+1} \right\}$. Eventually, for a suitable subsequence, we get

$$\mathbf{m}_\epsilon \to \mathbf{m} \text{ weak-(*)ly in } L^\infty(0,T; L^2 + L^{\grave{\gamma}}(\Omega; \mathbb{R}^3)).$$

Letting $\epsilon \to 0$ in the continuity equation, we infer the incompressibility condition in the weak sense, i.e.,

$$\int_0^\tau \int_\Omega \mathbf{m} \cdot \nabla_x \varphi \, \mathrm{d}x \, \mathrm{d}t = 0.$$

Define $\mathbf{u} = \frac{\mathbf{m}}{\hat{\varrho}}$. Multiplying momentum equation by ϵ and letting $\epsilon \to 0$, we obtain the diagnostic equation

$$\mathbf{b} \times \mathbf{u} + \nabla_x \left(P''(\hat{\varrho}) \varrho^{(1)} \right) = 0, \tag{4.3.35}$$

in the the sense of distributions.

Strong convergence using relative energy inequality

Let $(q, \mathbf{v})$ be a strong solution of the above system with initial data $(q_0, \mathbf{v}_0)$ satisfying (4.3.34) with $k \geq 4$. Our goal is to show that $(\varrho^{(1)}, \mathbf{u}) \equiv (q, \mathbf{v})$. Here we choose appropriate test functions and will show that $\lim_{\epsilon \to 0} \mathcal{E}_\epsilon(t) = 0$.
We consider the test functions for the relative energy inequality (4.3.15) as

$$\tilde{\mathbf{u}} = \mathbf{v} = (\mathbf{v}_h, 0), \; \tilde{\varrho} = \hat{\varrho} + \epsilon q.$$

Thus, we rewrite the relative energy inequality in the following form:

$$\begin{aligned}
\mathcal{E}_\epsilon(\tau) + &\int_{\overline{\Omega}} \mathrm{d}\mathfrak{R}_{e_\epsilon}(\tau) \\
&\leq \mathcal{E}_\epsilon(0) - \int_0^\tau \int_\Omega (\mathbf{m}_\epsilon - \varrho_\epsilon \mathbf{v}) \cdot (\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla_x) \mathbf{v}) \, \mathrm{d}x \, \mathrm{d}t \\
&\quad - \int_0^\tau \int_\Omega \left(\frac{(\mathbf{m}_\epsilon - \varrho_\epsilon \mathbf{v}) \otimes (\mathbf{m}_\epsilon - \varrho_\epsilon \mathbf{v})}{\varrho_\epsilon} \right) : \nabla_x \mathbf{v} \, \mathrm{d}x \, \mathrm{d}t \\
&\quad - \frac{1}{\epsilon^2} \int_0^\tau \int_\Omega (p(\varrho_\epsilon) - p(\tilde{\varrho})) \mathrm{div}_x \mathbf{v} \, \mathrm{d}x \, \mathrm{d}t - \frac{1}{\epsilon} \int_0^\tau \int_\Omega \mathbf{b} \times \mathbf{v} \cdot (\mathbf{m}_\epsilon - \hat{\varrho} \mathbf{v}) \, \mathrm{d}x \, \mathrm{d}t \\
&\quad + \frac{1}{\epsilon^2} \int_0^\tau \int_\Omega (\hat{\varrho} + \epsilon q - \varrho_\epsilon) P''(\hat{\varrho} + \epsilon q) \, \partial_t (\hat{\varrho} + \epsilon q) \, \mathrm{d}x \, \mathrm{d}t \\
&\quad + \frac{1}{\epsilon^2} \int_0^\tau \int_\Omega ((\hat{\varrho} + \epsilon q) \mathbf{v} - \mathbf{m}_\epsilon) \cdot \nabla_x (P'(\hat{\varrho} + \epsilon q) - P'(\hat{\varrho})) \, \mathrm{d}x \, \mathrm{d}t \\
&\quad - \int_0^\tau \int_{\overline{\Omega}} \nabla_x \mathbf{v} : \mathrm{d}\mathfrak{R}_{m_\epsilon}(t, \cdot) \, \mathrm{d}t \, .
\end{aligned}$$

Using the fact $\mathrm{div}_{x_h}\mathbf{v}=0$ and (4.3.32), we obtain

$$
\begin{aligned}
\mathcal{E}_\epsilon(\tau)+&\int_{\overline{\Omega}} \mathrm{d}\mathfrak{R}_{e_\epsilon}(\tau)\\
&\leq \mathcal{E}_\epsilon(0)-\int_0^\tau\int_\Omega (\mathbf{m}_\epsilon-\varrho_\epsilon\mathbf{v})\cdot(\partial_t\mathbf{v}+(\mathbf{v}\cdot\nabla_x)\mathbf{v})\ \mathrm{d}x\ \mathrm{d}t\\
&\quad+\frac{1}{\epsilon}\int_0^\tau\int_\Omega (\hat{\varrho}+\epsilon q-\varrho_\epsilon)P''(\hat{\varrho}+\epsilon q)\ \partial_t q\ \mathrm{d}x\ \mathrm{d}t\\
&\quad-\int_0^\tau\int_\Omega \left(\frac{(\mathbf{m}_\epsilon-\varrho_\epsilon\mathbf{v})\otimes(\mathbf{m}_\epsilon-\varrho_\epsilon\mathbf{v})}{\varrho_\epsilon}\right):\nabla_x\mathbf{v}\ \mathrm{d}x\ \mathrm{d}t\\
&\quad-\frac{1}{\epsilon^2}\int_0^\tau\int_\Omega (\hat{\varrho}\mathbf{v}-\mathbf{m}_\epsilon)\cdot\nabla_x(P'(\hat{\varrho}+\epsilon q)-P'(\hat{\varrho})-\epsilon P''(\hat{\varrho})q)\ \mathrm{d}x\ \mathrm{d}t\\
&\quad+\frac{1}{\epsilon}\int_0^\tau\int_\Omega q\mathbf{v}\cdot\nabla_x(P'(\hat{\varrho}+\epsilon q)-P'(\hat{\varrho}))\ \mathrm{d}x\ \mathrm{d}t\\
&\quad-\int_0^\tau\int_{\overline{\Omega}} \nabla_x\mathbf{v}:\mathrm{d}\mathfrak{R}_{m_\epsilon}(t,\cdot)\ \mathrm{d}t\ =\Sigma_{i=1}^7\mathcal{L}_i.
\end{aligned}
$$

Now we want to estimate each term $\mathcal{L}_i$ for $i=1,\cdots,7$. First we notice that, consideration of the *well prepared data*(4.3.34) yields

$$
\mathcal{E}_\epsilon(\varrho_{0,\epsilon},\mathbf{m}_{0,\epsilon}\mid\bar{\varrho}+\epsilon q_0,\mathbf{v}_0)\leq\left\|\frac{\mathbf{m}_{0,\epsilon}}{\varrho_{0,\epsilon}}-\mathbf{v}_0\right\|^2_{L^2(\Omega)}+\left\|\varrho^{(1)}_{0,\epsilon}-q_0\right\|^2_{L^2(\Omega)}.
$$

This implies

$$
|\mathcal{L}_1|\leq c(\epsilon). \tag{4.3.36}
$$

Here $c(\epsilon)$ is a generic function such that $c(\epsilon)\to 0$ as $\epsilon\to 0$.
First, we rewrite two terms of $\mathcal{L}_2$ and $\mathcal{L}_3$ as

$$
\mathbf{m}_\epsilon-\varrho_\epsilon\mathbf{v}=(\mathbf{m}_\epsilon-\bar{\varrho}\mathbf{u})+((\bar{\varrho}-\varrho_\epsilon)\mathbf{u})+(\varrho_\epsilon(\mathbf{u}-\mathbf{v}))
$$

and

$$
P''(\bar{\varrho}+\epsilon q)=P''(\bar{\varrho}+\epsilon q)-P''(\bar{\varrho})+P''(\bar{\varrho}).
$$

Using the weak convergence of the variables, we obtain

$$
\begin{aligned}
|\mathcal{L}_2+\mathcal{L}_3|\leq& c(\epsilon)+\int_0^\tau\int_\Omega(\hat{\varrho}\mathbf{v}-\mathbf{m})\cdot(\partial_t\mathbf{v}+(\mathbf{v}\cdot\nabla_x)\mathbf{v})\ \mathrm{d}x\ \mathrm{d}t\\
&+\int_0^\tau\int_\Omega(q-\varrho^{(1)})\ \partial_t(P''(\hat{\varrho})q)\ \mathrm{d}x\ \mathrm{d}t\ .
\end{aligned}
$$

We claim that

$$
\int_0^\tau\int_\Omega(\hat{\varrho}\mathbf{v}-\mathbf{m})\cdot(\partial_t\mathbf{v}+(\mathbf{v}\cdot\nabla_x)\mathbf{v})\ \mathrm{d}x\ \mathrm{d}t+\int_0^\tau\int_\Omega(q-\varrho^{(1)})\ \partial_t(P''(\hat{\varrho})q)\ \mathrm{d}x\ \mathrm{d}t=0. \tag{4.3.37}
$$

To prove the above claim, first we observe that

$$\mathbf{v}\cdot\partial_t(\hat{\varrho}\mathbf{v})+q\partial_t(P'(\hat{\varrho})q)=\frac{1}{2}\partial_t\left(\hat{\varrho}|\mathbf{v}|^2+P'(\hat{\varrho})q^2\right).$$

Since, $(q,\mathbf{v})$ solves (4.3.21), multiplying (4.3.33) by q we get

$$\int_0^\tau\int_\Omega \mathbf{v}\cdot\partial_t(\hat{\varrho}\mathbf{v})+q\partial_t(P'(\hat{\varrho})q)\ \mathrm{d}x\ \mathrm{d}t\ =0.$$

Now we use (4.3.35) and (4.3.32) to deduce

$$\int_0^\tau\int_\Omega\Big(\mathbf{m}\cdot\partial_t\mathbf{v}+\varrho^{(1)}\partial_t\left(P''(\hat{\varrho})q\right)\Big)\ \mathrm{d}x\ \mathrm{d}t\ =\int_0^\tau\int_\Omega \mathbf{b}\times\mathbf{m}\cdot\left(P''(\hat{\varrho})\mathbf{v}\right)\ \mathrm{d}x\ \mathrm{d}t$$

and

$$\begin{aligned}&\int_0^\tau\int_\Omega \mathbf{b}\times\mathbf{m}\cdot\left(P''(\hat{\varrho})\mathbf{v}\right)\ \mathrm{d}x\ \mathrm{d}t\ +\int_0^\tau\int_\Omega \mathbf{m}\cdot(\mathbf{v}\cdot\nabla_x)\mathbf{v}\ \mathrm{d}x\ \mathrm{d}t\\&=\int_0^\tau \mathbf{m}\cdot\nabla_{x_h}\left(\frac{1}{2}|\mathbf{v}|^2\right)\ \mathrm{d}x\ \mathrm{d}t\ =0.\end{aligned}$$

Hence we achieve (4.3.37) and it implies

$$|\mathcal{L}_2+\mathcal{L}_3|\le c(\epsilon). \tag{4.3.38}$$

From the definition of relative energy, we obtain

$$|\mathcal{L}_4|\le\int_0^\tau \mathcal{E}_\epsilon(t)\ \mathrm{d}t\ . \tag{4.3.39}$$

Since $0<\hat{\varrho}\in C^3([0,1])$, we verify that

$$\frac{1}{\epsilon}\nabla_x\big(P'(\hat{\varrho}+\epsilon q)-P'(\hat{\varrho})\big)\to\nabla_x\left(P''(\hat{\varrho})q\right)\ \text{ in } L^\infty(0,T;L^\infty\cap L^2(\Omega))$$

and

$$\frac{1}{\epsilon^2}\nabla_x(P'(\hat{\varrho}+\epsilon q)-P'(\hat{\varrho})-\epsilon P''(\hat{\varrho})q)\to\nabla_x\left(\frac{1}{2}P'''(\hat{\varrho})\right)\ \text{ in } L^\infty(0,T;L^\infty\cap L^2(\Omega)).$$

The above relation implies

$$\lim_{\epsilon\to 0}\mathcal{L}_5=\int_0^\tau\int_\Omega(\hat{\varrho}\mathbf{v}-\mathbf{m})\cdot\nabla_x\left(\frac{1}{2}P'''(\hat{\varrho})\right)\ \mathrm{d}x\ \mathrm{d}t\ =0 \tag{4.3.40}$$

and

$$\lim_{\epsilon\to 0}\mathcal{L}_6=\int_0^\tau\int_\Omega q\mathbf{v}\to\nabla_x\left(P''(\hat{\varrho})q\right)\ \mathrm{d}x\ \mathrm{d}t\ =0 \tag{4.3.41}$$

We obtain

$$|\mathcal{L}_7| \leq C \int_0^\tau \int_{\overline{\Omega}} \mathbb{I} : \mathrm{d}\mathfrak{C}_{m_\epsilon}(t,\cdot)\ \mathrm{d}t\ . \tag{4.3.42}$$

Combining (4.3.36)-(4.3.42), we have that

$$\mathcal{E}_\epsilon(\tau) + \int_{\overline{\Omega}} \mathrm{d}\mathfrak{C}_{e_\epsilon} \leq c(\epsilon) + \int_0^\tau \mathcal{E}_\epsilon(t)\ \mathrm{d}t\ + C \int_0^\tau \int_{\overline{\Omega}} \mathrm{d}\ \mathrm{Tr}(\mathfrak{C}_{m_\epsilon})\ \mathrm{d}t\ , \tag{4.3.43}$$

for a.e. $\tau \in (0,T)$. Finally, using the compatibility of *turbulent defect measures* and Grönwall's lemma (1.1.7), we infer

$$\mathcal{E}_\epsilon(\tau) + \int_{\overline{\Omega}} \mathrm{d}\mathfrak{C}_{e_\epsilon} \leq c(\epsilon) \leq c(\epsilon)C(T), \tag{4.3.44}$$

where $c(\epsilon) \to 0$ as $\epsilon \to 0$. Therefore, we obtain our desired result,

$$\lim_{\epsilon \to 0} \mathcal{E}_\epsilon(\tau) = 0. \tag{4.3.45}$$

Now using coerceivity of the relative energy functional (4.3.14), we say

$$\varrho^{(1)} = q \text{ and } \mathbf{v} = \mathbf{u}.$$

Moreover, we use coercievity together with (4.3.45) to conclude

$$\begin{aligned} &\frac{\varrho_\epsilon - \bar{\varrho}}{\epsilon} \to q \textit{ strongly} \text{ in } L^1_{\mathrm{loc}}((0,T) \times \Omega), \\ &\frac{\mathbf{m}_\epsilon}{\sqrt{\varrho_\epsilon}} \to \sqrt{\bar{\varrho}}\mathbf{v} \textit{ strongly} \text{ in } L^1_{\mathrm{loc}}((0,T) \times \Omega; \mathbb{R}^3). \end{aligned}$$

This completes the proof of the Theorem 4.3.14.

4.3.5 Case III: Low Mach and Rossby number limit in the presence of low stratification

In this case we consider $\Lambda = 1, \frac{m}{2} > n \geq 1$.

Properties of a static solution

First, we notice that a static solution $(\tilde{\varrho}_\epsilon, \mathbf{0})$ satisfies

$$\nabla_x p(\tilde{\varrho}_\epsilon) = \epsilon^{2(m-n)} \tilde{\varrho}_\epsilon \nabla_x G.$$

In terms of the pressure potential, we rewrite the above equation as

$$\nabla_x P'(\tilde{\varrho}_\epsilon) = \epsilon^{2(m-n)} \nabla_x G.$$

So, we obtain

$$P'(\tilde{\varrho}_\epsilon) = -\epsilon^{2(m-n)} x_3 + C,$$

where C is a constant. As a consequence of $G = (0, 0, -x_3)$, we have $\tilde{\varrho}_\epsilon(x) = \tilde{\varrho}_\epsilon(x_3)$. Without loss of generality, we consider $C = 1$. We know that $P'(s) \approx s^{\gamma-1}$ for $s \geq 0$. To reduce complication, here we assume $P'(s) = s^{\gamma-1}$, for $s \geq 0$. We also have

$$P''(\tilde{\varrho}_\epsilon)\nabla_x \tilde{\varrho}_\epsilon = \epsilon^{2(m-n)}.$$

For $0 < \epsilon < \frac{1}{2}$, we observe that a *static solution* $\tilde{\varrho}_\epsilon$ satisfies the following property:

$$\begin{aligned} &0 < \tilde{\varrho}_\epsilon \in C^\infty([0,1]), \\ &\sup_{x_3 \in [0,1]} |\tilde{\varrho}_\epsilon(x_3) - 1| \leq \epsilon^{\frac{2(m-n)}{\gamma-1}}, \quad \sup_{x_3 \in [0,1]} |\nabla_x \tilde{\varrho}_\epsilon(x_3)| \leq \epsilon^{2(m-n)}. \end{aligned} \tag{4.3.46}$$

Remark 4.3.15. Since, we are interested for the case $\epsilon \to 0$, thus consideration of $0 < \epsilon < \frac{1}{2}$ is justified. Furthermore, if $\gamma > 2$ and $\epsilon < 1$ we have

$$\sup_{x_3 \in [0,1]} |\tilde{\varrho}_\epsilon(x_3) - 1| \leq \epsilon^{2(m-n)}$$

As $m > n$, asymptotically, the static solution approaches the constant state $\tilde{\varrho} = 1$ as $\epsilon \to 0$.

Existence results for the target system

We recall the expected target system, the 2D Euler equation, i.e.

$$\begin{aligned} &\mathrm{div}_{x_h} \mathbf{v}_h = 0, \text{ in } \mathbb{R}^2, \\ &\partial_t \mathbf{v}_h + (\mathbf{v}_h \cdot \nabla_{x_h})\mathbf{v}_h + \nabla_{x_h}\Pi = 0, \text{ in } \mathbb{R}^2. \end{aligned} \tag{4.3.47}$$

The result stated below by Kato and Lai [89] ensures the existence and uniqueness for the incompressible Euler system in $\mathbb{R}^2$ for sufficiently smooth initial data.

Proposition 4.3.16. *Let*

$$\mathbf{v}_0 \in W^{k,2}(\mathbb{R}^2; \mathbb{R}^2), \ k \geq 3, \ div_{x_h} \mathbf{v}_0 = 0$$

be given. Then the system (4.3.47) *supplemented with initial data* $\mathbf{v}_h(0) = \mathbf{v}_0$ *admits regular solution* $(\mathbf{v}_h, \Pi)$, *unique in the class*

$$\begin{aligned} &\mathbf{v}_h \in C([0,T]; W^{k,2}(\mathbb{R}^2; \mathbb{R}^2)), \ \partial_t \mathbf{v}_h \in C([0,T]; W^{k-1,2}(\mathbb{R}^2; \mathbb{R}^2)), \\ &\Pi \in C([0,T]; W^{k,2}(\mathbb{R}^2)), \end{aligned} \tag{4.3.48}$$

with $div_{x_h} \mathbf{v}_h = 0$.

Alternatively, we write the system (4.3.47) as

$$\partial_t \, \mathrm{Curl}_{x_h} \mathbf{v}_h + \mathbf{v}_h \cdot \nabla_{x_h} \mathrm{Curl}_x \mathbf{v}_h = 0, \text{ in } \mathbb{R}^2.$$

Well-prepared data

We say that the set of initial data $\{(\varrho_{0,\epsilon}, \mathbf{m}_{0,\epsilon})\}_{\{\epsilon>0\}}$ is well-prepared if

$$\begin{aligned}
&\varrho_\epsilon(0,\cdot) = \varrho_{0,\epsilon} = \tilde{\varrho}_\epsilon + \epsilon^m \varrho^{(1)}_{0,\epsilon} \text{ with } \{\varrho^{(1)}_{0,\epsilon}\}_{\epsilon>0} \text{ is bounded in } L^2 \cap L^\infty(\Omega) \\
&\text{ and } \varrho^{(1)}_{0,\epsilon} \to 0 \text{ in } L^2(\Omega), \\
&\mathbf{u}_{0,\epsilon} = \frac{\mathbf{m}_{0,\epsilon}}{\varrho_{0,\epsilon}} \to \mathbf{v}_0 = \left(\mathbf{v}_0^{(1)}, \mathbf{v}_0^{(2)}, 0\right) \text{ in } L^2(\Omega;\mathbb{R}^3) \text{ with } \mathrm{div}_{x_h}\mathbf{v}_0 = 0.
\end{aligned} \tag{4.3.49}$$

Main Theorem

We provide the main result for this case.

Theorem 4.3.17. *Let $(\varrho_\epsilon, \mathbf{m}_\epsilon)$ be a dissipative solution of the system* (4.3.2)-(4.3.10) *with $\Lambda = 1$ and $\frac{m}{2} > n \geq 1$. Moreover, we assume that the initial data is well-prepared, i.e. it satisfies* (4.3.34) *and $\mathbf{v}_0 \in W^{k,2}(\mathbb{R}^2)$ with $k \geq 3$. Then,*

$$\begin{aligned}
&\mathit{ess} \sup_{t\in(0,T)} \|\varrho_\epsilon - \tilde{\varrho}_\epsilon\|_{(L^2+L^{\gamma'})(\Omega)} \leq \epsilon^m c \\
&\frac{\mathbf{m}_\epsilon}{\sqrt{\varrho_\epsilon}} \to \mathbf{v} \begin{cases} \textit{weak-(*)ly in} L^\infty(0,T;L^2(\Omega;\mathbb{R}^3)), \\ \textit{strongly in } L^1_{loc}((0,T)\times\Omega;\mathbb{R}^3), \end{cases}
\end{aligned}$$

where $\gamma' = \min\{2,\gamma\}$ and $\mathbf{v} = (\mathbf{v}_h, 0)$ is the unique solution of the incompressible Euler system with initial data $\mathbf{v}_0$ in $\mathbb{R}^2$.

In the remaining subsection, we give the proof.

Uniform bound and weak convergence

First, we note that $\tilde{\mathbf{u}} = 0$ and $\tilde{\varrho} = \tilde{\varrho}_\epsilon$ satisfy (4.3.12). Hence, we use them as test functions in the relative energy inequality(4.3.15). One the other hand, the choice of (4.3.49) ensures that the initial energy$E_{0,\epsilon}$ is uniformly bounded. Thus we have the following bounds

$$\begin{aligned}
&\text{ess} \sup_{t\in(0,T)} \left\| \frac{\mathbf{m}_\epsilon}{\sqrt{\varrho_\epsilon}} \right\|_{L^2(\Omega;\mathbb{R}^3)} \leq C, \\
&\text{ess} \sup_{t\in(0,T)} \left\| \left[\frac{\varrho_\epsilon - \bar{\varrho}}{\epsilon^m} \right]_{\text{ess}} \right\|_{L^2(\Omega)} \leq C, \\
&\text{ess} \sup_{t\in(0,T)} \|[\varrho_\epsilon]_{\text{res}}\|^\gamma_{L^\gamma(\Omega)} + \text{ess} \sup_{t\in(0,T)} \|[1]_{\text{res}}\|^\gamma_{L^\gamma(\Omega)} \leq \epsilon^{2m} C,
\end{aligned} \tag{4.3.50}$$

where C is independent of ϵ. We consider $\gamma' = \min\{2,\gamma\}$. The estimate (4.3.50) and the fact $\gamma' \leq 2$ imply

$$\text{ess} \sup_{t\in(0,T)} \|\varrho_\epsilon - \tilde{\varrho}_\epsilon\|_{(L^2+L^{\gamma'})(\Omega)} \leq (\epsilon^m + \epsilon^{\frac{2m}{\gamma'}})C \leq \epsilon^m C. \tag{4.3.51}$$

The equation (4.3.51), together with (4.3.46) yield

$$\varrho_\epsilon \to 1 \text{ in } L^\infty(0,T;L^q_{\text{loc}}(\Omega)) \text{ for any } 1 \le q < \gamma'. \tag{4.3.52}$$

Also, from the uniform bound (4.3.50) and (4.3.52) imply

$$\frac{\mathbf{m}_\epsilon}{\sqrt{\varrho_\epsilon}} \to \mathbf{u} \text{ weak-(*)ly in } L^\infty(0,T;L^2(\Omega;\mathbb{R}^3)),$$

and

$$\mathbf{m}_\epsilon \to \mathbf{m} \text{ weak-(*)ly in } L^\infty(0,T;L^2 + L^{\grave{\gamma}}(\Omega;\mathbb{R}^3)),$$

passing to suitable subsequence, where $\grave{\gamma} = \min\{\frac{4}{3}, \frac{2\gamma}{\gamma+1}\}$. The strong convergence of the density (4.3.52) helps to obtain $\mathbf{m} = \mathbf{u}$ in the weak sense.
Finally, we may let $\epsilon \to 0$ in the continuity equation to deduce that,

$$\int_0^\tau \int_\Omega \mathbf{u} \cdot \nabla_x \varphi \,\mathrm{d}x\,\mathrm{d}t = 0, \ \forall \varphi \in C_c^\infty(\Omega).$$

Strong convergence

Here we choose proper test functions and prove that $\lim_{\epsilon \to 0} \mathcal{E}_\epsilon(t) = 0$.
Taking motivation from (4.3.22), we consider another equation that describes a non-oscillatory part described by a variable q_ϵ, that satisfies

$$\partial_t(\Delta_{x_h} q_\epsilon - (\epsilon^{m-1})^2 q_\epsilon) + \frac{1}{\epsilon^{m-1}} \nabla^\perp_{x_h} q_\epsilon \cdot \nabla_{x_h}(\Delta_{x_h} q_\epsilon - (\epsilon^{m-1})^2 q_\epsilon) = 0, \tag{4.3.53}$$

in $\mathbb{R}^2$ supplemented with initial data $q_\epsilon(0,\cdot) = q_{0,\epsilon}$ such that

$$-\Delta_{x_h} q_{0,\epsilon} + (\epsilon^{2(m-1)}) q_{0,\epsilon} = \epsilon^{m-1} \mathrm{Curl} P_h(\mathbf{v}_0) \tag{4.3.54}$$

Let us introduce another variable $\mathbf{v}_\epsilon$ such that $\mathbf{v}_\epsilon$ and q_ϵ are interrelated by

$$\nabla_x q_\epsilon + \epsilon^{m-1} \mathbf{b} \times \mathbf{v}_\epsilon = 0. \tag{4.3.55}$$

Thus initial data for $\mathbf{v}_\epsilon$ satisfy

$$-\nabla_{x_h} q_{0,\epsilon} = \epsilon^{m-1} \mathbf{b} \times \mathbf{v}_{0,\epsilon}.$$

From the hypothesis on initial data in the Theorem 4.3.17, we have

$$\mathbf{v}_0 \in W^{k,2}(\mathbb{R}^d), \text{ with } k \ge 3.$$

We observe that $\|q_{0,\epsilon}\|_{L^2(\mathbb{R}^2)} \le C$ and $\|\nabla_x q_{0,\epsilon}\|_{L^2(\mathbb{R}^2)} \le \epsilon^{m-1} C$. Therefore, we can consider $\{q_{0,\epsilon}\}_{\epsilon > 0}$ such that $q_{0,\epsilon} \to 0$ in $L^2(\mathbb{R}^2)$ as $\epsilon \to 0$. Furthermore, we also note that $\mathbf{v}_{0,\epsilon} \to P_h(\mathbf{v}_0)$ as $\epsilon \to 0$.

In order to have a simplified notation, we consider $\omega = \epsilon^{m-1}$ and $\tilde{q}_\epsilon = \frac{q_\epsilon}{\omega}$. We rewrite (4.3.53) as

$$\partial_t(\Delta_{x_h}\tilde{q}_\epsilon - \omega^2\tilde{q}_\epsilon) + \nabla^\perp_{x_h}\tilde{q}_\epsilon \cdot \nabla_{x_h}(\Delta_{x_h}\tilde{q}_\epsilon - \omega^2 q_\epsilon) = 0, \tag{4.3.56}$$

We notice that the equation (4.3.56) has a similar structure to (4.3.19). Thus we apply the Proposition 4.3.10 to ensure the existence and uniqueness of solution $\tilde{q}_\epsilon$.

In order to obtain a uniform estimate independent of ϵ we multiply the (4.3.56) by q_ϵ and performing integration by parts, we get

$$\int_{\mathbb{R}^2} \left(|\nabla_{x_h}\tilde{q}_\epsilon|^2 + \omega^2|\tilde{q}_\epsilon|^2\right)(t,\cdot)\ \mathrm{d}x = \int_{\mathbb{R}^2} \left(|\nabla_{x_h}\tilde{q}_{0,\epsilon}|^2 + \omega^2|\tilde{q}_{0,\epsilon}|^2\right)\ \mathrm{d}x, \tag{4.3.57}$$

for a.e. $t \in (0,T)$. As the initial data for $\tilde{q}_\epsilon$ depends only on $\mathbf{v}_0$, we deduce that

$$\{-\Delta_{x_h}\tilde{q}_\epsilon + \omega^2\tilde{q}_\epsilon\}_{\epsilon>0} \text{ is bounded in } C^1([0,T];W^{k-2,2}(\mathbb{R}^2)) \cap C([0,T];W^{k-1,2}(\mathbb{R}^2)).$$

Now, from (4.3.55), we also get

$$\{\mathbf{v}_\epsilon\}_{\epsilon>0} \text{ is bounded in } C([0,T];W^{k,2}(\mathbb{R}^2)) \cap C^1([0,T];W^{k-1,2}(\mathbb{R}^2)).$$

It is easy to verify that $\partial_t\tilde{q}_\epsilon$ satisfies the equation

$$\partial_t q_\epsilon = (\Delta_{x_h} - \omega^2)^{-1}(\mathbf{v}_\epsilon \mathrm{Curl}_{x_h}\mathbf{v}_\epsilon)$$

Consequently, it yields

$$\{\partial_t\tilde{q}_\epsilon\}_{\epsilon>0} \text{ is bounded in } C([0,T];W^{k-1,2}(\mathbb{R}^2)),$$

and

$$\{\partial_t\mathbf{v}_\epsilon\}_{\epsilon>0} \text{ is bounded in } C([0,T];W^{k,2}(\mathbb{R}^2;\mathbb{R}^2))$$

This bounds are independent of ϵ.

Therefore, we obtain the following weak convergence:

$$\mathbf{v}_\epsilon \to \mathbf{v} \text{ weakly in } C([0,T];W^{k,2}(\mathbb{R}^2)),$$

and

$$\partial_t\mathbf{v}_\epsilon \to \partial_t\mathbf{v} \text{ weakly in } C([0,T];W^{k-1,2}(\mathbb{R}^2)).$$

Since $k \geq 4$, applying Sobolev embedding theorem, we obtain

$$\mathbf{v}_\epsilon \to \mathbf{v} \text{ in } L^q(0,T;L^q_{\mathrm{loc}}(\mathbb{R}^2)). \tag{4.3.58}$$

We rewrite (4.3.56) as

$$\partial_t(\mathrm{Curl}_{x_h}\mathbf{v}_\epsilon) - \omega^2\partial_t\tilde{q}_\epsilon + \mathbf{v}_\epsilon\nabla_x(\mathrm{Curl}_{x_h}\mathbf{v}_\epsilon) = 0. \tag{4.3.59}$$

From (4.3.58), we infer that

$$\partial_t(\mathrm{Curl}_{x_h}\mathbf{v}) + \mathbf{v}\cdot\nabla_x(\mathrm{Curl}_{x_h}\mathbf{v}) = 0.$$

This is similar to (4.3.47). We also have

$$\Pi \in C([0,T]; W^{k,2}(\mathbb{R}^2)).$$

Clearly we have the following estimates

$$\begin{aligned} &\|q_\epsilon\|_{L^\infty(0,T;L^2(\Omega))} \leq C \\ &\|\partial_t q_\epsilon\|_{L^\infty(0,T;L^q(\Omega))} + \|\nabla_{x_h} q_\epsilon\|_{L^\infty(0,T;L^2(\Omega))} \leq \epsilon^{m-1} C, \end{aligned} \tag{4.3.60}$$

for $q \geq 2$. Also $\mathbf{v}_\epsilon \in C([0,T]; W^{k-1,2})$ with $k \geq 4$ implies

$$\|q_\epsilon\|_{L^\infty((0,T)\times\mathbb{R}^2)} \leq C. \tag{4.3.61}$$

Now, we consider a suitable test function for the relative energy inequality(4.3.15) as

$$\tilde{\mathbf{u}} = \mathbf{V}_\epsilon = (\mathbf{v}_\epsilon, 0), \ \tilde{\varrho} = \tilde{\varrho}_\epsilon + \epsilon^m q_\epsilon, \tag{4.3.62}$$

where $(q_\epsilon, \mathbf{v}_3)$ satisfies (4.3.53) ans (4.3.55) and $\tilde{\varrho}_\epsilon$ is a static solution satisfies (4.3.46). We use the relation between q_ϵ and $\mathbf{v}_\epsilon$ and obtain

$$\begin{aligned} &\mathcal{E}_\epsilon(\tau) + \int_{\overline{\Omega}} \mathrm{d}\, \mathfrak{R}_{e_\epsilon}(\tau, \cdot) \\ &\leq \mathcal{E}_\epsilon(0) - \int_0^\tau \int_\Omega (\mathbf{m}_\epsilon - \varrho_\epsilon \mathbf{V}_\epsilon) \cdot (\partial_t \mathbf{V}_\epsilon + (\mathbf{V}_\epsilon \cdot \nabla_x)\mathbf{V}_\epsilon) \ \mathrm{d}x \ \mathrm{d}t \\ &- \int_0^\tau \int_\Omega \left(\frac{(\mathbf{m}_\epsilon - \varrho_\epsilon \mathbf{V}_\epsilon) \otimes (\mathbf{m}_\epsilon - \varrho_\epsilon \mathbf{V}_\epsilon)}{\varrho_\epsilon} \right) : \nabla_x \mathbf{V}_\epsilon \ \mathrm{d}x \ \mathrm{d}t \\ &+ \frac{1}{\epsilon^{2m}} \int_0^\tau \int_\Omega (\tilde{\varrho} - \varrho_\epsilon) \partial_t P'(\tilde{\varrho}) \ \mathrm{d}x \ \mathrm{d}t \ + \frac{1}{\epsilon^m} \int_0^\tau \int_\Omega \mathbf{m}_\epsilon \cdot \nabla_x q_\epsilon (P''(\tilde{\varrho}) - P''(1)) \ \mathrm{d}x \, \mathrm{d}t \\ &+ \frac{1}{\epsilon^{2m}} \int_0^\tau \int_\Omega \mathbf{m}_\epsilon \cdot (P''(\tilde{\varrho}) - P''(\tilde{\varrho}_\epsilon)) \nabla_x \tilde{\varrho}_\epsilon \ \mathrm{d}x \ \mathrm{d}t \\ &- \frac{1}{\epsilon^{2n}} \int_0^\tau \int_\Omega (\varrho_\epsilon - \tilde{\varrho}) \nabla_x G \cdot \mathbf{V}_\epsilon \ \mathrm{d}x \ \mathrm{d}t \ - \int_0^\tau \int_{\overline{\Omega}} \nabla_x \tilde{\mathbf{u}} : \mathrm{d}\mathfrak{R}_{m_\epsilon}(t, \cdot) \ \mathrm{d}t \ = \Sigma_{i=1}^8 \mathcal{L}_i. \end{aligned} \tag{4.3.63}$$

Here we compute each term $\mathcal{L}_i$, $i = 1(1)8$ of (4.3.63). For term $\mathcal{L}_1$ we have

$$\mathcal{E}_\epsilon(\varrho_{0,\epsilon}, (\varrho\mathbf{u})_{0,\epsilon} \mid \tilde{\varrho}_\epsilon + \epsilon^m q_{0,\epsilon}, \mathbf{v}_0) \leq \left\| \frac{(\varrho\mathbf{u})_{0,\epsilon}}{\varrho_{0,\epsilon}} - \mathbf{v}_0 \right\|_{L^2(\Omega)}^2 + \left\| \varrho_{0,\epsilon}^{(1)} - q_{0,\epsilon} \right\|_{L^2(\Omega)}^2.$$

Consideration of *well prepared data* yields,

$$|\mathcal{L}_1| \leq \xi(\epsilon). \tag{4.3.64}$$

From now on we use this generic function $\xi(\cdot)$, such that $\lim_{\epsilon\to 0}\xi(\epsilon)=0$.
We rewrite $\mathcal{L}_2$ as

$$\begin{aligned}\mathcal{L}_2 =& -\int_0^\tau\int_\Omega (\mathbf{m}_\epsilon - \varrho_\epsilon \mathbf{V}_\epsilon)\cdot(\partial_t \mathbf{V}_\epsilon + (\mathbf{V}_\epsilon\cdot\nabla_x)\mathbf{V}_\epsilon)\ \mathrm{d}x\ \mathrm{d}t\\
=& -\int_0^\tau\int_\Omega \mathbf{m}_\epsilon\cdot(\partial_t \mathbf{V}_\epsilon + (\mathbf{V}_\epsilon\cdot\nabla_x)\mathbf{V}_\epsilon)\ \mathrm{d}x\ \mathrm{d}t\\
&+\int_0^\tau\int_\Omega (\varrho_\epsilon - 1)\mathbf{V}_\epsilon\cdot\partial_t\mathbf{V}_\epsilon\ \mathrm{d}x\ \mathrm{d}t\ +\int_0^\tau\int_\Omega \partial_t\mathbf{V}_\epsilon\cdot\mathbf{V}_\epsilon\ \mathrm{d}x\ \mathrm{d}t\\
=&\mathcal{L}_{2,1}+\mathcal{L}_{2,2}+\mathcal{L}_{2,3}.\end{aligned}$$

Using (4.3.60) and (4.3.46) we obtain

$$|\mathcal{L}_{2,2}|\le \xi(\epsilon). \tag{4.3.65}$$

We claim

$$\mathcal{L}_{2,1}\to -\int_0^\tau\int_\Omega \mathbf{u}\cdot\nabla_{x_h}\Pi\ \mathrm{d}x\ \mathrm{d}t\ =0,$$

as $\epsilon\to 0$. Let, K be a compact subset of $\mathbb{R}^2$. We use (4.3.58) to deduce

$$\int_0^\tau\int_{K\times(0,1)} \mathbf{m}_\epsilon\cdot(\partial_t\mathbf{V}_\epsilon+(\mathbf{V}_\epsilon\cdot\nabla_x)\mathbf{V}_\epsilon)\ \mathrm{d}x\ \mathrm{d}t\ \to \int_0^\tau\int_{K\times(0,1)} \mathbf{u}\cdot(\partial_t\mathbf{v}+(\mathbf{v}\cdot\nabla_x)\mathbf{v})\ \mathrm{d}x\ \mathrm{d}t\ ,$$

where $\mathbf{v}=(\mathbf{v}_h,0)$. Using the fact that $\Pi\in C([0,T];W^{k,2}(\mathbb{R}^2))$ with $k\ge 3$, we have

$$|\mathcal{L}_{2,1}|\le\xi(\epsilon).$$

We want to estimate the term $\mathcal{L}_4$. First we rewrite it as,

$$\begin{aligned}\mathcal{L}_4 =&\frac{1}{\epsilon^m}\int_0^\tau\int_\Omega(\tilde\varrho-\varrho_\epsilon)P''(\tilde\varrho)\partial_t q_\epsilon\ \mathrm{d}x\ \mathrm{d}t\\
=&\int_0^\tau\int_\Omega(P''(\tilde\varrho)-P''(\tilde\varrho_\epsilon))\left(q_\epsilon-\frac{\varrho_\epsilon-\tilde\varrho_\epsilon}{\epsilon^m}\right)\partial_t q_\epsilon\ \mathrm{d}x\ \mathrm{d}t\\
&+\int_0^\tau\int_\Omega(P''(\tilde\varrho_\epsilon)-P''(1))\left(q_\epsilon-\frac{\varrho_\epsilon-\tilde\varrho_\epsilon}{\epsilon^m}\right)\partial_t q_\epsilon\ \mathrm{d}x\ \mathrm{d}t\\
&+\int_0^\tau\int_\Omega q_\epsilon\partial_t q_\epsilon\ \mathrm{d}x\ \mathrm{d}t\ -\int_0^\tau\int_\Omega\frac{\varrho_\epsilon-\tilde\varrho_\epsilon}{\epsilon^m}\partial_t q_\epsilon\ \mathrm{d}x\ \mathrm{d}t\\
=&\mathcal{L}_{4,1}+\mathcal{L}_{4,2}+\mathcal{L}_{4,3}+\mathcal{L}_{4,4}.\end{aligned}$$

First we have for each $x\in\Omega$,

$$(P''(\tilde\varrho)-P''(\tilde\varrho_\epsilon))\le C\epsilon^m|q_\epsilon(x)|.$$

We observe that

$$
\begin{aligned}
|\mathcal{L}_{4,1}| \leq \epsilon^m \sup_{t\in(0,T)} & \left\| \left[\frac{\varrho_\epsilon - \bar{\varrho}}{\epsilon^m}\right]_{\text{ess}} - q_\epsilon \right\|_{L^2(\Omega)} \|\partial_t q_\epsilon\|_{L^\infty([0,T]\times\mathbb{R}^d)} \|q_\epsilon\|_{L^\infty(0,T;L^2(\Omega))} \\
&+ \epsilon^m \sup_{t\in(0,T)} \left\| \left[\frac{\varrho_\epsilon - \bar{\varrho}}{\epsilon^m}\right]_{\text{res}} \right\|_{L^{\gamma'}(\Omega)} \|q_\epsilon\|_{L^\infty(0,T;L^{\gamma'^*}(\Omega))} \|\partial_t q_\epsilon\|_{L^\infty([0,T]\times\mathbb{R}^d)},
\end{aligned}
$$

where $\frac{1}{\gamma'} + \frac{1}{\gamma'^*} = 1$.
Similarly, using (4.3.46) we have

$$
\begin{aligned}
|\mathcal{L}_{4,2}| \leq & \epsilon^{\frac{2(m-n)}{\gamma-1}} \sup_{t\in(0,T)} \left\| \left[\frac{\varrho_\epsilon - \bar{\varrho}}{\epsilon^m}\right]_{\text{ess}} - q_\epsilon \right\|_{L^2(\Omega)} \|\partial_t q_\epsilon\|_{L^\infty(0,T;L^2(\Omega))} \\
&+ \epsilon^{\frac{2(m-n)}{\gamma-1}} \sup_{t\in(0,T)} \left\| \left[\frac{\varrho_\epsilon - \bar{\varrho}}{\epsilon^m}\right]_{\text{res}} \right\|_{L^{\gamma'}(\Omega)} \|\partial q_\epsilon\|_{L^\infty(0,T;L^{\gamma'^*}(\Omega))},
\end{aligned}
$$

for $1 < \gamma \leq 2$, and

$$
\begin{aligned}
|\mathcal{L}_{4,2}| \leq & \epsilon^{2(m-n)} \sup_{t\in(0,T)} \left\| \left[\frac{\varrho_\epsilon - \bar{\varrho}}{\epsilon^m}\right]_{\text{ess}} - q_\epsilon \right\|_{L^2(\Omega)} \|\partial_t q_\epsilon\|_{L^\infty(0,T;L^2(\Omega))} \\
&+ \epsilon^{2(m-n)} \sup_{t\in(0,T)} \left\| \left[\frac{\varrho_\epsilon - \bar{\varrho}}{\epsilon^m}\right]_{\text{res}} \right\|_{L^{\gamma'}(\Omega)} \|\partial q_\epsilon\|_{L^\infty(0,T;L^{\gamma'^*}(\Omega))},
\end{aligned}
$$

for $\gamma > 2$.
Analogously, we deduce

$$
|\mathcal{L}_{4,4}| \leq \left\| \frac{\varrho_\epsilon - \tilde{\varrho}_\epsilon}{\epsilon^m} \right\|_{L^\infty(0,T;L^2+L^{\gamma'}(\Omega))} \|\partial_t q_\epsilon\|_{L^\infty(0,T;L^2\cap L^{\gamma'^*}(\Omega))},
$$

where $\frac{1}{\gamma'} + \frac{1}{\gamma'^*} = 1$. We use estimate (4.3.60) to conclude

$$
|\mathcal{L}_{4,1}| + |\mathcal{L}_{4,2}| + |\mathcal{L}_{4,4}| \leq \xi(\epsilon). \tag{4.3.66}
$$

The equation (4.3.57) implies

$$
\mathcal{L}_{2,3} + \mathcal{L}_{4,4} = \int_0^\tau \int_\Omega \left(\partial_t \mathbf{v}_\epsilon \cdot \mathbf{v}_\epsilon + q_\epsilon \partial_t q_\epsilon\right) \, \mathrm{d}x \, \mathrm{d}t \; = 0 \tag{4.3.67}
$$

Therefore, combining all estimates we get

$$
|\mathcal{L}_2 + \mathcal{L}_4| \leq \xi(\epsilon). \tag{4.3.68}
$$

It is easy to verify that

$$
|\mathcal{L}_3| \leq \|\nabla_{x_h} \mathbf{v}_h\|_{L^\infty(0,\tau;L^\infty(\Omega))} \int_0^\tau \mathcal{E}_\epsilon(t) \, \mathrm{d}t \, . \tag{4.3.69}
$$

For the term $\mathcal{L}_5$, the first observation is for $x \in \Omega$ we have

$$P''(\tilde{\varrho}(x)) - P''(1) = (\tilde{\varrho} - 1)P'''(\eta(x)),$$

and, $\eta(x) \in (\min\{1, \tilde{\varrho}\}, \max\{1, \tilde{\varrho}\})$. From the choice of $\tilde{\varrho} = \tilde{\varrho}_\epsilon + \epsilon^m q_\epsilon$ we have

$$\sup_{x\in\Omega} |P'''(\eta(x))| \leq C,$$

where C is dependent only on $\mathbf{v}_0$.
We rewrite $\mathcal{L}_5$ as

$$\begin{aligned}
\mathcal{L}_5 &= \frac{1}{\epsilon^m} \int_0^\tau \int_\Omega (\tilde{\varrho} - 1)P'''(\eta(x))\mathbf{m}_\epsilon \cdot \nabla_x q_\epsilon \, \mathrm{d}x \, \mathrm{d}t \\
&= \frac{1}{\epsilon^m} \int_0^\tau \int_\Omega \mathbf{m}_\epsilon \cdot \nabla_x q_\epsilon (\tilde{\varrho}_\epsilon - 1)P'''(\eta(x)) \, \mathrm{d}x \, \mathrm{d}t \\
&\quad + \int_0^\tau \int_\Omega q_\epsilon \mathbf{m}_\epsilon \cdot \nabla_x q_\epsilon P'''(\eta(x)) \, \mathrm{d}x \, \mathrm{d}t \ ,
\end{aligned}$$

By using (4.3.46) we observe,

$$\begin{aligned}
|\mathcal{L}_5| \leq \epsilon^{m-2n} &\|\mathbf{m}_\epsilon\|_{L^\infty(0,T;L^2+L^{4/3}(\Omega;\mathbb{R}^3))} \|\nabla_x q_\epsilon\|_{L^\infty(0,T;L^2\cap L^4(\Omega;\mathbb{R}^3))} \\
&+ \|\mathbf{m}_\epsilon\|_{L^\infty(0,T;L^2+L^{4/3}(\Omega;\mathbb{R}^3))} \|q_\epsilon \nabla_x q_\epsilon\|_{L^\infty(0,T;L^2\cap L^4(\Omega;\mathbb{R}^3))},
\end{aligned} \tag{4.3.70}$$

for $\gamma > 2$, and

$$\begin{aligned}
|\mathcal{L}_5| \leq \epsilon^{\frac{2(m-n)}{\gamma-1}} &\|\mathbf{m}_\epsilon\|_{L^\infty(0,T;L^2+L^{2\gamma/\gamma+1}(\Omega;\mathbb{R}^3))} \|\nabla_x q_\epsilon\|_{L^\infty(0,T;L^2\cap L^{(2\gamma/\gamma+1)'}(\Omega;\mathbb{R}^3))} \\
&+ \|\mathbf{m}_\epsilon\|_{L^\infty(0,T;L^2+L^{2\gamma/\gamma+1}(\Omega;\mathbb{R}^3))} \|q_\epsilon \nabla_x q_\epsilon\|_{L^\infty(0,T;L^2\cap L^{(2\gamma/\gamma+1)'}(\Omega;\mathbb{R}^3))},
\end{aligned} \tag{4.3.71}$$

for $1 < \gamma \leq 2$, where $\frac{2\gamma}{\gamma+1}' = \frac{2\gamma}{\gamma-1}$.
In particular, $\frac{m}{2} > n \geq 1$, (4.3.50) and (4.3.60) imply

$$|\mathcal{L}_5| \leq \epsilon^{2(m-n)-1} C, \ \text{for } \gamma > 2,$$

and

$$|\mathcal{L}_5| \leq \epsilon^{\frac{(m-2n)+(m-\gamma+1)}{\gamma-1}} C, \ \text{for } 1 < \gamma < 2,$$

where C is a constant depending on $\mathbf{v}_0$ in both cases. Finally we obtain

$$\mathcal{L}_5 \leq \xi(\epsilon). \tag{4.3.72}$$

Similarly, we rewrite the term $\mathcal{L}_6$ as

$$\begin{aligned}
\mathcal{L}_6 &= \frac{1}{\epsilon^{2m}} \int_0^\tau \int_\Omega \mathbf{m}_\epsilon \cdot (\tilde{\varrho} - \tilde{\varrho}_\epsilon)P'''(\zeta(x))\nabla_x \tilde{\varrho}_\epsilon \, \mathrm{d}x \, \mathrm{d}t \\
&= \frac{1}{\epsilon^m} \int_0^\tau \int_\Omega \mathbf{m}_\epsilon \cdot q_\epsilon P'''(\zeta(x))\nabla_x \tilde{\varrho}_\epsilon \, \mathrm{d}x \, \mathrm{d}t \ ,
\end{aligned}$$

where for each x, $\zeta(x) \in (\min\{\tilde{\varrho}_\epsilon, \tilde{\varrho}\}, \max\{\tilde{\varrho}_\epsilon, \tilde{\varrho}\})$. Using arguments similar to $\mathcal{L}_5$, we have

$$\begin{aligned} |\mathcal{L}_6| &\leq \epsilon^{m-2n} \|\mathbf{m}_\epsilon\|_{L^\infty(0,T;L^2+L^{2\gamma/\gamma+1}(\Omega;\mathbb{R}^3))} \|q_\epsilon\|_{L^\infty(0,T;L^2\cap L^{(2\gamma/\gamma+1)'}(\Omega))} \\ &\leq \xi(\epsilon). \end{aligned} \tag{4.3.73}$$

Now, the choice of G implies

$$\mathcal{L}_7 = 0. \tag{4.3.74}$$

The compatibility of the defect measures yield

$$|\mathcal{L}_8| \leq C \int_0^\tau \int_{\overline{\Omega}} \mathrm{d}\mathfrak{R}_{e_\epsilon} \ \mathrm{d}t\ . \tag{4.3.75}$$

Therefore, combining all estimates (4.3.64)-(4.3.75), we get

$$\mathcal{E}_\epsilon(\tau) + \int_{\overline{\Omega}} \mathrm{d}\, \mathfrak{R}_{e_\epsilon}(\tau,\cdot) \leq \xi(\epsilon) + c\int_0^\tau \mathcal{E}_\epsilon(t)\ \mathrm{d}t\ + C\int_0^\tau \int_{\overline{\Omega}} \mathrm{d}\mathfrak{R}_{e_\epsilon}\ \mathrm{d}t\ . \tag{4.3.76}$$

We use Grönwall's lemma (1.1.7) to infer

$$\mathcal{E}_\epsilon(\tau) + \int_{\overline{\Omega}} \mathrm{d}\, \mathfrak{R}_{e_\epsilon}(\tau,\cdot) \leq \xi(\epsilon) C(T), \tag{4.3.77}$$

where $\xi(\epsilon) \to 0$ as $\epsilon \to 0$. The coercivity of the relative energy functional helps to deduce

$$\limsup_{\epsilon\to 0} \int_K \left| \frac{\mathbf{m}_\epsilon}{\sqrt{\varrho_\epsilon}} - \mathbf{v} \right|^2 \ \mathrm{d}x \leq C(T) \limsup_{\epsilon\to 0} \xi(\epsilon),$$

where, $K \subset \Omega$ is a compact set. Thus, we conclude that $\mathbf{u} = \mathbf{v}_h$. Also, we obtain

$$\frac{\mathbf{m}_\epsilon}{\sqrt{\varrho_\epsilon}} \to \mathbf{v} \ \textit{strongly} \text{ in } L^1_{\mathrm{loc}}((0,T)\times\Omega;\mathbb{R}^3).$$

It ends proof of the theorem 4.3.17.

4.4 Concluding remark

In this chapter, we consider only the *well-prepared* data and expect that the results are valid for the *ill-prepared* data as well. Then we can consider the *well-prepared* case as a special case of the *ill-prepared* case. The analysis is a bit difficult, since we need to consider appropriate Rossby-acoustic wave equations and suitable dispersive estimates in this context. Identifying the domain $\mathbb{R}^2 \times (0,1)$ with $\mathbb{R}^2 \times \mathbb{T}^1$ satisfying (2.6.29) will help us to obtain the estimates.

We also note that in the Section 4.2, we first obtain a dissipative solution of the target system and then use the properties of the strong solution to get the desired result. We expect that the same procedure can work for the problems in the Section 4.3, although it is not yet verified.

It is worth noting that for a rotating fluid we consider the gravitational potential as (4.3.10), and ignore the effect of the centrifugal force. There is a possibility that we may get similar results if we consider a more appropriate gravitational potential. In Subsection 4.3.2 we pointed out the importance of the choice of G to obtain the target system. Thus, for a different G, we may not obtain the exact system, but some similar systems.

Here we focus mainly on the inviscid fluid. For its viscous counterpart, there are some results with additional consideration of the high Reynolds number limit. For rotating fluids in the domain $\mathbb{R}^2 \times (0,1)$ there are some results, see Feireisl, Gallagher and Novotný [55], Feireisl et al. [54], Feireisl and Novotný [75, 74], Feireisl, Lu and Novotný [77], Li [94], to name a few. These results are based on weak solutions of the compressible system Navier–Stokes with monotone pressure law. Therefore, there is some restriction on the adiabatic exponent γ as $\gamma > \frac{3}{2}$. We have the definition dissipative solution for Navier-Stokes in the Definition 2.6.9 for the physically relevant adiabatic range $\gamma \geq 1$. Thus, this limitation can be overcome and most of the above results can be reproduced.

There are several results on the stratification of rotating fluids for the complete Euler system by considering measure-valued solutions, see Březina and Mácha [25]. Also, a singular limit problem for the complete compressible Euler system in the low Mach and strong stratification regime is considered by Bruell and Feireisl [23]. For singular limit problems with the Navier–Stokes–Fourier system, we recommend the monograph by Feireisl and Novotný [72].

Chapter 5

Convergence of a consistent approximation to the complete Euler system

5.1 Introduction

In this chapter our goal is to study the weak convergence of suitable approximation schemes of the complete Euler system. In the context of weak solutions, we have already mentioned several ill-posedness results for both barotropic Euler system and complete Euler system, see Chiodaroli and Kreml [36]. Now it is worthwhile to study in particular the solutions of the Euler system coming from the vanishing viscosity limit of the Navier–Stokes system. In [58] Feireisl and Hofmanová established that in the whole space ($\mathbb{R}^d$) the vanishing viscosity limit of the barotropic Navier–Stokes system either converges strongly or its weak limit is not a weak solution for the corresponding barotropic Euler system. In this chapter, we will investigate whether the similar phenomenon holds for the complete Euler system.

There have been many advances in the study of solutions of the barotropic Euler system coming from the vanishing viscosity limit of the compressible barotropic Navier–Stokes system. If compressible barotropic Euler system admits a smooth solution, the unconditional convergence of the vanishing viscosity limit of the Navier-Stokes system was established by Sueur[114]. Recently, Basarić [11] identified the vanishing viscosity limit of the Navier-Stokes system with a measure valued solution of the barotropic Euler system on an unbounded domain. However in a bounded domain, the choice of a boundary condition for the Navier-Stokes system plays a crucial role in avoiding the *boundary layer* difficulties. Feireisl in [51] showed that the vanishing viscosity limit of the Navier–Stokes–Fourier system in the class of general weak solutions yields the complete Euler system, provided that the latter admits a smooth solution in the bounded domain. Wang and Zhu [120] establish a similar result in bounded domain with complete slip boundary condition.

In this chapter we deal not only with the vanishing viscosity approximation of the complete Euler system but also with general approximations. Approximate solutions can be viewed as a kind of numerical approximation to the complete Euler system. Here we consider a more general class of approximate solutions, namely *consistent approximate solutions*, following DiPerna and Majda [44]. In the context of the complete Euler system in $\mathbb{R}^d$, the approximate solution arising from the vanishing viscosity and vanishing heat conduction approximation from the Navier–Stokes–Fourier system is a good candidate for an approximation scheme. One can also consider approximate solutions that come from *Brenner's two velocity model.* Both schemes have certain advantages and disadvantages. The existence of a weak solution for the Navier-Stokes-Fourier system with Boyle-Mariotte pressure law is still open. Therefore, one has to consider an additional radiation pressure as described in Feireisl and Novotny[72]. A discussion of these models is presented in Březina and Feireisl [26].

The consistent approximations typically generate the so–called measure–valued solutions. For the complete Euler system existence of a measure valued solution was proved by Brezina and Feireisl using Young measures, see[20], [26]. Later in [16], Breit, Feireisl and Hofmanová define dissipative solutions for the same system, by suitably modifying the measure-valued solutions.

Our main goal is to prove that in $\mathbb{R}^d$, if approximate solutions converge weakly to a weak solution of the complete Euler system, the convergence of the state variables will be strong, at least pointwise almost everywhere. Approximate solutions from the *Brenner's model* satisfy the *minimal principle for entropy* i.e., if the initial entropy $s_n(0,\cdot) \geq s_0$ in $\mathbb{R}^d$ for a constant s_0, then $s_n(t,x) \geq s_0$ for a.e. $(t,x) \in (0,T) \times \mathbb{R}^d$. Meanwhile this principle is not available for approximate solutions from the Navier–Stokes–Fourier system. Here we consider both types of approximate solutions. As we will see, the absence of the entropy minimum principle will significantly weaken the available uniform bounds for the approximate sequence. Nevertheless, we are able to establish strong a.e. convergence. In the context of the approximate solutions satisfying a suitable minimal principle for entropy a local strong convergence can be established.

Another important feature of our result is that we only assume that the initial energy is bounded. In fact, Feireisl and Hofmanová [58] obtained a similar result by considering a strong convergence of the initial energy.

We recall the complete Euler system in the physical space $\mathbb{R}^d$ with $d = 2, 3$, describing the time evolution of the density $\varrho = \varrho(t,x)$, the momentum $\mathbf{m} = \mathbf{m}(t,x)$ and the energy $\mathrm{e} = \mathrm{e}(t,x)$ of a compressible inviscid fluid in the space time cylinder $Q_T = (0,T) \times \mathbb{R}^d$:

$$
\begin{aligned}
&\partial_t \varrho + \mathrm{div}_x \mathbf{m} = 0, \\
&\partial_t \mathbf{m} + \mathrm{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) + \nabla_x p = 0, \\
&\partial_t \mathrm{e} + \mathrm{div}_x \left((\mathrm{e} + p) \frac{\mathbf{m}}{\varrho} \right) = 0.
\end{aligned}
$$

As we have already mentioned, by considering the entropy s or the total entropy S, the energy balance can be replaced by the entropy balance

$$\partial_t(\varrho s) + \mathrm{div}_x(s\mathbf{m}) = 0,$$

or by the total entropy balance

$$\partial_t S + \mathrm{div}_x\left(S\frac{\mathbf{m}}{\varrho}\right) = 0.$$

The other hypotheses are specified in the following way.

- **Consitutive relation:**The equation of state is given by Boyle-Mariotte law, i.e.,

$$e = c_v\vartheta,\ c_v = \frac{1}{\gamma-1},\ \text{where } \gamma > 1 \text{ is the adiabatic constant,} \tag{5.1.1}$$

  with internal energy ϱe. The total entropy helps us to rewrite the pressure p and e in terms of ϱ and S as

$$p = p(\varrho, S) = \varrho^\gamma \exp\left(\frac{S}{c_v\varrho}\right),\ e = e(\varrho, S) = \frac{1}{\gamma-1}\varrho^{\gamma-1}\exp\left(\frac{S}{c_v\varrho}\right).$$

- **Initial data:** The initial state of the fluid is given through the conditions

$$\varrho(0,\cdot) = \varrho_0,\ \mathbf{m}(0,\cdot) = \mathbf{m}_0,\ S(0,\cdot) = S_0. \tag{5.1.2}$$

- **Far field condition:** We introduce the *far field condition* as,

$$\varrho \to \varrho_\infty,\ \mathbf{m} \to \mathbf{m}_\infty,\ S \to S_\infty \text{ as } |x| \to \infty, \tag{5.1.3}$$

  with $\varrho_\infty > 0,\ \mathbf{m}_\infty \in \mathbb{R}^d$ and $S_\infty \in \mathbb{R}$.

The definition of an admissible weak solution of this system has been presented in the Section 2.4.

The present setting is more in the spirit of more general measure–valued solutions introduced in Březina and Feireisl [20]. As a matter of fact, considering weaker concept of generalized solutions makes our results stronger as the standard weak solutions are covered.

5.2 Approximate solutions of the complete Euler system

As we mentioned at the beginning of the chapter, our main results are related to the approximate problems of the complete Euler system. We assume that $(\varrho_\infty, \mathbf{m}_\infty, S_\infty) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}$ such that $\varrho_\infty > 0$. The *relative energy* with respect to $(\varrho_\infty, \mathbf{m}_\infty, S_\infty)$ is denoted by $\mathrm{e}(\cdot|\varrho_\infty, \mathbf{m}_\infty, S_\infty)$. It is defined as

$$\begin{aligned}&\mathrm{e}(\varrho, \mathbf{m}, S \,|\, \varrho_\infty, \mathbf{m}_\infty, S_\infty)\\ &= \mathrm{e}(\varrho, \mathbf{m}, S) - \partial\mathrm{e}(\varrho_\infty, \mathbf{m}_\infty, S_\infty)\cdot[(\varrho, \mathbf{m}, S) - (\varrho_\infty, \mathbf{m}_\infty, S_\infty)]\\ &\quad - \mathrm{e}(\varrho_\infty, \mathbf{m}_\infty, S_\infty),\end{aligned}$$

where e is the *total energy* with the following *energy extension* in $\mathbb{R}^{d+2}$:

$$(\varrho, \mathbf{m}, S) \mapsto \mathrm{e}(\varrho, \mathbf{m}, S) \equiv \begin{cases} \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + c_v \varrho^\gamma \exp\left(\frac{S}{c_v \varrho}\right), \text{ if } \varrho > 0, \\ 0, \text{ if } \varrho = \mathbf{m} = 0, \ S \leq 0, \\ \infty, \text{ otherwise} \end{cases} \tag{5.2.1}$$

5.2.1 Definition: Consistent approximation of the complete Euler system

We assume that for each $n \in \mathbb{N}$, $\varrho_{0,n}$ $\mathbf{m}_{0,n}$ and $S_{0,n}$ are measurable function in $\mathbb{R}^d$ such that

$$0 \leq \varrho_{0,n} \text{ and } \int_{\mathbb{R}^d} \mathrm{e}(\varrho_{0,n}, \mathbf{m}_{0,n}, S_{0,n} | \varrho_\infty, \mathbf{m}_\infty, S_\infty) \, \mathrm{d}x \leq C_n, \tag{5.2.2}$$

where $C_n < \infty$ is a constant.

We say that $\{(\varrho_n, \mathbf{m}_n, S_n = \varrho_n s_n)\}_{n\in\mathbb{N}}$ is a family of *admissible consistent approximate solutions* to the complete Euler system in $(0,T) \times \mathbb{R}^d$ with initial data $\{(\varrho_{0,n}, \mathbf{m}_{0,n}, S_{0,n} = \varrho_{0,n} s_{0,n})\}_{n\in\mathbb{N}}$ satisfying (5.2.2) if the following holds for each $n \in \mathbb{N}$:

- The variables $\varrho_n = \varrho_n(t,x)$, $\mathbf{m}_n = \mathbf{m}_n(t,x)$ and $S_n = S_n(t,x)$ are measurable function in $(0,T) \times \mathbb{R}^d$, with $\varrho_n \geq 0$;

- For any $\phi \in C_c^1([0,T) \times \mathbb{R}^d)$, we have

$$-\int_{\mathbb{R}^d} \varrho_{0,n} \phi(0,\cdot) \, \mathrm{d}x = \int_0^T \int_{\mathbb{R}^d} \left[\varrho_n \partial_t \phi + \mathbf{m}_n \cdot \nabla_x \phi\right] \mathrm{d}x \, \mathrm{d}t \ + \int_0^T \mathfrak{E}_{1,n}[\phi] \, \mathrm{d}t \ ; \tag{5.2.3}$$

- For any $\boldsymbol{\varphi} \in C_c^1([0,T) \times \mathbb{R}^d; \mathbb{R}^d)$, we have

$$-\int_{\mathbb{R}^d} \mathbf{m}_{0,n} \boldsymbol{\varphi}(0,\cdot) \, \mathrm{d}x = \int_0^T \int_{\mathbb{R}^d} \Bigg[\mathbf{m}_n \cdot \partial_t \boldsymbol{\varphi} + \mathbf{1}_{\{\varrho_n > 0\}} \frac{\mathbf{m}_n \otimes \mathbf{m}_n}{\varrho_n} : \nabla_x \boldsymbol{\varphi} + \mathbf{1}_{\{\varrho_n > 0\}} p(\varrho_n, S_n) \, \mathrm{div}_x \boldsymbol{\varphi}\Bigg] \mathrm{d}x \, \mathrm{d}t \ + \int_0^T \mathfrak{E}_{2,n}[\boldsymbol{\varphi}] \, \mathrm{d}t \ ; \tag{5.2.4}$$

- For a.e. $0 < \tau < T$, we have

$$\begin{aligned} &\int_{\mathbb{R}^d} \mathrm{e}(\varrho_n, \mathbf{m}_n, S_n \,|\, \varrho_\infty, \mathbf{m}_\infty, S_\infty)(\tau) \, \mathrm{d}x \\ &\leq \int_{\mathbb{R}^d} \mathrm{e}(\varrho_{0,n}, \mathbf{m}_{0,n}, S_{0,n} | \varrho_\infty, \mathbf{m}_\infty, S_\infty) \, \mathrm{d}x + \mathfrak{E}_{3,n}; \end{aligned} \tag{5.2.5}$$

- For any $\psi \in C_c^1((0,T)\times\mathbb{R}^d)$ with $\psi \geq 0$, we have

$$\begin{aligned}&\int_0^T \int_{\mathbb{R}^d} \left[S_n\,\partial_t\psi + \mathbf{1}_{\{\varrho_n>0\}}\frac{S_n}{\varrho_n}\mathbf{m}_n\cdot\nabla_x\psi \right] \mathrm{d}x\ \mathrm{d}t \\ &\leq -\int_{\mathbb{R}^d} \varrho_{0,n}s_{0,n}\psi(0,\cdot)\ \mathrm{d}x + \int_0^T \mathfrak{E}_{4,n}[\psi]\ \mathrm{d}t\ ;\end{aligned} \tag{5.2.6}$$

- Here, the terms $\mathfrak{E}_{1,n}[\phi]$, $\mathfrak{E}_{2,n}[\boldsymbol{\varphi}]$, $\mathfrak{E}_{3,n}$ and $\mathfrak{E}_{4,n}[\psi]$ represent *consistency errors*, i.e.,

$$\mathfrak{E}_{3,n},\ \mathfrak{E}_{4,n}[\psi] \geq 0$$

  and

$$\mathfrak{E}_{1,n}[\phi]\to 0,\ \mathfrak{E}_{2,n}[\boldsymbol{\varphi}]\to 0,\ \mathfrak{E}_{3,n}\to 0 \text{ and } \mathfrak{E}_{4,n}[\psi]\to 0 \text{ as } n\to\infty, \tag{5.2.7}$$

  for fixed ϕ, $\boldsymbol{\varphi}$ and $\psi(\geq 0)$ in $L^1(0,T)$.

Instead of (5.2.6), a renormalized version of the entropy inequality can be considered for the approximation problem:

$$\int_0^T \int_{\mathbb{R}^d} \left[\varrho_n\chi(s_n)\ \partial_t\psi + \chi(s_n)\mathbf{m}_n\cdot\nabla_x\psi \right] \mathrm{d}x\ \mathrm{d}t\ \leq -\int_{\mathbb{R}^d} \varrho_{0,n}\chi(s_{0,n})\psi(0,\cdot)\ \mathrm{d}x, \tag{5.2.8}$$

for any $\psi \in C_c^1((0,T)\times\mathbb{R}^d)$ with $\psi\geq 0$ and any χ,

$$\chi:\mathbb{R}\to\mathbb{R} \text{ a non–decreasing concave function, } \chi(s)\leq\bar{\chi} \text{ for all } s\in\mathbb{R}.$$

Remark 5.2.1. Obviously, one can recover the inequality (5.2.6) without error from the inequality (5.2.8). Moreover, considering the renormalized entropy inequality (5.2.8) leads to the conclusion that entropy is transported along streamlines, see Březina and Feireisl [20, Section 2.1.1]. We reformulate it by saying that *minimal principle for entropy* holds, i.e.

$$\text{for } s_0\in\mathbb{R},\ \text{if } s_n(0,\cdot)\geq s_0 \text{ then } s_n(\tau,\cdot)\geq s_0 \text{ in } \mathbb{R}^d \text{ for a.e. } 0<\tau<T. \tag{5.2.9}$$

Remark 5.2.2. In [20], it is shown that approximate solutions coming from the system Navier–Stokes–Fourier may not satisfy the renormalized version of the entropy balance (5.2.8), but only (5.2.6). Meanwhile, we note that the approximate solutions from Brenner's model satisfy (5.2.8), see [26, Section 4.1].

The above remark motivates us to consider two different approximation problems. From now on, we refer them as follows:

- **First approximation problem** : Approximate solutions satisfy (5.2.3)-(5.2.7);
- **Second approximation problem** : Approximate solutions satisfy (5.2.3)-(5.2.5), (5.2.7) and (5.2.8);

Assumption on the initial data

Now we prescribe an additional assumption on initial data. Together with (5.2.2), we assume that the initial relative energy is uniformly bounded, i.e.,

$$\varrho_{0,n} \geq 0 \text{ and} \int_{\mathbb{R}^d} \mathrm{e}(\varrho_{0,n}, \mathbf{m}_{0,n}, S_{0,n} | \varrho_\infty, \mathbf{m}_\infty, S_\infty) \, \mathrm{d}x \leq E_0, \tag{5.2.10}$$

where E_0 is independent of n. This assumption is shared by both approximate problems.

5.2.2 Young measure generated by approximate solutions

It is easy to prove that $(\varrho, \mathbf{m}) \mapsto \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho}$ is a strictly convex for $\varrho > 0$ and $\mathbf{m} \in \mathbb{R}^d$. We give the the following lemma from Breit et al. [16, Lemma 3.1] that ensures the convexity of pressure and eventually the internal energy.

Lemma 5.2.3. *The mapping*

$$(\varrho, S) \mapsto p(\varrho, S), \ \varrho > 0, S \in \mathbb{R}$$

is strictly convex.

Subsequently, we obtain $(\varrho, S) \mapsto e(\varrho, S), \ \varrho > 0, S \in \mathbb{R}$ is also strictly convex. Thus we conclude that the *total energy*

$$(\varrho, \mathbf{m}, S) \mapsto \mathrm{e}(\varrho, \mathbf{m}, S), \ \varrho \in \mathbb{R}, \ \mathbf{m} \in \mathbb{R}^d \text{ and } S \in \mathbb{R},$$

which follows the extension (5.2.1), is strictly convex when $\varrho > 0$, and convex elsewhere.

Thus using convexity of energy, we have

$$\mathrm{e}(\varrho, \mathbf{m}, S \,|\, \varrho_\infty, \mathbf{m}_\infty, S_\infty) \geq \begin{cases} (\varrho - \varrho_\infty)^2 + |\mathbf{m} - \mathbf{m}_\infty|^2 + (S - S_\infty)^2 \\ \quad \text{if } \frac{\varrho_\infty}{2} \leq \varrho \leq 2\varrho_\infty, \ |\mathbf{m} - \mathbf{m}_\infty| \leq \max\left\{1, \frac{|\mathbf{m}_\infty|}{2}\right\} \\ \quad \text{and } |S - S_\infty| \leq \max\left\{1, \frac{|S_\infty|}{2}\right\}, \\ |\varrho - \varrho_\infty| + |\mathbf{m} - \mathbf{m}_\infty| + |S - S_\infty|, \\ \quad \text{otherwise.} \end{cases} \tag{5.2.11}$$

Then (5.2.11) and (5.2.10) provide an uniform bound for the state variables. In particular we have

$$\begin{aligned} \|\varrho_n - \varrho_\infty\|_{L^\infty(0,T;L^1+L^2(\mathbb{R}^d))} &\leq C, \\ \| \ S_n - S_\infty\|_{L^\infty(0,T;L^1+L^2(\mathbb{R}^d))} &\leq C, \\ \|\mathbf{m}_n - \mathbf{m}_\infty\|_{L^\infty(0,T;L^1+L^2(\mathbb{R}^d;\mathbb{R}^d))} &\leq C. \end{aligned}$$

Following Ball [9, Section 2], we conclude that the sequence $(\varrho_n, \mathbf{m}_n, S_n)$ generates a Young measure $\{\mathcal{V}_{t,x}\}_{t\in(0,T)\times\mathbb{R}^d}$, passing to a subsequence if necessary. We denote the barycenter of the Young measure as $(\varrho, \mathbf{m}, S)$ i.e.,

$$\begin{aligned}&(\varrho(t,x), \mathbf{m}(t,x), S(t,x))\\ &= (\{(t,x)\mapsto \langle \mathcal{V}_{t,x}; \tilde{\varrho}\rangle\}, \{(t,x)\mapsto \langle \mathcal{V}_{t,x}; \tilde{\mathbf{m}}\rangle\}, \{(t,x)\mapsto \langle \mathcal{V}_{t,x}; \tilde{S}\rangle\}).\end{aligned}$$

From the Proposition 1.3.17, we also observe that

$$(\varrho, \mathbf{m}, S) \in L^\infty_{\text{weak-}(*)}(0,T; L^1_{\text{loc}}(\mathbb{R}^d)).$$

5.3 The first approximation problem

Hypothesis on the initial data

We recall the basic hypothesis that initial density is non-negative and initial relative energy is uniformly bounded, i.e.

$$\varrho_{0,n} \geq 0 \text{ and} \int_{\mathbb{R}^d} \mathrm{e}(\varrho_{0,n}, \mathbf{m}_{0,n}, S_{0,n} | \varrho_\infty, \mathbf{m}_\infty, S_\infty)\ \mathrm{d}x \leq E_0$$

with E_0 is independent of n. From (5.2.11) we deduce

$$\varrho_{0,n} - \varrho_\infty \in L^2 + L^1(\mathbb{R}^d) \text{ and } \varrho_{0,n} \to \varrho_0 \text{ weak-(*)ly in } \mathcal{M}^+_{\text{loc}}(\mathbb{R}^d) \text{ as } n\to\infty, \tag{5.3.1}$$

passing to a subsequence as the case may be. Here we will state the main theorem.

Theorem 5.3.1 (First approximation problem). *Let $d = 2,3$ and $\gamma > 1$. Let $(\varrho_n, \mathbf{m}_n, S_n = \varrho_n s_n)$ be a sequence of* admissible solutions *of the consistent approximation with uniformly bounded initial energy as in* (5.4.1) *and the initial densities satisfying* (5.3.1). *Suppose that the barycenter $(\varrho, \mathbf{m}, S)$ of the Young measure generated by the sequence $(\varrho_n, \mathbf{m}_n, S_n)$ is an admissible weak solution of the complete Euler system satisfying*

$$\varrho(0,x) = \varrho_0(x),\ S(t,x) = 0 \textit{ whenever } \varrho(t,x) = 0 \textit{ for a.e. } (t,x) \in (0,T)\times\mathbb{R}^d. \tag{5.3.2}$$

Then passing to a subsequence as the case may be, we have

$$\varrho_n \to \varrho,\ \mathbf{m}_n \to \mathbf{m} \textit{ and } S_n \to S \textit{ for a.e. } (t,x) \in (0,T)\times\mathbb{R}^d. \tag{5.3.3}$$

In the remainder of the section, our goal is to prove the Theorem 5.3.1.

5.3.1 Defect measures

We recall the relative energy bound

$$\|\mathrm{e}(\varrho_n, \mathbf{m}_n, S_n \mid \varrho_\infty, \mathbf{m}_\infty, S_\infty)\|_{L^\infty(0,T;L^1(\mathbb{R}^d))} \leq C, \tag{5.3.4}$$

and as a consequence we have

$$\begin{aligned} \|\varrho_n - \varrho_\infty\|_{L^\infty(0,T;L^1+L^2(\mathbb{R}^d))} + \left\|\mathbf{m}_n - \mathbf{m}_\infty\right\|_{L^\infty(0,T;L^1+L^2(\mathbb{R}^d;\mathbb{R}^d))} \\ + \|S_n - S_\infty\|_{L^\infty(0,T;L^1+L^2(\mathbb{R}^d))} \leq C. \end{aligned} \tag{5.3.5}$$

We also have a Young measure $\mathcal{V}$ generated by $\{(\varrho_n, \mathbf{m}_n, S_n)\}_{n\in\mathbb{N}}$ and

$$\mathcal{V} \in L^\infty_{\text{weak-(*)}}((0,T) \times R^d; \mathcal{P}(\mathbb{R} \times \mathbb{R}^d \times \mathbb{R})).$$

Defect measures for state variables $\varrho, \mathbf{m}$ and S

We have the following embedding

$$L^\infty(0,T; L^2 + L^1(\mathbb{R}^d)) \subset L^\infty_{\text{weak-(*)}}(0,T; L^2 + \mathcal{M}(\mathbb{R}^d)).$$

This gives

$$\varrho_n - \varrho_\infty \to \overline{\varrho} - \varrho_\infty \text{ as } n \to \infty \text{ in } L^\infty_{\text{weak-(*)}}(0,T; L^2 + \mathcal{M}(\mathbb{R}^d)).$$

We introduce the defect measure

$$\mathfrak{C}_\varrho = \overline{\varrho} - \{(t,x) \mapsto \langle \mathcal{V}_{t,x}; \tilde{\varrho}\rangle\}$$

Using the Remark 1.3.16 of Lemma 1.3.14, we obtain $\mathfrak{C}_\varrho \in L^\infty_{\text{weak-(*)}}(0,T;\mathcal{M}(\mathbb{R}^d))$. Similarly, for the sequences $\{(\mathbf{m}_n - \mathbf{m}_\infty)\}_{n\in\mathbb{N}}$ and $\{(S_n - S_\infty)\}_{n\in\mathbb{N}}$ we define the corresponding concentration defect measures as:

$$\mathfrak{C}_\mathbf{m} = \overline{\mathbf{m}} - \{(t,x) \mapsto \langle \mathcal{V}_{t,x}; \tilde{\mathbf{m}}\rangle\} \text{ and } \mathfrak{C}_S = \overline{S} - \{(t,x) \mapsto \langle \mathcal{V}_{t,x}; \tilde{S}\rangle\}.$$

From the fact $\varrho_n \geq 0$ we infer

$$\mathfrak{C}_\varrho \in L^\infty_{\text{weak-(*)}}(0,T;\mathcal{M}^+(\mathbb{R}^d)).$$

Relative energy defect

Let us remind ourselves that $L^\infty_{\text{weak-(*)}}(0,T;\mathcal{M}(\mathbb{R}^d))$ is the dual of $L^1(0,T;C_0(\mathbb{R}^d))$ and that the relative energy is uniformly bounded (5.2.10). Passing to a suitable subsequence, we obtain

$$\mathrm{e}(\varrho_n, \mathbf{m}_n, S_n \mid \varrho_\infty, \mathbf{m}_\infty, S_\infty) \to \overline{\mathrm{e}(\varrho, \mathbf{m}, S \mid \varrho_\infty, \mathbf{m}_\infty, S_\infty)} \text{ in } L^\infty_{\text{weak-(*)}}(0,T;\mathcal{M}(\mathbb{R}^d)).$$

We introduce defect measures:

- **Concentration defect for relative energy:**

$$\mathfrak{R}^{\mathrm{cd}} = \overline{\mathrm{e}(\varrho, \mathbf{m}, S \,|\, \varrho_\infty, \mathbf{m}_\infty, S_\infty)} - \langle \mathcal{V}_{t,x}; \mathrm{e}(\tilde{\varrho}, \tilde{\mathbf{m}}, \tilde{S} \,|\, \varrho_\infty, \mathbf{m}_\infty, S_\infty) \rangle,$$

- **Oscillation defect for relative energy:**

$$\mathfrak{R}^{\mathrm{od}} = \langle \mathcal{V}_{t,x}; \mathrm{e}(\tilde{\varrho}, \tilde{\mathbf{m}}, \tilde{S} \,|\, \varrho_\infty, \mathbf{m}_\infty, S_\infty) \rangle - \mathrm{e}(\varrho, \mathbf{m}, S \,|\, \varrho_\infty, \mathbf{m}_\infty, S_\infty),$$

- **Total relative energy defect:**

$$\mathfrak{R} = \mathfrak{R}^{\mathrm{cd}} + \mathfrak{R}^{\mathrm{od}}.$$

Remark 5.3.2. As a direct consequence of Lemma 1.3.14 and (5.2.10) we get

$$\|\mathfrak{C}_\varrho\|_{L^\infty(0,T;\mathcal{M}(\mathbb{R}^d))} \le \|\mathfrak{R}\|_{L^\infty(0,T;\mathcal{M}(\mathbb{R}^d))}.$$

Analogously, we have

$$\||\mathfrak{C}_\mathbf{m}|\|_{L^\infty(0,T;\mathcal{M}(\mathbb{R}^d))} + \||\mathfrak{C}_S|\|_{L^\infty(0,T;\mathcal{M}(\mathbb{R}^d))} \le \|\mathfrak{R}\|_{L^\infty(0,T;\mathcal{M}(\mathbb{R}^d))}.$$

Energy defect and its finiteness

First, we rewrite the relative energy as

$$\begin{aligned}
&\mathrm{e}(\varrho_n, \mathbf{m}_n, S_n) - \mathrm{e}(\varrho_\infty, \mathbf{m}_\infty, S_\infty) \\
&\ = \mathrm{e}(\varrho_n, \mathbf{m}_n, S_n \,|\, \varrho_\infty, \mathbf{m}_\infty, S_\infty) \\
&\ + \partial \mathrm{e}(\varrho_\infty, \mathbf{m}_\infty, S_\infty) \cdot (\varrho_n - \varrho_\infty, \mathbf{m}_n - \mathbf{m}_\infty, S_n - S_\infty),
\end{aligned}$$

Then the relative energy bound (5.2.10) together with (5.2.11) gives

$$\|\mathrm{e}(\varrho_n, \mathbf{m}_n, S_n) - \mathrm{e}(\varrho_\infty, \mathbf{m}_\infty, S_\infty)\|_{L^\infty(0,T;L^2+L^1(\mathbb{R}^d))} \le C.$$

In particular, we conclude that

$$\begin{aligned}
\mathrm{e}(\varrho_n, \mathbf{m}_n, S_n) - \mathrm{e}(\varrho_\infty, \mathbf{m}_\infty, S_\infty) &\to \overline{\mathrm{e}(\varrho, \mathbf{m}, S)} - \mathrm{e}(\varrho_\infty, \mathbf{m}_\infty, S_\infty) \\
&\qquad \text{weak-(*)ly in } L^\infty(0,T;L^2 + \mathcal{M}(\mathbb{R}^d)).
\end{aligned}$$

In a similar way, we consider the energy defect measures:

- **Concentration defect for energy:**

$$\mathfrak{R}^{\mathrm{cd}}_{\mathrm{eng}} = \overline{\mathrm{e}(\varrho, \mathbf{m}, S)} - \langle \mathcal{V}_{t,x}; \mathrm{e}(\tilde{\varrho}, \tilde{\mathbf{m}}, \tilde{S}) \rangle,$$

- **Oscillation defect for energy:**

$$\mathfrak{R}^{\mathrm{od}}_{\mathrm{eng}} = \langle \mathcal{V}_{t,x}; \mathrm{e}(\tilde{\varrho}, \tilde{\mathbf{m}}, \tilde{S}) \rangle - \mathrm{e}(\varrho, \mathbf{m}, S),$$

• **Total energy defect:**

$$\mathfrak{R}_{\text{eng}} = \mathfrak{R}^{\text{cd}}_{\text{eng}} + \mathfrak{R}^{\text{od}}_{\text{eng}}.$$

We observe that

$$\begin{aligned}
&\mathrm{e}(\varrho_n, \mathbf{m}_n S_n) - \mathrm{e}(\varrho, \mathbf{m}, S) \\
&= \mathrm{e}(\varrho_n, \mathbf{m}_n, S_n \,|\, \varrho_\infty, \mathbf{m}_\infty, S_\infty) - \mathrm{e}(\varrho, \mathbf{m}, S \,|\, \varrho_\infty, \mathbf{m}_\infty, S_\infty) \\
&+ \partial \mathrm{e}(\varrho_\infty, \mathbf{m}_\infty, S_\infty) \cdot (\varrho_n - \varrho, \mathbf{m}_n - \mathbf{m}, S_n - S)
\end{aligned}$$

The above equation together with the Remark 5.3.2 gives

$$\mathfrak{R}_{\text{eng}} \in L^\infty_{\text{weak-(*)}}(0, T; \mathcal{M}(\mathbb{R}^d)) \tag{5.3.6}$$

and

$$\|\mathfrak{R}_{\text{eng}}\|_{L^\infty(0,T;\mathcal{M}(\mathbb{R}^d))} \leq \|\mathfrak{R}\|_{L^\infty(0,T;\mathcal{M}(\mathbb{R}^d))}.$$

Specifically, we have

$$\mathfrak{R} = \mathfrak{R}_{\text{eng}} - \partial \mathrm{e}(\varrho_\infty, \mathbf{m}_\infty, S_\infty) \cdot (\mathfrak{C}_\varrho, \mathfrak{C}_{\mathbf{m}}, \mathfrak{C}_S).$$

From the observation that $(\varrho, \mathbf{m}, S) \mapsto \mathrm{e}(\varrho, \mathbf{m}, S)$ is a non-negative convex l.s.c function in $\mathbb{R}^{d+2}$, we obtain

$$\mathfrak{R}_{\text{eng}} \in L^\infty_{\text{weak-(*)}}(0, T; \mathcal{M}^+(\mathbb{R}^d)).$$

Remark 5.3.3. Suppose that the sequence $(\varrho_n, \mathbf{m}_n, S_n)$ has a weak or weak-(*) limit in the respective space, then the corresponding defect measure $(\mathfrak{C}_\varrho, \mathfrak{C}_{\mathbf{m}}, \mathfrak{C}_S)$ vanishes. As a consequence , we observe

$$\mathfrak{R} = \mathfrak{R}_{\text{eng}}.$$

Defect measures of the nonlinear terms in momentum equation

In the approximate momentum equation (5.2.4), we note the presence of two nonlinear terms

$$\mathbf{1}_{\varrho_n > 0} \frac{\mathbf{m}_n \otimes \mathbf{m}_n}{\varrho_n} \text{ and } \mathbf{1}_{\varrho_n > 0} p(\varrho_n, S_n).$$

Writing

$$\begin{aligned}
&\mathbf{1}_{\varrho_n > 0} \frac{\mathbf{m}_n \otimes \mathbf{m}_n}{\varrho_n} - \frac{\mathbf{m}_\infty \otimes \mathbf{m}_\infty}{\varrho_\infty} \\
&= \mathbf{1}_{\varrho_n > 0} \varrho_n \left(\left(\frac{\mathbf{m}_n}{\varrho_n} - \frac{\mathbf{m}_\infty}{\varrho_\infty} \right) \otimes \left(\frac{\mathbf{m}_n}{\varrho_n} - \frac{\mathbf{m}_\infty}{\varrho_\infty} \right) \right) \\
&\quad - \frac{(\mathbf{m}_n - \mathbf{m}_\infty) \otimes \mathbf{m}_\infty}{\varrho_\infty} - \frac{\mathbf{m}_\infty \otimes (\mathbf{m}_n - \mathbf{m}_\infty)}{\varrho_\infty} + (\varrho_n - \varrho_\infty) \frac{\mathbf{m}_\infty \otimes \mathbf{m}_\infty}{\varrho_\infty^2},
\end{aligned}$$

we obtain the following uniform bound

$$\left\| \mathbf{1}_{\varrho_n>0} \frac{\mathbf{m}_n \otimes \mathbf{m}_n}{\varrho_n} - \frac{\mathbf{m}_\infty \otimes \mathbf{m}_\infty}{\varrho_\infty} \right\|_{L^\infty(0,T;L^2+L^1(\mathbb{R}^d;\mathbb{R}^{d\times d}))} \leq C.$$

Hence, we have

$$\mathbf{1}_{\varrho_n>0} \frac{\mathbf{m}_n \otimes \mathbf{m}_n}{\varrho_n} - \frac{\mathbf{m}_\infty \otimes \mathbf{m}_\infty}{\varrho_\infty} \to \overline{\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho}} - \frac{\mathbf{m}_\infty \otimes \mathbf{m}_\infty}{\varrho_\infty}$$
$$\text{weak-(*)ly in } L^\infty_{\text{weak-(*)}}(0,T; L^2 + \mathcal{M}(\mathbb{R}^d; \mathbb{R}^{d\times d})).$$

Thus, we consider the *concentration defect* $\mathfrak{C}^{\text{eng,cd}}_{m_1}$ and the *oscillation defect* $\mathfrak{C}^{\text{eng,od}}_{m_1}$ as

$$\mathfrak{C}^{\text{eng,cd}}_{m_1} = \overline{\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho}} - \left\langle \mathcal{V}_{t,x}; \mathbf{1}_{\tilde{\varrho}>0} \frac{\tilde{\mathbf{m}} \otimes \tilde{\mathbf{m}}}{\tilde{\varrho}} \right\rangle$$

and

$$\mathfrak{C}^{\text{eng,od}}_{m_1} = \left\langle \mathcal{V}_{t,x}; \mathbf{1}_{\tilde{\varrho}>0} \frac{\tilde{\mathbf{m}} \otimes \tilde{\mathbf{m}}}{\tilde{\varrho}} \right\rangle - \mathbf{1}_{\varrho>0} \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho},$$

respectively. We notice that, for any $\xi \in \mathbb{R}^d$, we get

$$\left(\mathbf{1}_{\varrho_n>0} \frac{\mathbf{m}_n \otimes \mathbf{m}_n}{\varrho_n} - \frac{\mathbf{m}_\infty \otimes \mathbf{m}_\infty}{\varrho_\infty} \right) : (\xi \otimes \xi) \to \left(\overline{\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho}} - \frac{\mathbf{m}_\infty \otimes \mathbf{m}_\infty}{\varrho_\infty} \right) : (\xi \otimes \xi)$$
$$\text{weak-(*)ly in } L^\infty_{\text{weak-(*)}}(0,T; L^2 + \mathcal{M}(\mathbb{R}^d)).$$

Next, we note that for any $\xi \in \mathbb{R}^d$, the function

$$[\varrho, \mathbf{m}] \mapsto \begin{cases} \frac{|\mathbf{m}\cdot\xi|^2}{\varrho} \text{ if } \varrho > 0, \\ 0, \text{ if } \varrho = \mathbf{m} = 0, \\ \infty, \text{ otherwise} \end{cases} \tag{5.3.7}$$

is convex lower semi-continuous. It yields that

$$\mathfrak{C}^{\text{eng,cd}}_{m_1} + \mathfrak{C}^{\text{eng,od}}_{m_1} \in L^\infty_{\text{weak-(*)}}(0,T; \mathcal{M}^+(\mathbb{R}^d; \mathbb{R}^{d\times d}_{\text{sym}})).$$

To obtain this, we use the following observation:

$$\begin{aligned}(\mathfrak{C}^{\text{eng,cd}}_{m_1} + \mathfrak{C}^{\text{eng,od}}_{m_1}) \colon (\xi \otimes \xi) &= \lim_{n\to\infty} \mathbf{1}_{\varrho_n>0} \frac{\mathbf{m}_n \otimes \mathbf{m}_n}{\varrho_n} \colon (\xi \otimes \xi) - \mathbf{1}_{\varrho>0} \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho_n} \colon (\xi \otimes \xi) \\ &= \lim_{n\to\infty} \mathbf{1}_{\varrho_n>0} \frac{|\mathbf{m}_n \cdot \xi|^2}{\varrho_n} - \mathbf{1}_{\varrho>0} \frac{|\mathbf{m} \cdot \xi|^2}{\varrho} \text{ in } \mathcal{D}'((0,T) \times B)\end{aligned}$$

for any bounded open set $B \subset \mathbb{R}^d$ and eventually

$$(\mathfrak{C}^{\text{eng,d}}_{m_1} + \mathfrak{C}^{\text{eng,od}}_{m_1}) \colon (\xi \otimes \xi) \in L^\infty_{\text{weak-(*)}}(0,T; \mathcal{M}^+(\mathbb{R}^d)).$$

Analogously, for the pressure term $\mathbf{1}_{\varrho_n>0}p(\varrho_n, S_n)$ we define the *concentration defect* $\mathfrak{C}^{\text{eng,cd}}_{m_2}$ and the *oscillation defect* $\mathfrak{C}^{\text{eng,od}}_{m_2}$ as

$$\mathfrak{C}^{\text{eng,cd}}_{m_2} = \overline{p(\varrho, S)\mathbb{I}} - \left\langle \mathcal{V}_{t,x}; \mathbf{1}_{\tilde{\varrho}>0} p(\tilde{\varrho}, \tilde{S})\mathbb{I} \right\rangle$$

and

$$\mathfrak{C}^{\text{eng,od}}_{m_2} = \left\langle \mathcal{V}_{t,x}; \mathbf{1}_{\tilde{\varrho}>0} p(\tilde{\varrho}, \tilde{S})\mathbb{I} \right\rangle - \mathbf{1}_{\varrho>0} p(\varrho, S)\mathbb{I}.$$

Noticing that, for any $\xi \in \mathbb{R}^d$, $(\varrho, S) \mapsto p(\varrho, S)\mathbb{I} : (\xi \cdot \xi)$, with an extension

$$[\varrho, S] \mapsto \begin{cases} p(\varrho, S)|\xi|^2 \text{ if } \varrho > 0, \\ 0, \text{ if } \varrho = 0,\ S \leq 0 \\ \infty, \text{ otherwise} \end{cases} \tag{5.3.8}$$

is a convex lower semi-continuous function, we are able to conclude

$$\mathfrak{C}^{\text{eng,cd}}_{m_2} + \mathfrak{C}^{\text{eng,od}}_{m_2} \in L^\infty_{\text{weak-}(*)}(0, T; \mathcal{M}^+(\mathbb{R}^d; \mathbb{R}^{d\times d}_{\text{sym}})).$$

Finally, we consider the *total defect* as

$$\mathfrak{C}_{\text{eng}} = \mathfrak{C}^{\text{eng,cd}}_{m_1} + \mathfrak{C}^{\text{eng,od}}_{m_1} + \mathfrak{C}^{\text{eng,cd}}_{m_2} + \mathfrak{C}^{\text{eng,od}}_{m_2}.$$

Summerizing the above discussion we infer that

$$\mathfrak{C}_{\text{eng}} \in L^\infty_{\text{weak-}(*)}(0, T; \mathcal{M}^+(\mathbb{R}^d; \mathbb{R}^{d\times d}_{\text{sym}})).$$

Comparison of defect measures $\mathbf{Tr}(\mathfrak{C}_{\mathbf{eng}})$ and $\mathfrak{R}_{\mathbf{eng}}$

With the help of the following relation

$$\mathrm{Tr}\left(\frac{\mathbf{m}\otimes\mathbf{m}}{\varrho}\right) = \frac{|\mathbf{m}|^2}{\varrho} \text{ and } \mathrm{Tr}\left(\varrho^\gamma \exp\left(\frac{S}{c_v \varrho}\right)\mathbb{I}\right) = d\varrho^\gamma \exp\left(\frac{S}{c_v\varrho}\right)$$

we conclude the existence of $\Lambda_1, \Lambda_2 > 0$ such that

$$\Lambda_1 \mathfrak{R}_{\text{eng}} \leq \mathrm{Tr}(\mathfrak{C}_{\text{eng}}) \leq \Lambda_2 \mathfrak{R}_{\text{eng}}. \tag{5.3.9}$$

5.3.2 Limit passage

The main goal here is the limit passage in the continuity equation and the momentum equation.

Continuity equation

First, we perform the limit passage in the approximate continuity equation (5.2.3) and obtain

$$\int_0^T \int_{\mathbb{R}^d} \left[\partial_t \phi \ \mathrm{d}\overline{\varrho}(t) + \nabla_x \phi \cdot \mathrm{d}\overline{\mathbf{m}}\right] \ \mathrm{d}t \ = 0,$$

for $\phi \in C_c^1((0,T) \times \mathbb{R}^d)$. In a more suitable notation we write

$$\int_0^T \int_{\mathbb{R}^d} \left[\varrho \partial_t \phi + \mathbf{m} \cdot \nabla_x \phi\right] \ \mathrm{d}x \ \mathrm{d}t \ + \int_0^T \int_{\mathbb{R}^d} \left[\partial_t \phi \ \mathrm{d}\mathfrak{C}_\varrho + \nabla_x \phi \cdot \mathrm{d}\mathfrak{C}_{\mathbf{m}}\right] \ \mathrm{d}t \ = 0, \tag{5.3.10}$$

for $\phi \in C_c^1((0,T) \times \mathbb{R}^d)$. Further we prove that

$$\overline{\varrho} \in C_{\text{weak-(*)}}([0,T]; L^2 + \mathcal{M}(\mathbb{R}^d)).$$

Using (5.3.1) we conclude

$$\int_K \varrho_0 \psi \ \mathrm{d}x = \int_K \psi \mathrm{d}(\overline{\varrho}(0)), \tag{5.3.11}$$

for $K \subset \mathbb{R}^d$, K compact and $\psi \in C_c(K)$.

Local equi-integrability of $\{\varrho_n\}_{n\in\mathbb{N}}$ and $\{\mathbf{m}_n\}_{n\in\mathbb{N}}$

We assume that the triplet $(\varrho, \mathbf{m}, S)$ is a weak solution of the complete Euler system with initial data $(\varrho_0, \mathbf{m}_0, S_0)$, i.e. the continuity equation is

$$\int_0^T \int_{\mathbb{R}^d} \left[\varrho \partial_t \phi + \mathbf{m} \cdot \nabla_x \phi\right] \ \mathrm{d}x \ \mathrm{d}t \ = - \int_{\mathbb{R}^d} \varrho_0 \phi(0, \cdot) \ \mathrm{d}x, \tag{5.3.12}$$

for any $\phi \in C_c^1([0,T) \times \mathbb{R}^d)$.

Eventually, $\varrho \in L^1_{\text{loc}}((0,T) \times \mathbb{R}^d)$ and $\mathbf{m} \in L^1_{\text{loc}}((0,T) \times \mathbb{R}^d; \mathbb{R}^d)$ yield

$$\int_K \varrho_0 \psi \ \mathrm{d}x = \int_K \varrho(0, \cdot) \psi \ \mathrm{d}x, \tag{5.3.13}$$

for a compact subset $K \subset \mathbb{R}^d$ and $\psi \in C_c(K)$.

On the other hand, (5.3.12) together with (5.3.10) implies

$$\partial_t \mathfrak{C}_\varrho + \mathrm{div}_x \mathfrak{C}_{\mathbf{m}} = 0$$

in the sense of distributions in $(0,T) \times \mathbb{R}^d$. Considering the fact $\mathfrak{C}_\varrho \in L^\infty(0,T; \mathcal{M}(\mathbb{R}^d))$ and $\mathfrak{C}_{\mathbf{m}} \in L^\infty(0,T; \mathcal{M}(\mathbb{R}^d; \mathbb{R}^d))$, we write the above relation as,

$$\int_0^T \int_{\mathbb{R}^d} \partial_t \phi \ \mathrm{d}\mathfrak{C}_\varrho \ \mathrm{d}t \ + \int_0^T \int_{\mathbb{R}^d} \nabla_x \phi \cdot \ \mathrm{d}\mathfrak{C}_{\mathbf{m}} \ \mathrm{d}t \ = 0, \ \text{for } \phi \in \mathcal{D}((0,T) \times \mathbb{R}^d).$$

Let us consider $\phi(t,x) = \eta(t)\psi(x)$ with $\eta \in \mathcal{D}(0,T)$ and $\psi \in \mathcal{D}(\mathbb{R}^d)$. Then, we rewrite the above equation in the following form:

$$\int_0^T \left(\int_{\mathbb{R}^d} \psi \, \mathrm{d}\mathfrak{C}_\varrho \right) \eta'(t) \, \mathrm{d}t \, + \int_0^T \left(\int_{\mathbb{R}^d} \nabla_x \psi \cdot \, \mathrm{d}\mathfrak{C}_{\mathbf{m}} \right) \eta(t) \, \mathrm{d}t \, = 0.$$

Since the density and the momentum defects are finite, we have

$$\int_0^T \left(\int_{\mathbb{R}^d} \psi \, \mathrm{d}\mathfrak{C}_\varrho \right) \eta'(t) \, \mathrm{d}t \, + \int_0^T \left(\int_{\mathbb{R}^d} \nabla_x \psi \cdot \, \mathrm{d}\mathfrak{C}_{\mathbf{m}} \right) \eta(t) \, \mathrm{d}t \, = 0,$$

for $\eta \in \mathcal{D}(0,T)$, $\psi \in C^1(\mathbb{R}^d)$ and $\nabla_x \psi \in L^\infty(\mathbb{R}^d;\mathbb{R}^d)$. We consider $\psi = 1$ and obtain

$$\int_0^T \left(\int_{\mathbb{R}^d} \mathrm{d}\mathfrak{C}_\varrho \right) \eta'(t) \, \mathrm{d}t \, = 0.$$

From this we deduce that $t \mapsto \int_{\mathbb{R}^d} \mathrm{d}\mathfrak{C}_\varrho(t)$ is absolutely continuous in $(0,T)$ and the distributional derivative is 0. This along with (5.3.11) and (5.3.13) gives $\mathfrak{C}_\varrho(0,\cdot) = 0$ in $\mathbb{R}^d$. Finally, we get

$$\int_{\mathbb{R}^d} \mathrm{d}\mathfrak{C}_\varrho(t) = 0 \text{ for } t \in (0,T),$$

and $\mathfrak{C}_\varrho = 0$ for a.e. $t \in (0,T)$.

Let $B \subset (0,T) \times \mathbb{R}^d$ be a bounded Borel set. Since $\varrho_n \geq 0$ and $\mathfrak{C}_\varrho = 0$, we conclude that $\{\varrho_n\}_{n\in\mathbb{N}}$ is equi-integrable in B. We have

$$\mathbf{m}_n = \sqrt{\varrho_n} \frac{\mathbf{m}_n}{\sqrt{\varrho_n}},$$

and also $\frac{|\mathbf{m}_n|^2}{\varrho_n}$ is bounded in $L^1(B)$. As a consequence, we conclude $\{\mathbf{m}_n\}_{n\in\mathbb{N}}$ is equi-integrable in B.

Momentum equation with defect

Now, if we perform passage of limit in the momentum equation (5.2.4), we get

$$\begin{aligned}&\int_0^T \int_{\mathbb{R}^d} \left[\mathbf{m} \cdot \partial_t \boldsymbol{\varphi} + \mathbf{1}_{\{\varrho>0\}} \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} : \nabla_x \boldsymbol{\varphi} + \mathbf{1}_{\{\varrho>0\}} p(\varrho, S) \mathrm{div}_x \boldsymbol{\varphi} \right] \mathrm{d}x \, \mathrm{d}t \\ &+ \int_0^T \int_{\mathbb{R}^d} \nabla_x \boldsymbol{\varphi} : \mathrm{d}\mathfrak{C}_{\mathrm{eng}} = 0,\end{aligned} \tag{5.3.14}$$

for $\boldsymbol{\varphi} \in C_c((0,T) \times \mathbb{R}^d;\mathbb{R}^d)$.

Almost everywhere convergence

From our assumption that the barycenter of the Young measure is a weak solution of the complete Euler system, this implies

$$\int_{\mathbb{R}^d} \nabla_x \phi : \mathrm{d}\mathfrak{C}_{\mathrm{eng}} = 0 \text{ for any } \phi \in C_c^1(\mathbb{R}^d;\mathbb{R}^d) \text{ for a.e. } t \in (0,T).$$

Thus, from Proposition 1.3.20, we obtain

$$\mathfrak{C}_{\mathrm{eng}} = 0.$$

Eventually, the comparison of the defect measure (5.3.9) implies

$$\mathfrak{R}_{\mathrm{eng}} = 0.$$

As a consequence of the Theorem 1.3.26, we have

$$\mathrm{e}(\varrho_n, \mathbf{m}_n, S_n) \to \mathrm{e}(\varrho, \mathbf{m}, S) \text{ weakly in } L^1(B). \tag{5.3.15}$$

From this we deduce that

$$\overline{\mathrm{e}(\varrho, \mathbf{m}, S)} = \langle \mathcal{V}_{t,x}; \mathrm{e}(\tilde{\varrho}, \tilde{\mathbf{m}}, \tilde{S}) \rangle = \mathrm{e}(\varrho, \mathbf{m}, S) \text{ in } B.$$

Since e is convex and strictly convex in its domain of positivity, we use a *sharp form of the Jensen's inequality* as described in Lemma 1.3.30 to conclude that either

$$\mathcal{V}_{t,x} = \delta_{\{\varrho(t,x), \mathbf{m}(t,x), S(t,x)\}}$$

or

$$\mathrm{supp}[\mathcal{V}] \subset \{[\tilde{\varrho}, \tilde{\mathbf{m}}, \tilde{S}] | \tilde{\varrho} = 0,\ \tilde{\mathbf{m}} = 0,\ \tilde{S} \leq 0\}.$$

Here we recall the assumption (5.3.2), i.e.,

$$S(t,x) = 0 \text{ whenever } \varrho(t,x) = 0 \text{ for a.e. } (t,x) \in (0,T) \times \mathbb{R}^d.$$

It implies

$$\mathcal{V}_{t,x} = \delta_{\{\varrho(t,x), \mathbf{m}(t,x), S(t,x)\}}.$$

From Lemma 1.3.25, we conclude that $\{\varrho_n, \mathbf{m}_n, S_n\}$ converges to $(\varrho, \mathbf{m}, S)$ in measure. Passing to a suitable subsequence, we obtain

$$\varrho_n \to \varrho,\ \mathbf{m}_n \to \mathbf{m} \text{ and } S_n \to S \text{ a.e. in } (0,T) \times \mathbb{R}^d. \tag{5.3.16}$$

This completes the proof of the Theorem 5.3.1

5.4 The second approximation problem

Hypothesis on the initial data

We recall that the initial density is non-negative and the initial relative energy is uniformly bounded, i.e.,

$$\varrho_{0,n} \geq 0 \text{ and} \int_{\mathbb{R}^d} \mathrm{e}(\varrho_{0,n}, \mathbf{m}_{0,n}, S_{0,n} | \varrho_\infty, \mathbf{m}_\infty, S_\infty) \, \mathrm{d}x \leq E_0, \tag{5.4.1}$$

with E_0 is independent of n.

For the *second approximation problem*, we need an additional assumption that the initial entropy is bounded below, i.e., for some $s_0 \in \mathbb{R}$ we have

$$s_{0,n} \geq s_0 \text{ in } \mathbb{R}^d, \text{ for all } n \in \mathbb{N}. \tag{5.4.2}$$

Main Result

We state the main theorem for this approximation problem.

Theorem 5.4.1 (**Second approximation problem**). *Let $d = 2, 3$ and $\gamma > 1$ and $(\varrho_n, \mathbf{m}_n, S_n = \varrho_n s_n)$ be a sequence of* admissible solutions *of the consistent approximation with initial energy satisfying* (5.4.1) *and and the initial entropy satisfying* (5.4.2). *Suppose,*

$$\begin{aligned} &\varrho_n \to \varrho \textit{ in } \mathcal{D}'((0,T) \times \mathbb{R}^d), \ \mathbf{m}_n \to \mathbf{m} \textit{ in } \mathcal{D}'((0,T) \times \mathbb{R}^d; \mathbb{R}^d), \\ &S_n \to S \textit{ in } \mathcal{D}'((0,T) \times \mathbb{R}^d), \end{aligned} \tag{5.4.3}$$

where $(\varrho, \mathbf{m}, S)$ is a weak solution of the complete Euler system.
Then

$$\mathrm{e}(\varrho_n, \mathbf{m}_n, S_n \,|\, \varrho_\infty, \mathbf{m}_\infty, S_\infty) \to \mathrm{e}(\varrho, \mathbf{m}, S \,|\, \varrho_\infty, \mathbf{m}_\infty, S_\infty) \textit{ in } L^q(0,T; L^1_{\mathrm{loc}}(\mathbb{R}^d))$$

as $n \to \infty$ for any $1 \leq q < \infty$. Moreover,

$$\begin{aligned} &\varrho_n \to \varrho \textit{ in } L^q(0,T; L^\gamma_{\mathrm{loc}}(\mathbb{R}^d)), \mathbf{m}_n \to \mathbf{m} - \textit{ in } L^q(0,T; L^{\frac{2\gamma}{\gamma+1}}(\mathbb{R}^d; \mathbb{R}^d)) \\ &S_n \to S \textit{ in } L^q(0,T; L^\gamma_{\mathrm{loc}}(\mathbb{R}^d)), \end{aligned}$$

for any $1 \leq q < \infty$.

The remainder of this section is devoted to the proof of the Theorem 5.4.1. First, we note that the formulation of the *second approximation problem* and the hypothesis about the initial data (5.4.2) yield the minimal principle for the entropy(5.2.9), i.e., $s_n \geq s_0$ for a.e. $(t,x) \in (0,T) \times \mathbb{R}^d$. This helps us to obtain a finer estimate for the relative energy compared to (5.2.11), which is

$$
\mathrm{e}(\varrho, \mathbf{m}, S \mid \varrho_\infty, \mathbf{m}_\infty, S_\infty) \geq \begin{cases} (\varrho - \varrho_\infty)^2 + |\mathbf{m} - \mathbf{m}_\infty|^2 + (S - S_\infty)^2 \\ \quad \text{if } \frac{\varrho_\infty}{2} \leq \varrho \leq 2\varrho_\infty, \ |\mathbf{m} - \mathbf{m}_\infty| \leq \max\left\{1, \frac{|\mathbf{m}_\infty|}{2}\right\} \\ \quad \text{and } |S - S_\infty| \leq \max\left\{1, \frac{|S_\infty|}{2}\right\}, \\ (1 + \varrho^\gamma) + \frac{\mathbf{m}^2}{\varrho} + (1 + S^\gamma), \\ \quad \text{otherwise.} \end{cases} \tag{5.4.4}
$$

For a detailed discussion about of the above statement, see Breit et al. [16, Section 3]. Without loss of generality, we assume $s_0 \geq 0$, otherwise we need to rescale by taking the total entropy $S_n{=}\varrho_n(s_n - s_0)$.

Uniform bounds and weak convergence

Assumption (5.4.1) implies

$$
\|\mathrm{e}(\varrho_n, \mathbf{m}_n, S_n \mid \varrho_\infty, \mathbf{m}_\infty, S_\infty)\|_{L^\infty(0,T;L^1(\mathbb{R}^d))} \leq C.
$$

Together with (5.4.4), the above bound gives

$$
\begin{aligned} &\|\varrho_n - \varrho_\infty\|_{L^\infty(0,T;L^\gamma + L^2(\mathbb{R}^d))} \leq C, \\ &\left\|\mathbf{m}_n - \mathbf{m}_\infty\right\|_{L^\infty(0,T;L^{\frac{2\gamma}{\gamma+1}} + L^2(\mathbb{R}^d))} \leq C. \end{aligned} \tag{5.4.5}
$$

Eventually, recalling the total entropy S_n, we have

$$
\begin{aligned} &\|S_n - S_\infty\|_{L^\infty(0,T;L^\gamma + L^2(\mathbb{R}^d))} \leq C, \\ &\left\|\frac{S_n}{\sqrt{\varrho_n}}\right\|_{L^\infty(0,T;L^{2\gamma}(\mathbb{R}^d))} \leq C. \end{aligned} \tag{5.4.6}
$$

The above uniform bounds yield the following convergence:

$$
\begin{aligned} &\varrho_n - \varrho_\infty \to \varrho - \varrho_\infty \text{ weak-(*)ly in } L^\infty(0,T;L^\gamma + L^2(\mathbb{R}^d)), \\ &\mathbf{m}_n - \mathbf{m}_\infty \to \mathbf{m} - \mathbf{m}_\infty \text{ weak-(*)ly in } L^\infty(0,T;L^{\frac{2\gamma}{\gamma+1}} + L^2(\mathbb{R}^d)), \\ &S_n - S_\infty \to S - S_\infty \text{ weak-(*)ly in } L^\infty(0,T;L^\gamma + L^2(\mathbb{R}^d)), \end{aligned}
$$

passing to a suitable subsequence as the case may be. Here also one can consider a Young measure $\mathcal{V}$ generated by $(\varrho_n, \mathbf{m}_n, S_n)$ such that

$$
\mathcal{V} \in L^\infty_{\text{weak-(*)}}((0,T) \times \mathbb{R}^d; \mathcal{P}(\mathbb{R}^{d+2})). \tag{5.4.7}
$$

Since, Young measure captures the weak limit, we obtain

$$
\begin{aligned} &(\varrho(t,x), \mathbf{m}(t,x), S(t,x)) \\ &= (\{(t,x) \mapsto \langle \mathcal{V}_{t,x}; \tilde{\varrho} \rangle\}, \{(t,x) \mapsto \langle \mathcal{V}_{t,x}; \tilde{\mathbf{m}} \rangle\}, \{(t,x) \mapsto \langle \mathcal{V}_{t,x}; \tilde{S} \rangle\}). \end{aligned}
$$

5.4.1 Defect measures

Unlike Section 5.2, here we have the presence of a defect measure only in the nonlinear terms.

Relative energy defect

We know

$$L^\infty(0,T;L^1(\mathbb{R}^d)) \subset L^\infty_{\text{weak-(*)}}(0,T;\mathcal{M}(\mathbb{R}^d)).$$

Moreover, $L^\infty_{\text{weak-(*)}}(0,T;\mathcal{M}(\mathbb{R}^d))$ is the dual of $L^1(0,T;C_0(\mathbb{R}^d))$. Thus passing to a suitable subsequence, we obtain

$$\mathrm{e}(\varrho_n, \mathbf{m}_n, S_n \,|\, \varrho_\infty, \mathbf{m}_\infty, S_\infty) \to \overline{\mathrm{e}(\varrho, \mathbf{m}, S \,|\, \varrho_\infty, \mathbf{m}_\infty, S_\infty)} \text{ in } L^\infty_{\text{weak-(*)}}(0,T;\mathcal{M}(\mathbb{R}^d)).$$

In particular, we say

$$\mathrm{e}_{\text{kin}}(\varrho_n, \mathbf{m}_n|\varrho_\infty, \mathbf{m}_\infty) \to \overline{\mathrm{e}_{\text{kin}}(\varrho, \mathbf{m}|\varrho_\infty, \mathbf{m}_\infty)} \text{ in } L^\infty_{\text{weak-(*)}}(0,T;\mathcal{M}(\mathbb{R}^d))$$

and

$$\mathrm{e}_{\text{int}}(\varrho_n, S_n \,|\, \varrho_\infty, S_\infty) \to \overline{\mathrm{e}_{\text{int}}(\varrho, S \,|\, \varrho_\infty, S_\infty)} \text{ in } L^\infty_{\text{weak-(*)}}(0,T;\mathcal{M}(\mathbb{R}^d)).$$

We consider

$$\mathfrak{R}_e = \overline{\mathrm{e}(\varrho, \mathbf{m}, S \,|\, \varrho_\infty, \mathbf{m}_\infty, S_\infty)} - \mathbf{1}_{\{\varrho>0\}}\mathrm{e}(\varrho, \mathbf{m}, S \,|\, \varrho_\infty, \mathbf{m}_\infty, S_\infty).$$

Using convexity and lower semi-continuity of the relative energy, we have

$$\mathfrak{R}_e \in L^\infty_{\text{weak-(*)}}(0,T;\mathcal{M}^+(\mathbb{R}^d)). \tag{5.4.8}$$

Defects from the non linear terms in momentum equation

We consider a map $\mathbb{C}(\cdot,\cdot|\varrho_\infty, \mathbf{m}_\infty) : \mathbb{R}\times\mathbb{R}^d \to \mathbb{R}^{d\times d}$ as

$$\mathbb{C}(\varrho, \mathbf{m}|\varrho_\infty, \mathbf{m}_\infty) = \mathbf{1}_{\{\varrho>0\}}\varrho\left(\frac{\mathbf{m}}{\varrho} - \frac{\mathbf{m}_\infty}{\varrho_\infty}\right)\otimes\left(\frac{\mathbf{m}}{\varrho} - \frac{\mathbf{m}_\infty}{\varrho_\infty}\right).$$

For any $\xi \in \mathbb{R}^d$, we obtain that the map

$$(\varrho, \mathbf{m}) \mapsto \mathbb{C}(\varrho, \mathbf{m}|\varrho_\infty, \mathbf{m}_\infty) : (\xi\otimes\xi)$$

is a convex lower semi-continuous function with a possible extension

$$[\varrho, \mathbf{m}] \mapsto \begin{cases} \frac{|\mathbf{m}\cdot\xi|^2}{\varrho} \text{ if } \varrho > 0, \\ 0, \text{ if } \varrho = \mathbf{m} = 0, \\ \infty, \text{ otherwise }. \end{cases} \tag{5.4.9}$$

We have

$$\frac{\mathbf{m}_n \otimes \mathbf{m}_n}{\varrho_n} = \mathbb{C}(\varrho_n, \mathbf{m}_n | \varrho_\infty, \mathbf{u}_\infty) + \mathbf{m}_n \otimes \mathbf{u}_\infty + \mathbf{u}_\infty \otimes \mathbf{m}_n - \varrho_n \mathbf{u}_\infty \otimes \mathbf{u}_\infty,$$

with

$$\|\mathbb{C}(\varrho_n, \mathbf{m}_n | \varrho_\infty, \mathbf{u}_\infty)\|_{L^\infty(0,T;L^1(\mathbb{R}^d;\mathbb{R}^{d\times d}))} \leq C,$$

where $\mathbf{u}_\infty = \frac{\mathbf{m}_\infty}{\varrho_\infty}$. It implies

$$\mathbb{C}(\varrho_n, \mathbf{m}_n | \varrho_\infty, \mathbf{u}_\infty) \to \overline{\mathbb{C}(\varrho, \mathbf{m} | \varrho_\infty, \mathbf{u}_\infty)} \text{ weak-(*)ly in } L^\infty_{\text{weak-(*)}}(0,T; \mathcal{M}(\mathbb{R}^d; \mathbb{R}^{d\times d}_{\text{sym}})).$$

We introduce the *defect measure* as

$$\mathfrak{R}_{m_1} = \overline{\mathbb{C}(\varrho, \mathbf{m} | \varrho_\infty, \mathbf{u}_\infty)} - \mathbf{1}_{\{\varrho>0\}} \mathbb{C}(\varrho, \mathbf{m} | \varrho_\infty, \mathbf{u}_\infty) \tag{5.4.10}$$

Similarly, we define a map $\mathbb{P}(\cdot, \cdot | \varrho_\infty, S_\infty) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{d\times d}$ such that

$$\begin{aligned} &\mathbb{P}(\varrho, S | \varrho_\infty, S_\infty) \\ &= \left(p(\varrho, S) - \frac{\partial p}{\partial \varrho}(\varrho_\infty, S_\infty)(\varrho - \varrho_\infty) - \frac{\partial p}{\partial S}(\varrho_\infty, S_\infty)(S - S_\infty) - p(\varrho_\infty, S_\infty) \right) \mathbb{I}. \end{aligned}$$

Here, we define the defect measure

$$\mathfrak{R}_{m_2} = \overline{\mathbb{P}(\varrho, S | \varrho_\infty, S_\infty)} - \mathbf{1}_{\varrho>0} \mathbb{P}(\varrho, S | \varrho_\infty, S_\infty). \tag{5.4.11}$$

We use (5.4.9) to conclude

$$\mathfrak{R}_m = \mathfrak{R}_{m_1} + \mathfrak{R}_{m_2} \in L^\infty_{\text{weak-(*)}}(0,T; \mathcal{M}^+(\mathbb{R}^d; \mathbb{R}^{d\times d}_{\text{sym}}) \tag{5.4.12}$$

Comparison of defect measures

There exists scalars $\Lambda_1, \Lambda_2 > 0$ such that

$$\Lambda_1 \mathfrak{R}_e \leq \text{Tr}(\mathfrak{R}_m) \leq \Lambda_2 \mathfrak{R}_e. \tag{5.4.13}$$

Remark 5.4.2. It is clear that, we do not need to define the energy defect separately here as in Section 5.2. Basically, the weak convergence of the state variables implies that the energy defect coincides with the relative energy defect.

5.4.2 Limit passage

Now we pass to the limit in the equations of for approximate solutions and obtain

Equation of continuity:

$$\int_0^T \int_{\mathbb{R}^d} \left[\varrho \partial_t \phi + \mathbf{m} \cdot \nabla_x \phi \right] \,\mathrm{d}x \,\mathrm{d}t \; = 0, \tag{5.4.14}$$

for any $\phi \in C^1_c((0,T) \times \mathbb{R}^d)$,

Momentum equation with defect:

$$\begin{aligned}&\int_0^T \int_{\mathbb{R}^d} \left[\mathbf{m} \cdot \partial_t \boldsymbol{\varphi} + \mathbf{1}_{\{\varrho>0\}} \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} : \nabla_x \boldsymbol{\varphi} + \mathbf{1}_{\{\varrho>0\}}\, p(\varrho, S) \mathrm{div}_x \boldsymbol{\varphi} \right] \mathrm{d}x \ \mathrm{d}t \\ &+ \int_0^T \int_{\mathbb{R}^d} \nabla_x \boldsymbol{\varphi} : \mathrm{d}\mathfrak{R}_m = 0,\end{aligned} \tag{5.4.15}$$

for any $\boldsymbol{\varphi} \in C_c^1((0,T) \times \mathbb{R}^d; \mathbb{R}^d)$,

Relative energy:

$$\overline{\mathrm{e}(\varrho, \mathbf{m}, S \,|\, \varrho_\infty, \mathbf{m}_\infty, S_\infty)} = \mathrm{e}(\varrho, \mathbf{m}, S \,|\, \varrho_\infty, \mathbf{m}_\infty, S_\infty) + \mathfrak{R}_e. \tag{5.4.16}$$

Disappearance of defect measures

We assume that the triplet $(\varrho, \mathbf{m}, S)$ is an admissible weak solution of the complete Euler system, i.e., $(\varrho, \mathbf{m}, S)$ follows the Definition 2.4.1. It implies

$$\int_0^T \int_{\mathbb{R}^d} \nabla_x \boldsymbol{\varphi} : \mathrm{d}\mathfrak{R}_m = 0,$$

for any $\boldsymbol{\varphi} \in C_c^1([0,T] \times \mathbb{R}^d; \mathbb{R}^d)$. Thus, by applying Proposition (1.3.20) we conclude $\mathfrak{R}_m = 0$. Finally, using (5.4.13) we obtain $\mathfrak{R}_e = 0$.

Consequently, we also have

$$\begin{aligned}\mathrm{e}(\varrho_n, \mathbf{m}_n, S_n \,|\, \varrho_\infty, \mathbf{m}_\infty, S_\infty) &\to \mathrm{e}(\varrho, \mathbf{m}, S \,|\, \varrho_\infty, \mathbf{m}_\infty, S_\infty) \\ &\text{weak-(*)ly in } L^\infty_{\text{weak-(*)}}(0,T; \mathcal{M}(\mathbb{R}^d)).\end{aligned} \tag{5.4.17}$$

Almost everywhere convergence

Let $B \subset (0,T) \times \mathbb{R}^d$ be a compact set. Recall the Young measure generated by $\{(\varrho_n, \mathbf{m}_n, S_n)\}_{n \in \mathbb{N}}$ is $\mathcal{V}$. From $\mathfrak{R}_e = 0$ we infer that

$$\left\langle \mathcal{V}_{t,x}; \mathrm{e}(\tilde{\varrho}, \tilde{\mathbf{m}}, \tilde{S} \,|\, \varrho_\infty, \mathbf{m}_\infty, S_\infty) \right\rangle = \mathrm{e}(\varrho, \mathbf{m}, S \,|\, \varrho_\infty, \mathbf{m}_\infty, S_\infty) \text{ for a.e. } (0,T) \times \mathbb{R}^d.$$

We already have weak-(*) convergence of $\{\mathrm{e}(\varrho_n, \mathbf{m}_n, S_n \,|\, \varrho_\infty, \mathbf{m}_\infty, S_\infty)\}_{n \in \mathbb{N}}$, using Lemma 1.3.26 we deduce that

$$\mathrm{e}(\varrho_n, \mathbf{m}_n, S_n \,|\, \varrho_\infty, \mathbf{m}_\infty, S_\infty) \to \mathrm{e}(\varrho, \mathbf{m}, S \,|\, \varrho_\infty, \mathbf{m}_\infty, S_\infty) \text{ weakly in } L^1(B). \tag{5.4.18}$$

Now convexity of $\mathrm{e}(\cdot | \varrho_\infty, \mathbf{m}_\infty, S_\infty)$ and the Theorem 2.11 from Feireisl [50] helps us to conclude

$$\varrho_n \to \varrho, \mathbf{m}_n \to \mathbf{m} \text{ and } S_n \to S \text{ a.e. in } B. \tag{5.4.19}$$

Local strong convergence

We have $\{\mathrm{e}(\varrho_n, \mathbf{m}_n, S_n \,|\, \varrho_\infty, \mathbf{m}_\infty, S_\infty)\}_{n\in\mathbb{N}}$ is equi-integrable in B, in particular $\{\mathrm{e}_{\mathrm{int}}(\varrho_n, S_n)\}_{n\in\mathbb{N}}$ is equi-integrable in B. As a trivial consequence we obtain $\{(\varrho_n^\gamma, S_n^\gamma)\}_{n\in\mathbb{N}}$ is also equi-integrable. Above statement along with almost everywhere convergence gives

$$\varrho_n^\gamma \to \varrho^\gamma \text{ and } S_n^\gamma \to S^\gamma \text{weakly in } L^1(B).$$

It implies

$$\int_B \varrho_n^\gamma \,\mathrm{d}x\,\mathrm{d}t \to \int_B \varrho^\gamma \,\mathrm{d}x\,\mathrm{d}t \;\text{ and } \int_B S_n^\gamma \,\mathrm{d}x\,\mathrm{d}t \to \int_B S^\gamma \,\mathrm{d}x\,\mathrm{d}t\,. \tag{5.4.20}$$

These concludes the norm convergence i.e.,

$$|\varrho_n|_{L^\gamma(B)} \to |\varrho|_{L^\gamma(B)}.$$

Now weak convergence and norm convergence implies the strong convergence.

$$\varrho_n \to \varrho \;\text{ in } L^\gamma(B).$$

Similarly, for the total entropy we also obtain,

$$S_n \to S \text{ in } L^\gamma(B).$$

Strong convergence for the momentum follows exact steps as in part 5.4.5. Since $\varrho \in L^\gamma(B)$ we deduce that

$$\mathbf{m}_n \to \mathbf{m} \text{ in } L^{\frac{2\gamma}{\gamma+1}}(B; \mathbb{R}^d).$$

Relative energy is positive, lower semi-continuous and convex function. It implies

$$\mathrm{e}(\varrho_n, \mathbf{m}_n, S_n \,|\, \varrho_\infty, \mathbf{m}_\infty, S_\infty) \to \mathrm{e}(\varrho, \mathbf{m}, S \,|\, \varrho_\infty, \mathbf{m}_\infty, S_\infty) \text{ in } L^1(B).$$

We invoke the bounds (5.4.5) and (5.4.6) to conclude our desired strong convergences as stated in Theorem 5.4.1.

5.5 Concluding remark

In both theorems 5.3.1 and 5.4.1 we have the hypothesis that the barycenter of a Young measure $\mathcal{V}$, $(\varrho, \mathbf{m}, S)$ is an admissible weak solution of the complete Euler system. If we look closely at the proof, it is a matter of a small additional assumption. It suffices to assume that it solves the momentum equation and the continuity equation for suitable initial data in a weak sense to obtain the desired result.

The results in this chapter are based exclusively on the domain $\mathbb{R}^d$. The main stumbling block for bounded domain is the unavailability of the Proposition 1.3.20. Although there is modified version of the proposition for a bounded domain Ω.

Proposition 5.5.1. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain. Let $\mathbb{D} \in \mathcal{M}^+(\Omega; \mathbb{R}^{d\times d}_{sym}$ satisfying*

$$\int_\Omega \nabla_x \boldsymbol{\varphi} : d\mathbb{D} = 0 \text{ for any } \boldsymbol{\varphi} \in C^1_c(\Omega; \mathbb{R}^d),$$

and

$$\frac{1}{\delta} \int_{\{x \in \Omega |\, dist[x, \partial\Omega \leq \delta]\}} d(\mathit{Tr}(\mathbb{D})) \to 0 \text{ as } \delta \to 0.$$

Then $\mathbb{D} = 0$.

This is reflected as an additional hypothesis about energy as

$$\limsup_{\epsilon\to 0} \int_{x\in\Omega, \mathrm{dist}[x,\partial\Omega]\leq\delta} [\mathrm{e}(\varrho_\epsilon, \mathbf{m}_\epsilon, S_\epsilon) - \mathrm{e}(\varrho, \mathbf{m}, S)](\tau, \cdot)\ \mathrm{d}x$$

is of order $o(\delta)$ as $\delta \to 0$, for a.e. $\tau \in (0, T)$. For a detailed discussion the reader may consult Feireisl and Hoffmanová [58].

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