# Semiflow Selections and Vanishing Viscosity Limits in Fluid Dynamics

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# Abstract

Well-posedness of systems describing the motion of fluids in the class of strong and weak solutions represents one of the most challenging problems of the modern theory of partial differential equations.
To overcome the problem of existence, one suitable idea is to consider a larger class of solutions. In the first part of the thesis we identify a measure-valued solution, characterized by a parametrized Young measure, of the compressible Euler system with damping on a general (unbounded) domain as a vanishing viscosity limit for the compressible Navier-Stokes system. Afterwards, we establish the weak (measure-valued)–strong uniqueness principle, and, as a consequence, we obtain convergence of the weak solutions of the Navier-Stokes equations in the zero viscosity limit to the strong solution of the Euler system, as long as the latter exists.
To handle the problem of uniqueness, one possible way is to perform a semiflow selection, identifying, among all the solutions emanating from the same initial data, the one satisfying the semigroup property. In the second part of the thesis we study under which assumptions it is possible to guarantee the existence of a semiflow selection for autonomous and non-autonomous systems, choosing the Skorokhod space of càglàd functions as trajectory space. Subsequently, we adapt this abstract machinery to the compressible Navier-Stokes system, for which we will be able to prove the existence of a semiflow selection depending only on the initial density and momentum, and to models describing general non-Newtonian fluids. In this latter case, we prove the existence of dissipative solutions for a linear pressure and we will analyse under which conditions it is possible to guarantee the existence of weak solutions.

# ZUSAMMENFASSUNG

Die Wohlgestelltheit von Problemen, die aus der Beschreibung der Bewegung von Fluiden stammen, in der Klasse von starken oder schwachen Lösungen zu zeigen, ist eine der herausforderndsten Aufgabenstellungen in der modernen Theorie partieller Differentialgleichungen.
Eine Möglichkeit, um die Existenz von Lösungen zu zeigen, ist den Lösungsgebriff passend zu erweitern. Im ersten Teil dieser Arbeit identifizieren wir eine maßwertige Lösung der gedämpften kompressiblen Euler-Gleichungen auf einem allgemeinen (unbeschränkten) Gebiet, die durch ein parametrisiertes Young-Maß charakterisiert wird, als den Grenzwert der kompressiblen Navier-Stokes-Gleichungen bei verschwindender Viskosität. Anschließend zeigen wir ein Prinzip der schwach- beziehungsweise maßwertig-starken Einzigkeit und erhalten als Konsequenz die Konvergenz schwacher Lösungen der Navier-Stokes-Gleichungen bei verschwindender Viskosität gegen starke Lösungen der Euler-Gleichungen, falls letztere existieren.
Ein Weg die Frage nach der Einzigkeit von Lösungen zu behandeln ist die Auswahl eins Halbflusses, das heißt aus der Menge aller Lösungen zu gegebenen Anfangsdaten diejenige auszuwählen, die die Halbgruppeneigenschaft besitzt. Im zweiten Teil der Arbeit untersuchen wir, unter welchen Voraussetzungen eine solche Auswahl eines Halbflusses für autonome und nicht-autonome Systeme möglich ist, wobei wir als Raum für die Trajektorien den Skorochod-Raum der Càglàd-Funktionen wählen. Anschließend adaptieren wir die abstrakten Resultate für die kompressiblen Navier-Stokes-Gleichungen, für die wir die Existenz einer Halbfluss-Auswahl zeigen können, welche nur von den Anfangswerten der Dichte und Impulsdichte abhängt, sowie für eine Klasse von Modellen zur Beschreibung nichtnewtonscher Fluide. Für Letztere zeigen wir die Existenz dissipativer Lösungen unter Annahme eines linearen Drucks und untersuchen unter welchen Bedingungen die Existenz schwacher Lösungen gesichert ist.

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# INTRODUCTION

From the blood flowing in our bodies to the air we breathe, fluids play an essential role in our everyday lives. As the etymology of the word suggests (from Latin *fluere*,"to flow"), a fluid is a substance tending to flow or conform to the outline of its container, not resisting any shear force applied to it; the most common examples are gases, liquids and plasmas. In view of their importance for many real world problems and thus the need of a rigorous analytical description, it has already been figured out during the 18th century that the motion of fluids can be modelled through a system of partial differential equations, a mathematical transcription of (mainly) two physical conservation laws. Assuming that the fluid is a *continuum*, contained at a given time $t \in \mathbb{R}$ in a certain spatial domain $\Omega$ of the $d$-dimensional Euclidean space $\mathbb{R}^d$, the first principle, known as *conservation of mass*, gives birth to the following equation

$$\partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) = 0, \tag{1}$$

while the second one, known as *conservation of momentum*, has in general the following expression

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p = \operatorname{div}_x \mathbb{S}. \tag{2}$$

In both (1) and (2), the unknown quantities are the *density* $\varrho = \varrho(t, x)$ and the *velocity* $\mathbf{u} = \mathbf{u}(t, x)$ of the fluid, while the *pressure* $p$ and the *viscous stress* $\mathbb{S}$ are determined by the material properties of the fluid. Depending on the choice of $\mathbb{S}$, i.e. the capacity of the fluid to resist to deformation at a given rate, we obtain different systems. If we deal with an inviscid flow and thus $\mathbb{S} = 0$ in (2), we get the *compressible Euler system*; in this case, instead of velocity, the *momentum* $\mathbf{m} = (\varrho \mathbf{u})(t, x)$ is often considered as state function along with the density. Even though there are only few examples of inviscid fluids in the real world, better known as superfluids, Euler equations are a matter of great interest in fluid dynamics as they represent the optimal model to describe gases and the behaviour of some vortex-like phenomena with vanishing viscous forces, such as the formation of tornados. In ordinary conditions, several fluids are *isotropic*, i.e. their properties are the same in all directions, and satisfy a constitutive equation known as Newton's rheological law under which viscosity is a linear function of the velocity gradient. More precisely,

$$\mathbb{S} = \mathbb{S}(\nabla_x \mathbf{u}) = \mu \left( \nabla_x \mathbf{u} + \nabla_x^T \mathbf{u} - \frac{2}{d} (\operatorname{div}_x \mathbf{u}) \mathbb{I} \right) + \eta (\operatorname{div}_x \mathbf{u}) \mathbb{I},$$

where $\mu > 0$, $\eta \geq 0$ are constants. With this particular choice of $\mathbb{S}$ we obtain the *compressible Navier-Stokes system*. Water belongs to this class, even though it is modelled as an *incompressible* fluid, i.e. the density $\varrho$ is constant and thus equation (1) reduces to $\operatorname{div}_x \mathbf{u} = 0$. However, not all the fluids that we might encounter in nature are Newtonian, just think of blood, honey and paint to name some: viscosity can change to either more liquid or more solid when under force. We can then consider a *general compressible viscous fluid system*, assuming the viscous stress tensor $\mathbb{S}$ to be related to the symmetric velocity gradient

$$\mathbb{D}_x \mathbf{u} = \frac{1}{2} (\nabla_x \mathbf{u} + \nabla_x^T \mathbf{u})$$

through a general implicit rheological law

$$\mathbb{S} : \mathbb{D}_x \mathbf{u} = F(\mathbb{D}_x \mathbf{u}) + F^*(\mathbb{S}),$$

with $F$ a proper lower semi-continuous function and $F^*$ its conjugate. The physical background of writing the constitutive equation for $\mathbb{S}$ in this form is the fact that $\mathbb{S}$ is monotone in the velocity gradient and vice versa, as clearly explained in the recent survey on a new classification of incompressible fluids by Blechta, Málek and Rajagopal [10]. For more details on how to deduce equations (1) and (2) one can consult, for instance, the monograph by Chorin and Marsden [24].

According to Hadamard, every mathematical system modelling physical phenomena should be *well-posed*, meaning that the following three issues must be verified.

- *Existence*: for any fixed data, a solution exists.
- *Uniqueness*: for any fixed data, the solution is unique.
- *Stability*: small perturbations of the data should result in small variation of the corresponding solutions.

The *data* of our problem (1)–(2) are the initial and boundary conditions, i.e. the values of the state variables $\varrho$ and $\mathbf{u}$ (or $\mathbf{m}$ according to the system) at the initial time and on the boundary of the domain, respectively. In order to verify the aforementioned properties, it is first important to specify the notion of *solution*. Intuitively, one would say that the couple $(\varrho, \mathbf{u})$ of smooth, or at least continuously differentiable, functions constitutes a solution if it satisfies equations (1) and (2) pointwise. This is the definition of *strong* or *classical* solution. However, whether or not system (1)–(2) admits a unique strong solution for any fixed data is still an open question and it represents one of the most challenging problems of the contemporary theory of partial differential equations. The idea is then to look for a more general concept of solution, for which the derivatives may not all exist but which satisfy system (1)–(2) in some sense. Roughly speaking, we may multiply both equations (1) and (2) by *test functions*, i.e. smooth and compactly supported functions, integrate over our domain and through an integration by parts transfer all the derivatives to the test functions. In this way, we get the concept of *weak* or *distributional* solution. In order to establish some *a priori bounds* necessary to determine the function spaces framework where the distributional solutions are looked for, and in order to guarantee the problem to remain well-posed in the class of weak solutions, system (1)–(2) is often coupled with a third integral relation known as *energy inequality* and given by

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left[ \frac{1}{2} \varrho |\mathbf{u}|^2 + P(\varrho) \right] \mathrm{d}x + \int_\Omega \mathbb{S} : \nabla_x \mathbf{u} \, \mathrm{d}x \leq 0, \tag{3}$$

where the *pressure potential* $P$ is the solution of the following ordinary differential equation:

$$\varrho P'(\varrho) - P(\varrho) = p(\varrho).$$

It is worth noticing that equality holds in (3) if we deal with strong solutions, i.e. the density $\varrho$ and the velocity $\mathbf{u}$ are smooth functions, but it may not be true if we consider weak solutions. The point is that there may be some sources of dissipation due to possible singularities and, if this occurs, a non-negative measure must be added to the left-hand side of the energy equality, or we can equivalently consider the inequality as in (3). We say that the couple $(\varrho, \mathbf{u})$

constitutes a *dissipative weak* solution to system (1)–(2) if equations (1), (2) and inequality (3) hold in the distributional sense. Nevertheless, even with this weaker notion of solution, the problem of well-posedness has a lot of open tasks. Let us first focus on the Euler system. In 1965 Glimm [47] proved the existence of dissipative weak solutions when $d = 1$ and sufficiently small (in the $BV$-norm) initial data, while uniqueness and stability under the same conditions were shown later on by Bressan, Crasta and Piccoli [15]. There are, of course, a lot of available results but well-posedness in the context of dissipative weak solutions when $d \geq 2$ is far from being solved. Moreover, the recent developments achieved by means of the *convex integration* technique lead us to conclude that the barotropic Euler system is ill-posed in the class of dissipative weak solutions in the multidimensional setting. More precisely, De Lellis and Székelyhidi [28] proved the existence of bounded initial data for which there are infinitely many dissipative weak solutions, also known as *wild* solutions. Similarly, for a periodic domain, Chiodaroli [23] and Feireisl [34] showed with two different approaches that for any continuously differentiable initial density there exists a bounded initial velocity which generates wild solutions. The overall picture is quite different when we turn our attention to the Navier-Stokes system. Well-posedness for $d = 1$ was completely solved by Kazhikhov [52]. In higher dimension and with the *isentropic* pressure $p(\varrho) = a\varrho^\gamma$, existence of dissipative weak solutions for any finite initial data when the adiabatic exponent $\gamma$ is strictly greater than $d/2$ was proved by Lions [57] and Feireisl [33] with homogeneous boundary conditions, and by Girinon [46], Chang, Jin and Novotný [21] for general inflow-outflow boundary conditions, to cite few results. The case $d = 2$ and $\gamma = 1$ was solved by Plotnikov and Weigant [67]. However, as for the Euler system, uniqueness is still an open problem. Finally, not much can be said for systems describing general viscous fluids, if we look for global-in-time solutions with large data. The existence of large-time weak solutions was proved by Mamontov [60], [61] in the case of exponentially growing viscosity coefficients and linear pressure $p(\varrho) = a\varrho$, by Feireisl, Liao and Málek [40] in the case where the bulk viscosity $\lambda = \lambda(|\operatorname{div}_x \mathbf{u}|)$ becomes singular for a finite value of $|\operatorname{div}_x \mathbf{u}|$, and by Matušů-Nečasová and Novotný [62] in the case of linear pressure.

To overcome the problem of *existence*, one suitable idea is to consider a larger class of solutions. More precisely, we may look for a *measure-valued* solution, i.e. a map returning for every point in the domain a probability distribution of values and satisfying the equations only in an average sense. The advantage of relaying on this very weak concept of solution is that they can be easily identified as limits of weakly convergent subsequences of approximate solutions, even in presence of oscillations and concentrations, which actually become part of the definition. Moreover, recently they have been used in the analysis of convergence of certain numerical schemes, see e.g. [41]. In the context of the isentropic Euler system with $d = 1$, DiPerna [29] was able to show that a measure-valued solution obtained through convergence of approximated solutions is not just a measure but actually a weak solution. In higher dimension, the existence of *dissipative* measure-valued solutions, i.e. satisfying an analogous of the energy inequality (3), was first proved by Gwiazda, Świerczewska-Gwiazda and Wiedemann [48] and later by Breit, Feireisl and Hofmanová [14] with a slightly different approach involving energy defects instead of the description of concentrations via the Alibert–Bouchitté defect measures, cf. [4]. In the context of a general compressible viscous fluid system, Abbatiello, Feireisl and Novotný [2] showed the existence of *dissipative* solutions, i.e. weak solutions satisfying the system modulo a defect measure, with an isentropic pressure and adiabatic exponent $\gamma > 1$. However, the problem of uniqueness seems to be even harder to solve in such a wide class of solutions. Fundamental in this connection becomes the *weak-strong uniqueness* principle: if the system admits a solution in the classical sense then it must coincide with the measure-valued solution emanating from the same initial data. Because of its importance, the proof of existence of measure-valued solutions goes hand-in-hand with the one of weak-strong uniqueness and it is actually present in all the aforementioned works.

In the same spirit, in the first part of the present thesis we identify a class of measure-valued solutions of the isentropic Euler system with *damping*, an extra term modelling friction, as a vanishing viscosity limit for the Navier-Stokes system, cf. Chapter 2. The strategy here pursued can be seen as the "compressible" counterpart of the pioneering work from Di Perna and Majda [31] in the incompressible case. The compressible case was treated by Sueur [74] on a bounded domain by means of the relative energy method. Our goal is to propose an alternative approach based on the concept of dissipative measure-valued solutions and extend the result to a more general class of domains. A similar limit problem was also considered by Březina and Mácha [17] on the flat torus, where the starting system is the compressible Navier-Stokes one with some extra terms modelling non–local interaction forces, and by Chen and Glimm [22] on the whole domain $\mathbb{R}^3$, with a Kolmogorov–type hypothesis. It is worth noticing that, unlike the standard Euler system, the linear damping term guarantees the existence of a global-in-time strong solution at least for certain small initial data and therefore the result can be applied to concrete examples, cf. [65], [70] in the case of a bounded domain, and [50], [58], [72] for the whole space.

To handle the problem of *uniqueness*, one possible way is to perform a sort of selection in order to identify, among all the solutions emanating from the same initial data, a suitable one such that the resulting selection enjoys the *semigroup* (or *semiflow*) property: letting the system run from time $0$ to time $s$, restarting it and letting it run from time $s$ to time $t$ is equivalent in letting it run directly from time $0$ to time $t$. We refer to the procedure of identifying this kind of solution when possible as *semiflow selection*. The construction of the semigroup arises from the theory of Markov selection in order to study the well-posedness of certain system; it was first developed by Krylov [54] and later adapted by Flandoli and Romito [44], Cardona e Kapitanski [20] in the context of the incompressible Navier-Stokes system. In the same spirit, Breit, Feireisl and Hofmanová [14] proved the existence of the semiflow selection for dissipative measure-valued solution of the isentropic Euler system. Note that this strategy provides a suitable alternative to establish well-posedness in contrast to problems, where uniqueness can be achieved as a consequence of intrinsic stability, cf. e.g. DiPerna–Lions theory [30] and its extension by Ambrosio [5], or the theory of viscosity solutions for scalar parabolic equations, see Crandall, Ishii and Lions [26]. The main advantage of this approach is the possibility to identify the class of solutions that maximize the energy dissipation rate. They reflect many important properties with (hypothetical) smooth solutions in the long run: convergence to equilibria and/or certain stable waves, see Feireisl, Kwon and Novotný [39], Feireisl and Novotný [43].

The second part of this thesis will be entirely dedicated to the semiflow selection. More precisely, inspired by [14], [20], we will first analyse under which assumptions it is possible to guarantee the existence of a semiflow selection in an abstract setting, cf. Chapter 3. The main novelty of the present work is the choice of the Skhorohod space of cáglád, i.e. left-continuous and having right-hand limits, functions as trajectory space $\mathcal{T}$. Actually, one could think that a more natural choice for $\mathcal{T}$ would be the space of continuous functions as in [20]. However, this option can be too strong: if we want to apply the abstract setting to the typical systems arising from fluid dynamics, it is difficult to ensure the energy of the system to be continuous since it is at most a non-increasing quantity with possible jumps. For the aforementioned reason, in the context of the isentropic Euler system [14], the authors considered the energy in the $L^1$-space. But the space of integrable functions as trajectory space is still not an optimal choice as it is better to work with a space whose elements are well-defined at any point. Afterwards, we will adapt this abstract machinery to the compressible Navier-Stokes and general viscous fluid systems, cf. Chapters 4 and 5 respectively.

## STRUCTURE OF THE THESIS

We conclude this introductory part summarizing what will be treated in the present thesis.

In Chapter 1, we collect all the mathematical tools that will be used throughout the thesis, with close attention to the Orlicz and Skorokhod spaces, cf. Sections 1.2, 1.3 respectively, and to the Young measures, cf. Section 1.4. In particular, we derive a proper metric for the Skorokhod space on an unbounded domain and taking values in a separable Hilbert space, cf. Section 1.3.2.

The first main goal of Chapter 2 is to identify a class of generalized - *dissipative measure-valued* solutions - for the Euler system with damping on a general (unbounded) domain as a vanishing viscosity limit of the Navier-Stokes equations. More precisely, we will work on two fronts considering, on one side, a family of new domains obtained as intersection of the balls of radius $R$ with the primordial domain, and, on the other side, a family of viscous coefficients of the compressible Navier-Stokes system rescaled by $1/R$. For each fixed $R > 0$, the existence of dissipative weak solution in this context has already been established by Feireisl [33] for any finite energy initial data and thus, thanks to some a priori estimates, we will be able to perform the limit $R \to \infty$ simultaneously in the domain and in the viscosity. In particular, the dissipative weak solutions of the Navier-Stokes system will generate for $R \to \infty$ a Young measure, which will be identified with a dissipative measure-valued solution for the Euler system with damping, cf. Theorem 2.5.2. The second fundamental goal of Chapter 2 is the validity of the *weak-strong uniqueness* principle for the compressible Euler system with damping: if the system admits a strong solution, it must coincide with the dissipative measure-valued one emanating from the same initial data, cf. Theorem 2.6.1. Finally, combining these two achievements, we can conclude that the solutions of the Navier-Stokes system converge in the zero viscosity limit to the strong solution of the Euler system with damping as long as the latter exists, cf. Theorem 2.7.1. The work presented in this chapter can be found in [9].

In Chapter 3, we study under which conditions it is possible to guarantee the existence of a semiflow selection in an abstract setting. In the first part we will focus on an *autonomous system* for which the time variable does not appear explicitly in the equations. More precisely, denoting with $H$ a separable Hilbert space, with $X \subseteq H$ the phase space and with $\mathcal{T} = \mathfrak{D}([0,\infty); H)$ the trajectory space associated to the system, we will show the existence of a Borel-measurable map $\mathrm{U} : X \to \mathcal{T}$ such that for any initial data $\mathrm{u}_0 \in X$, $\mathrm{U}(\mathrm{u}_0)$ represents that particular solution satisfying the semigroup property:

$$\mathrm{U}(\mathrm{u}_0)(t_1 + t_2) = \mathrm{U}\left[\mathrm{U}(\mathrm{u}_0)(t_1)\right](t_2) \quad \text{for any } t_1, t_2 \geq 0,$$

cf. Theorem 3.1.2. Following Cardona and Kapitanski [20] and Breit, Feireisl and Hofmanová [14], the key point of the proof is to assume that the set-valued map $\mathcal{U} : X \to 2^{\mathcal{T}}$, associating to every initial data $u_0 \in X$ the family $\mathcal{U}(u_0)$ of solutions that arise from $u_0$, satisfies five properties: non-emptiness, compactness, Borel-measurability, shift-invariance and continuation. In the second part we will examine the case of a *non-autonomous system*, where the time variable appears explicitly in the equations and thus it is always necessary to specify the starting time $t_0 \geq 0$. Introducing the trajectory space $\mathcal{T}_{t_0} = \mathfrak{D}([t_0, \infty); H)$ for any $t_0 \geq 0$, we look for a semiprocess $\{P_{t_0}\}_{t_0 \geq 0}$, with $P_{t_0} : X \to \mathcal{T}_{t_0}$, such that for any initial data $v_0 \in X$, $P_{t_0}(v_0)$ represents that particular solution satisfying an analogous of semigroup property for autonomous systems:

$$P_{t_0}(v_0)(t_2) = P_{t_1}[P_{t_0}(v_0)(t_1)](t_2) \quad \text{for any } t_0 \leq t_1 \leq t_2. \tag{4}$$

As already done by Cardona and Kapitanski [19], the rather standard method how to prove the existence of such semiprocess is to convert the system into an autonomous one considering the time as a new unknown. However, there are possible drawbacks of this method. Let $h_0$ denote the time-dependent quantities appearing in the system.

1. If $h_0$ is independent of time, do we get the natural condition that $P_{t_0}(v_0)$ is independent of $t_0$ meaning
$$P_{t_0}(v_0) \simeq P_{t_1}(v_0) \quad \text{for every } t_0, t_1 \geq 0,$$
for every fixed initial data $v_0 \in X$?

2. If $h_0$ is periodic in time for some constant $T > 0$, do we get the natural condition that
$$P_{t_0+T}(v_0) \simeq P_{t_0}(v_0) \quad \text{for every } t_0 \geq 0,$$
for every fixed initial data $v_0 \in X$?

3. Let $h_0^1$ and $h_0^2$ be two different time-dependent quantities appearing in the system and let $\{P^i_{t_0}\}_{t_0 \geq 0}$ be the selection associated to $h_0^i$, with $i = 1, 2$. If
$$h_0^1(t) = h_0^2(t) \quad \text{for any } t \geq T > 0,$$
do we get the natural condition that
$$P_t^1(v_0) = P_t^2(v_0) \quad \text{for all } t \geq T > 0,$$
for every fixed initial data $v_0 \in X$?

In order to guarantee all the above properties, it is more convenient to consider various time dependent quantities $h_0 \in H_D$ represented by the forcing but also possibly boundary conditions as *data*, specifically not as quantities fixed from the beginning but as part of the (initial) data along with the initial condition $v_0 \in X$. More precisely, inspired by the works of Capelato, Samprogna and Simsen [18], [69], we will first define an *exact generalized semiprocess* as a family $\{\mathcal{G}_{t_0}\}_{t_0 \geq 0}$ of set-valued functions $\mathcal{G}_{t_0} : X \times H_D \to 2^{\mathcal{T}_{t_0}}$ that associated to every initial data $(v_0, h_0) \in X \times H_D$ the family $\mathcal{G}_{t_0}(v_0, h_0)$ of solutions arising from $(v_0, h_0)$ and satisfying the same five properties of the analogous set-valued map $\mathcal{U}$ for autonomous system. In this way, we will be able to prove the existence of a semiprocess $\{P_{t_0}\}_{t_0 \geq 0}$ associated to every exact generalized semiprocess, satisfying (4) and the three aforementioned properties, cf. Theorem 3.2.3 and Proposition 3.2.4. The first part on autonomous systems can be found in [8], while the second part on non-autonomous systems is new and does not appear in any published article.

Chapter 4 is entirely devoted to the compressible Navier-Stokes system. It is well-know that the system admits global in time dissipative weak solutions, see e.g. Lions [57] and Feireisl [33]; uniqueness, on the other side, is still an open problem. The idea is then to adapt the abstract machinery introduced in Chapter 3 to prove the existence of a semiflow selection in this context, cf. Theorem 4.3.1. Moreover, we will show that it is possible the select only the *admissible* solutions minimizing the total energy, cf. Definition 4.2.2. Following the same strategy performed by Breit, Feireisl and Hofmanová [14] to show existence of a semiflow selection for dissipative measure-valued solutions of the isentropic Euler system, at first we will consider the energy $E$ as a third state variable along with the density $\varrho$ and the momentum $\mathbf{m}$. However, while for the Euler flow the expression of the energy only in terms of $[\varrho, \mathbf{m}]$ is a delicate issue as it may contain defect due to possible oscillations/concentrations, such a problem does not occur for the Navier-Stokes system, where the energy is indeed a function of $[\varrho, \mathbf{m}]$ at least for a.e. $t \in (0, \infty)$. Due to the aforementioned reason, we will be able to prove the existence of a restricted selection not depending on the initial energy $E_0$, cf. Theorem 4.4.1. The work presented in this chapter can be found in [7].

The goal of Chapter 5 is to prove the existence of a semiflow selection for models of general compressible viscous fluid described by equations (1), (2) for which we assume a barotropic pressure $p(\varrho) = a\varrho^\gamma$, $\gamma \geq 1$, and the viscous stress tensor $\mathbb{S}$ to be related to the symmetric velocity gradient $\mathbb{D}_x\mathbf{u}$ through a general implicit rheological law of the type

$$\mathbb{S} : \mathbb{D}_x\mathbf{u} = F(\mathbb{D}_x\mathbf{u}) + F^*(\mathbb{S}),$$

with $F$ a proper lower semi-continuous function. It is worth noticing that Newton's rheological law 4.1.5 can be obtained choosing

$$F(\mathbb{D}_x\mathbf{u}) = \frac{\mu}{2}|\mathbb{D}_x\mathbf{u}|^2 + \frac{\lambda}{2}|\operatorname{div}_x \mathbf{u}|^2, \quad \text{with } \mu > 0,\ \frac{2}{d} + \lambda \geq 0,$$

and thus one may wonder why the Navier-Stokes system was studied in a separate chapter. The reason is that in this context we cannot guarantee the existence of weak solutions and a more general notion of solution must be considered, namely *dissipative solutions*, containing the defects arising from possible oscillations and/or concentrations, cf. Definition 5.2.1. In order to prove the existence of a semiflow selection in this context, there are two main difficulties we have to overcome: the *weak sequential stability* of the family of dissipative solutions and the *existence* of dissipative solutions, arising from a fixed initial data. While the first problem will be solved for any $\gamma \geq 1$, cf. Theorem 5.4.1, the second issue will be handled only for $\gamma = 1$, cf. Theorem 5.5.12, as the case $\gamma > 1$ was recently solved by Abbatiello, Feireisl and Novotný in [2]. It is interesting to note that for $\gamma = 1$ and particular choices of the function $F$, the defect in the momentum equation vanishes and the latter is satisfied in the sense of distributions, cf. Theorem 5.6.2. Thus our approach represents an alternative to the "standard" measure-valued framework applied in this context by Matušů-Nečasová and Novotný [62]. The work presented in this chapter can be found in [8], apart of Sections 5.5 and 5.6, which can be found in [6].

# NOTATION

| | |
|---|---|
| $\mathbb{R}^d$ | $d$-dimensional Euclidean space, $d \geq 1$ |
| $\Omega$ | domain - an open connected subset of $\mathbb{R}^d$ |
| $\overline{Q}$ | closure of a set $Q \subset \mathbb{R}^d$ |
| $\partial Q$ | boundary of a set $Q \subset \mathbb{R}^d$ |
| $\mathbf{a} = [a_1, \dots, a_d]$ | vector in $\mathbb{R}^d$, $d \geq 2$ |
| $\mathbb{A} = [a_{ij}]_{i,j=1}^d$ | square matrix in $\mathbb{R}^{d \times d}$, $d \geq 2$ |
| $\mathbb{A}^T = [a_{ji}]_{i,j=1}^d$ | transpose of a square matrix $\mathbb{A} = [a_{ij}]_{i,j=1}^d$ |
| $\mathrm{Tr}[\mathbb{A}] = \sum_{i=1}^d a_{ii}$ | trace of a square matrix $\mathbb{A} = [a_{ij}]_{i,j=1}^d$ |
| $\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^d a_i b_i$ | scalar product of two vectors $\mathbf{a} = [a_1, \dots, a_d]$, $\mathbf{b} = [b_1, \dots, b_d]$ |
| $\mathbb{A} : \mathbb{B} = \sum_{i,j=1}^d a_{ij} b_{ij}$ | scalar product of two matrices $\mathbb{A} = [a_{ij}]_{i,j=1}^d$, $\mathbb{B} = [b_{ij}]_{i,j=1}^d$ |
| $\mathbf{a} \otimes \mathbf{b} = [a_i b_j]_{i,j=1}^d$ | tensor product of two vectors $\mathbf{a} = [a_1, \dots, a_d]$, $\mathbf{b} = [b_1, \dots, b_d]$ |
| $\partial_{y_i} f = \frac{\partial f}{\partial y_i}$ | partial derivative of a function $f = f(y)$, $y = [y_1, \dots, y_d]$, with respect to the variable $y_i$ |
| $\partial^\alpha f = \partial_{y_1}^{\alpha_1} \dots \partial_{y_d}^{\alpha_d} f$ | partial derivative of order $\lvert\alpha\rvert = \sum_{i=1}^d \alpha_i$ of a function $f = f(y)$ |
| $\nabla_y f = [\partial_{y_1} f, \dots, \partial_{y_d} f]$ | gradient of a function $f = f(y)$ |
| $\mathbb{D}_y \boldsymbol{f} = \frac{1}{2}(\nabla_y \boldsymbol{f} + \nabla_y^T \boldsymbol{f})$ | symmetric gradient of a function $f = f(y)$ |
| $\mathrm{div}_y\, \mathbf{v} = \sum_{i=1}^d \partial_{y_i} v_i$ | divergence of a vector function $\mathbf{v}$, $\mathbf{v}(y) = [v_1(y), \dots, v_d(y)]$ |
| $\Delta_y f = \mathrm{div}_y \nabla_y f$ | Laplace operator of a function $f = f(y)$ |
| $2^X$ | family of all subsets of a space $X$ |
| $\lVert \cdot \rVert_X$ | norm on a normed linear space $X$ |
| $\langle \cdot ; \cdot \rangle_{X^*, X}$ | duality pairing between a vector space $X$ and its dual $X^*$ |
| $\mathrm{span}(M)$ | space of all linear finite combinations of vectors contained in $M \subset X$, with $X$ a vector space |
| $\mathrm{supp}(f) = \overline{\{y \in Q : \ f(y) \neq 0\}}$ | support of a function $f : Q \to \mathbb{R}$, $Q \subset \mathbb{R}^d$ |
| $\lesssim$ | inequality holds modulo a multiplicative constant |

# CONTENTS

# 1

# PRELIMINARIES

The aim of this chapter is to collect all the mathematical tools that will be used throughout the thesis. For the well-known results, a reference will be given without further details, while proofs will be provided for the ones derived for the specific purposes of this work.

## 1.1 COMMON FUNCTION SPACES

We start recalling the definitions and fundamental properties of the function spaces the reader will encounter during the treatment.

### 1.1.1 *Spaces of continuous functions*

Let $Q \subseteq \mathbb{R}^d$ be an open set and $X$ a Banach space. We denote with

- $C(Q;X)$ the space of continuous functions on $Q$ and ranging in $X$.

  If $Q$ is bounded, $C(\overline{Q};X)$ is a Banach space with norm
  $$\|f\|_{C(\overline{Q};X)} = \sup_{y\in\overline{Q}} \|f(y)\|_X;$$

- $C_{\mathrm{weak}}(Q;X)$ the space of functions defined on $Q$ and ranging in $X$ which are continuous with respect to the weak topology.

  If $Q$ is bounded, we say that
  $$f_n \to f \quad \text{in } C_{\mathrm{weak}}(\overline{Q};X) \quad \text{as } n \to \infty$$
  if for all $g \in X^*$
  $$\sup_{y\in\overline{Q}} |\langle g; f_n(y) - f(y)\rangle_{X^*,X}| \to 0 \quad \text{as } n \to \infty;$$

- $C^k(Q;X)$, with $k$ a non-negative integer, the space of $k$-times continuously differentiable functions on $Q$.

  If $Q$ is bounded, $C^k(\overline{Q};X)$ is a Banach space with norm
  $$\|f\|_{C^k(\overline{Q};X)} = \max_{|\alpha|\le k} \sup_{y\in\overline{Q}} \|\partial^\alpha f(y)\|_X.$$

  Moreover, we set
  $$C^\infty(Q;X) = \bigcap_{k=0}^{\infty} C^k(Q;X);$$

- $\mathcal{D}(Q;X) = C_c^\infty(Q;X)$ the space of functions belonging to $C^\infty(Q;X)$ and having compact support in $Q$;

- $C_0(Q)$ the completion of $C_c(Q)$ with respect to the uniform norm $\|\cdot\|_\infty$.

### 1.1.2 *Spaces of measures*

Let $Q \subseteq \mathbb{R}^d$ be an open set. We denote with

- $\mathcal{M}(Q;\mathbb{R}^m) = [C_0(Q;\mathbb{R}^m)]^*$ the space of vector-valued Radon measures, which can be identified as the dual space of $C_0(Q;\mathbb{R}^m)$.

  If $\Omega \subset \mathbb{R}^d$ is a bounded domain, then $\mathcal{M}(\overline{\Omega}) = [C(\overline{\Omega})]^*$.
- $\mathcal{M}^+(Q)$ the space of positive Radon measures;
- $\mathcal{M}^+(Q;\mathbb{R}^{d\times d}_{\mathrm{sym}})$ the space of tensor–valued Radon measures $\mathfrak{R}$ such that
$$\mathfrak{R} : (\xi \otimes \xi) \in \mathcal{M}^+(Q),$$
  for all $\xi \in \mathbb{R}^d$, and with components $\mathfrak{R}_{i,j} = \mathfrak{R}_{j,i}$;
- $\mathcal{D}'(Q;\mathbb{R}^m) = [C_c^\infty(Q;\mathbb{R}^m)]^*$ the space of distributions, which can be identified as the dual space of $\mathcal{D}(Q;\mathbb{R}^m)$.

### 1.1.3 *Lebesgue spaces*

Let $Q \subseteq \mathbb{R}^d$ be a measurable set and $X$ a Banach space. We denote with

- $L^p(Q;X)$, with $1 \le p \le \infty$, the Banach space of Bochner-measurable functions $f$ defined on $Q$ and ranging in $X$ such that the norm
$$\|f\|_{L^p(Q;X)} = \left(\int_Q \|f(y)\|_X^p \,\mathrm{d}x\right)^{\frac{1}{p}}, \quad \text{for } 1 \le p < \infty$$
$$\|f\|_{L^\infty(Q;X)} = \operatorname*{ess\,sup}_{y\in Q} \|f(y)\|_X$$
  is finite;
- $L^p_{\mathrm{loc}}(Q;X)$, with $1 \le p < \infty$, the vector space of locally $L^p$-integrable functions, meaning that $f \in L^p_{\mathrm{loc}}(Q;X)$ if $f \in L^p(K;X)$ for any compact set $K \subset Q$.

Let $Q \subseteq \mathbb{R}^d$ be a measurable set and $L^p(Q) := L^p(Q;\mathbb{R})$. Then, the following inequalities hold.

(i) *Hölder's inequality*:
$$\|fg\|_{L^r(Q)} \le \|f\|_{L^p(Q)}\|g\|_{L^q(Q)} \tag{1.1.1}$$
with
$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q}, \quad 1 \le p,q,r \le \infty,$$
for any $f \in L^p(Q)$ and $g \in L^q(Q)$ (see e.g. Adams [3], Theorem 2.3).

(ii) *Minkowski's inequality*:
$$\|f+g\|_{L^p(Q)} \le \|f\|_{L^p(Q)} + \|g\|_{L^p(Q)}, \tag{1.1.2}$$
for any $1 \le p < \infty$ (see e.g. Adams [3], Theorem 2.4).

(iii) *Interpolation inequality*:

$$\|h\|_{L^r(Q)} \leq \|h\|^{\theta}_{L^p(Q)} \|h\|^{1-\theta}_{L^q(Q)}, \tag{1.1.3}$$

with

$$\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}, \quad 1 \leq p < r < q \leq \infty,$$

for any $h \in L^p(Q) \cap L^q(Q)$ (it follows from (i) choosing $f = h^{\theta}$, $g = h^{1-\theta}$).

(iv) *Young's inequality*:

$$\|fg\|_{L^1(Q)} \leq \frac{1}{p}\|f\|^p_{L^p(Q)} + \frac{1}{q}\|g\|^q_{L^q(Q)} \tag{1.1.4}$$

with

$$1 = \frac{1}{p} + \frac{1}{q}, \quad 1 < p, q < \infty,$$

for any $f \in L^p(Q)$ and $g \in L^q(Q)$ (it easily follows using the concavity of the logarithm function

$$\log(|fg|) = \log(|f|) + \log(|g|) = \frac{1}{p}\log(|f|^p) + \frac{1}{q}\log(|g|^q) \leq \log\left(\frac{1}{p}|f|^p + \frac{1}{q}|g|^q\right),$$

exponentiating and integrating over $Q$).

There is another way how to introduce the Lebesgue spaces, namely to consider completion of the space $\mathfrak{D}(Q; X)$ with respect to the norm $\|\cdot\|_{L^p(Q;X)}$, when $1 \leq p < \infty$. To this end, we need the following definition.

**Definition 1.1.1.** A family of *regularizing kernels* in $\mathbb{R}^d$ is a sequence $\{\theta_\varepsilon\}_{\varepsilon>0}$ such that

$$\theta_\varepsilon(y) := \frac{1}{\varepsilon^d}\theta\left(\frac{y}{d}\right) \quad \text{for any } y \in \mathbb{R}^d,$$

where $\theta \in C_c^\infty(\mathbb{R}^d)$ is a non-negative, bell-shaped function such that

$$\theta(y) = \theta(|y|), \quad \int_{\mathbb{R}^d} \theta(y)\, \mathrm{d}y = 1.$$

Let $Q \subseteq \mathbb{R}^d$ be a measurable set. Given a function $f \in L^p(Q; X)$, with $1 \leq p \leq \infty$, the *convolution* of $\theta_\varepsilon$ with $f$ is given by

$$(\theta_\varepsilon * f)(y) = \int_{\mathbb{R}^d} \theta_\varepsilon(y-z) f(z)\, \mathrm{d}z \quad \text{for any } y \in \mathbb{R}^d,$$

where $f$ has been extended to be zero outside $Q$.

Lebesgue functions can be approximated by smooth functions, as stated in the following result.

**Theorem 1.1.2.** *Let $Q \subseteq \mathbb{R}^d$ be a measurable set, $X$ a separable Banach space and $\{\theta_\varepsilon\}_{\varepsilon>0}$ a family of regularizing kernels in $\mathbb{R}^d$.*

*(i) For any $f \in L^p(Q; X)$, with $1 \leq p < \infty$, extended to be zero outside $Q$,*

$$\theta_\varepsilon * f \in C_c^\infty(\mathbb{R}^d; X),$$

*and the derivatives can be transferred to the smooth term:*

$$\partial^\alpha(\theta_\varepsilon * f) = (\partial^\alpha \theta_\varepsilon) * f \quad \textit{for any partial derivative } \partial^\alpha.$$

*(ii) For any $f \in L^p(Q;X)$, with $1 \le p < \infty$,*

$$\theta_\varepsilon * f \to f \quad \text{in } L^p(Q;X)$$

*as $\varepsilon \to 0$.*

*Proof.* See e.g. Brezis [16], Proposition IV.20 and Theorem IV.22. □

We conclude reporting some properties of the $L^p$-spaces.

**Theorem 1.1.3.** *Let $Q \subseteq \mathbb{R}^d$ be a measurable set and $X$ a separable Banach space.*

*(i) $L^p(Q;X)$, with $1 \le p < \infty$, is separable.*
*Moreover, the space $C_c^\infty(Q;X)$ is dense in $L^p(Q;X)$, with $1 \le p < \infty$.*

*(ii) If $X$ is reflexive, then $L^p(Q;X)$, with $1 < p < \infty$, is reflexive.*
*More precisely, any continuous linear functional $\ell \in [L^p(Q;X)]^*$, with $1 \le p < \infty$, admits a unique representation $f_\ell \in L^{p'}(Q;X^*)$ such that*

$$\langle \ell; g \rangle_{[L^p(Q;X)]^*, L^p(Q;X)} = \int_Q \langle f_\ell(y); g(y) \rangle_{X^*,X} \,\mathrm{d}y,$$

*for all $g \in L^p(Q;X)$, where*

$$\frac{1}{p} + \frac{1}{p'} = 1. \tag{1.1.5}$$

*We say that $p'$ is the conjugate exponent of $p$ is (1.1.5) holds.*

*Proof.* (i) is a direct consequence of Theorem 1.1.2. For (ii), see [45], Theorem 1.14. □

### 1.1.4 *Sobolev spaces*

Let $Q \subseteq \mathbb{R}^d$ be an open set. We denote with

- $W^{k,p}(Q;\mathbb{R}^m)$, with $1 \le p \le \infty$ and $k$ a positive integer, the Banach space of functions $\boldsymbol{f}$ having all distributional derivatives $\partial^\alpha \boldsymbol{f}$ up to order $|\alpha| = k$ in $L^p(Q;\mathbb{R}^m)$, where $\partial^\alpha \boldsymbol{f} \in \mathcal{D}'(Q;\mathbb{R}^m)$ is defined as

$$\langle \partial^\alpha \boldsymbol{f}; \boldsymbol{\varphi} \rangle_{\mathcal{D}'(Q;\mathbb{R}^m), \mathcal{D}(Q;\mathbb{R}^m)} = (-1)^{|\alpha|} \int_Q \boldsymbol{f} \cdot \partial^\alpha \boldsymbol{\varphi} \,\mathrm{d}y,$$

  for any $\boldsymbol{\varphi} \in \mathcal{D}(Q;\mathbb{R}^m)$. The norm is defined as

$$\|\boldsymbol{f}\|_{W^{k,p}(Q;\mathbb{R}^m)} = \left( \sum_{i=1}^m \sum_{|\alpha| \le k} \|\partial^\alpha f_i\|_{L^p(Q)}^p \right)^{\frac{1}{p}}, \quad \text{for } 1 \le p < \infty$$
$$\|\boldsymbol{f}\|_{W^{k,\infty}(Q;\mathbb{R}^m)} = \max_{\substack{1 \le i \le m \\ |\alpha| \le k}} \|\partial^\alpha f_i\|_{L^\infty(Q)}.$$

  We will often write $H^k(Q;\mathbb{R}^m) := W^{k,2}(Q;\mathbb{R}^m)$;

- $W_0^{k,p}(Q;\mathbb{R}^m)$ the completion of $C_c^\infty(Q;\mathbb{R}^m)$ with respect to the norm $\|\cdot\|_{W^{k,p}(Q;\mathbb{R}^m)}$;

- $W^{-k,p'}(Q;\mathbb{R}^m)$, with $1 \le p < \infty$ and $k$ a positive integer, the dual space of $W_0^{k,p}(Q;\mathbb{R}^m)$;
- $W^{\mathbb{D},p}(Q;\mathbb{R}^d)$, with $1 \le p < \infty$, the Banach space of functions $\boldsymbol{f} \in L^p(Q;\mathbb{R}^d)$ whose trace-free distributional symmetric gradient

$$\mathbb{D}_y \boldsymbol{f} - \frac{1}{d}\operatorname{Tr}[\mathbb{D}_y \boldsymbol{f}]\mathbb{I}$$

belongs to $L^p(Q;\mathbb{R}^{d\times d})$. The norm is defined as

$$\|\boldsymbol{f}\|_{W^{\mathbb{D},p}(Q;\mathbb{R}^d)} = \|\boldsymbol{f}\|_{L^p(Q;\mathbb{R}^d)} + \left\|\mathbb{D}_y \boldsymbol{f} - \frac{1}{d}\operatorname{Tr}[\mathbb{D}_y \boldsymbol{f}]\mathbb{I}\right\|_{L^p(Q;\mathbb{R}^{d\times d})};$$

- $W_0^{\mathbb{D},p}(Q;\mathbb{R}^d)$ the completion of $C_c^\infty(Q;\mathbb{R}^d)$ with respect to the norm $\|\cdot\|_{W^{D,p}(Q;\mathbb{R}^d)}$.

Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and $1 \le p < \infty$. Then, the following two inequalities hold.

(i) *Poincaré inequality*: there exists a positive constant $c$ such that

$$\|\boldsymbol{f}\|_{L^p(\Omega;\mathbb{R}^d)} \le c\|\nabla_y \boldsymbol{f}\|_{L^p(\Omega;\mathbb{R}^{d\times d})} \tag{1.1.6}$$

holds for any $\boldsymbol{f} \in W_0^{1,p}(\Omega;\mathbb{R}^d)$ (see e.g. Ziemer [76], Theorem 4.5.1);

(ii) *Trace-free Korn inequality*: there exists a positive constant $c$ such that

$$\|\nabla_y \boldsymbol{f}\|_{L^p(\Omega;\mathbb{R}^{d\times d})} \le c\left\|\mathbb{D}_y \boldsymbol{f} - \frac{1}{d}\operatorname{Tr}[\mathbb{D}_y \boldsymbol{f}]\mathbb{I}\right\|_{L^p(Q;\mathbb{R}^{d\times d})} \tag{1.1.7}$$

holds for any $\boldsymbol{f} \in W_0^{\mathbb{D},p}(\Omega;\mathbb{R}^d)$ (see Breit, Cianchi and Diening [11], Theorem 3.1).

In the following result, we report some properties of the Sobolev spaces.

**Theorem 1.1.4.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain.*

*(i) $W^{k,p}(\Omega)$, with $1 \le p < \infty$, is separable.*
*Moreover, the space $C^k(\overline{\Omega})$ is dense in $W^{k,p}(\Omega)$, with $1 \le p < \infty$.*

*(ii) $W_0^{k,p}(\Omega)$, with $1 < p < \infty$, is reflexive.*
*$W^{-k,p'}(\Omega)$, with $1 \le p < \infty$ and $p'$ the conjugate exponent of $p$, is a proper subspace of the space of distributions $\mathcal{D}'(\Omega)$.*
*More precisely, any continuous linear functional $\ell \in W^{-k,p'}(\Omega)$, with $1 \le p < \infty$, admits a unique representation $f_\alpha \in L^{p'}(\Omega)$ such that*

$$\langle \ell; g\rangle_{W^{-k,p'}(\Omega),W_0^{k,p}(\Omega)} = \sum_{|\alpha|\le k} (-1)^{|\alpha|} \int_\Omega f_\alpha(y)\, \partial^\alpha g(y)\, \mathrm{d}y,$$

*for all $g \in W_0^{k,p}(\Omega)$.*

*Proof.* See e.g. Adams [3], Theorems 3.8 and 3.16. □

We report the *embedding* results connecting Sobolev and Lebesgue spaces.

**Theorem 1.1.5.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain, $k \geq 1$ and $1 \leq p \leq \infty$. The continuous embedding*

$$W^{k,p}(\Omega) \hookrightarrow L^q(\Omega) \tag{1.1.8}$$

*holds if*

*(i) $kp < d$ and $1 \leq q \leq p^*$ with*

$$p^* = \frac{dp}{d-kp}; \tag{1.1.9}$$

*(ii) $kp = d$ and $1 \leq q < \infty$;*

*(iii) $kp > d$ and $1 \leq q \leq \infty$.*

*Moreover, in all cases the embedding is compact except from $kp < d$ and $q = p^*$.*

*Proof.* See e.g. Ziemer [76], Theorem 2.5.1 and Remark 2.5.2. □

Consequently, we get the following embedding theorem for dual Sobolev spaces.

**Theorem 1.1.6.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, $k > 0$, $1 \leq p \leq \infty$ and $p'$ the conjugate exponent of $p$. The compact embedding*

$$L^q(\Omega) \hookrightarrow\hookrightarrow W^{-k,p'}(\Omega) \tag{1.1.10}$$

*holds if*

*(i) $kp < d$ and $\frac{p^*}{p^*-1} < q < \infty$ with $p^*$ defined as in (1.1.9);*

*(ii) $kp = d$ and $1 < q < \infty$;*

*(iii) $kp > d$ and $1 \leq q < \infty$.*

*Remark* 1.1.7. In Theorem 1.1.5, for $kp > d$ we have the stronger embedding

$$W^{k,p}(\Omega) \hookrightarrow C^{k-\left[\frac{d}{p}\right]-1,\,\nu}(\overline{\Omega}), \tag{1.1.11}$$

with

$$\nu \begin{cases} = \left[\frac{d}{p}\right] + 1 - \frac{d}{p} & \text{if } \frac{d}{p} \in \mathbb{Z}, \\ \text{arbitary value } \in (0,1) & \text{if } \frac{d}{p} \notin \mathbb{Z}, \end{cases}$$

where $[\cdot]$ denotes the integer part and $C^{m,\nu}(\overline{Q}) \subset C^m(\overline{Q})$ denotes the space of functions having their $m$-th derivative $\nu$-Hölder continuous in $\overline{\Omega}$. Moreover, the embedding is compact if $0 < \nu < \left[\frac{d}{p}\right] + 1 - \frac{d}{p}$.

### 1.1.5 *Space of functions of bounded variation*

Let $[a,b] \subset \mathbb{R}$. The *total variation* of a function $f$ defined on $[a,b]$ is the quantity

$$V_a^b(f) = \sup_{P \in \mathcal{P}} \sum_{i=0}^{n_P - 1} |f(t_{i+1}) - f(t_i)|,$$

where the supremum is taken over the set

$$\mathcal{P} = \big\{P = \{t_0, \dots, t_{n_P}\} \,|\, P \text{ is a partition of } [a,b],\ a = t_0 < t_1 < \cdots < t_{n_P} = b\big\}.$$

We say that $f$ is a function of *bounded variation*, and we write $f \in BV([a,b])$, if $V_a^b(f)$ is finite.

In particular, every monotone function $f$ defined on $[a,b]$ belongs to $BV([a,b])$. Indeed, for every partition $P = \{t_0, \dots, t_{n_P}\}$

$$\sum_{i=0}^{n_P-1} |f(t_{i+1}) - f(t_i)| = \pm \sum_{i=0}^{n_P-1} \big(f(t_{i+1}) - f(t_i)\big) = \pm\big(f(b) - f(a)\big)$$

and thus $V_a^b(f) = |f(b) - f(a)|$, which is finite.

We conclude reporting Helly's theorem.

**Theorem 1.1.8.** *Let $\{f_n\}_{n\in\mathbb{N}} \subset BV([a,b])$ be a uniformly bounded sequence, meaning that there exists a positive constant $c$ is independent of $n$ such that*

$$\sup_{n\in\mathbb{N}} V_a^b(f_n) \le c,$$

*Moreover suppose that there exists a positive constant $K$ such that*

$$\sup_{n\in\mathbb{N}} |f_n(t)| \le K \quad \text{for any } t \in [a,b].$$

*Then there exists $f \in BV([a,b])$ such that, passing to a suitable subsequence as the case may be,*

$$f_n(t) \to f(t) \quad \text{for any } t \in [a,b],$$

*as $n \to \infty$.*

*Proof.* See e.g. Natanson [63], Helly's First Theorem, page 222. □

## 1.2 ORLICZ SPACES

In this section we are going to recall the definition and basic properties of Orlicz spaces, which can be seen as a natural generalization of Lebesgue spaces. They will play a fundamental role in the study of a general compressible viscous fluid with linear pressure, cf. Chapter 5.

**Definition 1.2.1.** (i) We say that $\Phi$ is a *Young function* generated by $\varphi$ if

$$\Phi(t) = \int_0^t \varphi(s)\, \mathrm{d}s \quad \text{for any } t \ge 0,$$

where the real-valued function $\varphi$ defined on $[0,\infty)$ is non-negative, non-decreasing, left-continuous and such that

$$\varphi(0) = 0, \quad \lim_{s\to\infty} \varphi(s) = \infty.$$

(ii) Let $\Phi$ be a Young function generated by $\varphi$. We say that $\Psi$ is the *complementary* function to $\Phi$ if it is given by

$$\Psi(t) = \int_0^t \psi(s)\, \mathrm{d}s \quad \text{for any } t \geq 0,$$

where

$$\psi(s) := \sup_{\varphi(r) \leq s} r \quad \text{for any } s \geq 0.$$

(iii) A Young function $\Phi$ is said to satisfy the $\Delta_2$-*condition* if there exist a positive constant $K$ and $t_0 \geq 0$ such that

$$\Phi(2t) \leq K\Phi(t) \quad \text{for any } t \geq t_0. \tag{1.2.1}$$

(iv) Let $\Phi_1$ and $\Phi_2$ be two Young functions. We write

- $\Phi_1 \prec \Phi_2$ if there exists a positive constant $c$ and $t_0 \geq 0$ such that

$$\Phi_1(t) \leq \Phi_2(ct) \quad \text{for any } t \geq t_0;$$

- $\Phi_1 \prec\prec \Phi_2$ if

$$\lim_{t \to \infty} \frac{\Phi_1(t)}{\Phi_2(\lambda t)} = 0 \quad \text{for any } \lambda > 0.$$

**Lemma 1.2.2.** *A Young function $\Phi$ is continuous, non-negative, increasing and convex on $[0, \infty)$. Moreover*

$$\Phi(0) = 0, \quad \lim_{t \to \infty} \Phi(t) = \infty$$

$$\lim_{t \to 0^+} \frac{\Phi(t)}{t} = 0, \quad \lim_{t \to \infty} \frac{\Phi(t)}{t} = \infty.$$

*Proof.* See e.g. Kufner, John and Fučík [55], Lemma 3.2.2. □

*Example* 1.2.3. (i) $\Phi(t) = \frac{t^p}{p}$, with $1 < p < \infty$, is a Young function generated by $\varphi(s) = s^{p-1}$ and satisfying the $\Delta_2$-condition with $K = 2^p$ and $t_0 = 0$. Moreover $\Psi(t) = \frac{t^q}{q}$ is its complementary function, with $q$ the conjugate exponent of $p$;

(ii) $\Phi(t) = t \log^+ t$ is a Young function generated by

$$\varphi(s) = \begin{cases} 0 & \text{if } 0 \leq s \leq 1, \\ \log t + 1 & \text{if } t > 1. \end{cases}$$

Its complementary function is given by

$$\Psi(t) = \begin{cases} t & \text{if } 0 < t < 1, \\ e^{t-1} & \text{if } t \geq 1. \end{cases}$$

**Definition 1.2.4.** Let $Q \subset \mathbb{R}^d$ be an open bounded set, $\Phi$ a Young function and $\Psi$ its complementary one. We denote with

(i) $\widetilde{L}_\Phi(Q)$ the *Orlicz class* of all real-valued measurable functions $f$ defined a.e. on $Q$ such that the quantity

$$\sigma(f;\Phi) := \int_Q \Phi(|f(y)|)\, \mathrm{d}y$$

is finite;

(ii) $L_\Phi(Q)$ the *Orlicz space* of all real-valued measurable functions $f$ defined a.e. on $Q$ such that the quantity

$$\sup_{\substack{g\in\widetilde{L}_\Psi(Q)\\ \sigma(g;\Psi)\le 1}} \int_Q |f(y)g(y)|\,\mathrm{d}y \tag{1.2.2}$$

is finite. It is a vector space with norm $\|f\|_{L_\Phi}$ defined by (1.2.2) (see Kufner, John and Fučík [55], Theorem 3.6.4);

(iii) $E_\Phi(Q)$ is the closure with respect to the Orlicz norm $\|\cdot\|_{L_\Phi}$ of the set of all bounded measurable functions defined on $Q$.

We are now ready to state some properties of the Orlicz spaces.

**Theorem 1.2.5.** *Let $Q\subset\mathbb{R}^d$ be an open bounded set and $\Phi$ a Young function.*

*(i) $L_\Phi(Q)$ is a Banach space.*

*(ii) $E_\Phi(Q)$ is separable.*

*(iii) Any continuous linear functional $\ell\in[E_\Phi(Q)]^*$ admits a unique representation $f_\ell\in L_\Psi(Q)$ such that*

$$\langle \ell; g\rangle_{[E_\Phi(Q)]^*;E_\Phi(Q)} = \int_Q f(y)g(y)\,\mathrm{d}y,$$

*for all $g\in E_\Phi(Q)$, where $\Psi$ is the complementary function to $\Phi$.*

*Moreover, let Young function $\Phi$ satisfy the $\Delta_2$-condition.*

*(iv) $L_\Phi(Q)=E_\Phi(Q)$ and thus $L_\Phi(Q)$ is separable;*

*(v) $L_\Phi(Q)$ is reflexive if and only if the complementary function $\Psi$ satisfies the $\Delta_2$-condition, too.*

*Proof.* See Kufner, John and Fučík [55], Theorems 3.9.1, 3.12.9, 3.13.6, 3.13.9 and Lemma 3.12.3. □

We have the following comparison of Orlicz spaces.

**Theorem 1.2.6.** *Let $Q\subset\mathbb{R}^d$ be an open bounded set and $\Phi_1$, $\Phi_2$ be two Young functions.*

*(i) If $\Phi_1\prec\Phi_2$, then the following continuous embedding holds:*

$$L_{\Phi_2}(Q)\hookrightarrow L_{\Phi_1}(Q).$$

*(ii) If $\Phi_1\prec\prec\Phi_2$, then the following continuous embedding holds:*

$$L_{\Phi_2}(Q)\hookrightarrow E_{\Phi_1}(Q).$$

*Proof.* See Kufner, John and Fučík [55], Theorems 3.17.1, 3.17.5 and 3.17.7. □

As in the Lebegue-spaces case, we can approximate Orlicz functions with smooth functions.

**Theorem 1.2.7.** *Let $Q\subseteq\mathbb{R}^d$ be an open bounded set, $\Phi$ a Young function and $\{\theta_\varepsilon\}_{\varepsilon>0}$ a family of regularizing kernels in $\mathbb{R}^d$.*

*(i) For any $f \in E_\Phi(Q)$ extended to be zero outside $Q$*

$$\theta_\varepsilon * f \in C_c^\infty(\mathbb{R}^d).$$

*(ii) For any $f \in E_\Phi(Q)$*

$$\theta_\varepsilon * f \to f \quad \textit{in } E_\Phi(Q)$$

*as $\varepsilon \to 0$.*

*Proof.* See Kufner, John and Fučík [55], Theorem 3.18.1.1. □

### 1.2.1 *Compactness in Orlicz spaces*

We will need the following characterization of compactness in Orlicz spaces.

**Theorem 1.2.8.** *Let $Q \subset \mathbb{R}^d$ be an open bounded set and let $\Phi_1$, $\Phi_2$ be two Young functions such that $\Phi_1 \prec\prec \Phi_2$. Let $K$ be a bounded subset of $L_{\Phi_2}(Q)$ that is relatively compact in the sense of convergence in measure. Then $K$ is relatively compact in $L_{\Phi_1}(Q)$.*

*Proof.* See Kufner, John and Fučík [55], Theorems 3.14.11 and 3.17.8. □

We are now ready to prove the following result.

**Proposition 1.2.9.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and let $\Phi$ be a Young function. Then, for a fixed $q \geq 1$*

$$X = W_0^{1,q} \cap L^\infty(\Omega) \hookrightarrow\hookrightarrow L_\Phi(\Omega).$$

*Proof.* Let $K$ be a bounded set of $X$ and let $\Phi_1$ be a Young function such that $\Phi \prec\prec \Phi_1$. In particular, $K$ is bounded in the Orlicz space $L_{\Phi_1}(\Omega)$; indeed, denoting with $\Psi$ the complementary Young function of $\Phi$, we have that for every $f \in K$ and every $g$ belonging to the Orlicz class $\widetilde{L}_\Psi(\Omega)$

$$\int_\Omega |f(y)g(y)|\, \mathrm{d}y \leq \|f\|_{L^\infty(\Omega)} \|g\|_{L^1(\Omega)} \leq \|f\|_{L^\infty(\Omega)}\, \sigma(g;\Psi),$$

and thus

$$\|f\|_{L_{\Phi_1}(\Omega)} = \sup_{\substack{g \in \widetilde{L}_\Psi(\Omega) \\ \sigma(g;\Psi) \leq 1}} \int_\Omega |f(y)g(y)|\, \mathrm{d}y \leq \|f\|_{L^\infty(\Omega)} \leq \|f\|_X \leq c,$$

where $\|f\|_X = \max\{\|f\|_{W^{1,q}(\Omega)}, \|f\|_{L^\infty(\Omega)}\}$ and the constant $c$ is independent of the choice $f \in K$. Furthermore, since

$$W^{1,q}(\Omega) \hookrightarrow\hookrightarrow L^1(\Omega),$$

the set $K$ is relatively compact in $L^1(\Omega)$ and consequently it is relatively compact with respect to the convergence in measure. Now, it is sufficient to apply Theorem 1.2.8. □

### 1.2.2 *De la Vallée–Poussin criterion*

Strictly connected to the Dunford-Pettis theorem 1.5.4, the *De la Vallée–Poussin criterion* relates the *equi-integrability* of a sequence in $L^1$ with being uniformly bounded in an Orlicz space.

**Theorem 1.2.10.** *Let $Q \subset \mathbb{R}^d$ be a bounded measurable set and let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence in $L^1(Q)$. Then, the following statements are equivalent.*

*(i) The sequence $\{f_n\}_{n\in\mathbb{N}}$ is equi-integrable, meaning that for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that*

$$\int_M |f_n(y)|\, \mathrm{d}y < \varepsilon \quad \textit{for any } M \subset Q \textit{ such that } |M| < \delta,$$

*independently of $n$.*

*(ii) There exists a Young function $\Phi$ satisfying the $\Delta_2$-condition (1.2.1) such that the sequence $\{f_n\}_{n\in\mathbb{N}}$ is uniformly bounded in the Orlicz space $L_\Phi(Q)$.*

*Remark* 1.2.11. Condition (ii) can be replaced considering the following one, involving the notion of *superlinearity*.

(ii') *There exists a superlinear non-decreasing function $F : [0, \infty) \to [0, \infty)$ such that*

$$\sup_{n\in\mathbb{N}} \int_Q F(|f_n(y)|)\, \mathrm{d}y < \infty,$$

*where superlinearity means that*

$$\lim_{z\to+\infty} \frac{F(z)}{z} = +\infty.$$

*Proof.* (ii) $\Rightarrow$ (i) See Pedregal [66], Chapter 6, Lemma 6.4.

(i) $\Rightarrow$ (ii) We report the proof as we require the stronger condition, with respect to the standard version of the criterion, that the Young function $\Phi$ satisfies the $\Delta_2$-condition. For $n \in \mathbb{N}$ and $j \geq 1$ fixed, let

$$\mu_j(f_n) := |\{y \in Q : \ |f_n(y)| > j\}|.$$

As the sequence $\{f_n\}_{n\in\mathbb{N}}$ is equi-integrable, from the Dunford-Pettis theorem 1.5.4, condition (ii), there exists a strictly increasing sequence of positive integers $\{C_m\}_{m\in\mathbb{N}}$ such that for each $m$

$$\sup_{n\in\mathbb{N}} \int_{\{|f_n|>C_m\}} |f_n(y)|\, \mathrm{d}y \leq \frac{1}{2^m}.$$

For $n \in \mathbb{N}$ and $m \geq 1$ fixed

$$\int_{\{|f_n|>C_m\}} |f_n(y)|\, \mathrm{d}y = \sum_{j=C_m}^{\infty} \int_{\{j<|f_n|\leq j+1\}} |f_n(y)|\, \mathrm{d}y \geq \sum_{j=C_m}^{\infty} j\, [\mu_j(f_n) - \mu_{j+1}(f_n)] \geq \sum_{j=C_m}^{\infty} \mu_j(f_n).$$

In particular, we obtain

$$\sum_{m=1}^{\infty} \sum_{j=C_m}^{\infty} \mu_j(f_n) \leq \sum_{m=1}^{\infty} \int_{\{|f_n|>C_m\}} |f_n(y)|\, \mathrm{d}y \leq \sum_{m=1}^{\infty} \frac{1}{2^m} = 1.$$

For $m \geq 0$, we define

$$\alpha_m = \begin{cases} 0 & \text{if } m < C_1, \\ \max\{k : \ C_k \leq m\} & \text{if } m \geq C_1. \end{cases}$$

Notice that

$$\alpha_m \geq j \quad \Leftrightarrow \quad C_j \leq m. \tag{1.2.3}$$

It is straightforward that $\alpha_m \to \infty$ as $m \to \infty$. We define a step function $\varphi$ on $[0, \infty)$ by

$$\varphi(s) = \sum_{m=0}^{\infty} \alpha_m \chi_{(m,m+1]}(s) \quad \text{for any } 0 \leq s < \infty.$$

It is clear that $\varphi$ is non-negative, non-decreasing, left-continuous and such that $\varphi(0) = 0$, $\lim_{s\to\infty} \varphi(s) = \infty$. Then, we can define the Young function $\Phi$ generated by $\varphi$ as

$$\Phi(t) = \int_0^t \varphi(s)\, \mathrm{d}s, \quad \text{for any } 0 \leq t < \infty.$$

At this point, notice that we have the freedom to take the constants $C_j$, $j \geq 1$, as large as we want and consequently, the constants $\alpha_m$, $m \geq 1$, will be as small as we want. More precisely, we may find a positive constant $c$ such that

$$\alpha_{2m} \leq c\ \alpha_m \quad \text{for any } m \geq 1.$$

We then obtain, for all $s \in [0, \infty)$,

$$\varphi(2s) = \sum_{m=0}^{\infty} \alpha_m \chi_{\left(\frac{m}{2}, \frac{m+1}{2}\right)}(s) = \sum_{k=0}^{\infty} \alpha_{2k} \chi_{\left(k, k+\frac{1}{2}\right)}(s) \leq c \sum_{k=0}^{\infty} \alpha_k \chi_{\left(k, k+\frac{1}{2}\right)}(s) \leq c\ \varphi(s);$$

consequently, for all $t \in [0, \infty)$,

$$\Phi(2t) = \int_0^{2t} \varphi(s)\, \mathrm{d}s = 2 \int_0^t \varphi(2z)\, \mathrm{d}z \leq 2c \int_0^t \varphi(z)\, \mathrm{d}z = 2c\ \Phi(t),$$

and thus we get that the Young function $\Phi$ satisfies the $\Delta_2$-condition (1.2.1).

Finally, for $n \in \mathbb{N}$ fixed, using the fact that $\Phi(0) = \Phi(1) = 0$ and for $j \geq 1$, noticing that $\alpha_0 = 0$,

$$\Phi(j+1) = \int_0^{j+1} \varphi(s)\, \mathrm{d}s = \sum_{m=0}^{j} \int_m^{m+1} \varphi(s)\, \mathrm{d}s \leq \sum_{m=0}^{j} \varphi(m+1) = \sum_{m=0}^{j} \alpha_m = \sum_{m=1}^{j} \alpha_m,$$

we get

$$\begin{aligned}
\int_Q \Phi(|f_n(y)|)\, \mathrm{d}y &= \int_{\{|f_n|=0\}} \Phi(|f_n(y)|)\, \mathrm{d}y + \sum_{j=0}^{\infty} \int_{\{j<|f_n|\leq j+1\}} \Phi(|f_n(y)|)\, \mathrm{d}y \\
&\leq \sum_{j=1}^{\infty} [\mu_j(f_n) - \mu_{j+1}(f_n)]\ \Phi(j+1) \\
&\leq \sum_{j=1}^{\infty} [\mu_j(f_n) - \mu_{j+1}(f_n)] \sum_{m=1}^{j} \alpha_m \\
&= \sum_{m=1}^{\infty} \alpha_m \sum_{j=m}^{\infty} [\mu_j(f_n) - \mu_{j+1}(f_n)] \\
&= \sum_{m=1}^{\infty} \alpha_m \mu_m(f_n) = \sum_{m=1}^{\infty} \mu_m(f_n) \sum_{j=1}^{\alpha_m} 1 = \sum_{j=1}^{\infty} \sum_{m=C_j}^{\infty} \mu_m(f_n) \leq 1
\end{aligned}$$

where we used (1.2.3) in the last line. In particular, we obtain that the sequence $\{f_n\}_{n\in\mathbb{N}}$ is uniformly bounded in the Orlicz space $L_\Phi(Q)$. $\square$

## 1.3 SKOROKHOD SPACE

In this section we are going to study the Skorokhod space of càglàd (an acronym for "continue à gauche, limites à droite") functions, i.e., the space of left-continuous and having right-hand limits functions.

We will first focus on bounded domains and range the real line, defining a proper metric and characterizing the convergence with respect to it. For these purposes, we will follow the approach presented by Whitt [75]. It is worth noticing that, even if in [75] the author considered càdlàg ("continue à droite, limites à gauche") functions, the same construction works in our context as well: dealing with the completed graphs of the functions, which are obtained by adding segments joining the left and right limits at each discontinuity point to the graph, the actual value of the function at discontinuity points does not matter, provided that it falls appropriately between the left and right limits.

In the second part we will consider the unbounded domain $[0,\infty)$ and range a separable Hilbert space $H$, as the Skorokhod space $\mathfrak{D}([0,\infty);H)$ represents the suitable trajectory space for proving the existence of a semiflow selection, cf. Chapter 3. Unlike the case of continuous functions, the topology on the space of càglàd functions on an unbounded interval cannot be built up by simply considering functions being càglàd on any compact, see e.g. Jakubowski [49]. Our idea is then to see the Skorokhod space as a particular Fréchet space, cf. Definition 1.3.8.

### 1.3.1 *Bounded domain and range the real line*

**Definition 1.3.1.** Let $T>0$ be fixed. We define the *Skorokhod space* $\mathfrak{D}([0,T])$ as the space of the real-valued càglàd functions defined on $[0,T]$. More precisely, $\Phi$ belongs to the space $\mathfrak{D}([0,T])$ if it is left–continuous and has right–hand limits:

(i) for $0<t\le T$, $\Phi(t-)=\lim_{s\uparrow t}\Phi(s)$ exists and $\Phi(t-)=\Phi(t)$;

(ii) for $0\le t<T$, $\Phi(t+)=\lim_{s\downarrow t}\Phi(s)$ exists.

*Remark* 1.3.2. We can replace $[0,T]$ with any bounded domain of the type $[a,b]$, $a,b\in\mathbb{R}$. The space $\mathfrak{D}([a,b])$ is essentially the same as $\mathfrak{D}([0,T])$ since they are homeomorphic.

We are now ready to collect some properties of càglàd functions.

**Proposition 1.3.3.** *The following properties hold.*

*(i) Number of discontinuities: for each* $\Phi\in\mathfrak{D}([0,T])$*, the set*

$$Disc(\Phi):=\{t\in[0,T):\ \Phi(t)\neq\Phi(t+)\} \tag{1.3.1}$$

*is either finite or countably infinite.*

*(ii) Boundedness: each* $\Phi\in\mathfrak{D}([0,T])$ *is bounded, i.e., the uniform norm*

$$\|\Phi\|_\infty=\sup_{t\in[0,T]}|\Phi(t)|$$

*is finite.*

(iii) *Measurability: each* $\Phi \in \mathfrak{D}([0,T])$ *is Borel-measurable.*

*Proof.* See Whitt [75], Corollaries 12.2.1, 12.2.3 and 12.2.4. □

In order to define a proper metric, we first need to introduce few objects.

**Definition 1.3.4.** Let $\Phi \in \mathfrak{D}([0,T])$ be fixed. We define

(i) the *completed graph* of $\Phi$ as the set

$$\Gamma_\Phi := \{(t,z) \in [0,T] \times \mathbb{R} |\ z = \alpha\Phi(t) + (1-\alpha)\Phi(t+),\ \text{for some } \alpha \in [0,1]\},$$

with $\Phi(T+) := \Phi(T)$. In other words, the completed graph is a connected subset of the plane $\mathbb{R}^2$ containing the line segment joining $(t, \Phi(t))$ and $(t, \Phi(t+))$ for all discontinuity points;

(ii) an *order* on the completed graph $\Gamma_\Phi$ by saying that $(t_1, z_1) \le (t_2, z_2)$ if either

- $t_1 < t_2$;
- $t_1 = t_2$ and $|z_1 - \Phi(t_1+)| \le |z_2 - \Phi(t_1+)|$;

(iii) a *parametric representation* of the completed graph $\Gamma_\Phi$ (or of the function $\Phi$) as a continuous non-decreasing function $(r,u)$ mapping $[0,1]$ onto $\Gamma_\Phi$, with $r \in C([0,1];[0,T])$ being the time component and $u \in C([0,1];\mathbb{R})$ the spatial component. The parametric is non-decreasing using the order introduced above;

(iv) $\Pi(\Phi)$ the set of all parametric representations of $\Phi$.

We are now ready to introduce a proper metric on $\mathfrak{D}([0,T])$.

**Definition 1.3.5.** For $\Phi_1, \Phi_2 \in \mathfrak{D}([0,T])$, we define

$$d_T(\Phi_1,\Phi_2) = \inf_{\substack{(r_j,u_j)\in\Pi(\Phi_j)\\ j=1,2}} \max\{\|r_1-r_2\|_\infty, \|u_1-u_2\|_\infty\};$$

it constitutes a metric on $\mathfrak{D}([0,T])$ (see Whitt [75], Theorem 12.3.1).

In the following result we give some characterizations of the $d_T$-convergence.

**Theorem 1.3.6.** *Let* $\{\Phi^n\}_{n\in\mathbb{N}}$ *be a sequence in* $\mathfrak{D}([0,T])$ *endowed with the* $d_T$*-metric. If*

$$\Phi^n \to \Phi \quad \text{in } \mathfrak{D}([0,T]) \tag{1.3.2}$$

*as* $n \to \infty$*, then*

$$\Phi_n(t) \to \Phi(t) \quad \text{for all } t \notin Disc(\Phi),\ \text{including } t=0 \text{ and } t=T \tag{1.3.3}$$

*as* $n \to \infty$*, where* $Disc(\Phi)$ *defined in* (1.3.1) *is a dense subset of* $[0,T]$*. Moreover*

*(i) if* $\Phi^n$ *are monotone for all* $n \in \mathbb{N}$*, conditions* (1.3.2) *and* (1.3.3) *are equivalent;*

*(ii) if* $\Phi^n$ *are continuous for all* $n \in \mathbb{N}$*, condition* (1.3.2) *is equivalent to*

$$\sup_{t\in[0,T]} |\Phi_n(t) - \Phi(t)| \to 0 \quad \text{as } n \to \infty.$$

*Proof.* See Whitt [75], Theorem 12.5.1, Lemma 12.5.1 and Corollary 12.5.1. □

### 1.3.2 *Unbounded domain and range a Hilbert space*

**Definition 1.3.7.** Let $H$ be a given separable Hilbert space. We define the *Skorokhod space* $\mathfrak{D}([0,\infty);H)$ as the space of the càglàd functions defined on $[0,\infty)$ taking values in $H$. More precisely, $\Phi$ belongs to the space $\mathfrak{D}([0,\infty);H)$ if it is left–continuous and has right–hand limits:

(i) for $t>0$, $\Phi(t-)=\lim_{s\uparrow t}\Phi(s)$ exists and $\Phi(t-)=\Phi(t)$;

(ii) for $t\geq 0$, $\Phi(t+)=\lim_{s\downarrow t}\Phi(s)$ exists.

**Definition 1.3.8.** Let $\{e_k\}_{k\in\mathbb{N}}$ be a basis of the separable Hilbert space $H$. For every $\Phi\in\mathfrak{D}([0,\infty);H)$ we define

$$\widehat{\Phi}_T := \begin{cases}\Phi(0) & \text{for } t\in[-1,0],\\ \Phi(t) & \text{for } t\in(0,T),\\ \Phi(T) & \text{for } t\in[T,T+1],\end{cases} \tag{1.3.4}$$

and for every $\Phi,\Psi\in\mathfrak{D}([0,\infty);H)$ we define

$$d_\infty(\Phi,\Psi) := \sum_{M=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{2^M}\frac{1}{2^k}\frac{d_M\left(\left\langle\widehat{\Phi}_M;e_k\right\rangle,\left\langle\widehat{\Psi}_M;e_k\right\rangle\right)}{1+d_M\left(\left\langle\widehat{\Phi}_M;e_k\right\rangle,\left\langle\widehat{\Psi}_M;e_k\right\rangle\right)}, \tag{1.3.5}$$

where, for all $M\in\mathbb{N}$, $d_M$ denotes the Skorokhod metric on the space $\mathfrak{D}([-1,M+1])$, cf. Definition 1.3.5.

It is easy to verify that $d_\infty$ is a metric on $\mathfrak{D}([0,\infty);H)$. Moreover, we have the following result.

**Proposition 1.3.9.** *Let $\{\Phi^n\}_{n\in\mathbb{N}}$ be a sequence in $\mathfrak{D}([0,\infty);H)$ endowed with the $d_\infty$-metric. If*

$$\Phi^n\to\Phi \quad \textit{in } \mathfrak{D}([0,\infty);H) \tag{1.3.6}$$

*as $n\to\infty$, then for all $k\in\mathbb{N}$*

$$\langle\Phi^n(t);e_k\rangle\to\langle\Phi(t);e_k\rangle \quad \textit{for a.e. } t\in(0,\infty) \tag{1.3.7}$$

*as $n\to\infty$. Moreover*

*(i) if $\Phi^n$ are monotone for all $n\in\mathbb{N}$, conditions* (1.3.6) *and* (1.3.7) *are equivalent;*

*(ii) if $\Phi^n$ are continuous for all $n\in\mathbb{N}$, condition* (1.3.6) *is equivalent to*

$$\sup_{t\in[0,M]}|\langle\Phi^n(t)-\Phi(t);e_k\rangle|\to 0$$

*as $n\to\infty$, for all $k,M\in\mathbb{N}$.*

*Proof.* First of all, we will show that (1.3.6) is equivalent to

$$\left\langle\widehat{\Phi^n}_M;e_k\right\rangle\to\left\langle\widehat{\Phi}_M;e_k\right\rangle \quad \text{in } \mathfrak{D}([-1,M+1]) \tag{1.3.8}$$

as $n \to \infty$, for all $k, M \in \mathbb{N}$. Indeed, if (1.3.6) holds, let $\varepsilon > 0$, $k, M \in \mathbb{N}$ be fixed and choose a positive $\tilde{\varepsilon} = \tilde{\varepsilon}(\varepsilon)$ such that

$$\frac{2^{k+M}\tilde{\varepsilon}}{1 - 2^{k+M}\tilde{\varepsilon}} < \varepsilon. \tag{1.3.9}$$

From (1.3.6), there exists $n_0 = n_0(\tilde{\varepsilon})$ such that

$$\frac{1}{2^{k+M}} \frac{d_M\left(\left\langle \widehat{\Phi^n}_M; e_k \right\rangle, \left\langle \widehat{\Phi}_M; e_k \right\rangle\right)}{1 + d_M\left(\left\langle \widehat{\Phi^n}_M; e_k \right\rangle, \left\langle \widehat{\Phi}_M; e_k \right\rangle\right)} \leq d_\infty(\Phi^n, \Phi) < \tilde{\varepsilon}, \quad \text{for all } n \geq n_0,$$

which, combined with (1.3.9), implies

$$d_M\left(\left\langle \widehat{\Phi^n}_M; e_k \right\rangle, \left\langle \widehat{\Phi}_M; e_k \right\rangle\right) < \varepsilon \quad \text{for all } n \geq n_0.$$

Vice versa, if (1.3.8) holds, let $\varepsilon > 0$ be fixed and choose $N = N(\varepsilon) \geq 2$, such that $1/2^N < \varepsilon/2$. From (1.3.8), there exists $n_0 = n_0(\varepsilon)$ such that

$$\max_{k+M \leq N} d_M\left(\left\langle \widehat{\Phi^n}_M; e_k \right\rangle, \left\langle \widehat{\Phi}_M; e_k \right\rangle\right) < \varepsilon \quad \text{for all } n \geq n_0.$$

For every $n \geq n_0$ we obtain

$$d_\infty(\Phi^n, \Phi) \leq \sum_{2 \leq k+M \leq N} \frac{\varepsilon}{2^{k+M}} + \sum_{k+M > N} \frac{1}{2^{k+M}} = \frac{\varepsilon}{2}\left(1 - \frac{1}{2^{N-1}}\right) + \frac{1}{2^N} < \varepsilon.$$

Let now (1.3.8) hold and let $k \in \mathbb{N}$ be fixed. From Theorem 1.3.6, for each $M \in \mathbb{N}$

$$\left\langle \widehat{\Phi^n}_M(t); e_k \right\rangle \to \left\langle \widehat{\Phi}_M(t); e_k \right\rangle \quad \text{for all } t \notin Disc(\Phi), \text{ including } t = -1,\ t = M+1 \tag{1.3.10}$$

as $n \to \infty$, implying in particular from (1.3.4) the uniform convergence of $\widehat{\Phi^n}_M$ to $\widehat{\Phi}_M$ on $[-1, 0]$ and $[M, M+1]$. We then recover that for all $M \in \mathbb{N}$

$$\left\langle \Phi^n|_{[0,M]}(t); e_k \right\rangle \to \left\langle \Phi|_{[0,M]}(t); e_k \right\rangle \quad \text{for all } t \notin Disc(\Phi)$$

as $n \to \infty$. Introducing the set

$$Disc(\Phi) = \bigcup_{M \in \mathbb{N}} Disc_M(\Phi),$$

then trivially $Disc_M(\Phi) \subseteq Disc_{M+1}(\Phi)$ for all $M \in \mathbb{N}$ and

$$\langle \Phi^n(t); e_k \rangle \to \langle \Phi(t); e_k \rangle \quad \text{for all } t \notin Disc(\Phi)$$

as $n \to \infty$. Since by Proposition 1.3.3 each $Disc_M(\Phi)$ is either finite or countable, the set $Disc(\Phi)$ is at most countable and thus we get (1.3.7). Furthermore, conditions (i) and (ii) follow easily from the fact that

(i) if $\Phi^n$ is monotone for every $n \in \mathbb{N}$ then (1.3.8) is equivalent to (1.3.10) for all $k, M \in \mathbb{N}$ by Theorem 1.3.6;

(ii) if $\Phi^n$ is continuous for every $n \in \mathbb{N}$ then (1.3.8) reduces to uniform convergence on the interval $[-1, M+1]$ for all $k, M \in \mathbb{N}$.

□

## 1.4 YOUNG MEASURES

In this section we recall some useful definitions and results concerning the theory of Young measures, which will play a central role in the definition of *dissipative measure-valued solution* for the compressible Euler system with damping, cf. Chapter 2.

**Definition 1.4.1.** Let $Q \subseteq \mathbb{R}^n$ be an open set.

(i) The mapping $\nu : Q \to \mathcal{M}(\mathbb{R}^m)$ is said to be *weak-∗ measurable*, if for all $F \in L^1(Q; C_0(\mathbb{R}^m))$ the function

$$Q \ni y \mapsto \langle \nu_y, F(y, \cdot) \rangle = \int_{\mathbb{R}^m} F(y, \lambda) \, d\nu_y(\lambda),$$

is measurable. Here and in the sequel we use the standard notation $\nu_y = \nu(y)$, as if measures $\nu_y$ were parametrized by $y$.

(ii) $L^\infty_{\text{weak}}(Q; \mathcal{M}(\mathbb{R}^m))$ denotes the vector space of all weak-∗ measurable $\nu : Q \to \mathcal{M}(\mathbb{R}^m)$ such that the quantity

$$\operatorname*{ess\,sup}_{y \in Q} \|\nu_y\|_{\mathcal{M}(\mathbb{R}^m)} = \operatorname*{ess\,sup}_{y \in Q} \sup_{\substack{f \in C_0(\mathbb{R}^m) \\ \|f\|_\infty \le 1}} |\langle \nu_y, f \rangle|$$

is finite. The norm is defined as

$$\|\nu\|_{L^\infty_{\text{weak}}(Q; \mathcal{M}(\mathbb{R}^m))} = \operatorname*{ess\,sup}_{y \in Q} \|\nu_y\|_{\mathcal{M}(\mathbb{R}^m)}.$$

The following result states that the space $L^\infty_{\text{weak}}(Q; \mathcal{M}(\mathbb{R}^m))$ can be identified as the dual space of $L^1(Q; C_0(\mathbb{R}^m))$.

**Theorem 1.4.2.** *Let $Q \subseteq \mathbb{R}^n$ be an open set. Any continuous linear functional $\ell \in (L^1(Q; C_0(\mathbb{R}^m)))^*$ admits a unique representation $\nu \in L^\infty_{\text{weak}}(Q; \mathcal{M}(\mathbb{R}^m))$ such that*

$$\langle \ell; F \rangle_{(L^1(Q;C_0(\mathbb{R}^m)))^*; L^1(Q;C_0(\mathbb{R}^m))} = \int_Q \langle \nu_y, F(y) \rangle \, dy, \tag{1.4.1}$$

*for all $F \in L^1(Q; C_0(\mathbb{R}^m))$, and*

$$\|\ell\|_{(L^1(Q;C_0(\mathbb{R}^m)))^*} = \|\nu\|_{L^\infty_{\text{weak}}(Q;\mathcal{M}(\mathbb{R}^m))}.$$

*Proof.* See Málek, Nečas, Rokyta and Ružička [59], Chapter 3, Theorem 2.11. □

**Theorem 1.4.3.** *Let $Q \subset \mathbb{R}^n$ be a measurable set and let $\{z^R\}_{R>0}$ be a sequence of measurable functions such that $z^R : Q \to \mathbb{R}^m$, $R > 0$. Then there exists a subsequence, still denoted by $z^R$, and a measure-valued function $\nu$, called the Young measure associated to $\{z^R\}_{R>0}$, satisfying the following properties.*

*(i) $\nu \in L^\infty_{\text{weak}}(Q; \mathcal{M}(\mathbb{R}^m))$ is such that*

$$\|\nu_y\|_{\mathcal{M}(\mathbb{R}^m)} \le 1 \quad \text{for a.e. } y \in Q,$$

*and for every $\varphi \in C_0(\mathbb{R}^m)$*

$$\varphi(z^R) \overset{*}{\rightharpoonup} \overline{\varphi} \quad \text{in } L^\infty(Q),$$

*as $R \to \infty$, with*

$$\overline{\varphi}(y) = \langle \nu_y, \varphi \rangle \quad \text{for a.e. } y \in Q.$$

*(ii) If*

$$\lim_{k\to\infty}\sup_{R>0}|\{y\in Q\cap B_r:\ |z^R(y)|\geq k\}|=0 \tag{1.4.2}$$

*for every $r>0$, where $B_r\equiv\{y\in Q:\ |y|\leq r\}$, then*

$$\|\nu_y\|_{\mathcal{M}(\mathbb{R}^m)}=1\quad \textit{for a.e. } y\in Q.$$

*(iii) Let $\Phi:[0,\infty)\to\mathbb{R}$ be a Young function satisfying the $\Delta_2$-condition* (1.2.1)*. If condition* (1.4.2) *holds and*

$$\sup_{R>0}\int_Q\Phi(|\tau(z^R)|)dy<\infty, \tag{1.4.3}$$

*for some continuous function $\tau:\mathbb{R}^m\to\mathbb{R}$, then*

$$\tau(z^R)\overset{*}{\rightharpoonup}\overline{\tau}\quad \textit{in } L_\Phi(Q)$$

*as $R\to\infty$, with*

$$\overline{\tau}(y)=\langle\nu_y,\tau\rangle\quad \textit{for a.e. } y\in Q.$$

*Proof.* See See Málek, Nečas, Rokyta and Ružička [59], Chapter 4, Theorem 2.1.
We reproduce the proof of the first point in order to give an idea of the explicit construction of the Young measure associated to a measurable sequence $\{z^R\}_{R>0}$. First, for every $R$ we define the mapping

$$\nu^R:Q\to\mathcal{M}(\mathbb{R}^m)$$

defined for a.e. $y\in Q$ by

$$\nu_y^R=\delta_{z^R(y)},$$

where $\delta_a$ is the Dirac measure supported at $a\in\mathbb{R}^m$. Hence, for every $\psi\in L^1(Q;C_0(\mathbb{R}^m))$ the function

$$y\mapsto\langle\nu_y^R,\psi(y)\rangle$$

is measurable; indeed

$$\langle\nu_y^R,\psi(y)\rangle=\int_{\mathbb{R}^m}\psi(y,\cdot)\,\mathrm{d}\nu_y^R=\int_{\mathbb{R}^m}\psi(y,\cdot)\,\mathrm{d}\delta_{z^R(y)}=\psi(y,z^R(y)),$$

and thus

$$\int_Q|\langle\nu_y^R,\psi(y)\rangle|\,\mathrm{d}y\leq\int_Q\sup_{\lambda\in\mathbb{R}^m}|\psi(y,\lambda)|\,\mathrm{d}y=\|\psi\|_{L^1(Q;C_0(\mathbb{R}^m))}.$$

The mapping $\nu^R$ is weakly-$*$ measurable with

$$\|\nu^R\|_{L^\infty_{\mathrm{weak}}(Q;\mathcal{M}(\mathbb{R}^m))}=\operatorname*{ess\,sup}_{y\in Q}\|\nu_y^R\|_{\mathcal{M}(\mathbb{R}^m)}=\|\delta_{z^R(y)}\|_{\mathcal{M}(\mathbb{R}^m)}=1.$$

Therefore, $\{\nu^R\}_{R>0}$ is uniformly bounded in $L^\infty_{\mathrm{weak}}(Q;\mathcal{M}(\mathbb{R}^m))$, which by Theorem 1.4.2 is the dual space of the separable space $L^1(Q;C_0(\mathbb{R}^m))$; by Banach-Alaoglu theorem 1.5.3, there exists $\nu\in L^\infty_{\mathrm{weak}}(Q;\mathcal{M}(\mathbb{R}^m))$ such that, passing to a suitable subsequence as the case may be,

$$\nu^R\overset{*}{\rightharpoonup}\nu\quad \text{in } L^\infty_{\mathrm{weak}}(Q;\mathcal{M}(\mathbb{R}^m))$$

as $R \to \infty$. This means that for all $\psi \in L^1(Q; C_0(\mathbb{R}^m))$

$$\int_Q \psi(y, z^R(y))\, \mathrm{d}y = \int_Q \langle \nu_y^R, \psi(y) \rangle\, \mathrm{d}y \to \int_Q \langle \nu_y, \psi(y) \rangle\, \mathrm{d}y$$

as $R \to \infty$. If we now choose $\psi(y, \lambda) = g(y)\varphi(\lambda)$ with $g \in L^1(Q)$, $\varphi \in C_0(\mathbb{R}^m)$, the last limit tells us that

$$\int_Q g(t,x)\varphi(z^R(y))\, \mathrm{d}y = \int_Q g(y)\langle \nu_y^R, \varphi \rangle\, \mathrm{d}y \to \int_Q g(y)\langle \nu_y, \varphi \rangle\, \mathrm{d}y$$

as $R \to \infty$. Then, for every $\varphi \in C_0(\mathbb{R}^m)$, knowing that

$$\varphi(z^R) \overset{*}{\rightharpoonup} \overline{\varphi} \quad \text{in } L^\infty(Q)$$

as $R \to \infty$, we can deduce that

$$\overline{\varphi}(y) = \langle \nu_y, \varphi \rangle \quad \text{for a.e. } y \in Q.$$

From the weak-$*$ lower semi-continuity of the norm we also have that

$$\|\nu_y\|_{\mathcal{M}(\mathbb{R}^m)} \leq \liminf_{R \to \infty} \|\nu_y^R\|_{\mathcal{M}(\mathbb{R}^m)} = 1 \quad \text{for a.e. } y \in Q.$$

□

*Remark* 1.4.4. If $z^R$ are uniformly bounded in $L^p(Q; \mathbb{R}^m)$ for some $1 \leq p < \infty$, the condition (1.4.2) is satisfied. Indeed, denoting $A_k^R \equiv \{y \in Q \cap B_r; |z^R(y)| \geq k\}$, we have

$$|A_k^R| k^p \leq \int_{A_k^R} |z^R(y)|^p \mathrm{d}y \leq \int_Q |z^R(y)|^p \mathrm{d}y \leq c.$$

Since $c$ is independent of both $R$ and $k$, we obtain

$$\sup_{R>0} |A_k^R| \leq \frac{c}{k^p},$$

which implies (1.4.2).

### 1.4.1 *Concentration measures and dissipation defect*

The following three lemmas enable the introduction of the *concentration measures* and *dissipation defect* for a dissipative measure-valued solution, cf. Chapter 2.

**Lemma 1.4.5.** *Let $Q \subseteq \mathbb{R}^n$ be a measurable set and let $\{z^R\}_{R>0}$, $z^R : Q \to \mathbb{R}^m$ be a sequence of measurable functions generating a Young measure $\nu \in L^\infty_{\mathrm{weak}}(Q; \mathcal{M}(\mathbb{R}^m))$. For every continuous function $H : \mathbb{R}^m \to \mathbb{R}$ such that*

$$\|H(z^R)\|_{L^1(Q)} \leq c, \quad \textit{uniformly in } R,$$

*$\langle \nu_y; H \rangle$ is finite for a.e. $y \in Q$.*

*Proof.* Without loss of generality, we can consider $|H|$ or, equivalently, assume that $H \geq 0$. We take a family of cut-off functions

$$T_k(z) = \min\{z, k\};$$

Then $T_k(H) \in C_0(\mathbb{R}^m)$ and thus, for any fixed $k \in \mathbb{N}$,

$$T_k(H(\boldsymbol{z}^R)) \overset{*}{\rightharpoonup} \overline{T_k(H)} \quad \text{in } L^\infty(Q)$$

as $R \to \infty$, with

$$\overline{T_k(H)}(y) = \langle \nu_y, T_k(H) \rangle \quad \text{for a.e. } y \in Q,$$

as a consequence of Theorem 1.4.3, condition (i). On the other hand we have that

$$T_k(H)(\lambda) \nearrow H(\lambda), \text{ for any } \lambda \in \mathbb{R}^m$$

as $k \to \infty$; by monotone convergence theorem, we have that

$$\langle \nu_y, T_k(H) \rangle = \int_{\mathbb{R}^m} T_k(H)(\lambda)\, \mathrm{d}\nu_y(\lambda) \to \int_{\mathbb{R}^m} H(\lambda)\, \mathrm{d}\nu_y(\lambda) \quad \text{for a.e. } y \in Q$$

as $k \to \infty$. Hence $H$ is $\nu_y$-integrable but the integral can also be infinite. However, by the weak-$*$ lower semi-continuity of the norm

$$\|\langle \nu, T_k(H)\|_{L^1(Q)} \leq \liminf_{R\to\infty} \|T_k(H(\boldsymbol{z}^R))\|_{L^1(Q)} \leq \liminf_{R\to\infty} \|H(\boldsymbol{z}^R)\|_{L^1(Q)} \leq c,$$

uniformly in $k$. Then, since

(i) $\lim_{k\to\infty} \langle \nu_y; T_k(H) \rangle = \langle \nu_y; H \rangle$ for a.e. $y \in Q$;

(ii) $\sup_{k\in\mathbb{N}} \|\langle \nu; T_k(H) \rangle\|_{L^1(Q)} \leq c$,

applying Fatou's lemma we get that $\|\langle \nu; H \rangle\|_{L^1(Q)} \leq c$. Then $\langle \nu_y; H \rangle$ is finite for a.e. $y \in Q$. ☐

**Lemma 1.4.6.** *Let $Q \subseteq \mathbb{R}^n$ be a measurable set and let $\{\boldsymbol{z}^R\}_{R>0}$, $\boldsymbol{z}^R : Q \to \mathbb{R}^m$ be a sequence of measurable functions generating a Young measure $\nu \in L^\infty_{\text{weak}}(Q; \mathcal{M}(\mathbb{R}^m))$. Let $F : \mathbb{R}^m \to [0, \infty)$ be a continuous function such that*

$$\|F(\boldsymbol{z}^R)\|_{L^1(Q)} < \infty \quad \textit{uniformly in } R,$$

*and let $G : \mathbb{R}^m \to \mathbb{R}$ be a continuous function such that*

$$|G(\boldsymbol{z})| \lesssim F(\boldsymbol{z}) \quad \textit{for all } \boldsymbol{z} \in \mathbb{R}^m.$$

*Denote*

$$F_\infty := \overline{F(\boldsymbol{z})} - \langle \nu; F \rangle,$$
$$G_\infty := \overline{G(\boldsymbol{z})} - \langle \nu; G \rangle,$$

*where $\overline{F(\boldsymbol{z})}, \overline{G(\boldsymbol{z})} \in \mathcal{M}(Q)$ are the weak-$*$ limits of $\{F(\boldsymbol{z}^R)\}_{R>0}$, $\{\boldsymbol{G}(\boldsymbol{z}^R)\}_{R>0}$ in $\mathcal{M}(Q)$, respectively. Then*

$$|G_\infty| \lesssim F_\infty. \tag{1.4.4}$$

*Proof.* See Feireisl, Gwiazda, Świerczewska-Gwiazda and Wiedemann [35], Lemma 2.1; we report the proof in details. We have seen that the Young measure $\{\nu_y\}_{y\in Q}$ is such that for all $\psi \in L^1(Q; C_0(\mathbb{R}^m))$

$$\int_Q \psi(y, z^R(y))\, \mathrm{d}y \to \int_Q \langle \nu_y, \psi(y)\rangle\, \mathrm{d}y = \int_Q \int_{\mathbb{R}^m} \psi(y,\lambda)\, \mathrm{d}\nu_y(\lambda)\mathrm{d}y$$

as $R \to \infty$. Now, from the fact that

$$\begin{aligned} F(z^R) &\overset{*}{\rightharpoonup} \overline{F(z)} \quad \text{in } \mathcal{M}(Q), \\ G(z^R) &\overset{*}{\rightharpoonup} \overline{G(z)} \quad \text{in } \mathcal{M}(Q) \end{aligned}$$

as $R \to \infty$, we have that for any $\varphi \in C_0(Q)$

$$\langle \overline{F(z)}, \varphi\rangle = \lim_{R\to\infty} \int_Q F(z^R)\varphi\, \mathrm{d}y = \lim_{R\to\infty} \int_{\{|z^R|\le M\}} F(z^R)\varphi\, \mathrm{d}y + \lim_{R\to\infty} \int_{\{|z^R|> M\}} F(z^R)\varphi\, \mathrm{d}y.$$

Now, we can write

$$\int_{\{|z^R|\le M\}} F(z^R(y))\varphi(y)\, \mathrm{d}y = \int_Q \psi(y, z^R(y))\, \mathrm{d}y,$$

with

$$\psi(y,\lambda) = F(\lambda)\varphi(y)\chi_{\{|\lambda|\le M\}};$$

then, we have that $\psi \in L^1(Q; C_0(\mathbb{R}^m))$; indeed, calling $K = \mathrm{supp}(\varphi)$ we have

$$\int_Q \|\psi(y,\cdot)\|_{C_0(\mathbb{R}^m)}\, \mathrm{d}y = \int_K |\varphi(y)| \sup_{|\lambda|\le M} |F(\lambda)| \le |K| \sup_{y\in K} |\varphi(y)| \sup_{|\lambda|\le M} |F(\lambda)| \le c,$$

since both $\varphi$ and $F$ are continuous functions and so they admit maximum on compact sets. Then, for what we have told previously, we have

$$\begin{aligned} \lim_{R\to\infty} \int_{\{|z^R|\le M\}} F(z^R)\varphi\, \mathrm{d}y &= \lim_{R\to\infty} \int_Q \psi(y, z^R(y))\, \mathrm{d}y \\ &= \int_Q \langle \nu_y, \psi(y)\rangle\, \mathrm{d}y = \int_Q \int_{\mathbb{R}^m} \psi(y,\lambda)\, \mathrm{d}\nu_y(\lambda)\mathrm{d}y \\ &= \int_Q \int_{\{|\lambda|\le M\}} F(\lambda)\varphi(y)\, \mathrm{d}\nu_y(\lambda)\mathrm{d}y. \end{aligned}$$

Applying now Lebesgue theorem we have

$$\lim_{M\to\infty} \left( \lim_{R\to\infty} \int_{\{|z^R|\le M\}} F(z^R)\varphi\, \mathrm{d}y \right) = \int_Q \left( \int_{\mathbb{R}^m} F(\lambda)\mathrm{d}\nu_y(\lambda) \right) \varphi\, \mathrm{d}y = \int_Q \langle \nu_y; F\rangle \varphi\, \mathrm{d}y.$$

Similarly, for any $\varphi \in C_0(Q)$

$$\lim_{M\to\infty} \left( \lim_{R\to\infty} \int_{\{|z^R|\le M\}} G(z^R)\varphi\, \mathrm{d}y \right) = \int_Q \langle \nu_y; G\rangle \varphi\, \mathrm{d}y.$$

Then, we deduce

$$\langle F_\infty, \varphi \rangle = \lim_{M\to\infty} \left( \lim_{R\to\infty} \int_{\{|z^R|>M\}} F(z^R)\varphi \, dy \right),$$
$$\langle G_\infty, \varphi \rangle = \lim_{M\to\infty} \left( \lim_{R\to\infty} \int_{\{|z^R|>M\}} G(z^R)\varphi \, dy \right).$$

From condition $|G| \lesssim F$ we obtain (1.4.4). $\square$

However, in some cases it is impossible to guarantee the function $F$ to be continuous in Lemma 1.4.6. We might use the following result, where continuity is replaced by lower semi-continuity.

**Lemma 1.4.7.** *Under the same hypothesis of Lemma 1.4.6 with $F : \mathbb{R}^m \to [0, \infty]$ being lower semi-continuous and such that*

$$F(z) \gtrsim |z| \quad \text{as } |z| \to \infty,$$

*$G : \mathbb{R}^m \to \mathbb{R}$ being continuous and such that*

$$\limsup_{|z|\to\infty} |G(z)| < \liminf_{|z|\to\infty} F(z), \tag{1.4.5}$$

*relation* (1.4.4) *still holds.*

*Proof.* First of all, there exist a sequence $\{E_n\}_{n\in\mathbb{N}} \subset C^(\mathbb{R}^m)$ of continuous functions such that

$$0 \le E_n(z) \le F(z), \quad E_n(z) \nearrow F(z) \;\text{ as } n \to \infty, \quad \text{for any } z \in \mathbb{R}^m,$$

and, in view of (1.4.5), a constant $r > 0$ such that

$$|G(z)| < F(z) \quad \text{for any } |z| > r.$$

Consider now a function $T \in C^\infty(\mathbb{R}^m)$ such that

$$T(z) \begin{cases} = 0 & \text{if } |z| \le r, \\ \in (0,1) & \text{if } r < |z| < r+1, \\ = 1 & \text{if } |z| \ge r+1, \end{cases}$$

and construct a sequence $\{F_n\}_{n\in\mathbb{N}} \subset C^(\mathbb{R}^m)$ of continuous functions such that

$$F_n(z) := T(z) \max\{|G(z)|, E_n(z)\}.$$

In particular, notice that

$$F_n(z) = \begin{cases} 0 & \text{if } |z| \le r, \\ \max\{|G(z)|, E_n(z)\} & \text{if } |z| \ge r+1, \end{cases}$$

and thus

$$0 \le F_n(z) \le F(z), \quad F_n(z) \ge |G(z)| \;\text{ if } |z| \ge r+1.$$

We can apply Lemma 1.4.6 to get

$$\overline{G(z)} - \langle \nu; G \rangle \lesssim \overline{F_n(z)} - \langle \nu; F_n \rangle$$

for any $n \in \mathbb{N}$, and thus the proof reduces to show

$$\overline{F_n(z)} - \langle \nu; F_n \rangle \le \overline{F(z)} - \langle \nu; F \rangle,$$

or equivalently, to show that for any lower semi-continuous function $H : \mathbb{R}^m \to [0, \infty]$

$$\overline{H(z)} - \langle \nu; H \rangle \ge 0. \tag{1.4.6}$$

Repeating the same arguments, we can find a sequence $\{H_n\}_{n\in\mathbb{N}} \subset C_0(\mathbb{R}^m)$ of bounded continuous functions such that

$$0 \le H_n(z) \le H(z), \quad H_n(z) \nearrow H(z) \text{ as } n \to \infty, \quad \text{for any } z \in \mathbb{R}^m. \tag{1.4.7}$$

For any fixed $n \in \mathbb{N}$, we have that

$$H_n(z^R) \overset{*}{\rightharpoonup} \overline{H_n(z)} \quad \text{in } L^\infty(Q)$$

as $R \to \infty$, with

$$\overline{H_n(z)}(y) = \langle \nu_y, H_n \rangle \quad \text{for a.e. } y \in Q,$$

as a consequence of Theorem 1.4.3, condition (i). On the other side, from (1.4.7) and monotone convergence theorem, we have that

$$\langle \nu_y, H_n \rangle = \int_{\mathbb{R}^m} H_n(\lambda)\, \mathrm{d}\nu_y(\lambda) \to \int_{\mathbb{R}^m} H(\lambda)\, \mathrm{d}\nu_y(\lambda) = \langle \nu_y, H \rangle \quad \text{for a.e. } y \in Q$$

as $n \to \infty$. Finally, we obtain

$$0 \le \overline{H(z)} - \overline{H_n(z)} = \overline{H(z)} - \langle \nu; H_n \rangle \to \overline{H(z)} - \langle \nu; H \rangle$$

as $n \to \infty$, which proves (1.4.6). □

## 1.5 MISCELLANEOUS RESULTS

In this last section, we collect various fundamental theorems that will be widely used throughout the thesis.

1. We start recalling the *Arzelá-Ascoli theorem*.

   **Theorem 1.5.1.** *Let $Q \subset \mathbb{R}^d$ be compact and let $X$ be a compact topological metric space endowed with a metric $d_X$. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of functions in $C(Q; X)$ that is equicontinuous, meaning that for any $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that*

   $$d_X\left[f_n(y), f_n(z)\right] \le \varepsilon \quad \textit{provided } |y - z| < \delta,$$

   *independently of $n$. Then, there exists $f \in C(Q; X)$ such that, passing to a suitable subsequence as the case may be,*

   $$\sup_{y\in Q} d_X\left[f_n(y), f(y)\right] \to 0$$

   *as $n \to \infty$.*

   *Proof.* See e.g. Kelley [51], Chapter 7, Theorem 17. □

2. We now report the following version of *Gronwall's Lemma*.

**Lemma 1.5.2.** *Let $f = f(t)$ and $\beta = \beta(t)$ be continuous and non-negative functions defined on $[0, T]$, and let $\alpha = \alpha(t)$ be a continuous, positive and non-decreasing function defined on $[0, T]$. If*

$$f(\tau) \leq \alpha(\tau) + \int_0^\tau \beta(t) f(t)\, \mathrm{d}t \quad \textit{for all } \tau \in [0, T],$$

*then*

$$f(\tau) \leq \alpha(\tau) \exp\left(\int_0^\tau \beta(t)\, \mathrm{d}t\right) \quad \textit{for all } \tau \in [0, T].$$

*Proof.* See e.g. Pachpatte [64], Theorem 1.3.1. □

3. Next, we report the *Banach-Alaoglu theorem*, which provides a sufficient condition to get weakly relatively compact sequences.

**Theorem 1.5.3.** *Let $X$ be a normed vector space. Then, the closed unit ball in the continuous dual space $X^*$, endowed with its usual operator norm, is compact with respect to the weak-$*$ topology. Moreover,*

*(i) if $X$ is a separable Banach space, then every uniformly bounded sequence has a weakly-$*$ convergent subsequence.*

*In particular, if $\{f_n\}_{n\in\mathbb{N}}$ is uniformly bounded in $L^\infty(Q; X^*)$, with $Q \subseteq \mathbb{R}^d$ a measurable set, then, passing to a suitable subsequence as the case may be,*

$$f_n \overset{*}{\rightharpoonup} f \quad \textit{in } L^\infty(Q; X^*)$$

*as $n \to \infty$, meaning that*

$$\int_Q \langle f_n; g\rangle_{X^*,X}\, \mathrm{d}y \to \int_Q \langle f; g\rangle_{X^*,X}\, \mathrm{d}y$$

*as $n \to \infty$, for all $g \in L^1(Q; X)$;*

*(ii) if $X$ is a reflexive Banach space, then every uniformly bounded sequence in $X$ has a weakly convergent subsequence.*

*Proof.* See e.g. Rudin [68], Theorem 3.15. □

4. Since the $L^1$-space is neither reflexive nor the dual of a Banach space, uniformly bounded sequences in $L^1$ are in general not weakly relatively compact. However, the *Dunford-Pettis theorem* gives a necessary and sufficient condition to get weak compactness.

**Theorem 1.5.4.** *Let $Q \subset \mathbb{R}^d$ be a bounded measurable set and let $\{f_n\}_{n\in\mathbb{N}}$ be an uniformly bounded sequence in $L^1(Q)$. Then, the following statements are equivalent.*

*(i) $\{f_n\}_{n\in\mathbb{N}}$ is weakly precompact in $L^1(Q)$: there exists $f \in L^1(Q)$ such that, passing to a suitable subsequence as the case may be,*

$$f_n \rightharpoonup f \quad \textit{in } L^1(Q)$$

*as $n \to \infty$.*

(ii) *For any* $\varepsilon > 0$ *there exists* $C = C(\varepsilon) > 0$ *such that*

$$\int_{\{|f_n|>C\}} |f_n(y)| \, \mathrm{d}y < \varepsilon,$$

*independently of* $n$.

(iii) *The sequence* $\{f_n\}_{n\in\mathbb{N}}$ *is equi-integrable, cf. condition (i) of Theorem 1.2.10.*

*Proof.* See e.g. Ekeland and Temam [32], Chapter 8, Theorem 1.3. □

5. Subsequently, we report some properties of *lower semi-continuous* and *convex* functions.

**Theorem 1.5.5.** *Let* $\Phi : \mathbb{R}^m \to (-\infty, +\infty]$ *be a lower semi-continuous and convex function.*

(i) *The epigraph of* $\Phi$ *defined as*

$$\mathrm{Epi}[\Phi] := \{(a, z) \in \mathbb{R} \times \mathbb{R}^m | \; \Phi(z) \leq a\}$$

*is closed and convex.*

*Moreover, let* $Q \subseteq \mathbb{R}^d$ *be a measurable set and* $\{f_n\}_{n\in\mathbb{N}}$ *a sequence of functions in* $L^1(Q; \mathbb{R}^m)$ *such that*

$$f_n \rightharpoonup f \quad \text{in } L^1(Q; \mathbb{R}^m)$$

*as* $n \to \infty$.

(ii) *If* $\Phi(f_n) \in L^1(Q)$ *for any* $n \in \mathbb{N}$ *and*

$$\Phi(f_n) \rightharpoonup \overline{\Phi(f)} \quad \text{in } L^1(Q)$$

*as* $n \to \infty$, *then*

$$\Phi(f) \leq \overline{\Phi(f)} \quad \text{a.e. on } Q. \tag{1.5.1}$$

*If moreover* $\Phi$ *is strictly convex on an open set* $\mathcal{U} \subseteq \mathbb{R}^m$ *and the equality holds in* (1.5.1) *a.e. on* $Q$, *then, passing to a suitable subsequence as the case may be,*

$$f_n(y) \to f(y) \quad \text{for a.e. } y \in \{x \in Q | \; f(x) \in \mathcal{U}\}$$

*as* $n \to \infty$.

(iii) $\Phi(f) : Q \to \mathbb{R}$ *is integrable and*

$$\int_Q \Phi(f(y)) \, \mathrm{d}y \leq \liminf_{n\to\infty} \int_Q \Phi(f_n(y)) \, \mathrm{d}y.$$

*Proof.* For (i), see Ekeland and Temam [32], Chapter 1, Propositions 2.1 and 2.3.
For (ii) and (iii), see Feireisl [33], Theorem 2.11 and Corollary 2.2. □

*Remark* 1.5.6. Here and in what follows, if $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of functions in $L^1(Q; \mathbb{R}^m)$, with $Q \subseteq \mathbb{R}^d$ measurable, such that

$$f_n \rightharpoonup f \quad \text{in } L^1(Q; \mathbb{R}^m)$$

as $n \to \infty$, then any weak $L^1$-limit of a composition $\Phi(f_n)$ will be denoted by $\overline{\Phi(f)}$.

6. The following result states uniqueness of the inverse Laplace transform, better know as *Lerch's theorem*.

   **Theorem 1.5.7.** *If a function $F = F(s)$ has the inverse Laplace transform $f = f(t)$, meaning that*
   $$F(s) = \int_0^\infty e^{-st} f(t)\, \mathrm{d}t \quad \text{for any } s \geq s_0$$
   *with $f$ continuous, then $f$ is uniquely determined, considering functions which differ from each other on a point set having Lebesgue measure zero the same.*

   *Proof.* See e.g. Cohen [25], Theorem 2.1. □

7. Next, we report the *Banach-Caccioppoli fixed-point theorem*.

   **Theorem 1.5.8.** *Let $X$ be a Banach space with norm $\|\cdot\|_X$, $Z \subseteq X$ a closed convex subset of $X$ and let $F : Z \to Z$ be a contractive map, meaning that there exists a constant $k \in [0, 1)$ such that*
   $$\|F(z_1) - F(z_2)\|_X \leq k\|z_1 - z_2\|_X \quad \text{for any } z_1, z_2 \in Z.$$
   *Then there exists a unique fixed point $z \in Z$ of the mapping $F$, meaning that*
   $$z = F(z).$$

   *Proof.* See e.g. Schwartz [71], Chapter XII. □

8. The following result represents an useful criteria to get the Borel-measurability of a given map.

   **Theorem 1.5.9.** *Let $X$ be a metric space and $\mathcal{B}$ its Borel $\sigma$-field. Let $x \mapsto K_x$ map $X$ into comp(Y) for some separable metric space $Y$, where comp(Y) denotes the space of all compact subsets of $Y$. Suppose that for any sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $x_n \to x$ in $X$ as $n \to \infty$ and any $y_n \in K_{x_n}$ there exists $y \in K_x$ such that $y_n \to y$ in $Y$ as $n \to \infty$. Then, the map $x \mapsto K_x$ is a Borel map of $X$ into comp(Y).*

   *Proof.* See Stroock and Varadhan [73], Lemma 12.1.8. □

9. We conclude with a topological result known as *Cantor's intersection theorem*.

   **Theorem 1.5.10.** *Let $S$ be a Hausdorff space. A decreasing nested sequence of non-empty compact subsets of $S$ is non-empty. In other words, supposing $\{C_n\}_{n\in\mathbb{N}}$ is a sequence of non-empty compact subsets of $S$ satisfying*
   $$C_0 \supseteq C_1 \supseteq C_2 \supseteq \cdots \supseteq C_n \supseteq \ldots$$
   *it follows that*
   $$\bigcap_{n\in\mathbb{N}} C_n \neq \emptyset.$$

   *Proof.* See e.g. Lewin [56], Section 7.8. □

# Part I

# EXISTENCE: MEASURE-VALUED SOLUTIONS FOR THE COMPRESSIBLE EULER SYSTEM WITH DAMPING

# 2

# VANISHING VISCOSITY LIMIT

The goal of this chapter is to show that the dissipative weak solutions of the compressible Navier-Stokes system converge in the zero viscosity limit to the strong solution of the compressible Euler system with damping defined on a general spatial domain, as long as the latter exists, cf. Theorem 2.7.1. The proof is a direct consequence of two results: first, we identify the measure-valued solution of the Euler system as a vanishing viscosity limit of the Navier-Stokes system, cf. Theorem 2.5.2; secondly, we prove the weak-strong uniqueness principle for the Euler system, i.e. the measure-valued solution coincides with the classical one emanating from the same initial data on the lifespan of the strong solution, cf. Theorem 2.6.1.

The chapter is organized as follows. In Section 2.1 we introduce the models, in particular

(i) the Euler system in terms of $[\varrho, \mathbf{m}]$ on $(0,T) \times \Omega$, where $\Omega$ can be a bounded or an exterior domain, or the whole space $\mathbb{R}^d$;

(ii) for any fixed $R > 0$, the Navier-Stokes system in terms of $[\varrho_R, \mathbf{u}_R]$ on $(0,T) \times \Omega_R$, where $\Omega_R$ is the intersection of $\Omega$ with the ball centred at origin and radius $R$, and with the viscous stress terms rescaled by $1/R$.

In section 2.2 we derive the weak formulation of the compressible Navier-Stokes system, providing the definition of a dissipative weak solution $[\varrho_R, \mathbf{u}_R]$, cf. Definition 2.2.1. Assuming the initial energy to be bounded, in Section 2.3 we recover all the a priori estimates necessary to perform the limit $R \to \infty$, which generates the Young measure associated to the Euler system, analysed in Section 2.4, and leading to the definition of a dissipative measure-valued solution for the compressible Euler system with damping in Section 2.5, cf. Definition 2.5.1. Section 2.6 is devoted to the proof of the weak-strong uniqueness principle for the Euler system, showing that the dissipative measure-valued solutions satisfy an extended version of the energy inequality, known as relative energy inequality, for smooth and compactly supported $[r, \mathbf{U}]$, and extending it for any strong solution $[r, \mathbf{U}]$ of the Euler system through a density argument. Finally, in Section 2.7 we prove Theorem 2.7.1.

## 2.1 FROM THE EULER TO THE NAVIER-STOKES SYSTEM

Let us consider the *compressible Euler system with damping*, described by the following couple of equations

$$\partial_t \varrho + \operatorname{div}_x \mathbf{m} = 0, \tag{2.1.1}$$

$$\partial_t \mathbf{m} + \operatorname{div}_x \left( \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) + \nabla_x p(\varrho) + a\mathbf{m} = 0; \tag{2.1.2}$$

here $\varrho = \varrho(t,x)$ denotes the density, $\mathbf{m} = \mathbf{m}(t,x)$ the momentum - with the convection that the convective term is equal to zero whenever $\varrho = 0$ - and $p = p(\varrho)$ the pressure. The term $a\mathbf{m}$, with $a \geq 0$, represents "friction". We will study the system on the set

$$(t,x) \in (0,T) \times \Omega,$$

where $T > 0$ is a fixed time and $\Omega \subseteq \mathbb{R}^d$, with $d = 2, 3$, can be a bounded or unbounded domain, along with the boundary condition

$$\mathbf{m} \cdot \mathbf{n}|_{\partial\Omega} = 0, \tag{2.1.3}$$

for all $t \in [0, T]$; if $\Omega$ is unbounded, we impose the condition at infinity

$$\varrho \to \overline{\varrho}, \quad \mathbf{m} \to 0 \quad \text{as } |x| \to \infty, \tag{2.1.4}$$

with a constant $\overline{\varrho} \geq 0$. We also consider the following initial data

$$\varrho(0, \cdot) = \varrho_0, \quad \mathbf{m}(0, \cdot) = \mathbf{m}_0, \tag{2.1.5}$$

with $\varrho_0 \geq 0$. We finally assume that the pressure $p$ is given by the isentropic state equation

$$p(\varrho) = A\varrho^\gamma, \tag{2.1.6}$$

where $\gamma > 1$ is the adiabatic exponent and $A > 0$ is a constant.

Our goal is to identify a class of generalized *dissipative measure-valued* solutions for the Euler system (2.1.1), (2.1.2) as a vanishing viscosity limit of the Navier-Stokes equations. More specifically, we start considering the set

$$\Omega_R = \Omega \cap B_R, \quad B_R = \{x \in \mathbb{R}^d : |x| < R\},$$

where we assume $\Omega_R$ to be at least a Lipschitz domain, and we consider the *compressible Navier-Stokes system*

$$\partial_t \varrho_R + \operatorname{div}_x(\varrho_R \mathbf{u}_R) = 0, \tag{2.1.7}$$

$$\partial_t(\varrho_R \mathbf{u}_R) + \operatorname{div}_x(\varrho_R \mathbf{u}_R \otimes \mathbf{u}_R) + \nabla_x p(\varrho_R) = \frac{1}{R} \operatorname{div}_x \mathbb{S}(\nabla_x \mathbf{u}_R) - a\varrho_R \mathbf{u}_R; \tag{2.1.8}$$

now $\mathbf{u}_R = \mathbf{u}_R(t, x)$ is the velocity and $\mathbb{S}_R = \mathbb{S}(\nabla_x \mathbf{u}_R)$ is the viscous stress, which we assume to be a linear function of the velocity gradient, more specifically to satisfy Newton's rheological law

$$\mathbb{S}_R = \mathbb{S}(\nabla_x \mathbf{u}_R) = \mu \left( \nabla_x \mathbf{u}_R + \nabla_x^T \mathbf{u}_R - \frac{2}{d} (\operatorname{div}_x \mathbf{u}_R) \mathbb{I} \right) + \lambda (\operatorname{div}_x \mathbf{u}_R) \mathbb{I}, \tag{2.1.9}$$

where $\mu > 0$, $\lambda \geq 0$ are constants.

As our goal is to perform the vanishing viscosity limit for the Navier-Stokes system, we impose the complete slip boundary conditions on $\partial\Omega$:

$$\mathbf{u}_R \cdot \mathbf{n}|_{\partial\Omega} = 0, \ (\mathbb{S}_R \cdot \mathbf{n}) \times \mathbf{n}|_{\partial\Omega} = 0, \tag{2.1.10}$$

and the no–slip boundary conditions on $\partial B_R$:

$$\mathbf{u}_R|_{\partial B_R} = 0, \tag{2.1.11}$$

for all $t \in [0, T]$. Of course, conditions (2.1.10) and (2.1.11) may be not compatible but they do not give rise to any extra analytical problem assuming that $\partial B_R \cap \partial\Omega = \emptyset$ for $R$ large enough, meaning that $\partial\Omega$ is a compact set. That is $\Omega$ is either

(i) a bounded domain,

(ii) an exterior domain,

(iii) the whole space $\mathbb{R}^d$.

For the sake of simplicity, we restrict ourselves to these three cases. Finally, we impose the initial conditions independent of $R$:

$$\varrho_R(0,\cdot) = \varrho_0, \quad (\varrho_R \mathbf{u}_R)(0,\cdot) = \mathbf{m}_0 \quad \text{in } \Omega_R, \tag{2.1.12}$$

where $\varrho_0, \mathbf{m}_0$ are the initial conditions of the Euler system as in (2.1.5).

## 2.2 WEAK FORMULATION OF THE NAVIER-STOKES SYSTEM

First of all, choosing a constant background density $\overline{\varrho} \geq 0$, we can notice that the Navier-Stokes system (2.1.7), (2.1.8) can be rewritten as

$$\partial_t(\varrho_R - \overline{\varrho}) + \operatorname{div}_x(\varrho_R \mathbf{u}_R) = 0, \tag{2.2.1}$$

$$\partial_t(\varrho_R \mathbf{u}_R) + \operatorname{div}_x(\varrho_R \mathbf{u}_R \otimes \mathbf{u}_R) + \nabla_x[p(\varrho_R) - p(\overline{\varrho})] = \frac{1}{R} \operatorname{div}_x \mathbb{S}(\nabla_x \mathbf{u}_R) - a\varrho_R \mathbf{u}_R. \tag{2.2.2}$$

### 2.2.1 *Energy balance*

Multiplying the balance of momentum (2.2.2) by $\mathbf{u}_R$ and noticing that each term of this product can be rewritten as

$$\begin{aligned}
\partial_t(\varrho_R \mathbf{u}_R) \cdot \mathbf{u}_R &= \partial_t \left( \frac{1}{2} \varrho_R |\mathbf{u}_R|^2 \right) + \frac{1}{2} |\mathbf{u}_R|^2 \partial_t \varrho_R, \\
\operatorname{div}_x(\varrho_R \mathbf{u}_R \otimes \mathbf{u}_R) \cdot \mathbf{u}_R &= \operatorname{div}_x \left( \frac{1}{2} \varrho_R |\mathbf{u}_R|^2 \mathbf{u}_R \right) + \frac{1}{2} |\mathbf{u}_R|^2 \operatorname{div}_x(\varrho_R \mathbf{u}_R), \\
\nabla_x[p(\varrho_R) - p(\overline{\varrho})] \cdot \mathbf{u}_R &= \operatorname{div}_x([p(\varrho_R) - p(\overline{\varrho})]\mathbf{u}_R) - [p(\varrho_R) - p(\overline{\varrho})] \operatorname{div}_x \mathbf{u}_R, \\
\operatorname{div}_x \mathbb{S}(\nabla_x \mathbf{u}_R) \cdot \mathbf{u}_R &= \operatorname{div}_x(\mathbb{S}(\nabla_x \mathbf{u}_R)\mathbf{u}_R) - \mathbb{S}(\nabla_x \mathbf{u}_R) : \nabla_x \mathbf{u}_R,
\end{aligned}$$

we get the following equality

$$\begin{aligned}
\partial_t &\left( \frac{1}{2} \varrho_R |\mathbf{u}_R|^2 \right) + \operatorname{div}_x \left( \frac{1}{2} \varrho_R |\mathbf{u}_R|^2 \mathbf{u}_R \right) \\
&\quad + \operatorname{div}_x([p(\varrho_R) - p(\overline{\varrho})]\mathbf{u}_R) - [p(\varrho_R) - p(\overline{\varrho})] \operatorname{div}_x \mathbf{u}_R \\
&= \frac{1}{R} \operatorname{div}_x(\mathbb{S}(\nabla_x \mathbf{u}_R)\mathbf{u}_R) - \frac{1}{R} \mathbb{S}(\nabla_x \mathbf{u}_R) : \nabla_x \mathbf{u}_R - a\varrho_R |\mathbf{u}_R|^2,
\end{aligned} \tag{2.2.3}$$

where we used the fact that

$$\frac{1}{2} |\mathbf{u}_R|^2 (\partial_t \varrho_R + \operatorname{div}_x(\varrho_R \mathbf{u}_R)) = 0$$

by the continuity equation (2.1.7). Integrating (2.2.3) over $\Omega_R$, through an integration by parts and keeping in mind that $\mathbf{u}_R$ satisfies the boundary conditions (2.1.10), (2.1.11), we recover that terms

$$\operatorname{div}_x \left( \frac{1}{2} \varrho_R |\mathbf{u}_R|^2 \mathbf{u}_R \right), \ \operatorname{div}_x([p(\varrho_R) - p(\overline{\varrho})]\mathbf{u}_R), \ \operatorname{div}_x(\mathbb{S}(\nabla_x \mathbf{u}_R)\mathbf{u}_R)$$

vanish, and we get

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_R}\left(\frac{1}{2}\varrho_R|\mathbf{u}_R|^2\right)\,\mathrm{d}x - \int_{\Omega_R}[p(\varrho_R)-p(\overline{\varrho})]\,\mathrm{div}_x\,\mathbf{u}_R\,\mathrm{d}x \\ + a\int_{\Omega_R}\varrho_R|\mathbf{u}_R|^2\,\mathrm{d}x + \frac{1}{R}\int_{\Omega_R}\mathbb{S}(\nabla_x\mathbf{u}_R):\nabla_x\mathbf{u}_R\,\mathrm{d}x = 0. \end{aligned} \tag{2.2.4}$$

Introducing the *pressure potential* $P = P(\varrho)$ as a solution of

$$\varrho P'(\varrho) - P(\varrho) = p(\varrho), \tag{2.2.5}$$

we can write

$$-[p(\varrho_R)-p(\overline{\varrho})] = P(\varrho_R) - P'(\overline{\varrho})(\varrho_R-\overline{\varrho}) - P(\overline{\varrho}) - \varrho_R[P'(\varrho_R)-P'(\overline{\varrho})].$$

Noticing that

$$\begin{aligned} &\mathrm{div}_x\left[\big(P(\varrho_R)-P'(\overline{\varrho})(\varrho_R-\overline{\varrho})-P(\overline{\varrho})\big)\mathbf{u}_R\right] \\ &\quad= \left[P(\varrho_R)-P'(\overline{\varrho})(\varrho_R-\overline{\varrho})-P(\overline{\varrho})\right]\mathrm{div}_x\,\mathbf{u}_R + [P'(\varrho_R)-P'(\overline{\varrho})]\nabla_x\varrho_R\cdot\mathbf{u}_R \\ &\quad= \big(P(\varrho_R)-P'(\overline{\varrho})(\varrho_R-\overline{\varrho})-P(\overline{\varrho})-\varrho_R[P'(\varrho_R)-P'(\overline{\varrho})]\big)\,\mathrm{div}_x\,\mathbf{u}_R \\ &\qquad+ [P'(\varrho_R)-P'(\overline{\varrho})]\,\mathrm{div}_x(\varrho_R\mathbf{u}_R), \end{aligned}$$

and that, multiplying (2.2.1) by $P'(\varrho_R) - P'(\overline{\varrho})$,

$$\begin{aligned} [P'(\varrho_R)-P'(\overline{\varrho})]\,\mathrm{div}_x(\varrho_R\mathbf{u}_R) &= -[P'(\varrho_R)-P'(\overline{\varrho})]\partial_t(\varrho-\overline{\varrho}) \\ &= -\partial_t[P(\varrho_R)-P'(\overline{\varrho})(\varrho_R-\overline{\varrho})-P(\overline{\varrho})] \end{aligned}$$

we obtain

$$\begin{aligned} -[p(\varrho_R)-p(\overline{\varrho})]\,\mathrm{div}_x\,\mathbf{u}_R = {} & \mathrm{div}_x\left[\big(P(\varrho_R)-P'(\overline{\varrho})(\varrho_R-\overline{\varrho})-P(\overline{\varrho})\big)\mathbf{u}_R\right] \\ & + \partial_t[P(\varrho_R)-P'(\overline{\varrho})(\varrho_R-\overline{\varrho})-P(\overline{\varrho})]. \end{aligned}$$

Again, keeping in mind that $\mathbf{u}_R$ satisfies the boundary conditions (2.1.10), (2.1.11), we have

$$\int_{\Omega_R}\mathrm{div}_x\left[\big(P(\varrho_R)-P'(\overline{\varrho})(\varrho_R-\overline{\varrho})-P(\overline{\varrho})\big)\mathbf{u}_R\right]\,\mathrm{d}x = 0$$

and thus, (2.2.4) can be rewritten as

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_R}\left[\frac{1}{2}\varrho_R|\mathbf{u}_R|^2 + P(\varrho_R) - P'(\overline{\varrho})(\varrho_R-\overline{\varrho}) - P(\overline{\varrho})\right]\mathrm{d}x \\ + a\int_{\Omega_R}\varrho_R|\mathbf{u}_R|^2\,\mathrm{d}x + \frac{1}{R}\int_{\Omega_R}\mathbb{S}(\nabla_x\mathbf{u}_R):\nabla_x\mathbf{u}_R\,\mathrm{d}x = 0. \end{aligned} \tag{2.2.6}$$

### 2.2.2 *Dissipative weak solution*

We are now ready to give the definition of a dissipative weak solution.

**Definition 2.2.1.** The pair of functions $[\varrho_R, \mathbf{u}_R]$ is called *dissipative weak solution* of the Navier-Stokes system (2.1.7)–(2.1.12) if the following holds:

(i) *regularity class*:

$$\begin{aligned} \varrho_R &\in C_{\text{weak}}([0,T]; L^\gamma(\Omega_R)), \\ \varrho_R \mathbf{u}_R &\in C_{\text{weak}}([0,T]; L^{\frac{2\gamma}{\gamma+1}}(\Omega_R;\mathbb{R}^d)), \\ \mathbf{u}_R &\in L^2(0,T; W^{1,2}(\Omega_R;\mathbb{R}^d)),\ \mathbf{u}_R|_{\partial B_R} = 0, \end{aligned}$$

and $\varrho_R$ is a non-negative function a.e. in $(0,T)\times\Omega_R$;

(ii) *weak formulation of the continuity equation*: the integral identity

$$\left[\int_{\Omega_R} (\varrho_R - \overline{\varrho})\varphi(t,\cdot)\,\mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau \int_{\Omega_R} [(\varrho_R - \overline{\varrho})\partial_t\varphi + \varrho_R\mathbf{u}_R\cdot\nabla_x\varphi]\,\mathrm{d}x\mathrm{d}t \tag{2.2.7}$$

holds for any $\tau\in[0,T]$ and any $\varphi\in C_c^1([0,T]\times\overline{\Omega}_R)$, where $\varrho_R(0,\cdot)=\varrho_0$;

(iii) *weak formulation of the balance of momentum*: the integral identity

$$\begin{aligned} \left[\int_{\Omega_R} \varrho_R\mathbf{u}_R\cdot\boldsymbol{\varphi}(t,\cdot)\,\mathrm{d}x\right]_{t=0}^{t=\tau} &= \int_0^\tau\int_{\Omega_R} (\varrho_R\mathbf{u}_R\cdot\partial_t\boldsymbol{\varphi} + \varrho_R\mathbf{u}_R\otimes\mathbf{u}_R : \nabla_x\boldsymbol{\varphi})\ \mathrm{d}x\mathrm{d}t \\ &\quad + \int_0^\tau\int_{\Omega_R} [p(\varrho_R) - p(\overline{\varrho})]\mathrm{div}_x\boldsymbol{\varphi}\,\mathrm{d}x\mathrm{d}t \\ &\quad - \int_0^\tau\int_{\Omega_R} \left[\frac{1}{R}\mathbb{S}(\nabla_x\mathbf{u}_R) : \nabla_x\boldsymbol{\varphi} + a\varrho_R\mathbf{u}_R\cdot\boldsymbol{\varphi}\right]\ \mathrm{d}x\mathrm{d}t \end{aligned} \tag{2.2.8}$$

holds for any $\tau\in[0,T]$ and any $\boldsymbol{\varphi}\in C_c^1([0,T]\times\overline{\Omega}\cap B_R;\mathbb{R}^d)$ with $\boldsymbol{\varphi}\cdot\mathbf{n}|_{\partial\Omega}=0$, where $(\varrho_R\mathbf{u}_R)(0,\cdot)=\mathbf{m}_0$;

(iv) *energy inequality*: the integral inequality

$$\begin{aligned} &\int_{\Omega_R}\left[\frac{1}{2}\varrho_R|\mathbf{u}_R|^2 + P(\varrho_R) - P'(\overline{\varrho})(\varrho_R-\overline{\varrho}) - P(\overline{\varrho})\right](\tau,\cdot)\,\mathrm{d}x \\ &\quad + a\int_0^\tau\int_{\Omega_R}\varrho_R|\mathbf{u}_R|^2\,\mathrm{d}x\mathrm{d}t + \frac{1}{R}\int_0^\tau\int_{\Omega_R}\mathbb{S}(\nabla_x\mathbf{u}_R):\nabla_x\mathbf{u}_R\,\mathrm{d}x\mathrm{d}t \\ &\quad \le \int_{\Omega_R}\left[\frac{1}{2}\frac{|\mathbf{m}_0|^2}{\varrho_0} + P(\varrho_0) - P'(\overline{\varrho})(\varrho_0-\overline{\varrho}) - P(\overline{\varrho})\right]\ \mathrm{d}x \end{aligned} \tag{2.2.9}$$

holds for a.e. $\tau>0$.

The integral identities (2.2.7) and (2.2.8) can be easily deduced multiplying equations (2.1.7) and (2.1.8) respectively by test functions, integrating over $(0,\tau)\times\Omega_R$ and performing an integration by parts, while the energy inequality (2.2.9) follows from (2.2.6). Further details on the compressible Navier-Stokes system can be found in Chapter 4.

### 2.2.3 *Existence of weak solutions*

To guarantee the existence of weak solutions, we can now use the following result.

**Theorem 2.2.2.** *Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ be a domain with compact Lipschitz boundary such that $\partial\Omega \cap \partial B_R = \emptyset$ for $R$ large enough, let $\Omega_R = \Omega \cap B_R$ and let $T > 0$ be arbitrary. Suppose that the initial data satisfy*

$$\varrho_0 \in L^\gamma(\Omega_R), \quad \varrho_0 \geq 0 \text{ a.e. in } \Omega_R, \quad \frac{|(\varrho \mathbf{u})_0|^2}{\varrho_0} \in L^1(\Omega_R).$$

*Let the pressure $p$ satisfy* (2.1.6) *with*

$$\gamma > \frac{d}{2}.$$

*Then the Navier-Stokes system* (2.1.7)-(2.1.12) *admits a dissipative weak solution $[\varrho_R, \mathbf{u}_R]$ in $(0, T) \times \Omega_R$ in the sense of Definition 2.2.1.*

The proof follows the same line as in [33], Theorem 7.1. The fact that the boundary conditions are different on $\partial\Omega$ and $\partial B_R$ does not present any extra difficulty as the closures of these two components of the boundary are disjoint.

## 2.3 LIMIT PASSAGE

**Proposition 2.3.1.** *Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ be a domain with compact Lipschitz boundary such that $\partial\Omega \cap \partial B_R = \emptyset$ for $R$ large enough, and let $\overline{\varrho} \geq 0$ be a given far field density if $\Omega$ is unbounded. Let $\{\varrho_R, \mathbf{m}_R = \varrho_R \mathbf{u}_R\}_{R>0}$ be a family of dissipative weak solutions to the Navier-Stokes system* (2.1.7) – (2.1.12) *in*

$$(0, T) \times \Omega_R, \ \Omega_R = \Omega \cap B_R,$$

*in the sense of Definition 2.2.1. Let the corresponding initial data $\varrho_0$, $\boldsymbol{m}_0$ be independent of $R$ satisfying*

$$\varrho_0 \geq 0, \ \int_\Omega \left[ \frac{1}{2} \frac{|\mathbf{m}_0|^2}{\varrho_0} + P(\varrho_0) - P'(\overline{\varrho})(\varrho_0 - \overline{\varrho}) - P(\overline{\varrho}) \right] \mathrm{d}x \leq E_0. \tag{2.3.1}$$

*Moreover, suppose that $\boldsymbol{u}_R$ is extended to be zero and $\varrho_R$ as $\overline{\varrho}$ outside $B_R$, for every $R > 0$. Then, passing to a suitable subsequence as the case may be,*

$$\varrho_R - \overline{\varrho} \to \varrho - \overline{\varrho} \quad \text{in } C_{\text{weak}}([0, T]; L^2 + L^\gamma(\Omega)), \tag{2.3.2}$$

$$\mathbf{m}_R \to \mathbf{m} \quad \text{in } C_{\text{weak}}([0, T]; L^2 + L^{\frac{2\gamma}{\gamma+1}}(\Omega; \mathbb{R}^d)), \tag{2.3.3}$$

$$p(\varrho_R) - p(\overline{\varrho}) \overset{*}{\rightharpoonup} \overline{p(\varrho)} - p(\overline{\varrho}) \quad \text{in } L^\infty(0, T; L^2(\Omega) + \mathcal{M}(\overline{\Omega})), \tag{2.3.4}$$

$$\mathbb{1}_{\varrho_R > 0} \frac{\mathbf{m}_R \otimes \mathbf{m}_R}{\varrho_R} \overset{*}{\rightharpoonup} \overline{\mathbb{1}_{\varrho > 0} \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho}} \quad \text{in } L^\infty(0, T; \mathcal{M}(\overline{\Omega}; \mathbb{R}^{d\times d}_{\text{sym}})), \tag{2.3.5}$$

$$\frac{|\mathbf{m}_R|^2}{\varrho_R} \overset{*}{\rightharpoonup} \overline{\frac{|\mathbf{m}|^2}{\varrho}} \quad \text{in } L^\infty(0, T; \mathcal{M}(\overline{\Omega})), \tag{2.3.6}$$

$$\frac{1}{2} \frac{|\mathbf{m}_R|^2}{\varrho_R} + P(\varrho_R) - P'(\overline{\varrho})(\varrho_R - \overline{\varrho}) - P(\overline{\varrho}) \overset{*}{\rightharpoonup} \overline{E} \quad \text{in } L^\infty(0, T; \mathcal{M}(\overline{\Omega})) \tag{2.3.7}$$

*as $R \to \infty$.*

*Proof.* First of all, we can replace $\Omega_R$ by $\Omega$ in the previous integrals (2.2.7), (2.2.8) and (2.2.9); more precisely, from now on we will consider

$$\left[ \int_\Omega (\varrho_R - \overline{\varrho}) \varphi(t, \cdot) \ \mathrm{d}x \right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega [(\varrho_R - \overline{\varrho}) \partial_t \varphi + \mathbf{m}_R \cdot \nabla_x \varphi] \ \mathrm{d}x\mathrm{d}t, \tag{2.3.8}$$

for any $\tau \in [0,T]$ and all $\varphi \in C^1_c([0,T]\times\overline{\Omega})$,

$$\left[\int_\Omega \mathbf{m}_R \cdot \boldsymbol{\varphi}(t,\cdot)\,\mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \left[\mathbf{m}_R \cdot \partial_t \boldsymbol{\varphi} + \mathbb{1}_{\varrho_R>0} \frac{\mathbf{m}_R \otimes \mathbf{m}_R}{\varrho_R} : \nabla_x \boldsymbol{\varphi}\right] \mathrm{d}x\mathrm{d}t$$
$$+ \int_0^\tau \int_\Omega [p(\varrho_R) - p(\overline{\varrho})]\mathrm{div}_x \boldsymbol{\varphi}\,\mathrm{d}x\mathrm{d}t \tag{2.3.9}$$
$$- \int_0^\tau \int_\Omega \left[\frac{1}{R}\mathbb{S}(\nabla_x \mathbf{u}_R) : \nabla_x \boldsymbol{\varphi} + a\mathbf{m}_R \cdot \boldsymbol{\varphi}\right] \mathrm{d}x\mathrm{d}t,$$

for any $\tau \in [0,T]$ and all $\boldsymbol{\varphi} \in C^1_c([0,T]\times\overline{\Omega};\mathbb{R}^d)$, $\boldsymbol{\varphi}\cdot\mathbf{n}|_{\partial\Omega} = 0$, and

$$\int_\Omega \left[\frac{1}{2}\frac{|\mathbf{m}_R|^2}{\varrho_R} + P(\varrho_R) - P'(\overline{\varrho})(\varrho_R - \overline{\varrho}) - P(\overline{\varrho})\right](\tau,\cdot)\,\mathrm{d}x$$
$$+ a\int_0^\tau \int_\Omega \frac{|\mathbf{m}_R|^2}{\varrho_R}\,\mathrm{d}x\mathrm{d}t + \frac{1}{R}\int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}_R) : \nabla_x \mathbf{u}_R\,\mathrm{d}x\mathrm{d}t \tag{2.3.10}$$
$$\leq \int_\Omega \left[\frac{1}{2}\frac{|\mathbf{m}_0|^2}{\varrho_0} + P(\varrho_0) - P'(\overline{\varrho})(\varrho_0 - \overline{\varrho}) - P(\overline{\varrho})\right] \mathrm{d}x,$$

for a.e. $\tau \in [0,T]$. Note that this is correct for $R$ large enough as the test functions are compactly supported in $\Omega_R$. From (2.3.1) and the energy inequality (2.3.10), we can easily deduce

$$\left\|\frac{\mathbf{m}_R}{\sqrt{\varrho_R}}\right\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^d))} \leq c(E_0), \tag{2.3.11}$$
$$\|P(\varrho_R) - P'(\overline{\varrho})(\varrho_R - \overline{\varrho}) - P(\overline{\varrho})\|_{L^\infty(0,T;L^1(\Omega))} \leq c(E_0), \tag{2.3.12}$$
$$\frac{1}{R}\int_0^T \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}_R) : \nabla_x \mathbf{u}_R\,\mathrm{d}x\mathrm{d}t \leq c(E_0), \tag{2.3.13}$$

where the bounds are independent of $R$. Next, from (2.3.13), we can deduce that

$$\frac{1}{R}\int_0^T \|\mathbb{S}(\nabla_x \mathbf{u}_R)(t,\cdot)\|^2_{L^2(\Omega;\mathbb{R}^{d\times d})}\,\mathrm{d}t \leq c(E_0). \tag{2.3.14}$$

Following Feireisl, Jin and Novotný [37], we can now use relation

$$P(\varrho) - P'(\overline{\varrho})(\varrho - \overline{\varrho}) - P(\overline{\varrho}) \geq c(\overline{\varrho}) \begin{cases} (\varrho - \overline{\varrho})^2 & \text{for } \frac{\overline{\varrho}}{2} < \varrho < 2\overline{\varrho} \\ (1+\varrho^\gamma) & \text{otherwise,} \end{cases} \tag{2.3.15}$$

with a positive constant $c(\overline{\varrho})$. More precisely, following [42], Section 4.7, we introduce the decomposition of an integrable function $h_R$ into its *essential* and *residual* parts:

$$h_R = [h_R]_{\mathrm{ess}} + [h_R]_{\mathrm{res}}, \tag{2.3.16}$$

where

$$[h_R]_{\mathrm{ess}} = \chi(\varrho_R)h_R, \quad [h_R]_{\mathrm{res}} = (1-\chi(\varrho_R))h_R,$$

$$\chi \in C^\infty_c(0,\infty), \quad 0 \leq \chi \leq 1, \quad \chi(r) = 1 \text{ for } r \in \left[\frac{\overline{\varrho}}{2}, 2\overline{\varrho}\right].$$

Then, from (2.3.12) and (2.3.15), we have

$$\begin{aligned}\|[\varrho_R-\overline{\varrho}]_{\mathrm{ess}}\|_{L^\infty(0,T;L^2(\Omega))} &= \operatorname*{ess\,sup}_{t\in(0,T)}\int_\Omega(\varrho_R-\overline{\varrho})^2\chi(\varrho_R)(t,\cdot)\;\mathrm{d}x\\ &\le\frac{1}{c(\overline{\varrho})}\|(P(\varrho_R)-P'(\overline{\varrho})(\varrho_R-\overline{\varrho})-P(\overline{\varrho}))\|_{L^\infty(0,T;L^1(\Omega))}\\ &\le c(E_0),\end{aligned}\tag{2.3.17}$$

and

$$\begin{aligned}\|[\varrho_R-\overline{\varrho}]_{\mathrm{res}}\|_{L^\infty(0,T;L^\gamma(\Omega))} &= \operatorname*{ess\,sup}_{t\in(0,T)}\int_\Omega|\varrho_R-\overline{\varrho}|^\gamma(1-\chi(\varrho_R))(t,\cdot)\;\mathrm{d}x\\ &\lesssim\operatorname*{ess\,sup}_{t\in(0,T)}\int_\Omega(1+\varrho^\gamma)(1-\chi(\varrho_R))(t,\cdot)\;\mathrm{d}x\\ &\le\frac{1}{c(\overline{\varrho})}\|(P(\varrho_R)-P'(\overline{\varrho})(\varrho_R-\overline{\varrho})-P(\overline{\varrho}))\|_{L^\infty(0,T;L^1(\Omega))}\\ &\le c(E_0).\end{aligned}\tag{2.3.18}$$

In particular the Banach-Alaoglu thorem 1.5.3 implies that, passing to a suitable subsequence as the case may be,

$$[\varrho_R-\overline{\varrho}]_{\mathrm{ess}}\overset{*}{\rightharpoonup} f_{\varrho-\overline{\varrho}}\quad\text{in } L^\infty(0,T;L^2(\Omega)),\tag{2.3.19}$$

$$[\varrho_R-\overline{\varrho}]_{\mathrm{res}}\overset{*}{\rightharpoonup} g_{\varrho-\overline{\varrho}}\quad\text{in } L^\infty(0,T;L^\gamma(\Omega))\tag{2.3.20}$$

as $R\to\infty$; defining

$$\varrho-\overline{\varrho}:=f_{\varrho-\overline{\varrho}}+g_{\varrho-\overline{\varrho}},$$

we have that

$$\varrho_R-\overline{\varrho}\overset{*}{\rightharpoonup}\varrho-\overline{\varrho}\quad\text{in } L^\infty(0,T;L^2+L^\gamma(\Omega))$$

as $R\to\infty$, which can be strengthened to (2.3.2), as a consequence of the Arzelá-Ascoli theorem 1.5.1. We can repeat the same procedure for the momenta; indeed, in view of (2.3.11) we obtain

$$\begin{aligned}\|[\mathbf{m}_R]_{\mathrm{ess}}\|_{L^\infty(0,T;L^2(\Omega))} &= \operatorname*{ess\,sup}_{t\in(0,T)}\int_\Omega\varrho_R\frac{|\mathbf{m}_R|^2}{\varrho_R}\chi(\varrho_R)(t,\cdot)\;\mathrm{d}x\\ &\le 2\overline{\varrho}\left\|\frac{\mathbf{m}_R}{\sqrt{\varrho_R}}(t,\cdot)\right\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^d))}\\ &\le c(E_0);\end{aligned}\tag{2.3.21}$$

from (2.3.12) and (2.3.15), we also have

$$\begin{aligned}\|[\sqrt{\varrho_R}]_{\mathrm{res}}\|_{L^\infty(0,T;L^{2\gamma}(\Omega))} &= \operatorname*{ess\,sup}_{t\in(0,T)}\int_\Omega\varrho_R^\gamma(1-\chi(\varrho_R))(t,\cdot)\;\mathrm{d}x\\ &\le\operatorname*{ess\,sup}_{t\in(0,T)}\int_\Omega(\varrho_R^\gamma+1)(1-\chi(\varrho_R))(t,\cdot)\;\mathrm{d}x\\ &\le c(E_0),\end{aligned}$$

which, together with (2.3.11) and Hölder's inequality (1.1.1) with $r = \frac{2\gamma}{\gamma+1}$, $p = 2\gamma$ and $q = 2$ gives

$$\|[\mathbf{m}_R]_{\mathrm{res}}\|_{L^\infty(0,T;L^{\frac{2\gamma}{\gamma+1}}(\Omega;\mathbb{R}^d))} \leq \operatorname*{ess\,sup}_{t\in(0,T)} \|[\sqrt{\varrho_R}]_{\mathrm{res}}(t,\cdot)\|_{L^{2\gamma}(\Omega)} \left\| \frac{\mathbf{m}_R}{\sqrt{\varrho_R}}(t,\cdot) \right\|_{L^2(\Omega;\mathbb{R}^d)} \leq c(E_0). \tag{2.3.22}$$

Then we obtain, passing to suitable subsequences as the case may be

$$\varrho_R \mathbf{u}_R \overset{*}{\rightharpoonup} \mathbf{m} \quad \text{in } L^\infty(0,T;L^2 + L^{\frac{2\gamma}{\gamma+1}}(\Omega;\mathbb{R}^d))$$

as $R \to \infty$; again, the last convergence can be strengthened to (2.3.3). In a similar way, in view of (2.3.17), we have

$$\begin{aligned} \|[p(\varrho_R) - p(\overline{\varrho})]_{\mathrm{ess}}\|_{L^\infty(0,T;L^2(\Omega))} &= \operatorname*{ess\,sup}_{t\in(0,T)} \int_\Omega |p(\varrho_R) - p(\overline{\varrho})|^2 \chi(\varrho_R)(t,\cdot)\, \mathrm{d}x \\ &\leq p'(2\overline{\varrho})\ \|[\varrho_R - \overline{\varrho}]_{\mathrm{ess}}\|_{L^\infty(0,T;L^2(\Omega))} \\ &\leq c(E_0), \end{aligned} \tag{2.3.23}$$

and thus, passing to a suitable subsequence as the case may be,

$$[p(\varrho_R) - p(\overline{\varrho})]_{\mathrm{ess}} \overset{*}{\rightharpoonup} f_{\overline{p(\varrho)} - p(\overline{\varrho})} \quad \text{in } L^\infty(0,T;L^2(\Omega)) \tag{2.3.24}$$

as $R \to \infty$. On the other side, from (2.3.12) and (2.3.15),

$$\begin{aligned} \|[p(\varrho_R) - p(\overline{\varrho})]_{\mathrm{res}}\|_{L^\infty(0,T;L^1(\Omega))} &= A \operatorname*{ess\,sup}_{t\in(0,T)} \int_\Omega |\varrho_R^\gamma - \overline{\varrho}^\gamma|(1-\chi(\varrho_R))(t,\cdot)\, \mathrm{d}x \\ &\leq A \max\{\overline{\varrho}^\gamma, 1\} \operatorname*{ess\,sup}_{t\in(0,T)} \int_\Omega (1+\varrho_R^\gamma)(1-\chi(\varrho_R))(t,\cdot)\, \mathrm{d}x \\ &\leq c(E_0), \end{aligned}$$

while from (2.3.11),

$$\left\| \mathbb{1}_{\varrho_R > 0} \frac{\mathbf{m}_R \otimes \mathbf{m}_R}{\varrho_R} \right\|_{L^\infty(0,T;L^1(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}}))} \leq c(E_0).$$

There is a disturbing phenomena that may occur to bounded sequences in $L^1$: *concentrations*. The idea is then to see $L^1(\Omega)$ as embedded in the space of bounded Radon measures $\mathcal{M}(\overline{\Omega})$, which in turn can be identified as the dual space of the separable space $C_c(\overline{\Omega})$. We get, passing to suitable subsequences as the case may be,

$$[p(\varrho_R) - p(\overline{\varrho})]_{\mathrm{res}} \overset{*}{\rightharpoonup} g_{\overline{p(\varrho)} - p(\overline{\varrho})} \quad \text{in } L^\infty(0,T;\mathcal{M}(\overline{\Omega})), \tag{2.3.25}$$

and convergences (2.3.5)–(2.3.7) as $R \to \infty$. Defining

$$\overline{p(\varrho)} - p(\overline{\varrho}) := f_{\overline{p(\varrho)} - p(\overline{\varrho})} + g_{\overline{p(\varrho)} - p(\overline{\varrho})},$$

we obtain (2.3.4). □

With Proposition 2.3.1 at hand, we are now ready to perform the limit $R \to \infty$ in the weak formulation of our initial problem (2.2.1), (2.2.2); notice that the $R$-dependent viscous stress tensor vanishes. Indeed, using (2.3.14) and Hölder's inequality (1.1.1) with $r = 1$ and $p = q = 2$ we get

$$\begin{aligned}
&\frac{1}{R}\int_0^T \int_\Omega |\mathbb{S}(\nabla_x \mathbf{u}_R) : \nabla_x \boldsymbol{\varphi}| \, \mathrm{d}x\mathrm{d}t \\
&\qquad \le \frac{1}{\sqrt{R}} \left\| \frac{1}{\sqrt{R}} \mathbb{S}(\nabla_x \mathbf{u}_R) \right\|_{L^2((0,T)\times\Omega;\mathbb{R}^{d\times d})} \|\nabla_x \boldsymbol{\varphi}\|_{L^2((0,T)\times\Omega;\mathbb{R}^{d\times d})} \\
&\qquad \le \frac{c(E_0)}{\sqrt{R}} \|\nabla_x \boldsymbol{\varphi}\|_{L^2((0,T)\times\Omega;\mathbb{R}^{d\times d})} .
\end{aligned}$$

Keeping in mind that the functions $\varrho$ and $\mathbf{m}$ are weakly continuous in time, we finally get

$$\left[ \int_\Omega \varrho \varphi(t,\cdot) \, \mathrm{d}x \right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega [\varrho \partial_t \varphi + \mathbf{m} \cdot \nabla_x \varphi] \, \mathrm{d}x\mathrm{d}t, \tag{2.3.26}$$

for any $\tau \in [0,T]$ and any $\varphi \in C^1_c([0,T] \times \overline{\Omega})$, and

$$\begin{aligned}
\left[ \int_\Omega \mathbf{m} \cdot \boldsymbol{\varphi}(t,\cdot) \, \mathrm{d}x \right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \left[ \mathbf{m} \cdot \partial_t \boldsymbol{\varphi} + \overline{\mathbb{1}_{\varrho>0} \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho}} : \nabla_x \boldsymbol{\varphi} + \overline{p(\varrho)} \mathrm{div}_x \boldsymbol{\varphi} \right] \, \mathrm{d}x\mathrm{d}t \\
- a \int_0^\tau \int_\Omega \mathbf{m} \cdot \boldsymbol{\varphi} \, \mathrm{d}x\mathrm{d}t
\end{aligned} \tag{2.3.27}$$

for any $\tau \in [0,T)$ and any $\boldsymbol{\varphi} \in C^1_c([0,T] \times \overline{\Omega}; \mathbb{R}^d)$, $\boldsymbol{\varphi} \cdot \mathbf{n}|_{\partial\Omega} = 0$. Finally, we may pass to the limit in the energy inequality (2.3.10) to get

$$\int_{\overline{\Omega}} \overline{E}(\tau,\cdot) \, \mathrm{d}x + a \int_0^\tau \int_{\overline{\Omega}} \overline{\frac{|\mathbf{m}|^2}{\varrho}} \, \mathrm{d}x\mathrm{d}t \le \int_\Omega \left[ \frac{1}{2} \frac{|\mathbf{m}_0|^2}{\varrho_0} + P(\varrho_0) - P'(\overline{\varrho})(\varrho_0 - \overline{\varrho}) - P(\overline{\varrho}) \right] \mathrm{d}x \tag{2.3.28}$$

for a.e. $\tau \in (0,T)$. Equations (2.3.26), (2.3.27), and (2.3.28) form a suitable platform for introducing the measure-valued solutions of the Euler system with damping. The exact definition requires the concept of Young measures; the interested reader can find all the details in Section 1.4.

## 2.4 YOUNG MEASURE FOR THE COMPRESSIBLE EULER SYSTEM

Our next goal is to adapt the abstract machinery presented in Section 1.4 in order to introduce the definition of a dissipative measure-valued solution for the compressible Euler system with damping. To this end, it is enough to take

- $Q = (0,T) \times \Omega \subset \mathbb{R}^{d+1}$;
- $m = d + 1$;
- $\mathbf{z}^R = (\varrho_R - \overline{\varrho}, \mathbf{m}_R)$,

in Theorem 1.4.3, where $(\varrho_R, \mathbf{m}_R = \varrho_R \mathbf{u}_R)$ are the weak solutions of the Navier-Stokes system (2.2.1), (2.2.2). First of all, notice that condition (1.4.2) is satisfied for $\mathbf{z}^R = (\varrho_R - \overline{\varrho}, \mathbf{m}_R)$; indeed, introducing the sets $A^R_k \equiv \{y \in Q \cap B_r; |\mathbf{z}^R(y)| \ge k\}$ we have, for $y \in A^R_k$

$$\begin{aligned}
k \le |(\varrho_R - \overline{\varrho}, \mathbf{m}_R)(y)| &\le |(\varrho_R - \overline{\varrho})(y)| + |\mathbf{m}_R(y)| \\
&\le |[\varrho_R - \overline{\varrho}]_{\mathrm{ess}}(y)| + |[\varrho_R - \overline{\varrho}]_{\mathrm{res}}(y)| + |[\mathbf{m}_R]_{\mathrm{ess}}(y)| + |[\mathbf{m}_R]_{\mathrm{res}}(y)|,
\end{aligned}$$

and hence at least one of the terms on the last line must be $\geq \frac{k}{4}$ so that

$$A_k^R \subseteq \underbrace{\left\{ y \in Q \cap B_r; |[\varrho_R - \overline{\varrho}]_{\text{ess}}(y)| \geq \frac{k}{4} \right\}}_{\equiv A_{k,1}^R} \cup \underbrace{\left\{ y \in Q \cap B_r; |[\varrho_R - \overline{\varrho}]_{\text{res}}(y)| \geq \frac{k}{4} \right\}}_{\equiv A_{k,2}^R}$$
$$\cup \underbrace{\left\{ y \in Q \cap B_r; |[\mathbf{m}_R]_{\text{ess}}(y)| \geq \frac{k}{4} \right\}}_{\equiv A_{k,3}^R} \cup \underbrace{\left\{ y \in Q \cap B_r; |[\mathbf{m}_R]_{\text{res}}(y)| \geq \frac{k}{4} \right\}}_{\equiv A_{k,4}^R}.$$

For $k$ large enough $(k \geq 4)$, we have

$$\begin{aligned}
|A_k^R|k &\leq 4 \sum_{i=1}^{4} |A_{k,i}^R| \frac{k}{4} \\
&\lesssim |A_{k,1}^R| \left(\frac{k}{4}\right)^2 + |A_{k,2}^R| \left(\frac{k}{4}\right)^\gamma + |A_{k,3}^R| \left(\frac{k}{4}\right)^2 + |A_{k,4}^R| \left(\frac{k}{4}\right)^{\frac{2\gamma}{\gamma+1}} \\
&\leq \int_{A_{k,1}^R} |[\varrho_R - \overline{\varrho}]_{\text{ess}}(y)|^2 \, \mathrm{d}y + \int_{A_{k,2}^R} |[\varrho_R - \overline{\varrho}]_{\text{res}}(y)|^\gamma \, \mathrm{d}y + \int_{A_{k,3}^R} |[\mathbf{m}_R]_{\text{ess}}(y)|^2 \, \mathrm{d}y \\
&+ \int_{A_{k,4}^R} |[\mathbf{m}_R]_{\text{res}}(y)|^{\frac{2\gamma}{\gamma+1}} \, \mathrm{d}y \\
&\leq \|[\varrho_R - \overline{\varrho}]_{\text{ess}}\|_{L^2(Q)}^2 + \|[\varrho_R - \overline{\varrho}]_{\text{res}}\|_{L^\gamma(Q)}^\gamma + \|[\mathbf{m}_R]_{\text{ess}}\|_{L^2(Q;\mathbb{R}^d)}^2 + \|[\mathbf{m}_R]_{\text{res}}\|_{L^{\frac{2\gamma}{\gamma+1}}(Q;\mathbb{R}^d)}^{\frac{2\gamma}{\gamma+1}} \\
&\leq c(E_0),
\end{aligned}$$

where in particular the constant $c(E_0)$ is independent of $k$ and $R$ so that

$$\sup_{R>0} |A_k^R| \leq \frac{c}{k},$$

which implies (1.4.2).

Applying Theorem 1.4.3, condition (ii), we recover that the Young measure in our case is a parametrized family of probability measures supported on the set $[0, \infty) \times \mathbb{R}^d$, since the densities are supposed to be non-negative:

$$\nu_{t,x} : (t, x) \in (0, T) \times \Omega \to \mathcal{P}([0, \infty) \times \mathbb{R}^d),$$
$$\nu \in L^\infty_{\text{weak}}((0, T) \times \Omega; \mathcal{P}([0, \infty) \times \mathbb{R}^d)).$$

It is also easy to check that $\Psi(t) = t^p$ with $p > 1$ are Young functions that satisfy the $\Delta_2$-condition (1.2.1) with constant $2^p$, and in that case $L_\Psi(Q) = L^p(Q)$. Thus,

1. first, we can take $\Psi(t) = t^2$ and $\tau_1(z) = z_1 \, \chi(z_1 + \overline{\varrho})$, where $z = (z_1, z_2, z_3, z_4)$ in our case, to notice that condition (1.4.3) is equivalent in requiring that $[\varrho_R - \overline{\varrho}]_{\text{ess}}$ are uniformly bounded in $L^2((0, T) \times \Omega)$ which is true from (2.3.17). Then we obtain

$$\langle \nu_{t,x}; \tau_1 \rangle = f_{\varrho - \overline{\varrho}}(t, x) \quad \text{for a.e. } (t, x) \in (0, T) \times \Omega,$$

where $f_{\varrho-\overline{\varrho}}$ is the weak-$*$ limit found in (2.3.19); moreover, taking $\Psi(t) = t^\gamma$ and $\tau_2(z) = z_1(1 - \chi(z_1 + \overline{\varrho}))$, condition (1.4.3) is equivalent in requiring that $[\varrho_R - \overline{\varrho}]_{\text{res}}$ are uniformly bounded in $L^\gamma((0, T) \times \Omega)$ which is true from (2.3.18). Then we obtain

$$\langle \nu_{t,x}; \tau_2 \rangle = g_{\varrho - \overline{\varrho}}(t, x) \quad \text{for a.e. } (t, x) \in (0, T) \times \Omega,$$

where $g_\varrho$ is the weak-$*$ limit found in (2.3.20). Unifying the two results we get

$$\langle \nu_{t,x}; \tau_1 + \tau_2 \rangle = (\varrho - \overline{\varrho})(t,x) \quad \text{for a.e. } (t,x) \in (0,T) \times \Omega,$$

where $\varrho - \overline{\varrho}$ is the weak limit of the densities found in (2.3.2). We will write $\langle \nu_{t,x}; \varrho \rangle = \varrho(t,x)$ for almost every $(t,x) \in (0,T) \times \Omega$ just to make the notation readable;

2. secondly, we can take $\Psi(t) = t^2$ and $\tau_1(z) = z_i \, \chi(z_1 + \overline{\varrho})$ with $i = 2,3,4$ to see that condition (1.4.3) is equivalent in requiring that each component of $[\mathbf{m}_R]_{\text{ess}}$ is uniformly bounded in $L^2((0,T) \times \Omega)$ which is true from (2.3.21). Also, choosing $\Psi(t) = t^{\frac{2\gamma}{\gamma+1}}$ and $\tau_2(z) = z_i(1 - \chi(z_1 + \overline{\varrho}))$ with $i = 2,3,4$, condition (1.4.3) is equivalent in requiring that each component of $[\mathbf{m}_R]_{\text{res}}$ is uniformly bounded in $L^{\frac{2\gamma}{\gamma+1}}((0,T) \times \Omega)$ which is true from (2.3.22). Then we obtain

$$\langle \nu_{t,x}; \tau_1 + \tau_2 \rangle = m_i(t,x) \quad \text{for a.e. } (t,x) \in (0,T) \times \Omega,$$

which we will write $\langle \nu_{t,x}; \mathbf{m} \rangle = \mathbf{m}(t,x)$ for almost every $(t,x) \in (0,T) \times \Omega$, with $\mathbf{m}$ the weak limit of the momenta found in (2.3.3);

3. finally, we can take $\Psi(t) = t^2$ and $\tau_1(z) = [p(z_1 + \overline{\varrho}) - p(\overline{\varrho})] \, \chi(z_1 + \overline{\varrho})$ to notice that condition (1.4.3) is equivalent in requiring that $[p(\varrho_R) - p(\overline{\varrho})]_{\text{ess}}$ are uniformly bounded in $L^2((0,T) \times \Omega)$ which is true from (2.3.23). Then we obtain

$$\langle \nu_{t,x}; \tau_1 \rangle = f_{\overline{p(\varrho)-p(\overline{\varrho})}}(t,x) \quad \text{for a.e. } (t,x) \in (0,T) \times \Omega, \tag{2.4.1}$$

where $f_{\overline{p(\varrho)-p(\overline{\varrho})}}$ is the weak-$*$ limit found in (2.3.24).

Moreover, due to Lemma 1.4.5, it makes sense to introduce the following new measures:

$$\begin{aligned}
p_\infty &= \overline{p(\varrho)} - \langle \nu; p(\varrho) \rangle, \\
\mathbb{M}_\infty &= \overline{\mathbb{1}_{\varrho>0} \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho}} - \left\langle \nu; \mathbb{1}_{\varrho>0} \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right\rangle, \\
\sigma_\infty &= \overline{\frac{|\mathbf{m}|^2}{\varrho}} - \left\langle \nu; \frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} \right\rangle, \\
E_\infty &= \overline{E} - \left\langle \nu; \frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) - P'(\overline{\varrho})(\varrho - \overline{\varrho}) - P(\overline{\varrho}) \right\rangle.
\end{aligned}$$

Notice that $p_\infty$ is indeed a measure since, taking $\tau_2(z) = [p(z_1 + \overline{\varrho}) - p(\overline{\varrho})] \, (1 - \chi(z_1 + \overline{\varrho}))$, from (2.4.1), we have

$$p_\infty(t,x) = g_{\overline{p(\varrho)-p(\overline{\varrho})}}(t,x) - \langle \nu_{t,x}; \tau_2 \rangle \quad \text{for a.e. } (t,x) \in (0,T) \times \Omega,$$

where $g_{\overline{p(\varrho)-p(\overline{\varrho})}}$ is the weak-$*$ limit found in (2.3.25). Now, revisiting the momentum equation (2.3.27) and using the fact that

$$\operatorname{div}_x \boldsymbol{\varphi} = \mathbb{I} : \nabla_x \boldsymbol{\varphi},$$

we get

$$\begin{aligned}
\left[ \int_\Omega \mathbf{m} \cdot \boldsymbol{\varphi}(t,\cdot) \, \mathrm{d}x \right]_{t=0}^{t=\tau} &= \int_0^\tau \int_\Omega \left[ \mathbf{m} \cdot \partial_t \boldsymbol{\varphi} + \left( \left\langle \nu_{t,x}; \mathbb{1}_{\varrho>0} \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right\rangle + \mathbb{M}_\infty \right) : \nabla_x \boldsymbol{\varphi} \right] \mathrm{d}x\mathrm{d}t \\
&\quad + \int_0^\tau \int_\Omega [(\langle \nu_{t,x}; p(\varrho) \rangle + p_\infty) \mathbb{I} : \nabla_x \boldsymbol{\varphi} - a \mathbf{m} \cdot \boldsymbol{\varphi}] \, \mathrm{d}x\mathrm{d}t,
\end{aligned}$$

for all $\tau \in [0,T]$ and for all $\boldsymbol{\varphi} \in C^1_c([0,T]\times\overline{\Omega};\mathbb{R}^d)$, $\boldsymbol{\varphi}\cdot\mathbf{n}|_{\partial\Omega}=0$, which can be rewritten as

$$\left[\int_\Omega \mathbf{m}\cdot\boldsymbol{\varphi}(t,\cdot)\,\mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau\int_\Omega \left[\mathbf{m}\cdot\partial_t\boldsymbol{\varphi} + \left\langle \nu_{t,x}; \mathbb{1}_{\varrho>0}\frac{\mathbf{m}\otimes\mathbf{m}}{\varrho}\right\rangle : \nabla_x\boldsymbol{\varphi}\right]\mathrm{d}x\mathrm{d}t$$
$$+\int_0^\tau\int_\Omega [\langle \nu_{t,x}; p(\varrho)\rangle \operatorname{div}_x\boldsymbol{\varphi} - a\mathbf{m}\cdot\boldsymbol{\varphi}]\,\mathrm{d}x\mathrm{d}t + \int_0^\tau\int_{\overline{\Omega}}\nabla_x\boldsymbol{\varphi} : \mathrm{d}\mu_m,$$

for all $\tau \in [0,T]$ and for all $\boldsymbol{\varphi} \in C^1_c([0,T]\times\overline{\Omega};\mathbb{R}^d)$, $\boldsymbol{\varphi}\cdot\mathbf{n}|_{\partial\Omega}=0$, where

$$\mu_m = \mathbb{M}_\infty + p_\infty\mathbb{I} \in L^\infty_{\text{weak}}(0,T;\mathcal{M}(\overline{\Omega};\mathbb{R}^{d\times d}))$$

is a tensor-valued measure. Similarly, from (2.3.28) we get

$$\int_\Omega \left[\left\langle \nu_{\tau,x}; \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) - P'(\overline{\varrho})(\varrho-\overline{\varrho}) - P(\overline{\varrho})\right\rangle + E_\infty(\tau)\right]\mathrm{d}x$$
$$+a\int_0^\tau\int_\Omega\left[\left\langle \nu_{t,x}; \frac{|\mathbf{m}|^2}{\varrho}\right\rangle + \sigma_\infty\right]\mathrm{d}x\mathrm{d}t$$
$$\leq \int_\Omega\left[\left\langle \nu_{0,x}; \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) - P'(\overline{\varrho})(\varrho-\overline{\varrho})(\varrho-\overline{\varrho}) - P(\overline{\varrho})\right\rangle + E_\infty(0)\right]\mathrm{d}x,$$

for a.e. $\tau \in (0,T)$, which can be rewritten as

$$\int_\Omega \left\langle \nu_{\tau,x}; \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) - P'(\overline{\varrho})(\varrho-\overline{\varrho}) - P(\overline{\varrho})\right\rangle \mathrm{d}x$$
$$+a\int_0^\tau\int_\Omega\left\langle \nu_{t,x}; \frac{|\mathbf{m}|^2}{\varrho}\right\rangle\mathrm{d}x\mathrm{d}t + \mathcal{D}(\tau)$$
$$\leq \int_\Omega\left\langle \nu_{0,x}; \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) - P'(\overline{\varrho})(\varrho-\overline{\varrho}) - P(\overline{\varrho})\right\rangle\mathrm{d}x,$$

for a.e. $\tau \in (0,T)$, with $\mathcal{D} \in L^\infty(0,T)$ such that

$$\mathcal{D}(\tau) = \int_{\overline{\Omega}} E_\infty(\tau)\,\mathrm{d}x + a\int_0^\tau\int_{\overline{\Omega}}\sigma_\infty\,\mathrm{d}x\mathrm{d}t.$$

It can be deduced that

$$\int_0^\tau\int_{\overline{\Omega}}\mathrm{d}|\mu_m| \lesssim \int_0^\tau \mathcal{D}(t)\,\mathrm{d}t,$$

for a.e. $\tau \in (0,T)$. Indeed,

$$\int_0^\tau\int_{\overline{\Omega}}\mathrm{d}|\mu_m| \leq \int_0^\tau\int_{\overline{\Omega}}|\mathbb{M}_\infty|\,\mathrm{d}x\mathrm{d}t + \sum_{i,j=1}^d\int_0^\tau\int_{\overline{\Omega}}|p_\infty|\delta_{i,j}\,\mathrm{d}x\mathrm{d}t$$
$$= \int_0^\tau\int_{\overline{\Omega}}|\mathbb{M}_\infty|\,\mathrm{d}x\mathrm{d}t + d\int_0^\tau\int_{\overline{\Omega}}|p_\infty|\,\mathrm{d}x\mathrm{d}t.$$

Now it is sufficient to apply Lemma 1.4.6 with $F = P(\varrho) - P'(\overline{\varrho})(\varrho-\overline{\varrho}) - P(\overline{\varrho})$ and $G = p(\varrho) - p(\overline{\varrho})$, and noticing that

1. the function

$$[\varrho,\mathbf{m}] \mapsto \begin{cases} 0 & \text{if } \mathbf{m}=0, \\ \frac{|\mathbf{m}|^2}{\varrho} & \text{if } \varrho>0, \\ \infty & \text{otherwise} \end{cases}$$

is lower semi-continuous, we can apply Lemma 1.4.7 with $F = \frac{|\mathbf{m}|^2}{\varrho}$ and $G = 0$ to get

$$\overline{\frac{|\mathbf{m}|^2}{\varrho}} - \left\langle \nu; \frac{|\mathbf{m}|^2}{\varrho} \right\rangle \geq 0;$$

2. we have

$$\overline{\mathbb{1}_{\varrho>0} \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho}} - \left\langle \nu; \mathbb{1}_{\varrho>0} \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right\rangle \in L^\infty_{\text{weak}}(0, T; \mathcal{M}^+(\overline{\Omega}; \mathbb{R}^{d \times d})),$$

meaning that for any $\xi \in \mathbb{R}^d$

$$\left( \overline{\mathbb{1}_{\varrho>0} \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho}} - \left\langle \nu; \mathbb{1}_{\varrho>0} \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right\rangle \right) : (\xi \otimes \xi) \geq 0.$$

Indeed,

$$\begin{aligned}
&\overline{\mathbb{1}_{\varrho>0} \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho}} : (\xi \otimes \xi) - \left\langle \nu; \mathbb{1}_{\varrho>0} \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right\rangle : (\xi \otimes \xi) \\
&\quad = \lim_{R \to \infty} \mathbb{1}_{\varrho_R>0} \frac{\mathbf{m}_R \otimes \mathbf{m}_R}{\varrho_R} : (\xi \otimes \xi) - \left\langle \nu; \mathbb{1}_{\varrho>0} \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right\rangle : (\xi \otimes \xi) \\
&\quad = \lim_{R \to \infty} \mathbb{1}_{\varrho_R>0} \frac{|\mathbf{m}_R \cdot \xi|^2}{\varrho_R} - \left\langle \nu; \mathbb{1}_{\varrho>0} \frac{|\mathbf{m} \cdot \xi|^2}{\varrho} \right\rangle \\
&\quad = \overline{\mathbb{1}_{\varrho>0} \frac{|\mathbf{m} \cdot \xi|^2}{\varrho}} - \left\langle \nu; \mathbb{1}_{\varrho>0} \frac{|\mathbf{m} \cdot \xi|^2}{\varrho} \right\rangle
\end{aligned}$$

in $\mathcal{D}'((0, T) \times \mathcal{B})$ for any bounded open set $\mathcal{B} \subset \Omega$, where in the last line we used the fact that

$$\left\| \mathbb{1}_{\varrho_R>0} \frac{|\mathbf{m}_R \cdot \xi|^2}{\varrho_R} \right\|_{L^\infty(0,T;L^1(\Omega))} \leq c(E_0, \xi),$$

and thus the weak-$*$ limit in $L^\infty(0, T; \mathcal{M}(\overline{\Omega}))$ exists. Now it is enough to proceed as in step 1;

3. we can write

$$\operatorname{Tr} \left[ \overline{\mathbb{1}_{\varrho>0} \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho}} - \left\langle \nu; \mathbb{1}_{\varrho>0} \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right\rangle \right] = \overline{\mathbb{1}_{\varrho>0} \frac{|\mathbf{m}|^2}{\varrho}} - \left\langle \nu; \mathbb{1}_{\varrho>0} \frac{|\mathbf{m}|^2}{\varrho} \right\rangle,$$

finally, we conclude that

$$\int_0^\tau \int_{\overline{\Omega}} \mathrm{d}|\mu_m| \lesssim \int_0^\tau \int_{\overline{\Omega}} E_\infty \, \mathrm{d}x\mathrm{d}t \leq \int_0^\tau \mathcal{D}(t) \, \mathrm{d}t.$$

## 2.5 DISSIPATIVE MEASURE-VALUED SOLUTION

We are ready to introduce the concept of *dissipative measure-valued solution* to the compressible Euler system with damping. It can be seen as a generalization of a similar concept introduced by Gwiazda et al. [48]. While the definition in [48] is based on the description of concentrations via the Alibert–Bouchitté defect measures [4], our approach is motivated by [35], where the mere inequality (2.5.4) is required postulating the domination of the concentrations by the energy dissipation defect. This strategy seems to fit better the studies of singular limits on general physical domains performed in the present thesis.

**Definition 2.5.1.** A parametrized family of probability measures

$$\nu_{t,x} : (t,x) \in (0,T) \times \Omega \to \mathcal{P}([0,\infty) \times \mathbb{R}^d),$$
$$\nu \in L^\infty_{\text{weak}}((0,T) \times \Omega; \mathcal{P}([0,\infty) \times \mathbb{R}^d)),$$

is a *dissipative measure-valued solution* of problem (2.1.1)–(2.1.6) with initial condition $\{\nu_{0,x}\}_{x\in\Omega}$ if

1. the integral identity

$$\begin{aligned}
&\int_\Omega \langle \nu_{\tau,x}; \varrho\rangle \varphi(\tau,\cdot)\, \mathrm{d}x - \int_\Omega \langle \nu_{0,x}; \varrho\rangle \varphi(0,\cdot)\, \mathrm{d}x \\
&\qquad = \int_0^\tau \int_\Omega [\langle \nu_{t,x}; \varrho\rangle \partial_t \varphi + \langle \nu_{t,x}; \mathbf{m}\rangle \cdot \nabla_x \varphi]\, \mathrm{d}x\mathrm{d}t \\
&\qquad + \int_0^\tau \int_{\overline{\Omega}} \nabla_x \varphi \cdot \mathrm{d}\mu_c
\end{aligned} \tag{2.5.1}$$

holds for all $\tau \in [0,T]$ and for all $\varphi \in C^1_c([0,T] \times \overline{\Omega})$, with $\langle \nu_{0,x}; \varrho\rangle = \varrho_0$. Here

$$\mu_c \in L^\infty_{\text{weak}}(0,T; \mathcal{M}(\overline{\Omega}; \mathbb{R}^d))$$

is a vector–valued measure;

2. the integral identity

$$\begin{aligned}
&\int_\Omega \langle \nu_{\tau,x}; \mathbf{m}\rangle \cdot \boldsymbol{\varphi}(\tau,\cdot)\, \mathrm{d}x - \int_\Omega \langle \nu_{0,x}; \mathbf{m}\rangle \cdot \boldsymbol{\varphi}(0,\cdot)\, \mathrm{d}x \\
&\quad = \int_0^\tau \int_\Omega \left[ \langle \nu_{t,x}; \mathbf{m}\rangle \cdot \partial_t \boldsymbol{\varphi} + \left\langle \nu_{t,x}; \mathbb{1}_{\varrho>0} \frac{\mathbf{m}\otimes\mathbf{m}}{\varrho} \right\rangle : \nabla_x \boldsymbol{\varphi} \right] \mathrm{d}x\mathrm{d}t \\
&\quad + \int_0^\tau \int_\Omega [\langle \nu_{t,x}; p(\varrho)\rangle \operatorname{div}_x \boldsymbol{\varphi} - a\langle \nu_{t,x}; \mathbf{m}\rangle \cdot \boldsymbol{\varphi}]\, \mathrm{d}x\mathrm{d}t \\
&\quad + \int_0^\tau \int_{\overline{\Omega}} \nabla_x \boldsymbol{\varphi} : \mathrm{d}\mu_m,
\end{aligned} \tag{2.5.2}$$

holds for all $\tau \in [0,T]$ and for all $\boldsymbol{\varphi} \in C^1_c([0,T] \times \overline{\Omega}; \mathbb{R}^d)$, $\boldsymbol{\varphi} \cdot \mathbf{n}|_{\partial\Omega}$, with $\langle \nu_{0,x}; \mathbf{m}\rangle = \mathbf{m}_0$. Here

$$\mu_m \in L^\infty_{\text{weak}}(0,T; \mathcal{M}(\overline{\Omega}; \mathbb{R}^{d\times d}))$$

is a tensor–valued measure; both $\mu_c, \mu_m$ are called *concentration measures;*

3. the following inequality

$$\begin{aligned}
&\int_\Omega \left\langle \nu_{\tau,x}; \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) - P'(\overline{\varrho})(\varrho - \overline{\varrho}) - P(\overline{\varrho}) \right\rangle \mathrm{d}x \\
&\quad + a \int_0^\tau \int_\Omega \left\langle \nu_{t,x}; \frac{|\mathbf{m}|^2}{\varrho} \right\rangle \mathrm{d}x\mathrm{d}t + \mathcal{D}(\tau) \\
&\quad \le \int_\Omega \left\langle \nu_{0,x}; \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) - P'(\overline{\varrho})(\varrho - \overline{\varrho}) - P(\overline{\varrho}) \right\rangle \mathrm{d}x,
\end{aligned} \tag{2.5.3}$$

holds for a.e. $\tau \in (0,T)$, where

$$\mathcal{D} \in L^\infty(0,T), \quad \mathcal{D} \ge 0$$

is called *dissipation defect* of the total energy;

4. there exists a constant $C > 0$ such that
$$\int_0^\tau \int_{\overline{\Omega}} \mathrm{d}|\mu_c| + \int_0^\tau \int_{\overline{\Omega}} \mathrm{d}|\mu_m| \leq C \int_0^\tau \mathcal{D}(t) dt, \tag{2.5.4}$$
for a.e. $\tau \in (0, T)$.

Now, summarizing the discussion concerning the vanishing viscosity limit of the Navier-Stokes system, we can state the first result of the present thesis.

**Theorem 2.5.2.** *Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ be a domain with compact Lipschitz boundary and let $\overline{\varrho} \geq 0$ be a given far field density if $\Omega$ is unbounded. Let $[\varrho^R, \mathbf{u}^R]$ be a family of dissipative weak solutions to the Navier-Stokes system* (2.1.7) – (2.1.12) *in*
$$(0, T) \times \Omega_R,\ \Omega_R = \Omega \cap B_R.$$
*Let the corresponding initial data $\varrho_0$, $\mathbf{u}_0$ be independent of $R$ satisfying*
$$\varrho_0 \geq 0,\ \int_\Omega \left[\frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + P(\varrho_0) - P'(\overline{\varrho})(\varrho_0 - \overline{\varrho}) - P(\overline{\varrho})\right] \mathrm{d}x \leq E_0.$$

*Then the family $\{\varrho^R, \mathbf{m}^R = \varrho^R \mathbf{u}^R\}_{R>0}$ generates, as $R \to \infty$, a Young measure $\{\nu_{t,x}\}_{t\in(0,T);x\in\Omega}$ which is a dissipative measure-valued solution of the Euler system with damping* (2.1.1)–(2.1.6).

## 2.6 WEAK-STRONG UNIQUENESS

In this section we aim to prove the following result.

**Theorem 2.6.1.** *Let $[r, \mathbf{U}]$ be a strong solution of the compressible Euler system with damping* (2.1.1)–(2.1.6) *such that, for a fixed $m > \frac{d}{2} + 1$,*
$$\begin{aligned} r - \overline{\varrho} &\in C([0, T]; H^m(\Omega)) \\ \mathbf{U} &\in C([0, T]; H^m(\Omega; \mathbb{R}^d)) \end{aligned}$$
*with $r > 0$ and $\mathbf{U} \cdot \mathbf{n}|_{\partial\Omega} = 0$. Let $\{\nu_{t,x}\}_{(t,x)\in(0,T)\times\Omega}$ be a dissipative measure-valued solution of the same system (in terms of $\varrho$ and the momentum $\mathbf{m}$), with dissipation defect $\mathcal{D}$ in the sense of Definition 2.5.1 and such that*
$$\nu_{0,x} = \delta_{r(0,x),(r\mathbf{U})(0,x)} \quad \text{for a.e. } x \in \Omega.$$
*Then $\mathcal{D} = 0$ and*
$$\nu_{t,x} = \delta_{r(t,x),(r\mathbf{U})(t,x)} \quad \text{for a.e. } (t, x) \in (0, T) \times \Omega.$$

*Remark* 2.6.2. This theorem applies to the already know results concerning strong solutions; in particular

(i) if $\Omega$ is bounded, for local in time solutions see [70], and [65] for the global one;

(ii) if $\Omega = \mathbb{R}^3$, for local in time solution see for instance [50], [58], and [72] for the global one.

The proof will be divided in three steps:

1. first, in Section 2.6.1, introducing the *relative energy functional* $\mathcal{E}$, we will show that proving Theorem 2.6.1 is equivalently in showing that $\mathcal{D}(\tau) = 0$ and $\mathcal{E}(\tau) = 0$ for all $\tau \in (0, T)$, cf. Lemma 2.6.3;

2. secondly, in Section 2.6.2, we will prove that the dissipative measure-valued solutions of problem (2.1.1)–(2.1.6) satisfy an extended version of the energy inequality (2.5.3) known as *relative energy inequality* for smooth and compactly supported $[r, \mathbf{U}]$, cf. Lemma 2.6.4;

3. subsequently, in Section 2.6.3, through a density argument, we will show that the relative energy inequality holds for $[r, \mathbf{U}]$ as in the hypothesis of Theorem 2.6.1, cf. Lemma 2.6.7;

4. finally, in Section 2.6.4, we will prove that $\mathcal{D}(\tau) = 0$ and $\mathcal{E}(\tau) = 0$ for all $\tau \in (0, T)$, cf. Lemma 2.6.8.

### 2.6.1 *Relative energy*

Introducing the *relative energy functional*:

$$\mathcal{E}(\nu = \nu_{t,x}(\varrho, \mathbf{m}) | r, \mathbf{U}) = \int_\Omega \left\langle \nu_{t,x}; \frac{1}{2\varrho}(|\mathbf{m} - \varrho \mathbf{U}|^2) + P(\varrho) - P'(r)(\varrho - r) - P(r) \right\rangle \mathrm{d}x, \quad (2.6.1)$$

we can prove the following result.

**Lemma 2.6.3.** *Let* $\{\nu_{t,x}\}_{(t,x) \in (0,T) \times \Omega}$ *be a dissipative measure-valued solution of problem* (2.1.1)–(2.1.6) *in the sense of Definition 2.5.1, and let* $\varrho, r > 0$. *Then*

$$\mathcal{E}(\nu | \, r, \mathbf{U}) \geq 0,$$

*and the equality holds if and only if*

$$\nu_{t,x} = \delta_{r(t,x), r\mathbf{U}(t,x)} \quad \textit{for a.e. } (t,x) \in (0,T) \times \Omega.$$

*Proof.* If $\varrho \mapsto p(\varrho)$ is strictly increasing in $(0, \infty)$, which is true in our case, then the pressure potential $P$ is strictly convex; indeed

$$P''(\varrho) = \frac{p'(\varrho)}{\varrho} > 0.$$

For a differentiable function this is equivalent in saying that the function lies above all of its tangents:

$$P(\varrho) \geq P'(r)(\varrho - r) + P(r)$$

for all $\varrho, r \in (0, \infty)$, and the equality holds if and only if $\varrho = r$. In the latter case, it is also easy to deduce that $\mathbf{m} = r\mathbf{U}$. $\square$

### 2.6.2 *Relative energy inequality*

**Lemma 2.6.4.** *Let $\{\nu_{t,x}\}_{(t,x)\in(0,T)\times\Omega}$ be a dissipative measure-valued solution of the compressible Euler system with damping* (2.1.1)–(2.1.6) *with concentration measures $\mu_c$, $\mu_m$ and dissipation defect $\mathcal{D}$ in the sense of Definition* 2.5.1. *Then, for every function*

$$\begin{aligned} r-\overline{\varrho} &\in C_c^\infty([0,T]\times\overline{\Omega}), \\ \mathbf{U} &\in C_c^\infty([0,T]\times\overline{\Omega};\mathbb{R}^d), \end{aligned}$$

*in particular $\boldsymbol{U}\cdot\boldsymbol{n}|_{\partial\Omega}=0$, the following inequality holds:*

$$\begin{aligned} &[\mathcal{E}(\nu|r,\mathbf{U})]_{t=0}^{t=\tau} + a\int_0^\tau\int_\Omega \left\langle \nu_{t,x}; \frac{\mathbf{m}}{\varrho}\cdot(\mathbf{m}-\varrho\mathbf{U})\right\rangle \mathrm{d}x\mathrm{d}t + \mathcal{D}(\tau) \\ &\quad\le \int_0^\tau\int_\Omega \langle \nu_{t,x}; \varrho\mathbf{U}-\mathbf{m}\rangle\cdot[\partial_t\mathbf{U}+\nabla_x\mathbf{U}\cdot\mathbf{U}]\,\mathrm{d}x\mathrm{d}t \\ &\quad+\int_0^\tau\int_\Omega \left\langle \nu_{t,x}; \mathbb{1}_{\varrho>0}\frac{(\mathbf{m}-\varrho\mathbf{U})\otimes(\varrho\mathbf{U}-\mathbf{m})}{\varrho}\right\rangle : \nabla_x\mathbf{U}\,\mathrm{d}x\mathrm{d}t \\ &\quad-\int_0^\tau\int_\Omega \langle \nu_{t,x}; p(\varrho)-p(r)\rangle\,\mathrm{div}_x\,\mathbf{U}\;\;\mathrm{d}x\mathrm{d}t \\ &\quad-\int_0^\tau\int_\Omega [\langle \nu_{t,x}; (\varrho-r)\partial_t P'(r) + (\mathbf{m}-r\mathbf{U})\cdot\nabla_x P'(r)\rangle\,\mathrm{d}x\mathrm{d}t \\ &\quad-\int_0^\tau\int_{\overline{\Omega}} \nabla_x\mathbf{U}:\mathrm{d}\mu_m + \int_0^\tau\int_{\overline{\Omega}} \mathbf{U}\cdot\nabla_x\mathbf{U}\cdot\mathrm{d}\mu_c - \int_0^\tau\int_{\overline{\Omega}} \nabla_x P'(r)\cdot\mathrm{d}\mu_c. \end{aligned} \tag{2.6.2}$$

*Remark* 2.6.5. Relation (2.6.2) is know as *relative energy inequality*.

*Remark* 2.6.6. Note that we must have $\overline{\varrho}>0$ if $\Omega$ is unbounded in order to guarantee that $\varrho$ and $r$ are bounded below away from zero.

*Proof.* First of all, we can take $\mathbf{U}$ as a test function in the momentum equation (2.5.2) to obtain

$$\begin{aligned} \left[\int_\Omega \langle \nu_{t,x}; \mathbf{m}\rangle\cdot\mathbf{U}\,\mathrm{d}x\right]_{t=0}^{t=\tau} &= \int_0^\tau\int_\Omega \left[\langle \nu_{t,x}; \mathbf{m}\rangle\cdot\partial_t\mathbf{U} + \left\langle \nu_{t,x}; \mathbb{1}_{\varrho>0}\frac{\mathbf{m}\otimes\mathbf{m}}{\varrho}\right\rangle : \nabla_x\mathbf{U}\right]\mathrm{d}x\mathrm{d}t \\ &+\int_0^\tau\int_\Omega \langle \nu_{t,x}; p(\varrho)\rangle\mathrm{div}_x\mathbf{U}\,\mathrm{d}x\mathrm{d}t - a\int_0^\tau\int_\Omega \langle \nu_{t,x}; \mathbf{m}\rangle\cdot\mathbf{U}\,\mathrm{d}x\mathrm{d}t \\ &+\int_0^\tau\int_{\overline{\Omega}} \nabla_x\mathbf{U}:\mathrm{d}\mu_m, \end{aligned}$$

$\frac{1}{2}|\mathbf{U}|^2$ as a test function in the continuity equation (2.5.1) to get

$$\begin{aligned} \left[\frac{1}{2}\int_\Omega \langle \nu_{t,x}; \varrho\rangle|\mathbf{U}|^2\,\mathrm{d}x\right]_{t=0}^{t=\tau} &= \int_0^\tau\int_\Omega [\langle \nu_{t,x}; \varrho\rangle\mathbf{U}\cdot\partial_t\mathbf{U} + \langle \nu_{t,x}; \mathbf{m}\rangle\cdot\nabla_x\mathbf{U}\cdot\mathbf{U}]\,\mathrm{d}x\mathrm{d}t \\ &+\int_0^\tau\int_{\overline{\Omega}} \mathbf{U}\cdot\nabla_x\mathbf{U}\cdot\mathrm{d}\mu_c, \end{aligned}$$

and $P'(r) - P'(\overline{\varrho})$ as test function in (2.5.1) to get

$$\left[\int_\Omega \langle \nu_{t,x}; \varrho\rangle (P'(r)(t,\cdot) - P'(\overline{\varrho}))dx\right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \langle \nu_{t,x}; \varrho\rangle \partial_t P'(r)\ \mathrm{d}x\mathrm{d}t$$
$$+ \int_0^\tau \int_\Omega \langle \nu_{t,x}; \mathbf{m}\rangle \cdot \nabla_x P'(r)\ \mathrm{d}x\mathrm{d}t$$
$$+ \int_0^\tau \int_{\overline{\Omega}} \nabla_x P'(r) \cdot \mathrm{d}\mu_c.$$

Then, from the energy inequality (2.5.3), summing up all these terms we get

$$\begin{aligned}
&\left[\int_\Omega \left\langle \nu_{t,x}; \frac{1}{2\varrho}|\mathbf{m} - \varrho\mathbf{U}|^2 + P(\varrho) - \varrho P'(r) + \overline{\varrho}P'(\overline{\varrho}) - P(\overline{\varrho})\right\rangle \mathrm{d}x\right]_{t=0}^{t=\tau} \\
&\quad + a\int_0^\tau \int_\Omega \left\langle \nu_{t,x}; \frac{\mathbf{m}}{\varrho}\cdot(\mathbf{m} - \varrho\mathbf{U})\right\rangle \mathrm{d}x\mathrm{d}t + \mathcal{D}(\tau) \\
&\quad \le \int_0^\tau \int_\Omega \langle \nu_{t,x}; \varrho\mathbf{U} - \mathbf{m}\rangle \cdot [\partial_t \mathbf{U} + \nabla_x \mathbf{U}\cdot\mathbf{U}]\ \mathrm{d}x\mathrm{d}t \\
&\quad + \int_0^\tau \int_\Omega \left\langle \nu_{t,x}; \mathbb{1}_{\varrho>0}\frac{(\mathbf{m} - \varrho\mathbf{U})\otimes(\varrho\mathbf{U} - \mathbf{m})}{\varrho}\right\rangle : \nabla_x \mathbf{U}\ \mathrm{d}x\mathrm{d}t \\
&\quad - \int_0^\tau \int_\Omega \langle \nu_{t,x}; p(\varrho)\rangle\, \mathrm{div}_x\,\mathbf{U}\ \mathrm{d}x\mathrm{d}t \\
&\quad - \int_0^\tau \int_\Omega [\langle \nu_{t,x}; \varrho\rangle \partial_t P'(r) + \langle \nu_{t,x}; \mathbf{m}\rangle \cdot \nabla_x P'(r)]\ \mathrm{d}x\mathrm{d}t \\
&\quad - \int_0^\tau \int_{\overline{\Omega}} \nabla_x \mathbf{U} : \mathrm{d}\mu_m + \int_0^\tau \int_{\overline{\Omega}} \mathbf{U}\cdot\nabla_x\mathbf{U}\cdot \mathrm{d}\mu_c - \int_0^\tau \int_{\overline{\Omega}} \nabla_x P'(r)\cdot \mathrm{d}\mu_c.
\end{aligned}$$

Notice that the term

$$\frac{\mathbf{m}}{\varrho}\cdot(\mathbf{m} - \varrho\mathbf{U}) = \frac{|\mathbf{m}|^2}{\varrho} - \mathbf{m}\cdot\mathbf{U}$$

is well-defined and integrable. We have

$$P(\varrho) - \varrho P'(r) + \overline{\varrho}P'(\overline{\varrho}) - P(\overline{\varrho}) = P(\varrho) - P'(r)(\varrho - r) - P(r) - [p(r) - p(\overline{\varrho})],$$

where

$$\left[\int_\Omega \langle \nu_{t,x}; p(r) - p(\overline{\varrho})\rangle\ \mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \langle \nu_{t,x}; \partial_t(p(r) - p(\overline{\varrho}))\rangle\ \mathrm{d}x\mathrm{d}t$$
$$= \int_0^\tau \int_\Omega \langle \nu_{t,x}; \partial_t p(r)\rangle\ \mathrm{d}x\mathrm{d}t.$$

Using relation $p'(r) = rP''(r)$ along with the fact that

$$\int_\Omega \mathrm{div}_x[p(r)\mathbf{U}]\ \mathrm{d}x = \int_{\partial\Omega} p(r)\mathbf{U}\cdot\mathbf{n}\ \mathrm{d}S_x = 0,$$

we can deduce that

$$\int_\Omega \langle \nu_{t,x}; \partial_t p(r)\rangle\ \mathrm{d}x = \int_\Omega \langle \nu_{t,x}; r\partial_t P'(r) + \mathrm{div}_x[p(r)\mathbf{U}]\rangle\ \mathrm{d}x$$
$$= \int_\Omega \langle \nu_{t,x}; r\partial_t P'(r) + r\nabla_x P'(r)\cdot\mathbf{U} + p(r)\,\mathrm{div}_x\,\mathbf{U}\rangle\ \mathrm{d}x.$$

We finally obtain relation (2.6.2). $\square$

### 2.6.3 *Density argument*

**Lemma 2.6.7.** *Let $\{\nu_{t,x}\}_{(t,x)\in(0,T)\times\Omega}$ be a dissipative measure-valued solution of the compressible Euler system with damping* (2.1.1)–(2.1.6) *with the far field density $\overline{\varrho} > 0$ if $\Omega$ is unbounded, concentration measures $\mu_c$, $\mu_m$ and dissipation defect $\mathcal{D}$ in the sense of Definition 2.5.1. Then, the relative energy inequality* (2.6.2) *holds for any*

$$
\begin{aligned}
r - \overline{\varrho} &\in C([0,T]; H^m(\Omega)) \\
\mathbf{U} &\in C([0,T]; H^m(\Omega; \mathbb{R}^d))
\end{aligned}
$$

*where $m > \frac{d}{2} + 1$ is fixed, $r > 0$ and $\mathbf{U} \cdot \mathbf{n}|_{\partial\Omega} = 0$.*

*Proof.* Using the density of the compactly supported smooth functions in the Sobolev spaces, we can find two sequences $\{r_n - \overline{\varrho}\}_{n\in\mathbb{N}} \subset C_c^\infty([0,T]\times\overline{\Omega})$, $\{\mathbf{U}_n\}_{n\in\mathbb{N}} \subset C_c^\infty([0,T]\times\overline{\Omega};\mathbb{R}^d)$ such that

$$
\begin{aligned}
r_n - \overline{\varrho} \to r - \overline{\varrho} \quad &\text{in } C([0,T]; H^m(\Omega)), \\
\mathbf{U}_n \to \mathbf{U} \quad &\text{in } C([0,T]; H^m(\Omega; \mathbb{R}^d)).
\end{aligned}
$$

If we now fix $\varepsilon > 0$, we know that there exists $n_0 = n_0(\varepsilon)$ such that, for every $n \geq n_0$

$$
\sup_{t\in[0,T]} \|(r - r_n)(t,\cdot)\|_{H^m(\Omega)} < \varepsilon,
$$

$$
\sup_{t\in[0,T]} \|(\mathbf{U} - \mathbf{U}_n)(t,\cdot)\|_{H^m(\Omega;\mathbb{R}^d)} < \varepsilon.
$$

From now on, let $n \geq n_0$; for each $t \in [0,T]$ we have

$$
\begin{aligned}
&\int_\Omega \left\langle \nu_{t,x}; \frac{1}{2\varrho}|\mathbf{m} - \varrho\mathbf{U}|^2 \right\rangle \mathrm{d}x \\
&= \int_\Omega \left\langle \nu_{t,x}; \frac{1}{2\varrho}|\mathbf{m} - \varrho(\mathbf{U} - \mathbf{U}_n + \mathbf{U}_n)|^2 \right\rangle \mathrm{d}x \\
&= \int_\Omega \left\langle \nu_{t,x}; \frac{1}{2\varrho}|\mathbf{m} - \varrho\mathbf{U}_n|^2 \right\rangle \mathrm{d}x - \int_\Omega \langle \nu_{t,x}; \mathbf{m} - \varrho\mathbf{U}_n\rangle \cdot (\mathbf{U} - \mathbf{U}_n)(t,\cdot)\,\mathrm{d}x \\
&\quad + \frac{1}{2}\int_\Omega \langle \nu_{t,x}; \varrho\rangle |\mathbf{U} - \mathbf{U}_n|^2(t,\cdot)\,\mathrm{d}x \\
&= \int_\Omega \left\langle \nu_{t,x}; \frac{1}{2\varrho}|\mathbf{m} - \varrho\mathbf{U}_n|^2 \right\rangle \mathrm{d}x \\
&\quad - \int_\Omega \langle \nu_{t,x}; \mathbf{m} - (\varrho - \overline{\varrho})\mathbf{U}_n\rangle \cdot (\mathbf{U} - \mathbf{U}_n)(t,\cdot)\,\mathrm{d}x + \overline{\varrho}\int_\Omega \mathbf{U}_n \cdot (\mathbf{U} - \mathbf{U}_n)(t,\cdot)\,\mathrm{d}x \\
&\quad + \frac{1}{2}\int_\Omega \langle \nu_{t,x}; \varrho - \overline{\varrho}\rangle |\mathbf{U} - \mathbf{U}_n|^2(t,\cdot)\,\mathrm{d}x + \frac{\overline{\varrho}}{2}\int_\Omega |\mathbf{U} - \mathbf{U}_n|^2(t,\cdot)\,\mathrm{d}x.
\end{aligned}
$$

Revoking the splitting (2.3.16) of an integrable function, we can write

$$
\begin{aligned}
&\int_\Omega \langle \nu_{t,x}; \mathbf{m} - (\varrho - \overline{\varrho})\mathbf{U}_n\rangle \cdot (\mathbf{U} - \mathbf{U}_n)(t,\cdot)\,\mathrm{d}x \\
&\qquad = \int_\Omega \langle \nu_{t,x}; [\mathbf{m}]_{\mathrm{ess}} - [\varrho - \overline{\varrho}]_{\mathrm{ess}}\mathbf{U}_n\rangle \cdot (\mathbf{U} - \mathbf{U}_n)(t,\cdot)\,\mathrm{d}x \\
&\qquad + \int_\Omega \langle \nu_{t,x}; [\mathbf{m}]_{\mathrm{res}} - [\varrho - \overline{\varrho}]_{\mathrm{res}}\mathbf{U}_n\rangle \cdot (\mathbf{U} - \mathbf{U}_n)(t,\cdot)\,\mathrm{d}x;
\end{aligned}
$$

since

$$\langle \nu_{(t,\cdot)}; [\mathbf{m}]_{\mathrm{ess}} - [\varrho - \overline{\varrho}]_{\mathrm{ess}}\mathbf{U}_n \rangle, \ (\mathbf{U} - \mathbf{U}_n)(t,\cdot) \in L^2(\Omega; \mathbb{R}^d)$$

we can apply Hölder's inequality (1.1.1) with $r = 1$, $p = 2$ and $q = 2$ to get

$$\begin{aligned}
&\int_\Omega \langle \nu_{t,x}; [\mathbf{m}]_{\mathrm{ess}} - [\varrho - \overline{\varrho}]_{\mathrm{ess}}\mathbf{U}_n \rangle \cdot (\mathbf{U} - \mathbf{U}_n)(t,\cdot) \,\mathrm{d}x \\
&\quad \leq \sup_{t \in [0,T]} \|\langle \nu_{(t,\cdot)}; [\mathbf{m}]_{\mathrm{ess}} - [\varrho - \overline{\varrho}]_{\mathrm{ess}}\mathbf{U}_n \rangle\|_{L^2(\Omega;\mathbb{R}^d)} \|(\mathbf{U} - \mathbf{U}_n)(t,\cdot)\|_{L^2(\Omega;\mathbb{R}^d)} \\
&\quad \lesssim \sup_{t \in [0,T]} \|(\mathbf{U} - \mathbf{U}_n)(t,\cdot)\|_{H^m(\Omega;\mathbb{R}^d)} \\
&\quad \lesssim \varepsilon.
\end{aligned}$$

We also have that

$$\langle \nu_{(t,\cdot)}; [\varrho - \overline{\varrho}]_{\mathrm{res}}\mathbf{U}_n \rangle \in L^\gamma(K); \mathbb{R}^d)$$

with $K$ compact and since $\gamma > \frac{2\gamma}{\gamma+1}$ we obtain

$$\langle \nu_{(t,\cdot)}; [\mathbf{m}]_{\mathrm{res}} - [\varrho - \overline{\varrho}]_{\mathrm{res}}\mathbf{U}_n \rangle \in L^{\frac{2\gamma}{\gamma+1}}(\Omega; \mathbb{R}^d);$$

using the embedding of the Sobolev space into the Hölder one (1.1.11), we get that $(\mathbf{U} - \mathbf{U}_n)(t,\cdot) \in L^\infty(\Omega;\mathbb{R}^d)$ and hence, due to the interpolation inequality (1.1.3),

$$(\mathbf{U} - \mathbf{U}_n)(t,\cdot) \in L^p(\Omega;\mathbb{R}^d) \quad \text{for all } p \in [2,\infty].$$

Since $\frac{2\gamma}{\gamma-1} > 2$, we can again apply Hölder's inequality (1.1.1) with $r = 1$, $p = \frac{2\gamma}{\gamma+1}$ and $q = \frac{2\gamma}{\gamma-1}$ to get

$$\begin{aligned}
&\int_\Omega \langle \nu_{t,x}; [\mathbf{m}]_{\mathrm{res}} - [\varrho - \overline{\varrho}]_{\mathrm{res}}\mathbf{U}_n \rangle \cdot (\mathbf{U} - \mathbf{U}_n)(t,\cdot) \,\mathrm{d}x \\
&\quad \leq \sup_{t \in [0,T]} \|\langle \nu_{(t,\cdot)}; [\mathbf{m}]_{\mathrm{res}} - [\varrho - \overline{\varrho}]_{\mathrm{res}}\mathbf{U}_n \rangle\|_{L^{\frac{2\gamma}{\gamma+1}}(\Omega;\mathbb{R}^d)} \|(\mathbf{U} - \mathbf{U}_n)(t,\cdot)\|_{L^{\frac{2\gamma}{\gamma-1}}(\Omega;\mathbb{R}^d)} \\
&\quad \lesssim \sup_{t \in [0,T]} \|(\mathbf{U} - \mathbf{U}_n)(t,\cdot)\|_{H^m(\Omega;\mathbb{R}^d)} \\
&\quad \lesssim \varepsilon.
\end{aligned}$$

On the other side, we can apply Hölder's inequality (1.1.1) with $r = 1$, $p = 2$ and $q = 2$ to obtain

$$\begin{aligned}
\int_\Omega \mathbf{U}_n \cdot (\mathbf{U} - \mathbf{U}_n)(t,\cdot) \,\mathrm{d}x &\leq \sup_{t \in [0,T]} \|\mathbf{U}_n(t,\cdot)\|_{L^2(\Omega;\mathbb{R}^d)} \|(\mathbf{U} - \mathbf{U}_n)(t,\cdot)\|_{L^2(\Omega;\mathbb{R}^d)} \\
&\lesssim \sup_{t \in [0,T]} \|(\mathbf{U} - \mathbf{U}_n)(t,\cdot)\|_{H^m(\Omega;\mathbb{R}^d)} \\
&\lesssim \varepsilon.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_\Omega \langle \nu_{t,x}; \varrho - \overline{\varrho}\rangle |\mathbf{U} - \mathbf{U}_n|^2(t,\cdot)\,\mathrm{d}x \\
&\quad = \int_\Omega \langle \nu_{t,x}; [\varrho - \overline{\varrho}]_{\text{ess}} + [\varrho - \overline{\varrho}]_{\text{res}}\rangle |\mathbf{U} - \mathbf{U}_n|^2(t,\cdot)\,\mathrm{d}x \\
&\quad \leq \sup_{t\in[0,T]} \|\langle \nu_{(t,\cdot)}; [\varrho - \overline{\varrho}]_{\text{ess}}\rangle\|_{L^2(\Omega;\mathbb{R}^d)} \|(\mathbf{U} - \mathbf{U}_n)(t,\cdot)\|_{L^4(\Omega;\mathbb{R}^d)} \\
&\quad + \sup_{t\in[0,T]} \|\langle \nu_{(t,\cdot)}; [\varrho - \overline{\varrho}]_{\text{res}}\rangle\|_{L^\gamma(\Omega;\mathbb{R}^d)} \|(\mathbf{U} - \mathbf{U}_n)(t,\cdot)\|_{L^{\frac{2\gamma}{\gamma-1}}(\Omega;\mathbb{R}^d)} \\
&\quad \lesssim \varepsilon,
\end{aligned}$$

and

$$\int_\Omega |\mathbf{U} - \mathbf{U}_n|^2(t,\cdot)\,\mathrm{d}x \leq \sup_{t\in[0,T]} \|(\mathbf{U} - \mathbf{U}_n)(t,\cdot)\|_{H^m(\Omega;\mathbb{R}^d)} < \varepsilon.$$

Summarizing, so far we proved that there exists a positive constant $C$ such that

$$\int_\Omega \left\langle \nu_{t,x}; \frac{1}{2\varrho}|\mathbf{m} - \varrho\mathbf{U}|^2 \right\rangle \mathrm{d}x \leq \int_\Omega \left\langle \nu_{t,x}; \frac{1}{2\varrho}|\mathbf{m} - \varrho\mathbf{U}_n|^2 \right\rangle \mathrm{d}x + C\varepsilon.$$

Proceeding in a similar way, we can write

$$\begin{aligned}
&\int_\Omega \langle \nu_{t,x}; P(\varrho) - P'(r)(\varrho - r) - P(r)\rangle\,\mathrm{d}x \\
&\quad = \int_\Omega \langle \nu_{t,x}; P(\varrho) - P'(r_n)(\varrho - r_n) - P(r_n)\rangle\,\mathrm{d}x \\
&\quad + \int_\Omega \langle \nu_{t,x}; P'(r_n)(\varrho - r_n) - P'(r)(\varrho - r)\rangle\,\mathrm{d}x - \int_\Omega [P(r) - P(r_n)](t,\cdot)\,\mathrm{d}x \\
&\quad = \int_\Omega \langle \nu_{t,x}; P(\varrho) - P'(r_n)(\varrho - r_n) - P(r_n)\rangle\,\mathrm{d}x \\
&\quad + \int_\Omega [P(r_n) - P'(r)(r_n - r) + P(r)](t,\cdot)\,\mathrm{d}x - \int_\Omega \langle \nu_{t,x}; [P'(r) - P'(r_n)](\varrho - r_n)\rangle\,\mathrm{d}x \\
&\quad = \int_\Omega \langle \nu_{t,x}; P(\varrho) - P'(r_n)(\varrho - r_n) - P(r_n)\rangle\,\mathrm{d}x \\
&\quad + \frac{P''(\xi_1)}{2} \int_\Omega (r - r_n)^2(t,\cdot)\,\mathrm{d}x - P''(\xi_2) \int_\Omega \langle \nu_{t,x}; \varrho - \overline{\varrho}\rangle (r - r_n)\,\mathrm{d}x \\
&\quad + P''(\xi_2) \int_\Omega (r_n - \overline{\varrho})(r - r_n)(t,\cdot)\,\mathrm{d}x.
\end{aligned}$$

We can now focus on the last three terms: the first one is simply bounded as follows

$$\int_\Omega (r - r_n)^2(t,\cdot)\,\mathrm{d}x \leq \sup_{t\in[0,T]} \|(r - r_n)(t,\cdot)\|_{H^m(\Omega)} < \varepsilon;$$

the second one can be rewritten as

$$\begin{aligned}
&\int_\Omega \langle \nu_{t,x}; \varrho - \overline{\varrho} \rangle (r - r_n)(t,\cdot)\, \mathrm{d}x \\
&\quad = \int_\Omega \langle \nu_{t,x}; [\varrho - \overline{\varrho}]_{\mathrm{ess}} \rangle (r - r_n)(t,\cdot)\, \mathrm{d}x + \int_\Omega \langle \nu_{t,x}; [\varrho - \overline{\varrho}]_{\mathrm{res}} \rangle (r - r_n)(t,\cdot)\, \mathrm{d}x \\
&\quad \leq \sup_{t\in[0,T]} \| \langle \nu_{(t,\cdot)}; [\varrho - \overline{\varrho}]_{\mathrm{ess}} \rangle \|_{L^2(\Omega;\mathbb{R}^d)} \| (r - r_n)(t,\cdot) \|_{L^2(\Omega)} \\
&\quad + \sup_{t\in[0,T]} \| \langle \nu_{(t,\cdot)}; [\varrho - \overline{\varrho}]_{\mathrm{res}} \rangle \|_{L^\gamma(\Omega;\mathbb{R}^d)} \| (r - r_n)(t,\cdot) \|_{L^{\frac{\gamma}{\gamma-1}}(\Omega)} \\
&\quad \lesssim \varepsilon;
\end{aligned}$$

notice that, if $\gamma \in (1,2)$ we use the same argument as before while if $\gamma \in [2,\infty)$ we have to use the Sobolev embedding in the Lebesgue spaces (1.1.8). For the last term we can use Hölder's inequality (1.1.1) with $r = 1$, $p = 2$ and $q = 2$ to get

$$\begin{aligned}
\int_\Omega (r_n - \overline{\varrho})(r - r_n)(t,\cdot)\, \mathrm{d}x &\leq \sup_{t\in[0,T]} \| (r_n - \overline{\varrho})(t,\cdot) \|_{L^2(\Omega)} \| (r - r_n)(t,\cdot) \|_{L^2(\Omega)} \\
&\lesssim \sup_{t\in[0,T]} \| (r - r_n)(t,\cdot) \|_{H^m(\Omega)} \\
&\lesssim \varepsilon.
\end{aligned}$$

Again, summarizing, we proved the existence of a positive constant $C$ such that

$$\begin{aligned}
&\int_\Omega \langle \nu_{t,x}; P(\varrho) - P'(r)(\varrho - r) - P(r) \rangle\, \mathrm{d}x \\
&\quad \leq \int_\Omega \langle \nu_{t,x}; P(\varrho) - P'(r_n)(\varrho - r_n) - P(r_n) \rangle\, \mathrm{d}x + C\varepsilon.
\end{aligned}$$

Repeating the same steps for each term that appears in the relative energy inequality and introducing the operator

$$\begin{aligned}
\mathcal{L}(\nu|r,\mathbf{U})(\tau) &= a \int_0^\tau \int_\Omega \left\langle \nu_{t,x}; \frac{\mathbf{m}}{\varrho} \cdot (\mathbf{m} - \varrho\mathbf{U}) \right\rangle \mathrm{d}x\mathrm{d}t + \mathcal{D}(\tau) \\
&+ \int_0^\tau \int_\Omega \langle \nu_{t,x}; \mathbf{m} - \varrho\mathbf{U} \rangle \cdot [\partial_t \mathbf{U} + \nabla_x \mathbf{U} \cdot \mathbf{U}]\, \mathrm{d}x\mathrm{d}t \\
&- \int_0^\tau \int_\Omega \left\langle \nu_{t,x}; \mathbb{1}_{\varrho>0} \frac{(\mathbf{m} - \varrho\mathbf{U}) \otimes (\varrho\mathbf{U} - \mathbf{m})}{\varrho} \right\rangle : \nabla_x \mathbf{U}\, \mathrm{d}x\mathrm{d}t \\
&+ \int_0^\tau \int_\Omega \langle \nu_{t,x}; p(\varrho) - p(r) \rangle\, \mathrm{div}_x\, \mathbf{U}\, \mathrm{d}x\mathrm{d}t \\
&+ \int_0^\tau \int_\Omega [\langle \nu_{t,x}; (\varrho - r)\partial_t P'(r) + (\mathbf{m} - r\mathbf{U}) \cdot \nabla_x P'(r) \rangle\, \mathrm{d}x\mathrm{d}t \\
&+ \int_0^\tau \int_{\overline{\Omega}} \nabla_x \mathbf{U} : \mathrm{d}\mu_m + \int_0^\tau \int_{\overline{\Omega}} \mathbf{U} \cdot \nabla_x \mathbf{U} \cdot \mathrm{d}\mu_c - \int_0^\tau \int_{\overline{\Omega}} \nabla_x P'(r) \cdot \mathrm{d}\mu_c,
\end{aligned}$$

we have

$$[\mathcal{E}(\nu|r,\mathbf{U})(t)]_{t=0}^{t=\tau} + \mathcal{L}(\nu|r,\mathbf{U})(\tau) \leq [\mathcal{E}(\nu|r_n,\mathbf{U}_n)(t)]_{t=0}^{t=\tau} + \mathcal{L}(\nu|r_n,\mathbf{U}_n)(\tau) + C\varepsilon \leq C\varepsilon,$$

for some positive constant C, since for a test function we already proved that the relative energy inequality holds which is equivalent in saying that

$$[\mathcal{E}(\nu|r_n, \mathbf{U}_n)(t)]_{t=0}^{t=\tau} + \mathcal{L}(\nu|r_n, \mathbf{U}_n)(\tau) \leq 0.$$

By the arbitrary of $\varepsilon$ we can conclude that the relative energy inequality (2.6.2) holds for $[r, \mathbf{U}]$ as in our hypothesis. □

### 2.6.4 *Vanishing of the relative energy*

We are now ready to prove one of the main result: the weak-strong uniqueness principle.

**Lemma 2.6.8.** *Let* $[r, \mathbf{U}]$ *be a strong solution of the compressible Euler system with damping* (2.1.1)–(2.1.6) *such that, for a fixed* $m > \frac{d}{2} + 1$,

$$\begin{aligned} r - \overline{\varrho} &\in C([0,T]; H^m(\Omega)) \\ \mathbf{U} &\in C([0,T]; H^m(\Omega; \mathbb{R}^d)) \end{aligned}$$

*where* $r > 0$ *and the far field density* $\overline{\varrho} > 0$ *if the domain* $\Omega$ *is unbounded. Let* $\{\nu_{t,x}\}_{(t,x)\in(0,T)\times\Omega}$ *be a dissipative measure-valued solution of the same system (in terms of* $\varrho$ *and the momentum* $\mathbf{m}$*), with the same far field density* $\overline{\varrho}$*, concentration measures* $\mu_c$*,* $\mu_m$ *and dissipation defect* $\mathcal{D}$ *in the sense of Definition 2.5.1, such that*

$$\nu_{0,x} = \delta_{r(0,x),(r\mathbf{U})(0,x)} \quad \text{for a.e. } x \in \Omega.$$

*Then* $\mathcal{D}(\tau) = 0$ *and* $\mathcal{E}(\tau) = 0$ *for all* $\tau \in (0,T)$*, where the relative energy functional* $\mathcal{E}$ *is defined as in* (2.6.1).

*Proof.* We can use the fact that $[r, \mathbf{U}]$ is a strong solution: from the momentum equation (2.1.2) we can deduce that

$$\partial_t \mathbf{U} + \mathbf{U} \cdot \nabla_x \mathbf{U} = -\frac{1}{r} \nabla_x p(r) - a\mathbf{U} = -P''(r)\nabla_x r - a\mathbf{U} = -\nabla_x P'(r) - a\mathbf{U};$$

substituting in (2.6.2), we get

$$\begin{aligned} [\mathcal{E}(\nu|r, \mathbf{U})]_{t=0}^{t=\tau} &+ a \int_0^\tau \int_\Omega \left\langle \nu_{t,x}; \frac{1}{2\varrho} |\mathbf{m} - \varrho \mathbf{U}|^2 \right\rangle \mathrm{d}x\mathrm{d}t + \mathcal{D}(\tau) \\ &\leq \int_0^\tau \int_\Omega \left\langle \nu_{t,x}; \mathbb{1}_{\varrho>0} \frac{(\mathbf{m} - \varrho\mathbf{U}) \otimes (\varrho\mathbf{U} - \mathbf{m})}{\varrho} \right\rangle : \nabla_x \mathbf{U} \, \mathrm{d}x\mathrm{d}t \\ &- \int_0^\tau \int_\Omega \langle \nu_{t,x}; p(\varrho) - p(r) \rangle \operatorname{div}_x \mathbf{U} \, \mathrm{d}x\mathrm{d}t \\ &- \int_0^\tau \int_\Omega [\langle \nu_{t,x}; P''(r)(\varrho - r)[\partial_t r + \nabla_x r \cdot \mathbf{U}] \rangle \, \mathrm{d}x\mathrm{d}t \\ &- \int_0^\tau \int_{\overline{\Omega}} \nabla_x \mathbf{U} : \mathrm{d}\mu_m + \int_0^\tau \int_{\overline{\Omega}} \mathbf{U} \cdot \nabla_x \mathbf{U} \cdot \mathrm{d}\mu_c - \int_0^\tau \int_{\overline{\Omega}} \nabla_x P'(r) \cdot \mathrm{d}\mu_c. \end{aligned}$$

From the continuity equation (2.1.1) we also have

$$\partial_t r + \nabla_x r \cdot \mathbf{U} = -r \operatorname{div}_x \mathbf{U},$$

and thus, knowing that $rP''(r) = p'(r)$, we get

$$
\begin{aligned}
[\mathcal{E}(\nu|r,\mathbf{U})]_{t=0}^{t=\tau} + a\int_0^\tau \int_\Omega \left\langle \nu_{t,x}; \frac{1}{2\varrho}|\mathbf{m}-\varrho\mathbf{U}|^2 \right\rangle \,\mathrm{d}x\mathrm{d}t + \mathcal{D}(\tau) \\
\leq \int_0^\tau \int_\Omega \left\langle \nu_{t,x}; \mathbb{1}_{\varrho>0} \frac{(\mathbf{m}-\varrho\mathbf{U})\otimes(\varrho\mathbf{U}-\mathbf{m})}{\varrho} \right\rangle : \nabla_x \mathbf{U} \,\mathrm{d}x\mathrm{d}t \\
- \int_0^\tau \int_\Omega \langle \nu_{t,x}; p(\varrho) - p'(r)(\varrho - r) - p(r) \rangle \operatorname{div}_x \mathbf{U} \,\mathrm{d}x\mathrm{d}t \\
- \int_0^\tau \int_{\overline{\Omega}} \nabla_x \mathbf{U} : \mathrm{d}\mu_m + \int_0^\tau \int_{\overline{\Omega}} \mathbf{U}\cdot\nabla_x\mathbf{U}\cdot \mathrm{d}\mu_c - \int_0^\tau \int_{\overline{\Omega}} \nabla_x P'(r) \cdot \mathrm{d}\mu_c.
\end{aligned}
$$

Finally, using the fact that the initial data are the same and thus $\mathcal{E}(\nu|r,\mathbf{U})(0) = 0$, we end up to

$$
\begin{aligned}
\mathcal{E}(\nu|r,\mathbf{U})(\tau) + a\int_0^\tau \int_\Omega \left\langle \nu_{t,x}; \frac{1}{2\varrho}|\mathbf{m}-\varrho\mathbf{U}|^2 \right\rangle \mathrm{d}x\mathrm{d}t + \mathcal{D}(\tau) \\
\leq \int_0^\tau \int_\Omega \left\langle \nu_{t,x}; \left| \mathbb{1}_{\varrho>0} \frac{(\mathbf{m}-\varrho\mathbf{U})\otimes(\varrho\mathbf{U}-\mathbf{m})}{\varrho} \right| \right\rangle |\nabla_x \mathbf{U}| \,\mathrm{d}x\mathrm{d}t \\
+ \int_0^\tau \int_\Omega \langle \nu_{t,x}; |p(\varrho) - p'(r)(\varrho - r) - p(r)| \rangle |\operatorname{div}_x \mathbf{U}| \,\mathrm{d}x\mathrm{d}t \\
+ \int_0^\tau \int_{\overline{\Omega}} |\nabla_x \mathbf{U}| \cdot \mathrm{d}|\mu_m| + \int_0^\tau \int_{\overline{\Omega}} |\mathbf{U}\cdot\nabla_x\mathbf{U}| \cdot \mathrm{d}|\mu_c| \\
+ \int_0^\tau \int_{\overline{\Omega}} |\nabla_x P'(r)| \cdot \mathrm{d}|\mu_c|.
\end{aligned}
$$

Since $\mathbf{U}$ and $P'(r) - P(\overline{\varrho})$ are $L^\infty$-functions, we can control terms $|\nabla_x \mathbf{U}|$, $|\operatorname{div}_x \mathbf{U}|$, $|\mathbf{U}\cdot\nabla_x\mathbf{U}|$ and $|\nabla_x P'(r)|$ by some constants. It is also obvious that there exist a constant $c_1$ such that

$$
\left| \mathbb{1}_{\varrho>0} \frac{(\mathbf{m}-\varrho\mathbf{U})\otimes(\varrho\mathbf{U}-\mathbf{m})}{\varrho} \right| \leq \frac{c_1}{2\varrho}|\mathbf{m}-\varrho\mathbf{U}|^2,
$$

and a constant $c_2$ such that

$$
|p(\varrho) - p'(r)(\varrho - r) - p(r)| \leq c_2(P(\varrho) - P'(r)(\varrho - r) - P(r)).
$$

Thus

$$
\mathcal{E}(\varrho,\mathbf{m}|r,\mathbf{U})(\tau) + \mathcal{D}(\tau) \leq c\int_0^\tau [\mathcal{E}(\varrho,\mathbf{m}|r,\mathbf{U})(t) + \mathcal{D}(t)]\,\mathrm{d}t.
$$

By Gronwall lemma 1.5.2 we obtain

$$
\mathcal{E}(\varrho,\mathbf{m}|r,\mathbf{U})(\tau) + \mathcal{D}(\tau) \leq 0 \quad \text{for all } \tau \in (0,T).
$$

But since $\mathcal{E}, \mathcal{D} \geq 0$ this implies $\mathcal{D}(\tau) = 0$ and $\mathcal{E}(\tau) = 0$ for all $\tau \in (0,T)$. □

## 2.7 VANISHING VISCOSITY LIMIT

Unifying the two main results achieved in this chapter, namely Theorems 2.5.2 and 2.6.1, we conclude the first part of the present thesis proving our last theorem: the solutions of the Navier-Stokes system converge in the zero viscosity limit to the strong solution of the Euler system with damping on the life span of the latter.

**Theorem 2.7.1.** *Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ be a domain with compact Lipschitz boundary and $\overline{\varrho} > 0$ be a given far field density if $\Omega$ is unbounded. Let $\{\varrho_R, \mathbf{m}_R = \varrho_R \mathbf{u}_R\}_{R>0}$ be a family of dissipative weak solutions to the Navier-Stokes system* (2.1.7) – (2.1.12) *in*

$$(0, T) \times \Omega_R,\ \Omega_R = \Omega \cap B_R.$$

*Let the corresponding initial data $\varrho_0$, $\boldsymbol{m}_0$ be independent of $R$ satisfying*

$$\varrho_0 > 0,\ \int_\Omega \left[ \frac{1}{2} \frac{|\mathbf{m}_0|^2}{\varrho_0} + P(\varrho_0) - P'(\overline{\varrho})(\varrho_0 - \overline{\varrho}) - P(\overline{\varrho}) \right] \mathrm{d}x \leq E_0.$$

*Moreover, suppose that $\left( \varrho_0 - \overline{\varrho}, \frac{\mathbf{m}_0}{\varrho_0} \right) \in H^m(\Omega)$, $m > \frac{d}{2} + 1$, and that*

$$\begin{aligned} r - \overline{\varrho} &\in C([0, T]; H^m(\Omega)) \\ \mathbf{U} &\in C([0, T]; H^m(\Omega; \mathbb{R}^d)) \end{aligned}$$

*is the strong solution to the Euler system with damping* (2.1.1)–(2.1.6) *with initial data $\left( \varrho_0, \frac{\mathbf{m}_0}{\varrho_0} \right)$.*
*Then*

$$\begin{aligned} \varrho_R - \overline{\varrho} \to r - \overline{\varrho} \quad &\text{in } C_{\text{weak}}([0, T]; L^2 + L^\gamma(\Omega)) \text{ and in } L^1((0, T) \times K), \\ \mathbf{m}_R = \varrho_R \mathbf{u}_R \to r\mathbf{U} \quad &\text{in } C_{\text{weak}}([0, T]; L^2 + L^{\frac{2\gamma}{\gamma+1}}(\Omega; \mathbb{R}^d)) \text{ and in } L^1((0, T) \times K; \mathbb{R}^d) \end{aligned}$$

*as $R \to \infty$, for any compact $K \subset \Omega$.*

*Proof.* In the proof of Theorem 2.5.2, we showed that

$$\begin{aligned} \varrho_R - \overline{\varrho} \to \langle \nu;\ \varrho - \overline{\varrho} \rangle \quad &\text{in } C_{\text{weak}}([0, T]; L^2 + L^\gamma(\Omega)), \\ \mathbf{m}_R \to \langle \nu;\ \mathbf{m} \rangle \quad &\text{in } C_{\text{weak}}([0, T]; L^2 + L^{\frac{2\gamma}{\gamma+1}}(\Omega; \mathbb{R}^d)), \end{aligned}$$

where

$$\begin{aligned} \nu_{t,x} : (t, x) \in (0, T) \times \Omega \to \mathcal{P}([0, \infty) \times \mathbb{R}^d), \\ \nu \in L^\infty_{\text{weak}}((0, T) \times \Omega; \mathcal{P}([0, \infty) \times \mathbb{R}^d)), \end{aligned}$$

is the Young measure associated to the sequence $\{\varrho_R, \mathbf{m}_R\}_{R>0}$ but also the dissipative measure-valued solution to the Euler system with damping. Since

$$\nu_{0,x} = \delta_{\varrho_0(x), \mathbf{m}_0(x)} \quad \text{for a.e. } x \in \Omega,$$

we can apply Theorem 2.6.1 to get that

$$\nu_{t,x} = \delta_{r(t,x), r\mathbf{U}(t,x)} \quad \text{for a.e. } (t, x) \in (0, T) \times \Omega,$$

and hence we obtain the claim. ∎

Part II

# UNIQUENESS: SEMIFLOW SELECTION AND ITS APPLICATIONS TO FLUID DYNAMICS

# 3 ABSTRACT SETTING

Semiflow selection is an important tool when dealing with systems that lack uniqueness. In these contexts, a natural question is weather it is possible or not to construct a solution satisfying an important feature of systems with uniqueness, known as *semigroup* or *semiflow property*: letting the system run from time 0 to time $s$, restarting it and letting it run from time $s$ to time $t$ is equivalent in letting it run directly from time 0 to time $t$. The aim of this chapter is to prove the existence of a semiflow selection with range the *Skorokhod space* of càglàd, i.e. left–continuous and having right–hand limits, functions, introduced and studied in Section 1.3.

The chapter is organized as follows. In Section 3.1 we will focus on autonomous systems. More precisely, denoting with $X$ the phase space and with $\mathcal{T}$ the trajectory space, our goal is to prove the existence of a Borel-measurable map $\mathrm{U} : X \to \mathcal{T}$ satisfying the semigroup property: for any initial data $\mathrm{u}_0 \in X$, $\mathrm{U}(\mathrm{u}_0)$ is a solution of the system such that

$$\mathrm{U}(\mathrm{u}_0)(t_1 + t_2) = \mathrm{U}[\mathrm{U}(\mathrm{u}_0)(t_1)](t_2) \quad \text{for any } t_1, t_2 \geq 0,$$

cf. Theorem 3.1.2. In Section 3.2 we will focus on non-autonomous systems, for which we aim the prove the existence of a semiprocess $\{P_{t_0}\}_{t_0 \geq 0}$ such that, for any initial data $\mathrm{v}_0 \in X$, $P_{t_0}(\mathrm{v}_0)$ is a solution of the system satisfying

$$P_{t_0}(\mathrm{v}_0)(t_2) = P_{t_1}[P_{t_0}(\mathrm{v}_0)(t_1)](t_2) \quad \text{for every } t_0 \leq t_1 \leq t_2,$$

cf. Theorem 3.2.3. Considering the external force and the boundary condition not as quantities fixed from the beginning but as part of data, we will be able to show that some properties of the data can be transferred to the semiflow selection, cf. Proposition 3.2.4.

## 3.1 AUTONOMOUS SYSTEM

Let us first consider a general *autonomous* system:

$$\begin{cases} \partial_t \mathrm{u} + \mathrm{A}(\mathrm{u}) = 0 & \text{for } (t,x) \in (0,\infty) \times \Omega, \\ \mathrm{u} = 0 & \text{for } (t,x) \in (0,\infty) \times \partial\Omega, \\ \mathrm{u}(0) = \mathrm{u}_0. \end{cases} \tag{3.1.1}$$

The word "autonomous" refers to the fact that the time variable does not appear explicitly in the system. Consequently, (3.1.1) is time-shift invariant meaning that if $\mathrm{u} = \mathrm{u}(t,x)$ solves (3.1.1), $S_T \circ \mathrm{u}$ is a solution as well, where for every $T > 0$ the time-shift operator $S_T$ is defined as

$$S_T \circ \mathrm{u}(t) = \mathrm{u}(t + T) \quad \text{for all } t \geq 0.$$

In particular, in this context it is not important to specify the starting time $t_0 \geq 0$, which for simplicity can always be taken as $t_0 = 0$.

If there exists a unique solution for every fixed initial data, we can define a map U such that for every initial data $u_0$

$$\mathrm{U}(u_0)(t) \text{ is a solution of system (3.1.1) evaluated at time } t \geq 0.$$

If there exists a solution for every fixed initial data but it may not be unique, we can define a map U such that, among all the solutions arising from $u_0$, $\mathrm{U}(u_0)$ is the one satisfying the *semigroup property*; more precisely

$$\begin{gathered} \mathrm{U}(u_0)(t) \text{ is a solution of system (3.1.1) evaluated at time } t \geq 0 \text{ such that} \\ \mathrm{U}(u_0)(t_1 + t_2) = \mathrm{U}[\mathrm{U}(u_0)(t_1)](t_2) \text{ for every } t_1, t_2 \geq 0. \end{gathered}$$

We will refer to the procedure of finding a map satisfying such an important feature of systems with uniqueness as *semiflow selection*. The remaining part of this section will be dedicated to the proof of the existence of a semiflow selection for autonomous systems.

### 3.1.1 *Setting and main result*

In order to state the main result of this section, i.e. the existence of a semiflow selection, it is necessary to fix a proper setting. Let

- $H$ be a separable Hilbert space with a basis $\{e_k\}_{k\in\mathbb{N}}$;
- $X$ be the *phase space* associated to (3.1.1), which we suppose to be a closed convex subset of $H$;
- $\mathcal{T} = \mathfrak{D}([0,\infty); H)$ be the *trajectory space*, where $\mathfrak{D}([0,\infty); H)$ denotes the Skorokhod space of càglàd functions defined on $[0,\infty)$ and taking values in $H$, introduced and studied in Section 1.3.2;
- $\mathcal{U} : X \to 2^{\mathcal{T}}$ be the set-valued map that associates to every initial data $u_0 \in X$ the family of (classical, weak, measure-valued, ...) solutions to system (3.1.1) emanating from $u_0$; more precisely, for every $u_0 \in X$

$$\mathcal{U}(u_0) := \left\{ u \in \mathcal{T} : \begin{array}{l} \text{at any time } t \geq 0, \ u(t) \in X \text{ is the value} \\ \text{of a solution “in a certain sense” of} \\ \text{system (3.1.1) with initial data } u(0) = u_0 \end{array} \right\}. \tag{3.1.2}$$

  Furthermore, we suppose that $\mathcal{U}$ satisfies the following properties.

  (P1) *Non-emptiness*: for every $u_0 \in X$, $\mathcal{U}(u_0)$ is a non-empty subset of $\mathcal{T}$.

  (P2) *Compactness*: for every $u_0 \in X$, $\mathcal{U}(u_0)$ is a compact subset of $\mathcal{T}$.

  (P3) *Measurability*: the map $\mathcal{U} : X \to 2^{\mathcal{T}}$ is Borel-measurable.

  (P4) *Shift-invariance*: introducing the positive shift operator $S_T \circ u$ for every $T > 0$ and $u \in \mathcal{T}$ as
  $$S_T \circ u(t) := u(t+T), \quad \text{for all } t \geq 0, \tag{3.1.3}$$
  then, for any $T > 0$, $u_0 \in X$ and $u \in \mathcal{U}(u_0)$, we have
  $$S_T \circ u \in \mathcal{U}(u(T)).$$

(P5) *Continuation*: introducing the continuation operator $u_1 \cup_T u_2$ for every $T > 0$ and $u_1, u_2 \in \mathcal{T}$ as

$$u_1 \cup_T u_2(t) := \begin{cases} u_1(t) & \text{for } 0 \le t \le T, \\ u_2(t-T) & \text{for } t > T, \end{cases} \quad \text{for all } t \ge 0, \tag{3.1.4}$$

then, for any $T > 0$, $u_0 \in X$, $u_1 \in \mathcal{U}(u_0)$ and $u_2 \in \mathcal{U}(u_1(T))$, we have

$$u_1 \cup_T u_2 \in \mathcal{U}(u_0).$$

*Remark* 3.1.1. It is worth noticing that requirement $u(t) \in X$ for any $u_0 \in X$, $u \in \mathcal{U}(u_0)$ and any $t \ge 0$ in definition (3.1.2) is necessary in order to guarantee the validity of the shift-invariance and continuation properties.

We are now ready to state the following result; the proof is postponed to the next subsection.

**Theorem 3.1.2.** *Let the mapping $\mathcal{U} : X \subseteq H \to 2^{\mathcal{T}}$ satisfy properties (P1)–(P5) stated above. Then, there exists a Borel-measurable map*

$$U : X \to \mathcal{T}, \quad U(u_0) \in \mathcal{U}(u_0) \textit{ for every } u_0 \in X,$$

*satisfying the semigroup property: for any $u_0 \in X$ and any $t_1, t_2 \ge 0$*

$$U(u_0)(t_1 + t_2) = U[U(u_0)(t_1)](t_2).$$

### 3.1.2 *Proof of the existence of a semiflow selection*

The idea of the proof of Theorem 3.1.2 is to reduce iteratively the set $\mathcal{U}(u_0)$ for a fixed $u_0 \in X$, selecting the minimum points of particular functionals in order to obtain finally a single point in $\mathcal{T}$, which will define $U(u_0)$. The procedure has been proposed by Cardona and Kapitanski [20] in the context of continuous trajectories and later adapted to more general setting by Breit, Feireisl and Hofmanová [14].

Let us consider the functionals $I_{\lambda,k,f} : \mathcal{T} \to \mathbb{R}$ defined for every $u \in \mathcal{T}$ as

$$I_{\lambda,k,f}(u) = \int_0^\infty e^{-\lambda t} f\left(\langle u(t); e_k\rangle\right) \mathrm{d}t, \tag{3.1.5}$$

where $\lambda > 0$, $\{e_k\}_{k\in\mathbb{N}}$ is a basis in $H$ and $f : \mathbb{R} \to \mathbb{R}$ is continuous and bounded; this choice is justified by the fact that for a fixed $k \in \mathbb{N}$ we can see $I_{\lambda,k,f}$ as the Laplace transform of the function $f\left(\langle u(\cdot); e_k\rangle\right)$, an useful interpretation for the proof of the existence of the semiflow U.

The following result asserts the continuity of the map $I_{\lambda,k,f}$.

**Proposition 3.1.3.** *Let $\lambda > 0$, $k \in \mathbb{N}$ and $f : \mathbb{R} \to \mathbb{R}$ be continuous and bounded. Then, the map $I_{\lambda,k,f} : \mathcal{T} \to \mathbb{R}$ defined in (3.1.5) is continuous.*

*Proof.* Let $\{u_n\}_{n\in\mathbb{N}}$ in $\mathcal{T}$ be such that

$$u_n \to u \quad \text{in } \mathcal{T} = \mathfrak{D}([0,\infty); H) \tag{3.1.6}$$

as $n \to \infty$. By Proposition 1.3.9, (3.1.6) implies in particular that

$$\langle \mathrm{u}_n(t); e_k\rangle \to \langle \mathrm{u}(t); e_k\rangle \quad \text{for a.e. } t \in (0, \infty)$$

as $n \to \infty$. Due to continuity and boundedness of $f$, we get

$$I_{\lambda,k,f}(\mathrm{u}_n) = \int_0^\infty e^{-\lambda t} f(\langle \mathrm{u}_n(t); e_k\rangle)\, \mathrm{d}t \to \int_0^\infty e^{-\lambda t} f(\langle \mathrm{u}(t); e_k\rangle)\, \mathrm{d}t = I_{\lambda,k,f}(\mathrm{u}),$$

i.e., what we wanted to prove. □

We define the selection mapping for every $\mathrm{u}_0 \in X$ as

$$I_{\lambda,k,f} \circ \mathcal{U}(\mathrm{u}_0) = \{\mathrm{u} \in \mathcal{U}(\mathrm{u}_0) : \; I_{\lambda,k,f}(\mathrm{u}) \le I_{\lambda,k,f}(\widetilde{\mathrm{u}}) \text{ for all } \widetilde{\mathrm{u}} \in \mathcal{U}(\mathrm{u}_0)\}. \tag{3.1.7}$$

Notice, in particular, that the minimum exists since $I_{\lambda,k,f}$ is continuous on $\mathcal{T}$ and the set $\mathcal{U}(\mathrm{u}_0)$ is compact.

In the following result we will show that the set–valued mapping $I_{\lambda,k,f} \circ \mathcal{U}$ satisfies properties (P1)–(P5); the proof is an adaptation of Proposition 5.1 in [14].

**Proposition 3.1.4.** *Let $\lambda > 0$, $k \in \mathbb{N}$ and $f : \mathbb{R} \to \mathbb{R}$ be continuous and bounded. If the set–valued map $\mathcal{U} : X \to 2^{\mathcal{T}}$ satisfies properties (P1)–(P5), then the set–valued map*

$$I_{\lambda,k,f} \circ \mathcal{U} : X \to 2^{\mathcal{T}},$$

*defined for any $\mathrm{u}_0 \in X$ as in (3.1.7), enjoys properties (P1)–(P5) as well.*

*Proof.* (P1) Since the map $I_{\lambda,k,f}$ is continuous and the set $\mathcal{U}(\mathrm{u}_0)$ is non-empty and compact for any $\mathrm{u}_0 \in X$, we get that $I_{\lambda,k,f} \circ \mathcal{U}(\mathrm{u}_0)$ is non-empty for any $\mathrm{u}_0 \in X$.

(P2) As the set of minima of a continuous function is closed (it is the counterpart of a point) we get that $I_{\lambda,k,f} \circ \mathcal{U}(\mathrm{u}_0) \subseteq \mathcal{U}(\mathrm{u}_0)$ is closed in a compact set and hence compact itself for any $\mathrm{u}_0 \in X$.

(P3) Notice that, since $I_{\lambda,k,f} \circ \mathcal{U}(\mathrm{u}_0)$ is a compact subset of the separable metric space $\mathcal{T}$ for any $\mathrm{u}_0 \in X$, the Borel-measurability of the multivalued mapping

$$\mathrm{u}_0 \in X \mapsto I_{\lambda,k,f} \circ \mathcal{U}(\mathrm{u}_0) \in \mathcal{K} \subset 2^{\mathcal{T}}$$

corresponds to measurability with respect to the Hausdorff metric on the space of all compact subsets of $\mathcal{T}$.

In other words, let $d_H$ be the Hausdorff metric on the subspace $\mathcal{K} \subset 2^{\mathcal{T}}$ of all the compact subsets of $\mathcal{T}$:

$$d_H(K_1, K_2) = \inf_{\varepsilon \ge 0}\{K_1 \subset V_\varepsilon(K_2) \text{ and } K_2 \subset V_\varepsilon(K_1)\} \quad \text{for all } K_1, K_2 \in \mathcal{K},$$

where $V_\varepsilon(A)$ is the $\varepsilon$-neighborhood of the set $A$ in the topology of $\mathcal{T}$; then, it is enough to show that the mapping defined for all $K \in \mathcal{K}$ as

$$\mathcal{I}_{\lambda,k,f}[K] = \{z \in K | \; I_{\lambda,k,f}(z) \le I_{\lambda,k,f}(\tilde{z}) \text{ for all } \tilde{z} \in K\} = \left\{ z \in K | \; \min_{z \in K} I_{\lambda,k,f}(z) \right\}$$

is continuous as a mapping on $\mathcal{K}$ endowed with the Hausdorff metric $d_H$. In particular we want to show that if $K_n \xrightarrow{d_H} K$ with $K_n, K \in \mathcal{K}$ then $\mathcal{I}_{\lambda,k,f}[K_n] \xrightarrow{d_H} \mathcal{I}_{\lambda,k,f}[K]$ as $n \to \infty$. More precisely, it is enough to show that for every $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon)$ such that

$$\mathcal{I}_{\lambda,k,f}[K_n] \subset V_\varepsilon(\mathcal{I}_{\lambda,k,f}[K]) \quad \text{and} \quad \mathcal{I}_{\lambda,k,f}[K] \subset V_\varepsilon(\mathcal{I}_{\lambda,k,f}[K_n]) \tag{3.1.8}$$

for all $n \geq n_0$. First of all, notice that by the continuity of $I_{\lambda,k,f}$ we have

$$\min_{K_n} I_{\lambda,k,f} \to \min_K I_{\lambda,k,f} \quad \text{as } n \to \infty. \tag{3.1.9}$$

We start proving the first inclusion of (3.1.8). By contradiction, suppose that exists a sequence $\{z_n\}_{n\in\mathbb{N}}$ such that

$$z_n \in K_n, \quad I_{\lambda,k,f}(z_n) = \min_{K_n} I_{\lambda,k,f}, \quad z_n \to z \in K \setminus V_\varepsilon(\mathcal{I}_{\lambda,k,f}[K])$$

as $n \to \infty$; in particular, $I_{\lambda,k,f}(z) > \min_K I_{\lambda,k,f}$. By the continuity of $I_{\lambda,k,f}$ we have

$$\min_{K_n} I_{\lambda,k,f} = I_{\lambda,k,f}(z_n) \to I_{\lambda,k,f}(z) > \min_K I_{\lambda,k,f}$$

as $n \to \infty$; but this contradicts (3.1.9). Interchanging the roles of $K_n$ and $K$ we get the opposite inclusion in (3.1.8).

(P4) We want to prove the shift-invariance: for any $T > 0$, $\mathrm{u}_0 \in X$ and $\mathrm{u} \in I_{\lambda,k,f} \circ \mathcal{U}(\mathrm{u}_0)$

$$S_T \circ \mathrm{u} \in I_{\lambda,k,f} \circ \mathcal{U}(\mathrm{u}(T)).$$

Let $\mathrm{v} \in \mathcal{U}(\mathrm{u}(T))$; then, since in particular $\mathrm{u} \in \mathcal{U}(\mathrm{u}_0)$ and $\mathcal{U}$ satisfies property (P5), we get

$$\mathrm{u} \cup_T \mathrm{v} \in \mathcal{U}(\mathrm{u}_0).$$

From the choice of u, which minimizes $I_{\lambda,k,f}$ on $\mathcal{U}(\mathrm{u}_0)$, we obtain

$$I_{\lambda,k,f}(\mathrm{u}) \leq I_{\lambda,k,f}(\mathrm{u} \cup_T \mathrm{v}). \tag{3.1.10}$$

Hence, using (3.1.10) in the fourth line and the definition of $\cup_T$ in the fifth line,

$$\begin{aligned}
I_{\lambda,k,f}(S_T \circ \mathrm{u}) &= \int_0^\infty e^{-\lambda t} f\left(\langle \mathrm{u}(t+T); e_k\rangle\right) \mathrm{d}t \\
&= e^{\lambda T} \int_T^\infty e^{-\lambda s} f\left(\langle \mathrm{u}(s); e_k\rangle\right) \mathrm{d}s \\
&= e^{\lambda T} \left( I_{\lambda,k,f}(\mathrm{u}) - \int_0^T e^{-\lambda s} f\left(\langle \mathrm{u}(s); e_k\rangle\right) \mathrm{d}s \right) \\
&\leq e^{\lambda T} \left( I_{\lambda,k,f}(\mathrm{u} \cup_T \mathrm{v}) - \int_0^T e^{-\lambda s} f\left(\langle \mathrm{u}(s); e_k\rangle\right) \mathrm{d}s \right) \\
&= e^{\lambda T} \int_T^\infty e^{-\lambda s} f\left(\langle \mathrm{v}(s-T); e_k\rangle\right) \mathrm{d}s \\
&= e^{\lambda T} \int_0^\infty e^{-\lambda(t+T)} f\left(\langle \mathrm{v}(t); e_k\rangle\right) \mathrm{d}t \\
&= I_{\lambda,k,f}(\mathrm{v}).
\end{aligned}$$

As $\mathrm{v} \in \mathcal{U}(\mathrm{u}(T))$ was arbitrarily chosen, we obtained that $S_T \circ \mathrm{u}$ minimizes $I_{\lambda,k,f}$ on $\mathcal{U}(\mathrm{u}(T))$ and consequently belongs to $I_{\lambda,k,f} \circ \mathcal{U}(\mathrm{u}(T))$ for any $T > 0$.

(P5) We want to prove the continuation: for any $T > 0$, $u_0 \in X$, $u_1 \in I_{\lambda,k,f} \circ \mathcal{U}(u_0)$ and $u_2 \in I_{\lambda,k,f} \circ \mathcal{U}(u_1(T))$

$$u_1 \cup_T u_2 \in I_{\lambda,k,f} \circ \mathcal{U}(u_0).$$

Using the shift-invariance for $\mathcal{U}$ we obtain

$$S_T \circ u_1 \in \mathcal{U}(u_1(T));$$

since $u_2$ is a minimum of $I_{\lambda,k,f}$ on $\mathcal{U}(u_1(T))$ we get

$$I_{\lambda,k,f}(u_2) \leq I_{\lambda,k,f}(S_T \circ u_1). \tag{3.1.11}$$

Hence, using (3.1.11) in the fourth line,

$$\begin{aligned} I_{\lambda,k,f}(u_1 \cup_T u_2) &= \int_0^T e^{-\lambda t} f(\langle u_1(t); e_k\rangle)\, dt + \int_T^\infty e^{-\lambda t} f(\langle u_2(t-T); e_k\rangle)\, dt \\ &= \int_0^T e^{-\lambda t} f(\langle u_1(t); e_k\rangle)\, dt + e^{-\lambda T} \int_0^\infty e^{-\lambda s} f(\langle u_2(s); e_k\rangle)\, ds \\ &= \int_0^T e^{-\lambda t} f(\langle u_1(t); e_k\rangle)\, dt + e^{-\lambda T} I_{\lambda,k,f}(u_2) \\ &\leq \int_0^T e^{-\lambda t} f(\langle u_1(t); e_k\rangle)\, dt + e^{-\lambda T} I_{\lambda,k,f}(S_T \circ u_1) \\ &= \int_0^T e^{-\lambda t} f(\langle u_1(t); e_k\rangle)\, dt \\ &\quad + e^{-\lambda T} \int_0^\infty e^{-\lambda s} f(\langle u_1(s+T); e_k\rangle)\, ds \\ &= \int_0^T e^{-\lambda t} f(\langle u_1(t); e_k\rangle)\, dt + \int_T^\infty e^{-\lambda t} f(\langle u_1(t); e_k\rangle)\, dt \\ &= I_{\lambda,k,f}(u_1). \end{aligned}$$

On the other side, using the continuation property for $\mathcal{U}$ we know that $u_1 \cup_T u_2 \in \mathcal{U}(u_0)$ and, since $u_1$ is a minimum of $I_{\lambda,k,f}$ on $\mathcal{U}(u_0)$, we have

$$I_{\lambda,k,f}(u_1) \leq I_{\lambda,k,f}(u_1 \cup_T u_2).$$

The only possibility is then

$$I_{\lambda,k,f}(u_1 \cup_T u_2) = I_{\lambda,k,f}(u_1),$$

which in particular implies that $u_1 \cup_T u_2 \in I_{\lambda,k,f} \circ \mathcal{U}(u_0)$.

□

We are now ready to prove the existence of the semiflow selection U.

*Proof of Theorem 3.1.2.* Fixing a smooth, bounded, strictly increasing function $f : \mathbb{R} \to \mathbb{R}$ and a countable set $\{\lambda_j\}_{j\in\mathbb{N}}$ dense in $(0,\infty)$, we consider the functionals $I_{j,k} : \mathcal{T} \to \mathbb{R}$ such that

$$I_{j,k}(u) = \int_0^\infty e^{-\lambda_j t} f(\langle u(t); e_k\rangle)\, dt$$

for any $u \in \mathcal{T}$. Choosing an enumeration $\{j(i), k(i)\}_{i=1}^{\infty}$ of the all involved combinations of indices, we define the maps

$$\mathcal{U}^i := I_{j(i),k(i)} \circ \cdots \circ I_{j(1),k(1)} \circ \mathcal{U}, \quad i = 1, 2, \ldots$$

and

$$\mathcal{U}^{\infty} := \bigcap_{i=1}^{\infty} \mathcal{U}^i.$$

We will now show that the set-valued map

$$X \ni u_0 \mapsto \mathcal{U}^{\infty}(u_0) \in 2^{\mathcal{T}}$$

satisfies properties (P1)–(P5) as well.

(P1)-(P2) First, notice that for every fixed initial data $u_0 \in X$ the sets $\mathcal{U}^j(u_0)$ are nested:

$$\mathcal{U}^1(u_0) \supseteq \mathcal{U}^2(u_0) \supseteq \cdots \supseteq \mathcal{U}^j(u_0) \supseteq \ldots.$$

By Proposition 3.1.4 we obtain that $\mathcal{U}^1(u_0)$ is a non-empty compact subset of $\mathcal{T}$, and iterating this procedure we can deduce that the same holds for all $\mathcal{U}^j(u_0)$. Since $\mathcal{T}$ is a Hausdorff space, every compact set is closed; thus, the countable intersection of closed sets $\mathcal{U}^{\infty}(u_0)$ is a closed subset of the compact set $\mathcal{U}^1(u_0)$. Applying Cantor's intersection theorem 1.5.10 we get in particular that $\mathcal{U}^{\infty}(u_0)$ is a non-empty compact subset of $\mathcal{T}$ for every $u_0 \in X$ fixed.

(P3) As $\mathcal{U}^{\infty}$ is an intersection Borel-measurable maps, it is Borel-measurable itself.

(P4) In order to prove the shift-invariance property, let $u_0 \in X$ and $u \in \mathcal{U}^{\infty}(u_0)$; then, in particular $u \in \mathcal{U}^j(u_0)$ for every $j$. By Proposition 3.1.4, we can deduce that $\mathcal{U}^j$ satisfies the shift-invariance property for every $j$. This implies

$$S_T \circ u \in \mathcal{U}^j(u(T)), \text{ for all } j \text{ and all } T > 0.$$

Thus

$$S_T \circ u \in \mathcal{U}^{\infty}(u(T)), \text{ for all } T > 0.$$

(P5) In order to prove the continuation property, let $T > 0$, $u_0 \in X$, $u_1 \in \mathcal{U}^{\infty}(u_0)$ and $u_2 \in \mathcal{U}^{\infty}(u_1(T))$; then, in particular $u_1 \in \mathcal{U}^j(u_0)$ and $u_2 \in \mathcal{U}^j(u_1(T))$ for every $j$. By Proposition 3.1.4, we can deduce that $\mathcal{U}^j$ satisfies the continuation property for every $j$. This implies

$$u_1 \cup_T u_2 \in \mathcal{U}^j(u_0) \text{ for all } j.$$

Thus

$$u_1 \cup_T u_2 \in \mathcal{U}^{\infty}(u_0).$$

We now claim that for every $u_0 \in X$ the set $\mathcal{U}^{\infty}(u_0)$ is a singleton. Indeed, if $u_1, u_2 \in \mathcal{U}^{\infty}(u_0)$ for a fixed $u_0 \in X$, then

$$I_{j(i),k(i)}(u_1) = I_{j(i),k(i)}(u_2)$$

for all $i = 1, 2, \dots$. Since the integrals $I_{j(i),k(i)}$ can be seen as Laplace transforms of the functions $f(\langle \mathrm{u}(\cdot); e_k\rangle)$, we can apply Lerch's theorem 1.5.7 to deduce that

$$f(\langle \mathrm{u}_1(t); e_k\rangle) = f(\langle \mathrm{u}_2(t); e_k\rangle)$$

for all $k \in \mathbb{N}$ and a.e. $t \in (0, \infty)$. Since the function $f$ is strictly increasing, we obtain that

$$\langle \mathrm{u}_1(t); e_k\rangle = \langle \mathrm{u}_2(t); e_k\rangle$$

for all $k \in \mathbb{N}$ and a.e. $t \in (0, \infty)$; in particular, from (1.3.5) we get that $d_\infty(\mathrm{u}_1, \mathrm{u}_2) = 0$ and thus $\mathrm{u}_1 = \mathrm{u}_2$ in $\mathcal{T}$.

Finally, we define the semiflow selection u for all $\mathrm{u}_0 \in X$ as

$$\mathrm{U}(\mathrm{u}_0) := \mathcal{U}^\infty(\mathrm{u}_0) \in \mathcal{T};$$

mesurability follows from the property (P3) for $\mathcal{U}^\infty$, while the semigroup property follows from property (P4): for any $\mathrm{u}_0 \in X$ and any $t_1, t_2 \geq 0$

$$\mathrm{U}(\mathrm{u}_0)(t_1 + t_2) = S_{t_1} \circ \mathrm{U}(\mathrm{u}_0)(t_2) = \mathrm{U}[\mathrm{U}(\mathrm{u}_0)(t_1)](t_2).$$

□

## 3.2 NON-AUTONOMOUS SYSTEM

Let us now consider a general *non-autonomous* system

$$\begin{cases} \partial_t \mathrm{v} + \mathrm{A}(\mathrm{v}) = f(t), & \text{for } (t, x) \in (t_0, +\infty) \times \Omega, \\ \mathrm{v} = \mathrm{v}_b(t), & \text{for } (t, x) \in (t_0, +\infty) \times \partial\Omega, \\ \mathrm{v}(t_0) = \mathrm{v}_0, \end{cases} \tag{3.2.1}$$

where both the force $f = f(t, x)$ and the boundary condition $\mathrm{v}_b = \mathrm{v}_b(t, x)$ depend explicitly on the time variable. The system is then no longer time-shift independent and it is always necessary to specify the initial time $t_0 \geq 0$. We would like to point out that actually the specific form of the system is not important and never used. What really matters are the axioms satisfied by the solutions, as we have seen in the context of autonomous systems.

If there exists a unique solution v for every fixed initial time $t_0 \geq 0$, we can define a semiprocess $\{P_{t_0}\}_{t_0 \geq 0}$ such that for every initial data $\mathrm{v}_0$

$$P_{t_0}(\mathrm{v}_0)(t) \text{ is the solution of system (3.2.1) evaluated at time } t \geq t_0.$$

If there exists a solution v for every fixed initial time $t_0 \geq 0$ but it may not be unique, we can select a semiprocess $\{P_{t_0}\}_{t_0 \geq 0}$ such that, among all the solutions arising from an initial data $\mathrm{v}_0$, $P_{t_0}(\mathrm{v}_0)$ is the one satisfying an analogous of the semigroup property for autonomous systems; more precisely,

$$\begin{gathered} P_{t_0}(\mathrm{v}_0)(t) \text{ is the solution of system (3.2.1) evaluated at time } t \geq t_0 \text{ such that} \\ P_{t_0}(\mathrm{v}_0)(t_2) = P_{t_1}[P_{t_0}(\mathrm{v}_0)(t_1)](t_2) \text{ for every } t_0 \leq t_1 \leq t_2. \end{gathered} \tag{3.2.2}$$

### 3.2.1 *Setting and main result*

First of all, we will consider the couple $h_0 = [\mathrm{v}_b, f]$ not as a quantity fixed from the beginning but as the data along with the initial condition $\mathrm{v}_0$. Let

- $H$ be a separable Hilbert space;
- $X$ be the *phase space* associated to system (3.2.1), which we suppose to be a closed convex subset of $H$;
- $H_D$ be the *data space*, which we suppose to be a time-shift invariant subspace of $L^1_{\mathrm{loc}}([0,\infty); X_D)$ with $X_D$ a given Banach space, i.e., for every $h \in H_D$
$$S_T \circ h(t) = h(t+T) \in H_D$$
- $\mathcal{T}_{t_0} = \mathfrak{D}([t_0,\infty); H \times H_D)$ be the *trajectory space* for every fixed $t_0 \geq 0$;
- $\sigma_t : H_D \to H_D$ be the operator such that
$$\sigma_t(h_0)(r) = \begin{cases} h_0(t) & \text{for } 0 \leq r \leq t, \\ h_0(r) & \text{for } r > t; \end{cases} \tag{3.2.3}$$
in particular, for any $t \geq 0$, $\sigma_t(h_0)$ is the restriction of $h_0$ on $(t,\infty)$ extended to be the constant $h_0(t)$ on $[0,t]$.

We can now give the following definition.

**Definition 3.2.1.** Let the spaces $H$, $X$, $H_D$, $\mathcal{T}_{t_0}$ and the operator $\sigma_t$ be fixed as above. A *generalized semiprocess* $\{\mathcal{G}_{t_0}\}_{t_0\geq 0}$ in $X \times H_D$ is a family of set–valued functions
$$\mathcal{G}_{t_0} : X \times H_D \to 2^{\mathcal{T}_{t_0}} \quad \text{for every } t_0,$$
such that for every $\mathrm{v}_0 \in X$ and every $h_0 \in H_D$
$$\mathcal{G}_{t_0}(\mathrm{v}_0,h_0) = \left\{ (\mathrm{v},h) \in \mathcal{T}_{t_0} : \begin{array}{l} \text{at any time } t \geq t_0,\ (\mathrm{v}(t),h(t)) \in X \times H_D \text{ is} \\ \text{the value of a solution of system (3.2.1) with} \\ \text{initial condition } \mathrm{v}_0 \text{ and data } h(t) = \sigma_t(h_0) \end{array} \right\},$$
satisfying the following properties.

(P1) *Existence*: for every $t_0 \geq 0$, every $\mathrm{v}_0 \in X$ and every $h_0 \in H_D$, there exists at least one couple $(\mathrm{v},h) \in \mathcal{G}_{t_0}(\mathrm{v}_0,h_0)$.

(P2) *Compactness*: for every $t_0 \geq 0$, every $\mathrm{v}_0 \in X$ and every $h_0 \in H_D$, $\mathcal{G}_{t_0}(\mathrm{v}_0,h_0)$ is a compact subset of $\mathcal{T}_{t_0}$.

(P3) *Measurability*: for every $t_0 \geq 0$, $\mathcal{G}_{t_0} : X \times H_D \to 2^{\mathcal{T}_{t_0}}$ is Borel-measurable.

(P4) *Restriction-invariance*: introducing for every $\omega \in \Omega_{t_0}$ the restriction-operator $R_{t_1}$, with $t_1 \geq t_0$, as
$$R_{t_1} \circ \omega = \omega|_{[t_1,\infty)}.$$
then, for every $0 \leq t_0 \leq t_1$, every $\mathrm{v}_0 \in X$, every $h_0 \in H_D$ and every couple $(\mathrm{v},h) \in \mathcal{G}_{t_0}(\mathrm{v}_0,h_0)$ we have
$$R_{t_1} \circ (\mathrm{v},h) = (R_{t_1} \circ \mathrm{v},\ R_{t_1} \circ h) \in \mathcal{G}_{t_1}[\mathrm{v}(t_1),h(t_1)]. \tag{3.2.4}$$

Moreover, we say that a generalized semiprocess $\{\mathcal{G}_{t_0}\}_{t_0 \geq 0}$ is *exact* if the following property holds.

(P5) *Continuation*: introduction for every $\omega_1 \in \Omega_{t_0}$, $\omega_2 \in \Omega_{t_1}$ and $s \geq t_0$ the continuation operator $\cup_{t_0,t_1}$, with $0 \leq t_0 \leq t_1$, as

$$\omega_1 \cup_{t_0,t_1} \omega_2(s) = \begin{cases} \omega_1(s) & \text{for } s \in [t_0, t_1], \\ \omega_2(s) & \text{for } s \in (t_1, \infty). \end{cases}$$

then, for every $0 \leq t_0 \leq t_1$, every $\mathrm{v}_0 \in X$, every $h_0 \in H_D$ and

$$(\mathrm{v}_1, h_1) \in \mathcal{G}_{t_0}(\mathrm{v}_0, h_0),$$
$$(\mathrm{v}_2, h_2) \in \mathcal{G}_{t_1}[\mathrm{v}_1(t_1), h(t_1)],$$

we have

$$(\mathrm{v}_1, h_1) \cup_{t_0,t_1} (\mathrm{v}_2, h_2) = (\mathrm{v}_1 \cup_{t_0,t_1} \mathrm{v}_2,\ h_1 \cup_{t_0,t_1} h_2) \in \mathcal{G}_{t_0}(\mathrm{v}_0, h_0). \tag{3.2.5}$$

*Remark* 3.2.2. The validity of the restriction-invariance and continuation properties is tautological for the data terms. Indeed,

(P4) for every couple $(\mathrm{v}, h) \in \mathcal{G}_{t_0}(\mathrm{v}_0, h_0)$, every $(\widetilde{\mathrm{v}}, \widetilde{h}) \in \mathcal{G}_{t_1}[\mathrm{v}(t_1), h(t_1)]$, every $t \geq t_1$ and $r \geq 0$

$$\widetilde{h}(t)(r) = \sigma_t\left(h(t_1)\right)(r) = \sigma_t\left(\sigma_{t_1}(h_0)\right)(r) = \begin{cases} \sigma_{t_1}(h_0)(t) = h_0(t) & \text{for } r \in [0, t] \\ \sigma_{t_1}(h_0)(r) = h_0(r) & \text{for } r \in (t, \infty) \end{cases}$$

and thus

$$\widetilde{h}(t) = \sigma_t(h_0) = R_{t_1} \circ h(t) \quad \text{for every } t \geq t_1;$$

(P5) for every $(\mathrm{v}_1, h_1) \in \mathcal{G}_{t_0}(\mathrm{v}_0, h_0)$ and every $(\mathrm{v}_2, h_2) \in \mathcal{G}_{t_1}[\mathrm{v}_1(t_1), h_1(t_1)]$ we have

$$h_1 \cup_{t_0,t_1} h_2(t) = \begin{cases} h_1(t) = \sigma_t(h_0) & \text{for } t \in [t_0, t_1], \\ h_2(t) = \sigma_t(h_1)(t_1) = \sigma_t(h_0) & \text{for } t \in (t_1, \infty) \end{cases}$$

and thus

$$h_1 \cup_{t_0,t_1} h_2(t) = \sigma_t(h_0) \quad \text{for every } t \geq t_0.$$

We are now ready to state the following result; the proof is postponed to the next subsection.

**Theorem 3.2.3.** *Let $H$ be a separable Hilbert space, $X$ a closed convex subset of $H$, $H_D$ a time-shift invariant subspace of $L^1_{\mathrm{loc}}([0,\infty); X_D)$ with $X_D$ a given Banach space, and let $\{\mathcal{G}_{t_0}\}_{t_0 \in \mathbb{R}}$ be an exact generalized semiprocess in $X \times H_D$ in the sense of Definition 3.2.1. Then for every fixed $t_0 \geq 0$ there exists a Borel-measurable map*

$$P_{t_0} : X \times H_D \to \mathfrak{D}([t_0, \infty); H \times H_D)$$

*such that*

$$P_{t_0}[\mathrm{v}_0, h_0] \in \mathcal{G}_{t_0}(\mathrm{v}_0, h_0), \quad \textit{for every } (\mathrm{v}_0, h_0) \in X \times H_D$$

*satisfying the following property:*

$$P_{t_0}[\mathrm{v}_0, h_0](t_2) = P_{t_1}\left[P_{t_0}[\mathrm{v}_0, h_0](t_1)\right](t_2), \tag{3.2.6}$$

*for any $t_0 \leq t_1 \leq t_2$.*

### 3.2.2 *Proof of the existence of the semiflow*

In order to recycle the already existing results for autonomous systems and more precisely, in order to apply Theorem 3.1.2, we must first convert our initial system (3.2.1) into an autonomous one starting from 0. To this end, it is sufficient to think of the time $t = t(s)$ as a dependent variable of a new parameter $s$ such that $t(0) = t_0$; for simplicity, we consider the following translation

$$\begin{cases} \dot{t}(s) = 1 \\ t(0) = t_0 \end{cases} \quad \Rightarrow \quad t(s) = s + t_0.$$

Introducing the new variable $\mathrm{w} = \mathrm{w}(s, x)$ such that $\mathrm{w}(s, x) = [t(s),\ \mathrm{u}(s, x) = \mathrm{v}\,(t(s), x)]$, we have

$$\partial_s \mathrm{w} = [\dot{t},\ \dot{t} \cdot \partial_t \mathrm{v}] = [\dot{t},\ \partial_t \mathrm{v}] = [1,\ f - \mathrm{A}(\mathrm{v})]\,,$$

$$\mathrm{w}(0, \cdot) = [t(0),\ \mathrm{v}\,(t(0), \cdot)] = [t_0,\ \mathrm{v}\,(t_0, \cdot)] = [t_0, \mathrm{v}_0]$$

and thus (3.2.1) can be rewritten as

$$\begin{cases} \partial_s \mathrm{w} + \mathrm{B}(\mathrm{w}) = [0,\ f(s + t_0)], & \text{for } (s, x) \in (0, +\infty) \times \Omega, \\ \mathrm{w} = [0, \mathrm{v}_b(s + t_0)], & \text{for } (s, x) \in [0, +\infty) \times \partial\Omega, \\ \mathrm{w}(0, \cdot) = \mathrm{w}_0, & \end{cases} \tag{3.2.7}$$

with $\mathrm{B}(\mathrm{w}) = [-1,\ \mathrm{A}(\mathrm{v})]$ and $\mathrm{w}_0 = [t_0,\ \mathrm{v}_0]$. We are now ready to prove Theorem 3.2.3.

*Proof of Theorem 3.2.3.* Let

- $Y = [0, \infty) \times X \times H_D$;
- $K = [0, \infty) \times H \times H_D$;
- $\mathcal{T} = \mathfrak{D}([0, \infty); K)$ be the Skorokhod space of càglàd functions defined on $[0, \infty)$ and taking values in $K$;
- $\mathcal{U} : Y \to 2^{\mathcal{T}}$ be the set–valued map such that for every $(t_0, \mathrm{v}_0, h_0) \in Y$

$$\mathcal{U}(t_0, \mathrm{v}_0, h_0) = \left\{ (t, \mathrm{u}, g) \in \mathcal{T} : \begin{array}{l} \text{at any time } s \geq 0,\ (t(s), \mathrm{u}(s), g(s)) \in Y \text{ is} \\ \text{the value of the solution of system (3.2.7)} \\ \text{with initial condition } (t_0, \mathrm{v}_0) \text{ and data} \\ g(s) = \sigma_{s+t_0}(h_0) \end{array} \right\}. \tag{3.2.8}$$

In particular, it is easy to check that

$$(\mathrm{v}, h) \in \mathcal{G}_{t_0}(\mathrm{v}_0, h_0) \quad \Leftrightarrow \quad (t, \mathrm{u}, g) \in \mathcal{U}(t_0, \mathrm{v}_0, h_0), \tag{3.2.9}$$

where $t(s) = s + t_0$, $\mathrm{u}(s) = \mathrm{v}(s + t_0)$ and $g(s) = h(s + t_0)$ for every $s, t_0 \geq 0$. Consequently, we can deduce that $\mathcal{U}$ satisfies the following properties.

(P1) *Existence*: for every $(t_0, \mathrm{v}_0, h_0) \in Y$, $\mathcal{U}(t_0, \mathrm{v}_0, h_0)$ is a non-empty subset of $\mathcal{T}$.

(P2) *Compactness*: for every $(t_0, \mathrm{v}_0, h_0) \in Y$, $\mathcal{U}(t_0, \mathrm{v}_0, h_0)$ is a compact subset of $\mathcal{T}$.

(P3) *Measurability*: the set-valued map $\mathcal{U} : Y \to 2^{\mathcal{T}}$ is Borel-measurable.

(P4) *Shift-invariance*: for every $T > 0$, every $(t_0, \mathrm{v}_0, h_0) \in Y$ and every $(t, \mathrm{u}, g) \in \mathcal{U}(t_0, \mathrm{v}_0, h_0)$, we have
$$S_T \circ (t, \mathrm{u}, g) \in \mathcal{U}(t(T), \mathrm{u}(T), g(T)),$$
where we recall that the positive shift operator $S_T$ is defined as in (3.1.3).

(P5) *Continuation*: for every $T > 0$, every $(t_0, \mathrm{v}_0, h_0) \in Y$ and every
$$\begin{aligned} &(t_1, \mathrm{u}_1, g_1) \in \mathcal{U}(t_0, \mathrm{v}_0, h_0) \\ &(t_2, \mathrm{u}_2, g_2) \in \mathcal{U}(t_1(T), \mathrm{u}_1(T), g_1(T)) \end{aligned}$$
we have
$$(t_1, \mathrm{u}_1, g_1) \cup_T (t_2, \mathrm{u}_2, g_2) \in \mathcal{U}(t_0, \mathrm{v}_0, h_0),$$
where we recall that the continuation operator $\cup_T$ is defined as in (3.1.4).

Properties (P1)–(P5) are a direct consequence of $\{\mathcal{G}_{t_0}\}_{t_0 \geq 0}$ being an exact generalized semiprocess. Let us check the shift-invariance and continuation properties.

(P4) Let $T > 0$, $(t_0, \mathrm{v}_0, h_0) \in Y$ and $(t, \mathrm{u}, g) \in \mathcal{U}(t_0, \mathrm{v}_0, h_0)$ be fixed. Introducing $\mathrm{v}(t) := \mathrm{u}(t - t_0)$ and $h(t) := g(t - t_0)$ for every $t \geq t_0$, from (3.2.9) we obtain that $(\mathrm{v}, h) \in \mathcal{G}_{t_0}(\mathrm{v}_0, h_0)$; moreover, noticing that for every $s \geq 0$ and $t = s + t_0 + T$
$$\begin{aligned} S_T \circ (\mathrm{u}, g)(s) &= (\mathrm{u}(s+T), g(s+T)) \\ &= (\mathrm{v}(s + t_0 + T), h(s + t_0 + T)) \\ &= (\mathrm{v}(t), h(t)) \\ &= R_{t_0+T} \circ (\mathrm{v}, h)(t) \end{aligned}$$
for every $t \geq t_0 + T$, from (3.2.4) we have
$$R_{t_0+T} \circ (\mathrm{v}, h) \in \mathcal{G}_{t_0+T}(\mathrm{v}(t_0+T), h(t_0+T)) = \mathcal{G}_{t(T)}(\mathrm{u}(T), g(T)),$$
and thus, from (3.2.9) we can deduce that
$$S_T \circ (t, \mathrm{u}, g) \in \mathcal{U}(t(T), \mathrm{u}(T), g(T)).$$

(P5) Let $T > 0$, $(t_0, \mathrm{v}_0, h_0) \in Y$ and
$$\begin{aligned} &(t_1, \mathrm{u}_1, g_1) \in \mathcal{U}(t_0, \mathrm{v}_0, h_0), \\ &(t_2, \mathrm{u}_2, g_2) \in \mathcal{U}(t_1(T), \mathrm{u}_1(T), g_1(T)) \end{aligned}$$
be fixed. Introducing
$$\begin{aligned} &\mathrm{v}_1(t) := \mathrm{u}_1(t - t_0), &&h_1(t) := g_1(t - t_0) &&\text{for any } t \geq t_0, \\ &\mathrm{v}_2(t) := \mathrm{u}_2(t - t_0 - T), &&h_2(t) := g_2(t - t_0 - T) &&\text{for any } t \geq t_0 + T, \end{aligned}$$
from (3.2.9) we obtain that
$$\begin{aligned} &(\mathrm{v}_1, h_1) \in \mathcal{G}_{t_0}(\mathrm{v}_0, h_0) \\ &(\mathrm{v}_2, h_2) \in \mathcal{G}_{t(T)}(\mathrm{u}_1(T), g_1(T)) = \mathcal{G}_{t_0+T}(\mathrm{v}_1(t_0+T), h_1(t_0+T)). \end{aligned}$$

Furthermore, for every $s \geq 0$ and $t = s + t_0$

$$\mathrm{u}_1 \cup_T \mathrm{u}_2(s) = \begin{cases} \mathrm{u}_1(s) & \text{for } s \in [0,T] \\ \mathrm{u}_2(s-T) & \text{for } s \in (T,\infty) \end{cases} = \begin{cases} \mathrm{v}_1(t) & \text{for } t \in [t_0, t_0+T] \\ \mathrm{v}_2(t) & \text{for } t \in (t_0+T,\infty) \end{cases}$$
$$= \mathrm{v}_1 \cup_{t_0,t_0+T} \mathrm{v}_2(t),$$

for every $t \geq t_0$; similarly, for every $s \geq 0$ and $t = s + t_0$

$$g_1 \cup_T g_2(s) = h_1 \cup_{t_0,t_0+T} h_2(t).$$

Then, from (3.2.5) we obtain

$$(\mathrm{v}_1, h_1) \cup_{t_0,t_0+T} (\mathrm{v}_2, h_2) \in \mathcal{G}_{t_0}(\mathrm{v}_0, h_0)$$

and thus, from (3.2.9) we can deduce that

$$(t_1, \mathrm{u}_1, g_1) \cup_T (t_2, \mathrm{u}_2, g_2) \in \mathcal{U}(t_0, \mathrm{v}_0, h_0).$$

We can apply Theorem 3.1.2 to the set–valued map $\mathcal{U}$ to get the existence of a Borel-measurable map

$$\mathrm{U} : Y \to \mathcal{T}, \quad \mathrm{U}(t_0, \mathrm{v}_0, h_0) \in \mathcal{U}(t_0, \mathrm{v}_0, h_0) \text{ for every } (t_0, \mathrm{v}_0, h_0) \in Y,$$

satisfying the semigroup property: for any $(t_0, \mathrm{v}_0, h_0) \in Y$ and any $s_1, s_2 \geq 0$

$$\mathrm{U}(t_0, \mathrm{v}_0, h_0)(s_1 + s_2) = \mathrm{U}[\mathrm{U}(t_0, \mathrm{v}_0, h_0)(s_1)](s_2).$$

Writing for every $s \geq 0$

$$\mathrm{U}(t_0, \mathrm{v}_0, h_0)(s) = (t, \mathrm{u}, g)(s),$$

we can define the restriction mapping

$$\begin{aligned} \mathrm{V} : [0,\infty) \times X \times H_D &\longrightarrow \mathfrak{D}([0,\infty); H \times H_D) \\ (t_0, \mathrm{v}_0, h_0) &\longmapsto (\mathrm{u}, g)(s), \end{aligned}$$

and thus, to conclude it is enough to define for every $(t_0, \mathrm{v}_0, h_0) \in Y$ and every $t \geq t_0$

$$P_{t_0}(\mathrm{v}_0, h_0)(t) := \mathrm{V}(t_0, \mathrm{v}_0, h_0)(t - t_0).$$

Indeed, for every $t_0 \leq t_1 \leq t_2$ we have

$$\begin{aligned} P_{t_0}(\mathrm{v}_0, h_0)(t_2) &= \mathrm{V}(t_0, \mathrm{v}_0, h_0)(t_2 - t_0) \\ &= \mathrm{V}(t_0, \mathrm{v}_0, h_0)(t_2 - t_1 + t_1 - t_0) \\ &= \mathrm{V}[\ \mathrm{V}(t_0, \mathrm{v}_0, h_0)(t_1 - t_0)\ ](t_2 - t_1) \\ &= \mathrm{V}[P_{t_0}(\mathrm{v}_0, h_0)(t_1)](t_2 - t_1) \\ &= P_{t_1}[P_{t_0}(\mathrm{v}_0, h_0)(t_1)](t_2), \end{aligned}$$

where in the third line we used the fact that u satisfies the semigroup property. We get (3.2.6) and thus the claim. □

### 3.2.3 *Properties*

The inclusion of the data $h_0 = [v_b, f]$ appearing in (3.2.1) as part of a semiprocess guarantees the validity of some useful properties for the correspondent selected semiflow, summarized in the following result.

**Proposition 3.2.4.** *Let $H$, $X$ and $H_D$ be fixed as in the hypothesis of Theorem 3.1.2. Let $\{\mathcal{G}_{t_0}\}_{t_0 \in \geq 0}$ an exact generalized semiprocess in $X \times H_D$ in the sense of Definition 3.2.1, and let $\{P(t_0)\}_{t_0 \geq 0}$ be the semiflow selection associated to $\{\mathcal{G}_{t_0}\}_{t_0 \geq 0}$. Then the following properties hold.*

*(i) If $h_0 \in H_D$ is independent of time, then $P_{t_0}(\mathrm{v}_0, h_0)$ is independent of $t_0$ for every $\mathrm{v}_0 \in X$; in particular, for every $0 \leq t_0 \leq t_1$ and every $\mathrm{v}_0 \in X$ we have*

$$P_{t_0}(\mathrm{v}_0, h_0) \equiv S_{t_1 - t_0} \circ P_{t_1}(\mathrm{v}_0, h_0). \tag{3.2.10}$$

*(ii) If $h_0 \in H_D$ is periodic in time with period $T$, meaning $h_0(t+T) = h_0(t)$ for all $t \geq 0$, then for every $t_0 \geq 0$ and every $\mathrm{v}_0 \in X$*

$$P_{t_0}(\mathrm{v}_0, h_0) \equiv S_T \circ P_{t_0+T}(\mathrm{v}_0, h_0). \tag{3.2.11}$$

*(iii) If $h_1, h_2 \in H_D$ are such that $h_1(t) = h_2(t)$ for all $t \geq \tilde{t} > 0$, then for every $t_0 \geq \tilde{t}$ and every $\mathrm{v}_0 \in X$*

$$P_{t_0}(\mathrm{v}_0, h_1) \equiv P_{t_0}(\mathrm{v}_0, h_2). \tag{3.2.12}$$

*Proof.* (i) Let $P_{t_1}(\mathrm{v}_0, h_0) = (\mathrm{v}, h)$; then, in particular

$$\begin{aligned} \mathrm{v}(t_1) &= \mathrm{v}_0, \\ h(t) &= \sigma_t(h_0) \quad \text{for all } t \geq t_1, \end{aligned}$$

where $\sigma_t$ is defined as in (3.2.3). For every $t \geq t_0$ we have

$$\begin{aligned} S_{t_1 - t_0} \circ P_{t_1}(\mathrm{v}_0, h_0)(t) &= \left(S_{t_1 - t_0} \circ \mathrm{v}(t), S_{t_1 - t_0} \circ h(t)\right) \\ &= \left(\mathrm{v}(t + t_1 - t_0),\ S_{t_1 - t_0} \circ (\sigma_t(h_0))\right) \\ &= \left(\widetilde{\mathrm{v}}(t),\ \sigma_t(h_0)\right), \end{aligned}$$

where in the last line we used the fact that $h_0$ is independent of time. Noticing that

$$\widetilde{\mathrm{v}}(t_0) = \mathrm{v}(t_0 + t_1 - t_0) = \mathrm{v}(t_1) = \mathrm{v}_0,$$

we recover that

$$S_{t_1 - t_0} \circ P_{t_1}(\mathrm{v}_0, h_0) \in \mathcal{G}_{t_0}(\mathrm{v}_0, h_0).$$

Uniqueness of the semiflow selection lead us to (3.2.10).

(ii) Let $P_{t_0+T}(\mathrm{v}_0, h_0) = (\mathrm{v}, h)$; then, in particular

$$\begin{aligned} \mathrm{v}(t_0 + T) &= \mathrm{v}_0, \\ h(t) &= \sigma_t(h_0) \quad \text{for all } t \geq t_0 + T. \end{aligned}$$

Proceeding as before, for every $t \geq t_0$ we have

$$\begin{aligned} S_T \circ P_{t_0+T}(\mathrm{v}_0, h_0)(t) &= \big(S_T \circ \mathrm{v}(t), S_T \circ h(t)\big) \\ &= \big(\mathrm{v}(t+T),\ S_T \circ (\sigma_t(h_0))\,\big) \\ &= \big(\widetilde{\mathrm{v}}(t),\ S_T \circ (\sigma_t(h_0))\,\big). \end{aligned}$$

Noticing that

$$\widetilde{\mathrm{v}}(t_0) = \mathrm{v}(t_0 + T) = \mathrm{v}_0,$$

and that, using the periodicity of $h_0$, for every $r \geq 0$

$$S_T \circ (\sigma_t(h_0))\,(r) = \begin{cases} S_T \circ h_0(t) = h_0(t+T) = h_0(t) & \text{for } r \in [0,t], \\ S_T \circ h_0(r) = h_0(r+T) = h_0(r) & \text{for } r \in (t,\infty), \end{cases} = \sigma_t(h_0)(r),$$

we recover that

$$S_T \circ P_{t_0+T}(\mathrm{v}_0, h_0) \in \mathcal{G}_{t_0}(\mathrm{v}_0, h_0).$$

Proceeding as in the previous step, we obtain (3.2.11).

(iii) For every $t \geq t_0 \geq \tilde{t}$, $h_1(t) = h_2(t)$ and thus, for every $r \geq 0$ we have

$$\sigma_t(h_1)(r) = \begin{cases} h_1(t) = h_2(t) & \text{for } r \in [0,t] \\ h_1(r) = h_2(r) & \text{for } r \in (t,\infty) \end{cases} = \sigma_t(h_2)(r);$$

then, (3.2.12) easily follows.

□

# 4

# COMPRESSIBLE NAVIER-STOKES SYSTEM

The goal of this chapter is to prove the existence of a semiflow selction for the compressible Navier-Stokes system in the class of dissipative weak solutions, adapting the abstract machinery of the previous chapter to this context. Specifically, fixing a separable Hilbert space $H$, a proper phase space $X$ and letting $\mathcal{T} = \mathfrak{D}([0,\infty; H))$ be the trajectory space, our goal is to show the existence of a Borel-measurable map $\mathrm{V} : X \to \mathcal{T}$ associating to any initial data $[\varrho_0, \mathbf{m}_0] \in X$ the dissipative weak solution $\mathrm{V}[\varrho_0, \mathbf{m}_0]$ satisfying the semigroup property: for any $t_1, t_2 \geq 0$

$$\mathrm{V}[\varrho_0, \mathbf{m}_0](t_1 + t_2) = \mathrm{V}[\ \mathrm{V}[\varrho_0, \mathbf{m}_0](t_1)\ ](t_2).$$

The chapter is organized as follows. In Section 4.1 we introduce the system, while in Section 4.2 we recall the definition of a dissipative weak solution, assuming that the energy equals almost everywhere a càglàd function $E$, cf. Definition 4.2.1. In Section 4.3 we prove the existence of a semiflow selection depending on the initial density $\varrho_0$, momentum $\mathbf{m}_0$ and energy $E_0$, cf. Theorem 4.3.1; moreover, we show that it is possible to select only the admissible solutions, i.e. the ones minimizing the total energy, cf. Lemma 4.3.6. Due to the fact that the energy can be written in terms of the density and momentum at least almost everywhere, in Section 4.4 we prove the existence of a restricted selection not depending on $E_0$, cf. Theorem 4.4.1.

## 4.1 THE SYSTEM

We consider the *compressible Navier-Stokes system*, described by the following couple of equations

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{4.1.1}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}). \tag{4.1.2}$$

The unknown variables of the system are the *density* $\varrho = \varrho(t,x)$ and the *velocity* $\mathbf{u} = \mathbf{u}(t,x)$. The term $p = p(\varrho)$ denotes the pressure, which we assume to be of the type

$$p(\varrho) = a\varrho^\gamma \tag{4.1.3}$$

with $a > 0$ and the adiabatic exponent

$$\gamma > \frac{d}{2},$$

but more general types of pressure preserving the essential features of (4.1.3) are allowed, such as

$$p \in C[0,\infty) \cap C^1(0,\infty),\ p(0) = 0,\ p'(\varrho) > 0 \text{ for } \varrho > 0. \tag{4.1.4}$$

The term $\mathbb{S} = \mathbb{S}(\nabla_x \mathbf{u})$ denotes the viscous stress tensor, which is supposed to be a linear function of the velocity gradient, more precisely to satisfy Newton's rheological law

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left( \nabla_x \mathbf{u} + \nabla_x^T \mathbf{u} - \frac{2}{d} (\mathrm{div}_x\, \mathbf{u}) \mathbb{I} \right) + \lambda (\mathrm{div}_x\, \mathbf{u}) \mathbb{I}, \tag{4.1.5}$$

with $\mu > 0$ and $\lambda \geq 0$. We will consider the system on the set

$$(t, x) \in (0, \infty) \times \Omega,$$

where $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ is a bounded domain with $\partial\Omega$ of class $C^{2+\nu}$ for a certain $\nu > 0$. As our goal is to handle a potentially ill–posed problem, we have deliberately omitted the case $d = 1$, for which the problem is known to be well posed, see Kazhikhov [52]. Finally, we impose the no–slip boundary condition for the velocity

$$\mathbf{u}|_{\partial\Omega} = 0 \quad \text{for all } t \in [0, \infty), \tag{4.1.6}$$

and prescribe the initial conditions

$$\varrho(0, \cdot) = \varrho_0, \quad (\varrho\mathbf{u})(0, \cdot) = (\varrho\mathbf{u})_0. \tag{4.1.7}$$

### 4.1.1 *Renormalized continuity equation*

A crucial step for proving the existence of weak solutions to system (4.1.1)–(4.1.7) consists in rescaling the state variables in the continuity equation (4.1.1). Formally, we multiply (4.1.1) by $B'(\varrho)$, with $B$ a smooth function, to get

$$\partial_t B(\varrho) + \nabla_x B(\varrho) \cdot \mathbf{u} + \varrho B'(\varrho) \operatorname{div}_x \mathbf{u} = 0,$$

which can be rewritten as

$$\partial_t B(\varrho) + \operatorname{div}_x(B(\varrho)\mathbf{u}) + (\varrho B'(\varrho) - B(\varrho)) \operatorname{div}_x \mathbf{u} = 0. \tag{4.1.8}$$

Equation (4.1.8) is known as *renormalized continuity equation*. Even if in our analysis we mainly follow Feireisl [33], the idea of renormalization can be traced back to the pioneering work of Kružkov [53] and later DiPerna and Lions [30].

### 4.1.2 *Energy balance*

We recall that the compressible Navier-Stokes system (4.1.1)–(4.1.7) has already been introduced in Chapter 2. More precisely, multiplying the balance of momentum (4.1.2) by $\mathbf{u}$ and introducing the *pressure potential* $P = P(\varrho)$ as a solution of

$$\varrho P'(\varrho) - P(\varrho) = p(\varrho), \tag{4.1.9}$$

in Section 2.2.1 we have recovered the energy balance

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho)\right] \mathrm{d}x + \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \, \mathrm{d}x = 0. \tag{4.1.10}$$

The quantity

$$E(t) := \int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho)\right] \mathrm{d}x$$

represents the total *mechanical energy* of the fluid. Noticing that from (4.1.5) and (4.1.6)

$$\int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \, \mathrm{d}x = \mu \int_\Omega |\nabla_x \mathbf{u}|^2 \mathrm{d}x + (\lambda + \mu) \int_\Omega |\operatorname{div}_x \mathbf{u}|^2 \mathrm{d}x \geq 0,$$

then, from (4.1.10), we can deduce that $E = E(t)$ is a non-increasing function. As the First law of thermodynamics asserts that the total energy of a closed system is conserved, $E(t)$ cannot be the total energy of a viscous fluid at time $t$. In accordance with the Second law of thermodynamics and (4.1.10), a part of the mechanical energy is irreversibly converted to another form of internal energy associated with the production of heat as long as the fluid is in motion. In the framework of our simplified model, the resulting changes of temperature and their influence on the fluid motion are not taken into account. A mathematical theory for the full energetically complete system has been developed by Feireisl and Novotný [42].

Moreover, if we deal with the concept of strong solutions, i.e. if $\varrho$ and $\mathbf{u}$ are smooth functions solving equations (4.1.1) and (4.1.2) pointwise, we may integrate (4.1.10) over $(0,\tau)$ to get

$$\int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho)\right](\tau)\,\mathrm{d}x + \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\mathbf{u}\,\mathrm{d}x\mathrm{d}t = \int_\Omega \left[\frac{1}{2\varrho_0}|(\varrho\mathbf{u})_0|^2 + P(\varrho_0)\right]\mathrm{d}x, \tag{4.1.11}$$

for every $\tau \geq 0$.

## 4.2 DISSIPATIVE WEAK SOLUTION

We are now ready to give the definition of a dissipative weak solution to the compressible Navier-Stokes system.

**Definition 4.2.1.** The pair of functions $\varrho$, $\mathbf{u}$ is called *dissipative weak solution* of the Navier-Stokes system (4.1.1)–(4.1.7) with total energy $E$ and initial data

$$[\varrho_0, (\varrho\mathbf{u})_0, E_0] \in L^\gamma(\Omega) \times L^{\frac{2\gamma}{\gamma+1}}(\Omega;\mathbb{R}^d) \times [0,\infty)$$

if the following holds:

(i) *regularity class*:

$$[\varrho, \varrho\mathbf{u}, E] \in C_{\mathrm{weak,loc}}([0,\infty); L^\gamma(\Omega)) \times C_{\mathrm{weak,loc}}([0,\infty); L^{\frac{2\gamma}{\gamma+1}}(\Omega;\mathbb{R}^d)) \times \mathfrak{D}([0,\infty)),$$

with $\varrho \geq 0$;

(ii) *weak formulation of the renormalized continuity equation*: for any $\tau > 0$ and any functions

$$B \in C[0,\infty) \cap C^1(0,\infty),\ b \in C[0,\infty) \text{ bounded on } [0,\infty),$$
$$B(0) = b(0) = 0 \quad \text{and} \quad b(z) = zB'(z) - B(z) \text{ for any } z > 0,$$

the integral identity

$$\left[\int_\Omega B(\varrho)\varphi(t,\cdot)\,\mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega [B(\varrho)\partial_t\varphi + B(\varrho)\mathbf{u}\cdot\nabla_x\varphi + b(\varrho)\,\mathrm{div}_x\,\mathbf{u}\varphi]\,\mathrm{d}x\mathrm{d}t, \tag{4.2.1}$$

holds for any $\varphi \in C_c^1([0,\infty)\times\overline{\Omega})$, with $\varrho(0,\cdot) = \varrho_0$;

(iii) *weak formulation of the balance of momentum*: for any $\tau > 0$ the integral identity

$$\begin{aligned}\left[\int_\Omega \varrho\mathbf{u}\cdot\boldsymbol{\varphi}(t,\cdot)\,\mathrm{d}x\right]_{t=0}^{t=\tau} &= \int_0^\tau \int_\Omega [\varrho\mathbf{u}\cdot\partial_t\boldsymbol{\varphi} + (\varrho\mathbf{u}\otimes\mathbf{u}) : \nabla_x\boldsymbol{\varphi} + p(\varrho)\,\mathrm{div}_x\,\boldsymbol{\varphi}]\,\mathrm{d}x\mathrm{d}t,\\ &\quad - \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\boldsymbol{\varphi}\,\mathrm{d}x\mathrm{d}t\end{aligned} \tag{4.2.2}$$

holds for any $\boldsymbol{\varphi} \in C_c^1([0,\infty)\times\Omega;\mathbb{R}^d)$, with $(\varrho\mathbf{u})(0,\cdot) = (\varrho\mathbf{u})_0$;

(iv) *energy inequality*: the càglàd function $E$ is non-increasing in $[0, \infty)$, satisfies

$$\int_\Omega \left[ \frac{1}{2} \varrho |\mathbf{u}|^2 + P(\varrho) \right] (\tau, \cdot) \, \mathrm{d}x = E(\tau), \tag{4.2.3}$$

for a.e. $\tau \geq 0$, and the energy inequality

$$[E\psi]_{t=\tau_1-}^{t=\tau_2+} - \int_{\tau_1}^{\tau_2} E(t)\psi'(t) \, \mathrm{d}t + \int_{\tau_1}^{\tau_2} \psi \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \, \mathrm{d}x\mathrm{d}t \leq 0 \tag{4.2.4}$$

holds for any $0 \leq \tau_1 \leq \tau_2$, $\psi \in C_c^1[0, \infty)$, $\psi \geq 0$, with $E(0-) = E_0$.

Notice that conditions (i) of Definition 4.2.1 and $E(0-) = E_0$ come naturally from the assumption that the total mechanical energy of the system is bounded at the initial time $t = 0$, specifically

$$E(0+) \leq E_0.$$

The integral identities (4.2.1) and (4.2.2) can be easily deduced multiplying equations (4.1.8) and (4.1.2) respectively by test functions, integrating over $(0, \tau) \times \Omega$ and performing an integration by parts.

Regarding condition (iv) of Definition 4.2.1 a preliminary remark is necessary: it is an outstanding open problem if the energy equality (4.1.11), which has been derived under the assumption of smoothness of all quantities involved, holds for any weak solution satisfying (4.2.1) and (4.2.2). The problem is that the mechanical energy dissipation can be enhanced by singularities and, if this occurs, a non-negative measure must be added to the left-hand side of (4.1.10). Accordingly, we obtain the mechanical energy inequality in the form

$$E(\tau) + \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \, \mathrm{d}x\mathrm{d}t \leq E(0),$$

for a.e. $\tau \geq 0$, or its distributional form expressed by (4.2.4).

### 4.2.1 *Admissible solution*

Following Breit, Feireisl and Hofmanová [14], we introduce a subclass of dissipative weak solutions that reflect the physical principle of minimization of the total energy. At the present state, we retain the total energy $E$ as an integral part of the solution so we work with the triples $[\varrho, \mathbf{m}, E]$. Finally, in Section 4.4 we pass to the natural state variables $[\varrho, \mathbf{m}]$. To this end, let $[\varrho^i, \mathbf{m}^i, E^i]$, $i = 1, 2$, be two dissipative solutions starting from the same initial data $[\varrho_0, \mathbf{m}_0, E_0]$. We introduce the relation

$$[\varrho^1, \mathbf{m}^1, E^1] \prec [\varrho^2, \mathbf{m}^2, E^2] \;\Leftrightarrow\; E^1(\tau\pm) \leq E^2(\tau\pm) \text{ for any } \tau \in (0, \infty).$$

**Definition 4.2.2.** We say that a dissipative weak solution $[\varrho, \mathbf{m}, E]$ to problem (4.1.1)–(4.1.7) starting from the initial data $[\varrho_0, \mathbf{m}_0, E_0]$ in the sense of Definition 4.2.1 is *admissible* if it is minimal with respect to the relation $\prec$. Specifically, if

$$[\tilde{\varrho}, \tilde{\mathbf{m}}, \tilde{E}] \prec [\varrho, \mathbf{m}, E],$$

where $[\tilde{\varrho}, \tilde{\mathbf{m}}, \tilde{E}]$ is another dissipative solution starting from $[\varrho_0, \mathbf{m}_0, E_0]$, then

$$E = \tilde{E} \text{ in } [0, \infty).$$

In particular, such selection criterion guarantees that equilibrium states belong to the class of dissipative weak solutions (see [14], Section 6.3).

## 4.3 SEMIFLOW SELECTION

Our goal is to apply the abstract machinery introduced in the previous chapter in order to show the existence of a semiflow selection for system (4.1.1)–(4.1.7). More precisely, we aim to prove Theorem 4.3.1 below, fixing first a proper setting.

### 4.3.1 *Set-up*

First of all, we must fix the separable Hilbert space $H$, the phase space $X \subseteq H$ and the map $\mathcal{U}$ introduced at the beginning of Section 3.1.1. In this context

- $H := W^{-k,2}(\Omega) \times W^{-k,2}(\Omega;\mathbb{R}^d) \times \mathbb{R}$, where the natural number $k > \frac{d}{2}+1$ is fixed. This particular choice of $k$ guarantees the compact embedding
$$L^q(\Omega) \hookrightarrow\hookrightarrow W^{-k,2}(\Omega) \quad \text{for any } q \geq 1,$$
as clearly noticeable from (1.1.10);
- the phase space $X$ can be chosen as
$$X := \left\{ [\varrho_0, \mathbf{m}_0, E_0] \in H: \ \varrho_0 \in L^1(\Omega),\ \varrho_0 \geq 0,\ \mathbf{m}_0 \in L^1(\Omega;\mathbb{R}^d) \text{ satisfying (4.3.1)} \right\}$$
where
$$\int_\Omega \left[ \frac{1}{2}\frac{|\mathbf{m}_0|^2}{\varrho_0} + P(\varrho_0) \right] \mathrm{d}x \leq E_0; \tag{4.3.1}$$
- $\mathcal{U} : X \to 2^{\mathcal{T}}$, with the trajectory space
$$\mathcal{T} = \mathfrak{D}([0,\infty); H),$$
is the set–valued mapping that associate to every $[\varrho_0, \mathbf{m}_0, E_0] \in X$ the family of dissipative weak solutions in the sense of Definition 4.2.1 arising from the initial data $[\varrho_0, \mathbf{m}_0, E_0]$. More precisely, for every $[\varrho_0, \mathbf{m}_0, E_0] \in X$
$$\mathcal{U}[\varrho_0, \mathbf{m}_0, E_0] = \left\{ [\varrho, \mathbf{m}, E] \in \mathcal{T} : \begin{array}{l} \text{at any time } t \geq 0, [\varrho(t,\cdot), \mathbf{m}(t,\cdot), E(t)] \in X \\ \text{is a dissipative weak solution of} \\ \text{problem (4.1.1)–(4.1.7) in the sense of} \\ \text{Definition 4.2.1 with initial data } [\varrho_0, \mathbf{m}_0, E_0] \end{array} \right\}.$$

Notice that everything is well-defined; indeed, denoting with $L^1_+(\Omega)$ the space of non–negative integrable functions on $\Omega$, we can rewrite $X$ as
$$\{[\varrho_0, \mathbf{m}_0, E_0] \in L^1_+(\Omega) \times L^1(\Omega;\mathbb{R}^d) \times \mathbb{R} : \ g(\varrho_0, \mathbf{m}_0) \leq E_0\},$$
so that it coincides with the epigraph of the function $g : L^1_+(\Omega) \times L^1(\Omega;\mathbb{R}^d) \to [0,+\infty]$ defined as
$$g(\varrho_0, \mathbf{m}_0) = \int_\Omega \left[ \frac{1}{2}\frac{|\mathbf{m}_0|^2}{\varrho_0} + P(\varrho_0) \right] \mathrm{d}x.$$

From the fact that

$$[\varrho, \mathbf{m}] \mapsto \begin{cases} 0 & \text{if } \mathbf{m} = 0, \\ \frac{|\mathbf{m}|^2}{\varrho} & \text{if } \varrho > 0, \\ \infty & \text{otherwise,} \end{cases}$$

we get that the function $g$ is lower semi–continuous and convex and thus, from Theorem 1.5.5, condition (i), we deduce that its epigraph is a closed convex subset of $L^\gamma(\Omega) \times L^{\frac{2\gamma}{\gamma+1}}(\Omega; \mathbb{R}^d) \times \mathbb{R}$ for all $\gamma > \frac{d}{2}$.

From our choice of $k$, we can use the Sobolev embedding (1.1.10) with $p = 2$ to conclude that

$$C_{\text{weak,loc}}([0, \infty); L^r(\Omega)) \hookrightarrow C_{\text{loc}}([0, \infty); W^{-k,2}(\Omega)) \hookrightarrow \mathfrak{D}([0, \infty); W^{-k,2}(\Omega)),$$

for every $r \geq 1$. Furthermore, due to the weak continuity of the density $\varrho$ and the momentum $\mathbf{m}$, for every fixed $T > 0$ and every $t \in [0, T]$, from the energy inequality we can deduce that

$$\|\varrho(t, \cdot)\|_{L^\gamma(\Omega)} \leq \sup_{t \in [0,T]} \|\varrho(t, \cdot)\|_{L^\gamma(\Omega)} \leq c \sup_{t \in [0,T]} \|1 + P(\varrho)(t, \cdot)\|_{L^1(\Omega)} \leq c(E_0, \Omega),$$

$$\begin{aligned} \|\mathbf{m}(t, \cdot)\|_{L^{\frac{2\gamma}{\gamma+1}}(\Omega; \mathbb{R}^d)} &\leq \sup_{t \in [0,T]} \|\mathbf{m}(t, \cdot)\|_{L^{\frac{2\gamma}{\gamma+1}}(\Omega; \mathbb{R}^d)} \\ &\leq \operatorname*{ess\,sup}_{t \in (0,T)} \left\| \frac{\mathbf{m}}{\sqrt{\varrho}}(t, \cdot) \right\|_{L^2(\Omega; \mathbb{R}^d)} \|\sqrt{\varrho}(t, \cdot)\|_{L^{2\gamma}(\Omega)} \leq c(E_0, \Omega). \end{aligned}$$

Finally, from condition (i) of Definition 4.2.1 we also have that $\varrho(t, \cdot) \geq 0$ for all $t \geq 0$, while relation

$$\int_\Omega \left[ \frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \right] (t, \cdot) \, \mathrm{d}x \leq E(t-) = E(t)$$

holds for all $t \geq 0$ since the energy is convex and $\varrho$ and $\mathbf{m}$ are weakly continuous in time. In particular, we have that for every $t \geq 0$

$$[\varrho(t, \cdot), \mathbf{m}(t, \cdot), E(t)] \in X.$$

### 4.3.2 *Main result*

Keeping in mind the notation introduced in the previous section, we are now ready to state our main result.

**Theorem 4.3.1.** *The compressible Navier-Stokes system* (4.1.1)–(4.1.7) *admits a semiflow selection* U *in the class of dissipative weak solutions, i.e., there exists a Borel measurable map* $\mathrm{U} : X \to \mathcal{T}$ *such that*

$$\mathrm{U}[\varrho_0, \mathbf{m}_0, E_0] \in \mathcal{U}[\varrho_0, \mathbf{m}_0, E_0] \text{ for any } [\varrho_0, \mathbf{m}_0, E_0] \in X$$

*satisfying the semigroup property: for any* $[\varrho_0, \mathbf{m}_0, E_0] \in X$ *and any* $t_1, t_2 \geq 0$

$$\mathrm{U}[\varrho_0, \mathbf{m}_0, E_0](t_1 + t_2) = \mathrm{U}[\,\mathrm{U}[\varrho_0, \mathbf{m}_0, E_0](t_1)\,](t_2).$$

Theorem 4.3.1 is a consequence of Theorem 3.1.2 once we have verified that $\mathcal{U}$ satisfies properties (P1)–(P5). To this end, we emphasise the following points.

- Property (P1) is equivalent to showing the *existence* of a dissipative weak solution in the sense of Definition 4.2.1 for any fixed initial data $[\varrho_0, \mathbf{m}_0, E_0] \in X$. This is the main result achieved in Feireisl [33], Theorem 7.1, which we report for reader's convenience.

  **Proposition 4.3.2.** *Let* $[\varrho_0, \boldsymbol{m}_0, E_0] \in X$ *be given; then the Navier-Stokes system* (4.1.1)–(4.1.7) *admits a dissipative weak solution in the sense of Definition 4.2.1 with the initial data* $[\varrho_0, \boldsymbol{m}_0, E_0]$.

- Properties (P2) and (P3) hold true if we manage to prove the *weak sequential stability* of the solution set $\mathcal{U}[\varrho_0, \mathbf{m}_0, E_0]$ for every $[\varrho_0, \mathbf{m}_0, E_0] \in X$ fixed, since it will in particular imply compactness and the closed-graph property of the mapping

$$X \ni [\varrho_0, \mathbf{m}_0, E_0] \to \mathcal{U}[\varrho_0, \mathbf{m}_0, E_0] \in 2^{\mathcal{T}},$$

  and thus the Borel–measurality of $\mathcal{U}$, cf. Lemma 1.5.9. Again, the weak sequential stability of the solution set has already been proved in Feireisl [33], Theorems 6.1, 6.2, which we report for reader's convenience.

  **Proposition 4.3.3.** *Suppose that* $\{\varrho_{0,n}, \mathbf{m}_{0,n}, E_{0,n}\}_{n\in\mathbb{N}} \subset X$ *is a sequence of initial data giving rise to a family of dissipative weak solutions* $\{\varrho_n, \mathbf{m}_n, E_n\}_{n\in\mathbb{N}}$ *to problem* (4.1.1)–(4.1.7) *in the sense of Definition 4.2.1, that is,* $[\varrho_n, \mathbf{m}_n, E_n] \in \mathcal{U}[\varrho_{0,n}, \mathbf{m}_{0,n}, E_{0,n}]$ *for any* $n \in \mathbb{N}$. *Moreover, we assume that the initial densities converge strongly*

$$\varrho_{0,n} \to \varrho_0 \quad \text{in } L^\gamma(\Omega)$$

  *as* $n \to \infty$ *and there exists a constant* $\overline{E} > 0$ *such that* $E_{0,n} \le \overline{E}$ *for any* $n \in \mathbb{N}$.

  *Then, passing to suitable subsequences as the case may be,*

$$\begin{aligned} \mathbf{m}_{0,n} &\rightharpoonup \mathbf{m}_0 \quad \text{in } L^{\frac{2\gamma}{\gamma+1}}(\Omega; \mathbb{R}^d), \\ E_{0,n} &\to E_0 \end{aligned}$$

  *as* $n \to \infty$, *and*

$$\begin{aligned} \varrho_n &\to \varrho \quad \text{in } C_{\text{weak,loc}}([0,\infty); L^\gamma(\Omega)) \\ \mathbf{m}_n &\to \mathbf{m} \quad \text{in } C_{\text{weak,loc}}([0,\infty); L^{\frac{2\gamma}{\gamma+1}}(\Omega; \mathbb{R}^d)) \\ E_n &\to E \quad \text{in } \mathfrak{D}([0,\infty); \mathbb{R}) \end{aligned}$$

  *as* $n \to \infty$, *where*

$$[\varrho, \mathbf{m}, E] \in \mathcal{U}[\varrho_0, \mathbf{m}_0, E_0].$$

- Property (P4) is equivalent to showing the following lemma.

  **Lemma 4.3.4.** *Let* $[\varrho_0, \mathbf{m}_0, E_0] \in X$ *and* $[\varrho, \mathbf{m}, E] \in \mathcal{U}[\varrho_0, \mathbf{m}_0, E_0]$. *Then we have*

$$S_T \circ [\varrho, \mathbf{m}, E] \in \mathcal{U}[\varrho(T), \mathbf{m}(T), E(T)]$$

  *for any* $T > 0$.

  *Proof.* Clearly, a dissipative weak solution on the time interval $(0, \infty)$ solves also the same problem on $(T, \infty)$ with the initial data $[\varrho(T), \mathbf{m}(T), E(T+)]$. Shifting the test functions in the integrals, this implies

$$S_T \circ [\varrho, \mathbf{m}, E] \in \mathcal{U}[\varrho(T), \mathbf{m}(T), E(T+)].$$

Since the energy is non-increasing and caglad, we can choose $E(T)$ as initial energy; indeed, everything will be well defined

$$S_T \circ E(0-) = E(T) = E(T-) \geq E(T+).$$

□

- Property (P5) is equivalent to showing the following lemma.

**Lemma 4.3.5.** *Let* $[\varrho_0, \mathbf{m}_0, E_0] \in X$ *and*

$$[\varrho^1, \mathbf{m}^1, E^1] \in \mathcal{U}[\varrho_0, \mathbf{m}_0, E_0], \ [\varrho^2, \mathbf{m}^2, E^2] \in \mathcal{U}[\varrho^1(T), \mathbf{m}^1(T), E^1(T)].$$

*Then*

$$[\varrho^1, \mathbf{m}^1, E^1] \cup_T [\varrho^2, \mathbf{m}^2, E^2] \in \mathcal{U}[\varrho_0, \mathbf{m}_0, E_0].$$

*Proof.* We are simply pasting two solutions together at the time $T$, letting the second start from the point reached by the first one at the time $T$; thus the integral identities remain satisfied. Choosing the initial energy for $[\varrho^2, \mathbf{m}^2, E^2]$ equal to $E^1(T) = E^1(T-)$, the energy of the solution $[\varrho^1, \mathbf{m}^1, E^1] \cup_T [\varrho^2, \mathbf{m}^2, E^2]$ remains a non-increasing càglàd function defined on $[0, \infty)$. □

### 4.3.3 *Selection of admissible solutions*

If we want to select only the admissible solutions in the sense of Definition 4.2.2, it is sufficient to start the selection considering in (3.1.5) the functional $I_{1,k}$ with the function $f$ strictly increasing and such that

$$f\left(\langle \mathrm{u}(t); e_k \rangle\right) = f(E(t)) \quad \text{for all } t \geq 0,$$

where $\mathrm{u}(t) = [\varrho(t,\cdot), \mathbf{m}(t,\cdot), E(t)]$.

**Lemma 4.3.6.** *Let* $f : \mathbb{R} \to \mathbb{R}$ *be smooth, bounded and strictly increasing. Suppose that* $[\varrho, \mathbf{m}, E] \in \mathcal{U}[\varrho_0, \mathbf{m}_0, E_0]$ *satisfies*

$$\int_0^\infty e^{-t} f(E(t)) \, \mathrm{d}t \leq \int_0^\infty e^{-t} f(\tilde{E}(t)) \, \mathrm{d}t \tag{4.3.2}$$

*for any* $[\tilde{\varrho}, \tilde{\mathbf{m}}, \tilde{E}] \in \mathcal{U}[\varrho_0, \mathbf{m}_0, E_0]$. *Then* $[\varrho, \mathbf{m}, E]$ *is* $\prec$ *minimal, meaning, admissible.*

*Proof.* We proceed by contradiction. Let $[\tilde{\varrho}, \tilde{\mathbf{m}}, \tilde{E}] \in \mathcal{U}[\varrho_0, \mathbf{m}_0, E_0]$ be such that $[\tilde{\varrho}, \tilde{\mathbf{m}}, \tilde{E}] \prec [\varrho, \mathbf{m}, E]$, that is, $\tilde{E} \leq E$ in $[0, \infty)$. Then, since $f$ is strictly increasing, $f(\tilde{E}(t)) \leq f(E(t))$ for every $t \in [0, \infty)$, which implies that $e^{-t}[f(E(t)) - f(\tilde{E}(t))] \geq 0$. Using the monotonicity of the integral, we obtain

$$\int_0^\infty e^{-t} [f(E(t)) - f(\tilde{E}(t))] \, \mathrm{d}t \geq 0;$$

on the other side, condition (4.3.2) tells us that

$$\int_0^\infty e^{-t} [f(E(t)) - f(\tilde{E}(t))] \, \mathrm{d}t \leq 0.$$

The only possibility is to have the equality in both the integral relations above and thus $f(E(t)) = f(\tilde{E}(t))$ for a.e. $t \in (0, \infty)$; since $f$ is strictly increasing, this implies $E = \tilde{E}$ a.e. in $(0, \infty)$. □

## 4.4 RESTRICTION TO SEMIGROUP ACTING ONLY ON THE INITIAL DATA

As a matter of fact, the semiflow selection $\mathrm{U} = \mathrm{U}\{\varrho_0, \mathbf{m}_0, E_0\}$ is determined in terms of the *three* state variables: the density $\varrho$, the momentum $\mathbf{m}$, and the energy $E$. Introduction of the energy might be superfluous; indeed, as pointed out in (4.2.3)

$$E(\tau) = \int_\Omega \left[ \frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \right] (\tau, \cdot) \, \mathrm{d}x \quad \text{for a.e. } \tau \geq 0.$$

The point is that the equality holds with the exception of a zero measure set of times. More specifically, the energy $E(\tau)$ is a càglàd non-increasing function with well-defined right and left limits $E(\tau\pm)$, while

$$\int_\Omega \left[ \frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \right] (\tau, \cdot) \, \mathrm{d}x$$

is defined at *any* $\tau$ in terms of weakly continuous functions $t \mapsto \varrho(t, \cdot)$, $t \mapsto \mathbf{m}(t, \cdot)$. Due to the convexity of the superposition

$$[\varrho, \mathbf{m}] \mapsto \frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho)$$

the function

$$\tau \mapsto \int_\Omega \left[ \frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \right] (\tau, \cdot) \, \mathrm{d}x$$

is lower semi–continuous in $\tau$. In particular,

$$\int_\Omega \left[ \frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \right] (\tau, \cdot) \, \mathrm{d}x \leq E(\tau\pm) \quad \text{for any } \tau,$$

where equality holds with the exception of a set of times of measure zero.

We may introduce a new selection defined only in terms of the initial data $\varrho_0$, $\mathbf{m}_0$; however, the price to pay is that the semigroup property will hold almost everywhere in time. More specifically, we can state this final result.

**Theorem 4.4.1.** *Let* $\mathrm{U} : X \to \mathcal{T}$, $\mathrm{U} = \mathrm{U}[\varrho_0, \mathbf{m}_0, E_0]$ *be the semiflow selection associated to the compressible Navier-Stokes system* (4.1.1)–(4.1.7) *stated in Theorem 4.3.1. Consider the set of initial data*

$$\widetilde{X} = \left\{ [\varrho_0, \mathbf{m}_0] : \ \left[ \varrho_0, \mathbf{m}_0, \int_\Omega \left( \frac{1}{2} \frac{|\mathbf{m}_0|^2}{\varrho_0} + P(\varrho_0) \right) dx \right] \in X \right\}.$$

*Defining* $\mathrm{V} : \widetilde{X} \to \mathcal{T}$ *such that*

$$\mathrm{V}[\varrho_0, \mathbf{m}_0](t) = \mathrm{U} \left[ \varrho_0, \mathbf{m}_0, \int_\Omega \left( \frac{1}{2} \frac{|\mathbf{m}_0|^2}{\varrho_0} + P(\varrho_0) \right) \mathrm{d}x \right] (t)$$

*for all* $t \in (0, \infty)$, *then* V *will satisfy the semigroup property only almost everywhere; more precisely, calling* $\mathcal{T} \subset (0, \infty)$ *the set of times defined as*

$$\mathcal{T} = \left\{ \tau \in (0, \infty) : \ E(\tau) = \int_\Omega \left[ \frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \right] (\tau, \cdot) \, \mathrm{d}x \right\},$$

*then* $\mathcal{T}$ *is a set of full measure and*

$$\mathrm{V}[\varrho_0, \mathbf{m}_0](t_1 + t_2) = \mathrm{V}[\ \mathrm{V}[\varrho_0, \mathbf{m}_0](t_1)\ ](t_2)$$

*holds for all* $t_1, t_2 \in \mathcal{T}$.

# 5 GENERAL COMPRESSIBLE VISCOUS FLUID

In this chapter we aim to prove the existence of a semiflow selection for a compressible non-Newtonian fluid in the class of dissipative solutions. Specifically, we assume a barotropic pressure $p(\varrho) = a\varrho^\gamma$, $\gamma \geq 1$, and the viscous stress tensor $\mathbb{S}$ to be related to the symmetric velocity gradient $\mathbb{D}_x\mathbf{u}$ through a general implicit rheological law. To verify the validity of Theorem 3.1.2 in this context and more precisely, in order to verify the validity of properties (P1)–(P5) of Section 3.1.1 for the set-valued map $\mathcal{U}$ that associates to every initial data the family of dissipative solutions emanating from it, there are two main problems to handle: the weak sequential stability and the existence of dissipative solutions for $\gamma = 1$, which may be of independent interest.

The chapter is organized as follows. In Section 5.1 we introduce the system, while in Section 2.5 we give the definition of dissipative solution, which can be seen as a dissipative weak solution of the problem with an extra term appearing in the balance of momentum and energy inequality, representing a concentration measure. In Section 5.3 we fix a proper setting and state the existence of a semiflow selection, cf. Theorem 5.3.2. Section 5.4 is devoted to the proof of the weak sequential stability of the solution set, cf. Theorem 5.4.1, based on the family of a priori estimates that can deduced assuming the initial energies to be uniformly bounded. In Section 5.5 we focus on showing the existence of dissipative solutions when the pressure is a linear function of the density, cf. Theorem 5.5.12; more precisely, we perform a three-level approximation scheme: addition of artificial viscosity terms in the continuity equation and balance of momentum in order to convert the hyperbolic system into a parabolic one, regularization of the convex potential to make it continuously differentiable, approximation via the Faedo-Galerkin technique and a family of finite-dimensional spaces. Finally, in Section 5.6 we study under which conditions it is possible to guarantee the existence of dissipative weak solutions for $\gamma = 1$.

## 5.1 THE SYSTEM

A mathematical model of compressible viscous fluids can be described by the following system

$$\partial_t\varrho + \operatorname{div}_x(\varrho\mathbf{u}) = 0, \tag{5.1.1}$$

$$\partial_t(\varrho\mathbf{u}) + \operatorname{div}_x(\varrho\mathbf{u}\otimes\mathbf{u}) + \nabla_x p(\varrho) = \operatorname{div}_x \mathbb{S}. \tag{5.1.2}$$

Analogously to the compressible Navier-Stokes system examined in the previous chapter, the unknown variables are the density $\varrho = \varrho(t,x)$ and the velocity $\mathbf{u} = \mathbf{u}(t,x)$, while the viscous stress tensor $\mathbb{S}$ is assumed to be connected to the symmetric velocity gradient $\mathbb{D}_x\mathbf{u}$ through Fenchel's identity

$$\mathbb{S} : \mathbb{D}_x\mathbf{u} = F(\mathbb{D}_x\mathbf{u}) + F^*(\mathbb{S}), \tag{5.1.3}$$

where, denoting with $\mathbb{R}^{d\times d}_{\mathrm{sym}}$ the space of $d$-dimensional real symmetric tensors,

$$F : \mathbb{R}^{d\times d}_{\mathrm{sym}} \to [0,\infty) \text{ is convex and lower semi-continuous with } F(0) = 0, \tag{5.1.4}$$

and $F^*$ is its conjugate. Furthermore, following [2], Section 2.1.2, we will suppose $F$ to satisfy relation

$$F(\mathbb{D}) \geq \mu \left| \mathbb{D} - \frac{1}{d} \operatorname{Tr}[\mathbb{D}]\mathbb{I} \right|^q - c \quad \text{for all } \mathbb{D} \in \mathbb{R}^{d\times d}_{\text{sym}}, \tag{5.1.5}$$

for some $\mu > 0$, $c > 0$ and $q > 1$. Notice that condition (5.1.3) is equivalent in requiring

$$\mathbb{S} \in \partial F(\mathbb{D}_x \mathbf{u}),$$

where $\partial$ denotes the subdifferential of a convex function.

Regarding pressure, for simplicity we will consider the standard isentropic case

$$p(\varrho) = a\varrho^\gamma, \quad \gamma \geq 1, \tag{5.1.6}$$

with $a$ a positive constant; however, more general EOS preserving the essential features of (5.1.6) such as (4.1.4) can be considered. The pressure potential $P$, satisfying the ODE

$$\varrho P'(\varrho) - P(\varrho) = p(\varrho),$$

will be of the form

$$P(\varrho) = \begin{cases} a\ \varrho \log \varrho & \text{if } \gamma = 1, \\ \frac{a}{\gamma-1}\varrho^\gamma & \text{if } \gamma > 1; \end{cases} \tag{5.1.7}$$

in particular, this implies that

$$P \text{ is a strictly convex superlinear continuous function on } [0,\infty). \tag{5.1.8}$$

We will study the system on the set

$$(t,x) \in (0,\infty) \times \Omega,$$

where the physical domain $\Omega \subset \mathbb{R}^d$ is assumed to be bounded and Lipschitz, on the boundary of which we impose the no–slip condition

$$\mathbf{u}|_{\partial\Omega} = 0. \tag{5.1.9}$$

Finally, we fix the initial conditions

$$\varrho(0,\cdot) = \varrho_0, \quad (\varrho\mathbf{u})(0,\cdot) = \mathbf{m}_0. \tag{5.1.10}$$

We conclude this section with the following result, collecting the significant properties of the conjugate function $F^*$.

**Proposition 5.1.1.** *Let the function F satisfy conditions* (5.1.4). *Then, its conjugate*

$$F^* : \mathbb{R}^{d\times d}_{\text{sym}} \to [0,\infty] \text{ is convex, lower semi-continuous and superlinear.} \tag{5.1.11}$$

*Proof.* First of all, we recall that $F^*$ is defined for every $\mathbb{A} \in \mathbb{R}^{d\times d}_{\text{sym}}$ as

$$F^*(\mathbb{A}) := \sup_{\mathbb{B}\in\mathbb{R}^{d\times d}_{\text{sym}}} \{\mathbb{A} : \mathbb{B} - F(\mathbb{B})\}.$$

The non-negativity of $F^*$ is trivial if $F(0) = 0$ since

$$F^*(\mathbb{A}) \geq \mathbb{A} : 0 - F(0) = 0 \quad \text{for every } \mathbb{A} \in \mathbb{R}^{d\times d}_{\text{sym}}.$$

It is also well-know that the conjugate is convex and lower semi-continuous as it is the supremum of a family of affine functions. It remains to prove the superlinearity:

$$\lim_{|\mathbb{A}|\to\infty} \frac{F^*(\mathbb{A})}{|\mathbb{A}|} = +\infty. \tag{5.1.12}$$

Let $B_R(0)$ be the ball centred at origin and radius $R > 0$; using the fact that for any $\mathbb{A} \in \mathbb{R}^{d\times d}_{\text{sym}}$

$$\sup_{\mathbb{B}\in B_R(0)} \mathbb{A} : \mathbb{B} = \sup_{\mathbb{B}\in B_R(0)} \{\mathbb{A} : \mathbb{B} - F(\mathbb{B}) + F(\mathbb{B})\} \leq F^*(\mathbb{A}) + \sup_{\mathbb{B}\in B_R(0)} F(\mathbb{B})$$

we have

$$\frac{F^*(\mathbb{A})}{|\mathbb{A}|} \geq \sup_{\substack{0<r\leq R\\ |\mathbb{V}|\leq 1}} \left\{ r\frac{\mathbb{A}}{|\mathbb{A}|} : \mathbb{V} \right\} - \frac{1}{|\mathbb{A}|} \sup_{\mathbb{B}\in B_R(0)} F(\mathbb{B}) \geq R - \frac{c}{|\mathbb{A}|},$$

where we used the fact that $F(\mathbb{B})$ is finite for any $\mathbb{B} \in \mathbb{R}^{d\times d}_{\text{sym}}$. We conclude that

$$\liminf_{|\mathbb{A}|\to\infty} \frac{F^*(\mathbb{A})}{|\mathbb{A}|} \geq R,$$

and, since $R$ can be chosen arbitrarily large, we obtain (5.1.12). □

## 5.2 DISSIPATIVE SOLUTION

Following the work of Abbatiello, Feireisl and Novotný [2], we introduce to the concept of dissipative solutions. From now on, it is better to consider the density $\varrho$ and the momentum $\mathbf{m} = \varrho\mathbf{u}$ as state variables, since they are at least weakly continuous in time.

**Definition 5.2.1.** The pair of functions $[\varrho, \mathbf{m}]$ constitutes a *dissipative solution* to the problem (5.1.1)–(5.1.10) with the total energy $E$ and initial data

$$[\varrho_0, \mathbf{m}_0, E_0] \in L^\gamma(\Omega) \times L^{\frac{2\gamma}{\gamma+1}}(\Omega; \mathbb{R}^d) \times [0, \infty)$$

if the following holds:

(i) $\varrho \geq 0$ in $(0, \infty) \times \Omega$ and

$$[\varrho, \mathbf{m}, E] \in C_{\text{weak,loc}}([0, \infty); L^\gamma(\Omega)) \times C_{\text{weak,loc}}([0, \infty); L^{\frac{2\gamma}{\gamma+1}}(\Omega; \mathbb{R}^d)) \times \mathfrak{D}([0, \infty));$$

(ii) the integral identity

$$\left[ \int_\Omega \varrho\varphi(t, \cdot) \,\mathrm{d}x \right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega [\varrho\partial_t\varphi + \mathbf{m} \cdot \nabla_x\varphi] \,\mathrm{d}x\mathrm{d}t \tag{5.2.1}$$

holds for any $\tau > 0$ and any $\varphi \in C^1_c([0, \infty) \times \overline{\Omega})$, with $\varrho(0, \cdot) = \varrho_0$;

(iii) there exist

$$\mathbb{S} \in L^1_{\text{loc}}(0, \infty; L^1(\Omega; \mathbb{R}^{d\times d}_{\text{sym}})) \quad \text{and} \quad \mathfrak{R} \in L^\infty_{\text{weak}}(0, \infty; \mathcal{M}^+(\overline{\Omega}; \mathbb{R}^{d\times d}_{\text{sym}}))$$

such that the integral identity

$$\left[\int_{\Omega} \mathbf{m} \cdot \boldsymbol{\varphi}(t,\cdot)\,\mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \left[\mathbf{m} \cdot \partial_t \boldsymbol{\varphi} + \mathbb{1}_{\varrho>0} \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} : \nabla_x \boldsymbol{\varphi} + p(\varrho)\,\mathrm{div}_x\,\boldsymbol{\varphi}\right] \mathrm{d}x\mathrm{d}t$$
$$- \int_0^\tau \int_\Omega \mathbb{S} : \nabla_x \boldsymbol{\varphi}\,\mathrm{d}x\mathrm{d}t + \int_0^\tau \int_{\overline{\Omega}} \nabla_x \boldsymbol{\varphi} : \mathrm{d}\mathfrak{R}\,\mathrm{d}t \tag{5.2.2}$$

holds for any $\tau > 0$ and any $\boldsymbol{\varphi} \in C_c^1([0,\infty) \times \overline{\Omega}; \mathbb{R}^d)$, $\boldsymbol{\varphi}|_{\partial\Omega} = 0$, with $\mathbf{m}(0,\cdot) = \mathbf{m}_0$;

(iv) there exists

$$\mathbf{u} \in L^q_{\mathrm{loc}}(0,\infty; W_0^{1,q}(\Omega;\mathbb{R}^d)) \text{ such that } \mathbf{m} = \varrho\mathbf{u} \text{ a.e. in } (0,\infty)\times\Omega;$$

(v) there exist a constant $\lambda > 0$ and a càglàd function $E$, non–increasing in $[0,\infty)$, satisfying

$$\int_\Omega \left[\frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho)\right](\tau,\cdot)\,\mathrm{d}x + \frac{1}{\lambda}\int_{\overline{\Omega}} \mathrm{d}\,\mathrm{Tr}[\mathfrak{R}(\tau)] = E(\tau) \tag{5.2.3}$$

for a.e. $\tau > 0$, such that the energy inequality

$$[E(t)\psi(t)]_{t=\tau_1^-}^{t=\tau_2^+} - \int_{\tau_1}^{\tau_2} E\,\psi'\,\mathrm{d}t + \int_{\tau_1}^{\tau_2} \psi \int_\Omega [F(\mathbb{D}_x\mathbf{u}) + F^*(\mathbb{S})]\,\mathrm{d}x\mathrm{d}t \le 0 \tag{5.2.4}$$

holds for any $0 \le \tau_1 \le \tau_2$ and any $\psi \in C_c^1[0,\infty)$, $\psi \ge 0$, with $E(0-) = E_0 \ge E(0+)$.

*Remark* 5.2.2. As was proved by Abbatiello, Feireisl and Novotný [2], the notion of solution introduced in Definition 5.2.1 satisfies the *weak-strong uniqueness principle*: a dissipative solution of (5.1.1)–(5.1.10) coincides with the strong solution of the same problem emanating from the same initial data as long as the latter exists, cf. Theorem 6.3 in [2].

The notion of dissipative solution introduced in Definition 5.2.1 is a natural generalization of the original concept of dissipative measure-valued solution, constructed through Young measure and extensively analysed in the first part of this thesis for the compressible Euler system with damping, cf. Section 2.4. As clearly explained in Section 5.4, the concentration measure $\mathfrak{R}$, that we may call *Reynolds stress*, appearing in the weak formulation of the balance of momentum (5.2.2) arises from possible oscillations and/or concentrations in the convective and pressure terms

$$\mathbb{1}_{\varrho>0}\frac{\mathbf{m}\otimes\mathbf{m}}{\varrho} + p(\varrho)\mathbb{I}$$

when $\gamma > 1$, while for $\gamma = 1$, i.e. when the pressure is a linear function of the density $\varrho$, it is only the convective term that contributes to $\mathfrak{R}$. By consistency, we should have introduced the dissipation defect $\mathfrak{E} \in L^\infty_{\mathrm{weak}}(0,\infty;\mathcal{M}^+(\overline{\Omega}))$ of the total energy arising from possible concentrations and/or oscillations in the kinetic and potential energy terms

$$\frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho).$$

Instead of (5.2.3) we would then have

$$E(\tau) = \int_\Omega \left[\frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho)\right](\tau,\cdot)\,\mathrm{d}x + \int_{\overline{\Omega}} \mathrm{d}\mathfrak{E}(\tau), \tag{5.2.5}$$

satisfying the energy inequality (5.2.4). Choosing a positive constant $\lambda > 0$ such that

$$\mathrm{Tr}\left[\mathbb{1}_{\varrho>0}\frac{\mathbf{m}\otimes\mathbf{m}}{\varrho} + \chi(\gamma)p(\varrho)\mathbb{I}\right] = \frac{|\mathbf{m}|^2}{\varrho} + d(\gamma-1)P(\varrho) \le \lambda\left(\frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho)\right)$$

with

$$\chi(\gamma) = \begin{cases} 0 & \text{if } \gamma = 1, \\ 1 & \text{if } \gamma > 1, \end{cases}$$

and adapting Lemma 1.4.6, we could recover the compatibility condition

$$\frac{1}{\lambda}\mathrm{Tr}[\mathfrak{R}(\tau)] \le \mathfrak{E}(\tau) \quad \text{for a.e. } \tau > 0. \tag{5.2.6}$$

In this sense, our choice of the energy (5.2.3) makes the problem more general and easier to handle with only one free quantity instead of two; however, it reduces to (5.2.5) simply choosing a dissipation defect $\mathfrak{E}$ of the type

$$\mathfrak{E}(\tau) := \frac{1}{\lambda}\mathrm{Tr}[\mathfrak{R}(\tau)] \quad \text{for a.e. } \tau > 0.$$

## 5.3 SEMIFLOW SELECTION

Our goal is to prove the existence of a semiflow selection for system (5.1.1)–(5.1.10), adapting the construction of Chapter 3.

### 5.3.1 *Set–up*

As for the compressible Navier-Stokes system, we must fix the space $H$, the phase space $X \subseteq H$ and the map $\mathcal{U}$ introduced at the beginning of Section 3.1.1. Similarly to was done in Section 4.3.1:

- $H := W^{-k,2}(\Omega) \times W^{-k,2}(\Omega;\mathbb{R}^d) \times \mathbb{R}$, where the natural number $k > \frac{d}{2} + 1$ is fixed;
- the phase space $X$ can be chosen as

  $$X := \left\{[\varrho_0, \mathbf{m}_0, E_0] \in H : \ \varrho_0 \in L^1(\Omega),\ \varrho_0 \ge 0,\ \mathbf{m}_0 \in L^1(\Omega;\mathbb{R}^d) \text{ satisfying (5.3.1)}\right\}$$

  where

  $$\int_\Omega \left[\frac{1}{2}\frac{|\mathbf{m}_0|^2}{\varrho_0} + P(\varrho_0)\right]\mathrm{d}x \le E_0; \tag{5.3.1}$$

- $\mathcal{U} : X \to 2^{\mathcal{T}}$, with the trajectory space

  $$\mathcal{T} = \mathfrak{D}([0,\infty); H),$$

  is the set–valued mapping that associate to every $[\varrho_0, \mathbf{m}_0, E_0] \in X$ the family of dissipative solutions in the sense of Definition 5.2.1 arising from the initial data $[\varrho_0, \mathbf{m}_0, E_0]$. More precisely, for every $[\varrho_0, \mathbf{m}_0, E_0] \in X$

  $$\mathcal{U}[\varrho_0, \mathbf{m}_0, E_0] = \left\{ [\varrho, \mathbf{m}, E] \in \mathcal{T} : \begin{array}{l} \text{at any time } t \ge 0, [\varrho(t,\cdot), \mathbf{m}(t,\cdot), E(t)] \in X \\ \text{is a dissipative solution of} \\ \text{problem (5.1.1)–(5.1.10) in the sense of} \\ \text{Definition 5.2.1 with initial data } [\varrho_0, \mathbf{m}_0, E_0] \end{array} \right\}.$$

Notice that everything is well-defined; indeed, proceeding as in Section 4.3.1, $X$ is a closed convex subset of $L^{\gamma}(\Omega) \times L^{\frac{2\gamma}{\gamma+1}}(\Omega; \mathbb{R}^d) \times \mathbb{R}$ for all $\gamma \geq 1$ and for every $t \geq 0$

$$[\varrho(t, \cdot), \mathbf{m}(t, \cdot), E(t)] \in X.$$

### 5.3.2 *Main result*

Keeping in mind the notation introduced in the previous section, we are now ready to state our main result.

**Theorem 5.3.1.** *System* (5.1.1)–(5.1.10) *admits a semiflow selection* U *in the class of dissipative solutions, i.e., there exists a Borel measurable map* $\mathrm{U} : X \to \mathcal{T}$ *such that*

$$\mathrm{U}[\varrho_0, \mathbf{m}_0, E_0] \in \mathcal{U}[\varrho_0, \mathbf{m}_0, E_0] \quad \textit{for every } [\varrho_0, \mathbf{m}_0, E_0] \in X$$

*satisfying the semigroup property: for any* $[\varrho_0, \mathbf{m}_0, E_0] \in X$ *and any* $t_1, t_2 \geq 0$

$$\mathrm{U}[\varrho_0, \mathbf{m}_0, E_0](t_1 + t_2) = \mathrm{U}[\, \mathrm{U}[\varrho_0, \mathbf{m}_0, E_0](t_1) \,](t_2).$$

Theorem 5.3.1 is a consequence of Theorem 3.1.2 once we have verified that $\mathcal{U}$ satisfies properties (P1)–(P5). To this end, we emphasise the following points.

- Property (P1) is equivalent in showing the *existence* of a dissipative solution in the sense of Definition 5.2.1 for any fixed initial data $[\varrho_0, \mathbf{m}_0, E_0] \in X$, which, for $\gamma > 1$, is the main result achieved by Abbatiello, Feireisl and Novotný in [2], Section 3.
- Properties (P2) and (P3) hold true if we manage to prove the *weak sequential stability* of the solution set $\mathcal{U}[\varrho_0, \mathbf{m}_0, E_0]$ for every $[\varrho_0, \mathbf{m}_0, E_0] \in X$ fixed, since it will in particular imply compactness and the closed-graph property of the mapping

$$X \ni [\varrho_0, \mathbf{m}_0, E_0] \to \mathcal{U}[\varrho_0, \mathbf{m}_0, E_0] \in 2^{\mathcal{T}},$$

  and thus the Borel–measurality of $\mathcal{U}$, cf. Lemma 1.5.9.
- Properties (P4) and (P5) can be easily checked following the same arguments of Lemmas 4.3.4 and 4.3.5, respectively.

In conclusion, we are done if we show the existence of a dissipative solution for every $[\varrho_0, \mathbf{m}_0, E_0] \in X$ fixed when $\gamma = 1$ and the weak sequential stability of the solution set $\mathcal{U}[\varrho_0, \mathbf{m}_0, E_0]$ for every $[\varrho_0, \mathbf{m}_0, E_0] \in X$ fixed. The proofs being quite elaborated, they are postponed to the next sections.

*Remark* 5.3.2. As already pointed out for the compressible Navier-Stokes system in Section 4.3.3, among all the dissipative solutions emanating from the same initial data it is possible to select only the *admissible* ones, i.e. satisfying the physical principal of minimizing the total energy in the sense of Definition 4.2.2. Indeed, it is sufficient to start the selection considering in (3.1.5) the functional $I_{1,k}$ with the function $f$ such that

$$f\left(\langle \mathrm{u}(t); e_k \rangle\right) = f(E(t)) \quad \text{for all } t \geq 0,$$

where $\mathrm{u}(t) = [\varrho(t, \cdot), \mathbf{m}(t, \cdot), E(t)]$; see Lemma 4.3.6 for more details.

## 5.4 WEAK SEQUENTIAL STABILITY

This section will be entirely dedicated to the proof of the following result.

**Theorem 5.4.1.** *Let $\{[\varrho_n, \mathbf{m}_n]\}_{n\in\mathbb{N}}$ be a family of dissipative solutions with the corresponding total energies $\{E_n\}_{n\in\mathbb{N}}$ and initial data $\{[\varrho_{0,n}, \mathbf{m}_{0,n}, E_{0,n}]\}_{n\in\mathbb{N}}$ in the sense of Definition 5.2.1. If*

$$[\varrho_{0,n}, \mathbf{m}_{0,n}, E_{0,n}] \to [\varrho_0, \mathbf{m}_0, E_0] \quad \text{in } H,$$

*then, at least for suitable subsequences,*

$$[\varrho_n, \mathbf{m}_n, E_n] \to [\varrho, \mathbf{m}, E] \quad \text{in } \mathfrak{D}([0,\infty); H), \tag{5.4.1}$$

*where $H = W^{-k,2}(\Omega) \times W^{-k,2}(\Omega;\mathbb{R}^d) \times \mathbb{R}$ with the natural number $k > \frac{d}{2}+1$ fixed, and $[\varrho, \mathbf{m}]$ is another dissipative solution of the same problem with total energy $E$ and initial data $[\varrho_0, \mathbf{m}_0, E_0]$.*

The proof will be divided in four steps:

1. in Section 5.4.1 we will first deduce a family of uniform bounds and convergences, including the limits $\varrho$ of the densities, $\mathbf{m}$ of the momenta and $\mathbf{u}$ of the velocities;
2. in Section 5.4.2 we will pass to the limit in the weak formulation of the continuity equation and the balance of momentum;
3. in order to show that $\mathbf{m}$ can be written as the product $\varrho\mathbf{u}$, in Section 5.4.3 we will state and prove Lemma 5.4.2;
4. finally, in Section 5.4.4 we will focus on finding the limit $E$ of the energies.

### 5.4.1 *Uniform bounds and limits establishment*

Our first goal is to show the following convergences, passing to suitable subsequences as the case may be:

$$\varrho_n \to \varrho \quad \text{in } C_{\text{weak, loc}}([0,\infty); L^\gamma(\Omega)), \tag{5.4.2}$$

$$\mathbf{m}_n \to \mathbf{m} \quad \text{in } C_{\text{weak, loc}}([0,\infty); L^{\frac{2\gamma}{\gamma+1}}(\Omega;\mathbb{R}^d)), \tag{5.4.3}$$

$$\mathbf{u}_n \rightharpoonup \mathbf{u} \quad \text{in } L^q_{\text{loc}}(0,\infty; W^{1,q}_0(\Omega;\mathbb{R}^d)) \tag{5.4.4}$$

$$\mathbb{S}_n \rightharpoonup \mathbb{S} \quad \text{in } L^1_{\text{loc}}(0,\infty; L^1(\Omega;\mathbb{R}^{d\times d})), \tag{5.4.5}$$

$$\mathbb{1}_{\varrho_n>0}\frac{\mathbf{m}_n\otimes\mathbf{m}_n}{\varrho_n} \overset{*}{\rightharpoonup} \overline{\mathbb{1}_{\varrho>0}\frac{\mathbf{m}\otimes\mathbf{m}}{\varrho}} \quad \text{in } L^\infty(0,\infty; \mathcal{M}(\overline{\Omega};\mathbb{R}^{d\times d}_{\text{sym}})), \tag{5.4.6}$$

$$p(\varrho_n) \overset{*}{\rightharpoonup} \overline{p(\varrho)} \quad \text{in } L^\infty(0,\infty; \mathcal{M}(\overline{\Omega})) \quad \text{if } \gamma > 1, \tag{5.4.7}$$

$$\mathfrak{R}_n \overset{*}{\rightharpoonup} \widetilde{\mathfrak{R}} \quad \text{in } L^\infty(0,\infty; \mathcal{M}^+(\overline{\Omega};\mathbb{R}^{d\times d}_{\text{sym}})) \tag{5.4.8}$$

as $n\to\infty$. From our hypothesis, all the initial energies are uniformly bounded by a positive constant $\overline{E}$ independent of $n$; specifically,

$$\int_\Omega \left[\frac{1}{2}\frac{|\mathbf{m}_{0,n}|^2}{\varrho_{0,n}} + P(\varrho_{0,n})\right] \mathrm{d}x \le \overline{E}.$$

From (5.2.3) and the energy inequality (5.2.4) it is easy to deduce the following uniform bounds

$$\left\| \frac{\mathbf{m}_n}{\sqrt{\varrho_n}} \right\|_{L^\infty(0,\infty;L^2(\Omega;\mathbb{R}^d))} \leq c_1 = c_1(\overline{E}), \tag{5.4.9}$$

$$\|P(\varrho_n)\|_{L^\infty(0,\infty;L^1(\Omega))} \leq c(\overline{E}), \tag{5.4.10}$$

$$\| \operatorname{Tr}[\mathfrak{R}_n] \|_{L^\infty_{\text{weak}}(0,\infty;\mathcal{M}^+(\overline{\Omega}))} \leq c(\overline{E}), \tag{5.4.11}$$

$$\|F(\mathbb{D}_x \mathbf{u}_n)\|_{L^1((0,\infty)\times\Omega)} \leq c(\overline{E}), \tag{5.4.12}$$

$$\|F^*(\mathbb{S}_n)\|_{L^1((0,\infty)\times\Omega)} \leq c(\overline{E}). \tag{5.4.13}$$

*Convergences of $\varrho_n$ and $\boldsymbol{m}_n$*

For $\gamma > 1$, from (5.1.7), (5.4.10) and the Banach-Alaoglu theorem 1.5.3, we can easily deduce, passing to a suitable subsequence as the case may be,

$$\varrho_n \overset{*}{\rightharpoonup} \varrho \quad \text{in } L^\infty(0,\infty;L^\gamma(\Omega)) \tag{5.4.14}$$

as $n \to \infty$. Similarly, from (5.4.14), (5.4.9) and the fact that, using Hölder's inequality (1.1.1) with $r = \frac{2\gamma}{\gamma+1}$, $p = 2$, $q = 2\gamma$, for a.e. $t > 0$

$$\|\mathbf{m}_n(t,\cdot)\|_{L^{\frac{2\gamma}{\gamma+1}}(\Omega;\mathbb{R}^d)} \leq \left\| \frac{\mathbf{m}_n}{\sqrt{\varrho_n}}(t,\cdot) \right\|_{L^2(\Omega;\mathbb{R}^d)} \|\sqrt{\varrho_n}(t,\cdot)\|_{L^{2\gamma}(\Omega)} \leq c(\overline{E}),$$

passing to a suitable subsequence, we obtain

$$\mathbf{m}_n \overset{*}{\rightharpoonup} \mathbf{m} \quad \text{in } L^\infty(0,\infty;L^{\frac{2\gamma}{\gamma+1}}(\Omega;\mathbb{R}^d)) \tag{5.4.15}$$

as $n \to \infty$.

Since the $L^1$-space is not reflexive, for $\gamma = 1$ a more detailed analysis is needed. If we consider the Young function $\Phi(z) = z \log^+ z$, the densities $\{\varrho_n\}_{n\in\mathbb{N}}$ can be seen as uniformly bounded in $L^\infty(0,\infty;L_\Phi(\Omega))$, where $L_\Phi(\Omega)$ is the Orlicz space associated to $\Phi$; indeed, noticing that

$$\varrho \log^+ \varrho = \begin{cases} 0 & \text{if } 0 \leq \varrho < 1, \\ \varrho \log \varrho & \text{if } \varrho \geq 1, \end{cases} \quad \text{and} \quad -\frac{1}{e} \leq \varrho \log \varrho \leq 0 \quad \text{if } 0 \leq \varrho \leq 1,$$

from (5.4.10), for a.e. $\tau > 0$ we have

$$\begin{aligned} \int_\Omega \varrho \log^+ \varrho(\tau,\cdot) \, \mathrm{d}x &= \int_{\{\varrho \geq 1\}} \varrho \log \varrho(\tau,\cdot) \, \mathrm{d}x \\ &\leq \int_\Omega \varrho \log \varrho(\tau,\cdot) \, \mathrm{d}x - \int_{\{0 \leq \varrho < 1\}} \varrho \log \varrho(\tau,\cdot) \, \mathrm{d}x \\ &\leq c(\overline{E}) + \frac{1}{e}|\Omega|. \end{aligned}$$

As the function $\Phi$ satisfies the $\Delta_2$-condition (1.2.1), $L_\Phi(\Omega) = E_\Phi(\Omega)$ and it can be seen as the dual space of the separable space $E_\Psi(\Omega)$, cf. Theorem 1.2.5, where $\Psi$ denotes the complementary Young function of $\Phi$; hence by the Banach-Alaoglu theorem 1.5.3, passing to a suitable subsequence, we get

$$\varrho_n \overset{*}{\rightharpoonup} \varrho \quad \text{in } L^\infty(0,\infty; L_\Phi(\Omega)) \tag{5.4.16}$$

as $n \to \infty$. We are now able to prove the equi-integrability of the sequence $\{\mathbf{m}_n(t,\cdot)\}_{n\in\mathbb{N}} \subset L^1(\Omega;\mathbb{R}^d)$ for a.e. $t > 0$. More precisely, we want to show that for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon)$ such that for all $n \in \mathbb{N}$ and a.e. $t > 0$

$$\int_M |\mathbf{m}_n|(t,\cdot)\,\mathrm{d}x < \varepsilon \quad \text{for all } M \subset \Omega \text{ such that } |M| < \delta.$$

Fix $\varepsilon > 0$ and choose $\tilde{\varepsilon} = \tilde{\varepsilon}(\varepsilon)$ such that $\tilde{\varepsilon} = (\varepsilon/c_1)^2$, with $c_1$ as in (5.4.9). As a consequence of De la Vallée–Poussin criterion, cf. Theorem 1.2.10, we deduce that that the sequence $\{\varrho_n(t,\cdot)\}_{n\in\mathbb{N}} \subset L^1(\Omega)$ is equi-integrable for a.e. $t > 0$; thus, there exists $\delta = \delta(\tilde{\varepsilon})$ such that for all $n \in \mathbb{N}$ and a.e. $t > 0$,

$$\int_M \varrho_n(t,\cdot)\,\mathrm{d}x < \tilde{\varepsilon} \quad \text{for all } M \subset \Omega \text{ such that } |M| < \delta. \tag{5.4.17}$$

Fix $M \subset \Omega$ with $|M| < \delta$; applying Hölder's inequality (1.1.1) with $r = 1$ and $p = q = 2$, (5.4.9), (5.4.17) and writing

$$\mathbf{m} = \sqrt{\varrho}\,\frac{\mathbf{m}}{\sqrt{\varrho}},$$

we get that for all $n \in \mathbb{N}$ and a.e. $t > 0$

$$\int_M |\mathbf{m}_n|(t,\cdot)\,\mathrm{d}x \le \left(\int_M \varrho_n(t,\cdot)\,\mathrm{d}x\right)^{\frac{1}{2}} \left(\int_M \frac{|\mathbf{m}_n|^2}{\varrho_n}(t,\cdot)\,\mathrm{d}x\right)^{\frac{1}{2}} < c_1\tilde{\varepsilon}^{\frac{1}{2}} = \varepsilon.$$

Dunford–Pettis theorem 1.5.4 ensures that for a.e. $t > 0$ the sequence $\{\mathbf{m}_n(t,\cdot)\}_{n\in\mathbb{N}} \subset L^1(\Omega;\mathbb{R}^d)$ is relatively compact with respect to the weak topology; in particular, we have that

$$\mathbf{m}_n \rightharpoonup \mathbf{m} \quad \text{in } L^1_{\mathrm{loc}}(0,\infty; L^1(\Omega;\mathbb{R}^d))$$

as $n \to \infty$. Next, to get (5.4.2) from (5.4.14) and (5.4.16) we have to show that the family of $t$–dependent functions

$$f_n(t) := \int_\Omega \varrho_n(t,\cdot)\phi\,\mathrm{d}x$$

converges strongly in $C([a,b])$ for any $\phi \in C_c^\infty(\Omega)$ and any compact subset $[a,b] \subset (0,\infty)$. Recalling that the densities $\varrho_n$ and the momenta $\mathbf{m}_n$ are weakly continuous in time, the sequences $\{f_n\}_{n\in\mathbb{N}}$ and $\{f_n'\}_{n\in\mathbb{N}}$ are uniformly bounded in $[a,b]$, since for all $\gamma \ge 1$

$$\sup_{t\in[a,b]} |f_n(t)| \le \sup_{t\in[a,b]} \|\varrho_n(t,\cdot)\|_{L^\gamma(\Omega)} \|\phi\|_{L^{\gamma'}(\Omega)} \le c(\phi),$$

while from the uniform boundedness of the momenta $\mathbf{m}_n$ in $L^\infty(0,\infty; L^p(\Omega;\mathbb{R}^d))$ with $p = \frac{2\gamma}{\gamma+1}$ and $\gamma \ge 1$,

$$\sup_{t\in[a,b]} |f_n'(t)| \le \sup_{t\in[a,b]} \|\mathbf{m}_n(t,\cdot)\|_{L^p(\Omega;\mathbb{R}^d)} \|\nabla_x\phi\|_{L^{p'}(\Omega;\mathbb{R}^d)} \le c(\phi).$$

As a consequence of the Arzelà-Ascoli theorem 1.5.1, we get (5.4.2). A similar argument can be applied to get (5.4.3).

Finally, notice that, from the compact Sobolev embedding (1.1.10) with $p = 2$, convergence (5.4.2) becomes

$$\varrho_n \to \varrho \quad \text{in } C_{\text{loc}}([0,\infty); W^{-k,2}(\Omega))$$

as $n \to \infty$, and thus by Proposition 1.3.9 we get

$$\varrho_n \to \varrho \quad \text{in } \mathfrak{D}([0,\infty); W^{-k,2}(\Omega))$$

as $n \to \infty$. The same argument can be applied to show that

$$\mathbf{m}_n \to \mathbf{m} \quad \text{in } \mathfrak{D}([0,\infty); W^{-k,2}(\Omega; \mathbb{R}^d))$$

as $n \to \infty$.

*Convergences of $\mathbf{u}_n$ and $\mathbb{S}_n$*

From (5.1.5) and (5.4.12) we can also deduce that

$$\left\| \mathbb{D}_x \mathbf{u}_n - \frac{1}{d}(\operatorname{div}_x \mathbf{u}_n)\mathbb{I} \right\|_{L^q((0,\infty)\times\Omega;\mathbb{R}^{d\times d})} \leq c(\overline{E}).$$

Fixing a compact interval $[a,b] \subset (0,+\infty)$ and an open bounded interval $I$ such that $[a,b] \subset I$, the previous inequality combined with the $L^q$-version of the trace-free Korn's inequality (1.1.7) implies

$$\|\nabla_x \mathbf{u}_n\|_{L^q(I\times\Omega;\mathbb{R}^{d\times d})} \leq c(\overline{E});$$

the standard Poincaré inequality (1.1.6) ensures then

$$\|\mathbf{u}_n\|_{L^q(I;W_0^{1,q}(\Omega;\mathbb{R}^d))} \leq c(\overline{E}), \tag{5.4.18}$$

and thus we get convergence (5.4.4).

The superlinearity of $F^*$ (5.1.11) combined with (5.4.13), the De la Vallée–Poussin criterion (1.2.10) and the Dunford–Pettis theorem 1.5.4, gives convergence (5.4.5).

*Convergences of $p(\varrho_n)$, $\mathbb{1}_{\varrho_n>0}\frac{\mathbf{m}_n\otimes\mathbf{m}_n}{\varrho_n}$ and $\mathfrak{R}_n$.*

Notice that in (5.4.7) we don't consider the case $\gamma = 1$ because it reduces to (5.4.2). On the other side, when $\gamma > 1$, estimates (5.4.9) and (5.4.10), combined with the fact that

$$\left| \mathbb{1}_{\varrho>0} \frac{\mathbf{m}\otimes\mathbf{m}}{\varrho} \right| \lesssim \frac{|\mathbf{m}|^2}{\varrho}, \quad p(\varrho) \lesssim (1+P(\varrho)),$$

imply that the pressures $\{p(\varrho_n(t,\cdot))\}_{n\in\mathbb{N}}$ and the convective terms $\left\{\mathbb{1}_{\varrho_n>0}\frac{\mathbf{m}_n\otimes\mathbf{m}_n}{\varrho_n}(t,\cdot)\right\}_{n\in\mathbb{N}}$ are uniformly bounded in the non–reflexive $L^1$–space for a.e. $t > 0$. The idea is then to see the $L^1$–space as embedded in the space of bounded Radon measures $\mathcal{M}(\overline{\Omega})$, which in turn can be identified as the dual space of the separable space $C(\overline{\Omega})$. Accordingly, introducing the space $L^\infty_{\text{weak}}(0,\infty;\mathcal{M}(\overline{\Omega}))$, cf. Definition 1.4.1, we obtain convergences (5.4.6) and (5.4.7). Finally, estimate (5.4.11) guarantees convergence (5.4.8).

### 5.4.2 *Limit passage*

We are now ready to pass to the limit in the weak formulation of the continuity equation and the balance of momentum, obtaining that

$$\left[\int_\Omega \varrho\varphi(t,\cdot)\,\mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau\int_\Omega [\varrho\partial_t\varphi + \mathbf{m}\cdot\nabla_x\varphi]\,\mathrm{d}x\mathrm{d}t$$

holds $\tau > 0$ and any $\varphi \in C_c^1([0,\infty)\times\overline{\Omega})$, with $\varrho(0,\cdot) = \varrho_0$, and

$$\begin{aligned}\left[\int_\Omega \mathbf{m}\cdot\boldsymbol{\varphi}(t,\cdot)\,\mathrm{d}x\right]_{t=0}^{t=\tau} &= \int_0^\tau\int_\Omega \left[\mathbf{m}\cdot\partial_t\boldsymbol{\varphi} + \overline{\mathbb{1}_{\varrho>0}\frac{\mathbf{m}\otimes\mathbf{m}}{\varrho}} : \nabla_x\boldsymbol{\varphi} + \overline{p(\varrho)}\,\mathrm{div}_x\,\boldsymbol{\varphi}\right]\mathrm{d}x\mathrm{d}t\\ &\quad - \int_0^\tau\int_\Omega \mathbb{S} : \nabla_x\boldsymbol{\varphi}\,\mathrm{d}x\mathrm{d}t + \int_0^\tau\int_{\overline{\Omega}} \nabla_x\boldsymbol{\varphi} : \mathrm{d}\widetilde{\mathfrak{R}}\,\mathrm{d}t\end{aligned}$$

holds for any $\tau > 0$ and any $\boldsymbol{\varphi} \in C_c^1([0,\infty)\times\overline{\Omega};\mathbb{R}^d)$, $\boldsymbol{\varphi}|_{\partial\Omega} = 0$, with $\mathbf{m}(0,\cdot) = \mathbf{m}_0$. The last integral identity can be rewritten as

$$\begin{aligned}\left[\int_\Omega \mathbf{m}\cdot\boldsymbol{\varphi}(t,\cdot)\,\mathrm{d}x\right]_{t=0}^{t=\tau} &= \int_0^\tau\int_\Omega \left[\mathbf{m}\cdot\partial_t\boldsymbol{\varphi} + \mathbb{1}_{\varrho>0}\frac{\mathbf{m}\otimes\mathbf{m}}{\varrho} : \nabla_x\boldsymbol{\varphi} + p(\varrho)\,\mathrm{div}_x\,\boldsymbol{\varphi}\right]\mathrm{d}x\mathrm{d}t\\ &\quad - \int_0^\tau\int_\Omega \mathbb{S} : \nabla_x\boldsymbol{\varphi}\,\mathrm{d}x\mathrm{d}t + \int_0^\tau\int_{\overline{\Omega}} \nabla_x\boldsymbol{\varphi} : \mathrm{d}\check{\mathfrak{R}}\,\mathrm{d}t\end{aligned}$$

where $\check{\mathfrak{R}} \in L^\infty_{\text{weak}}(0,\infty;\mathcal{M}(\overline{\Omega};\mathbb{R}^{d\times d}_{\text{sym}}))$ is such that

$$\mathrm{d}\check{\mathfrak{R}} = \mathrm{d}\widetilde{\mathfrak{R}} + \left(\overline{\mathbb{1}_{\varrho>0}\frac{\mathbf{m}\otimes\mathbf{m}}{\varrho}} - \mathbb{1}_{\varrho>0}\frac{\mathbf{m}\otimes\mathbf{m}}{\varrho}\right)\mathrm{d}x + \left(\overline{p(\varrho)} - p(\varrho)\right)\chi(\gamma)\mathbb{I}\,\mathrm{d}x, \tag{5.4.19}$$

with

$$\chi(\gamma) = \begin{cases} 0 & \text{if } \gamma = 1,\\ 1 & \text{if } \gamma > 1.\end{cases}$$

We can prove the stronger condition

$$\check{\mathfrak{R}} \in L^\infty_{\text{weak}}(0,\infty;\mathcal{M}^+(\overline{\Omega};\mathbb{R}^{d\times d}_{\text{sym}})); \tag{5.4.20}$$

more precisely, we want to show that for all $\xi \in \mathbb{R}^d$, all open sets $\mathcal{B} \subset \Omega$ and a.e. $\tau > 0$

$$\check{\mathfrak{R}}(\tau) : (\xi\otimes\xi)(\mathcal{B}) \geq 0.$$

we can rewrite the term on the left–hand side as

$$\begin{aligned}\int_{\mathcal{B}} (\xi\otimes\xi) : \mathrm{d}\check{\mathfrak{R}}(\tau+) &= \lim_{d\to 0}\int_\tau^{\tau+d}\int_{\mathcal{B}} (\xi\otimes\xi) : \mathrm{d}\check{\mathfrak{R}}(t)\,\mathrm{d}t\\ &= \lim_{d\to 0}\int_0^\infty\int_{\overline{\Omega}} \mathbb{1}_{[\tau,\tau+d]\times\mathcal{B}}\,(\xi\otimes\xi) : \mathrm{d}\check{\mathfrak{R}}(t)\,\mathrm{d}t\end{aligned}$$

Since the indicator function $\mathbb{1}_{[\tau,\tau+d]\times\mathcal{B}}$ can be approximated by some non–negative test functions, it is enough to show that

$$\int_0^\infty\int_{\overline{\Omega}} \varphi\,(\xi\otimes\xi) : \mathrm{d}\check{\mathfrak{R}}(t)\,\mathrm{d}t \geq 0$$

holds for all $\varphi \in C_c^\infty((0,T)\times\Omega)$, $\varphi \geq 0$. We can notice that the first term on the right–hand side of (5.4.19) will obviously satisfy the above inequality since $\widetilde{\mathfrak{R}}$ itself belongs to $L^\infty_{\text{weak}}(0,\infty;\mathcal{M}^+(\overline{\Omega};\mathbb{R}^{d\times d}_{\text{sym}}))$, and

$$\int_0^\infty \int_{\overline{\Omega}} \left(\overline{p(\varrho)} - p(\varrho)\right) \mathbb{I} : (\xi\otimes\xi)\ \varphi\ \mathrm{d}x\mathrm{d}t = \int_0^\infty \int_{\overline{\Omega}} \left(\overline{p(\varrho)} - p(\varrho)\right) |\xi|^2\varphi\ \mathrm{d}x\mathrm{d}t \geq 0,$$

since $\varrho \mapsto p(\varrho)$ is a convex lower semi–continuous function and thus $\overline{p(\varrho)} \geq p(\varrho)$, cf. Theorem 1.5.5, condition (ii). Finally, following the same idea developed by Feireisl and Hofmanová [36], Section 3.2, as a consequence of (5.4.6) we can write

$$\begin{aligned}
\int_0^\infty \int_{\overline{\Omega}} &\left(\overline{\mathbb{1}_{\varrho>0}\frac{\mathbf{m}\otimes\mathbf{m}}{\varrho}} - \mathbb{1}_{\varrho>0}\frac{\mathbf{m}\otimes\mathbf{m}}{\varrho}\right) : (\xi\otimes\xi)\ \varphi\ \mathrm{d}x\mathrm{d}t \\
&= \lim_{n\to\infty}\int_0^\infty \int_{\overline{\Omega}} \left(\mathbb{1}_{\varrho_n>0}\frac{\mathbf{m}_n\otimes\mathbf{m}_n}{\varrho_n} - \mathbb{1}_{\varrho>0}\frac{\mathbf{m}\otimes\mathbf{m}}{\varrho}\right) : (\xi\otimes\xi)\ \varphi\ \mathrm{d}x\mathrm{d}t \\
&= \lim_{n\to\infty}\int_0^\infty \int_{\overline{\Omega}} \left(\mathbb{1}_{\varrho_n>0}\frac{|\mathbf{m}_n\cdot\xi|^2}{\varrho_n} - \mathbb{1}_{\varrho>0}\frac{|\mathbf{m}\cdot\xi|^2}{\varrho}\right)\ \varphi\ \mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{5.4.21}$$

The Cauchy–Schwarz inequality allows to write $|\mathbf{m}\cdot\xi|^2 \leq |\mathbf{m}|^2|\xi|^2$, and thus by (5.4.9) we obtain

$$\left\| \mathbb{1}_{\varrho_n>0}\frac{|\mathbf{m}_n\cdot\xi|^2}{\varrho_n}\right\|_{L^\infty(0,\infty;L^1(\Omega))} \leq c(\overline{E},\xi);$$

it is possible then to find the limit

$$\mathbb{1}_{\varrho_n>0}\frac{|\mathbf{m}_n\cdot\xi|^2}{\varrho_n} \overset{*}{\rightharpoonup} \overline{\mathbb{1}_{\varrho>0}\frac{|\mathbf{m}\cdot\xi|^2}{\varrho}} \quad \text{in } L^\infty(0,\infty;\mathcal{M}(\overline{\Omega}))$$

as $n\to\infty$, and rewrite the first line in (5.4.21) as

$$\int_0^\infty \int_{\overline{\Omega}} \left(\overline{\mathbb{1}_{\varrho>0}\frac{|\mathbf{m}\cdot\xi|^2}{\varrho}} - \mathbb{1}_{\varrho>0}\frac{|\mathbf{m}\cdot\xi|^2}{\varrho}\right)\ \varphi\ \mathrm{d}x\mathrm{d}t.$$

As in the previous passage, (5.4.20) will now follow from the weak lower semi–continuity on $X$ of the convex function

$$[\varrho,\mathbf{m}] \mapsto \begin{cases} 0 & \text{if } \mathbf{m}=0, \\ \frac{|\mathbf{m}\cdot\xi|^2}{\varrho} & \text{if } \varrho>0, \\ \infty & \text{otherwise} \end{cases}$$

for any $\xi \in \mathbb{R}^d$ fixed. We proved in particular that the pair of functions $[\varrho,\mathbf{m}]$ satisfies conditions (ii) and (iii) of Definition 5.2.1. However, $\widetilde{\mathfrak{R}}$ has to be slightly modified in order to get the energy (5.2.3), as we will see in Section 5.4.4.

### 5.4.3 *Auxiliary lemma*

In order to prove that

$$\mathbf{m} = \varrho\mathbf{u} \quad \text{a.e. in } (0,\infty)\times\Omega,$$

and in particular to show that $[\varrho,\mathbf{m}]$ satisfy condition (iv) of Definition 5.2.1, we need the following result.

**Lemma 5.4.2.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain. Suppose*

$$\{\varrho_n\}_{n\in\mathbb{N}} \text{ to be uniformly bounded in } L^\infty(0,\infty; L_\Phi(\Omega)),$$

*where $L_\Phi(\Omega)$ is the Orlicz space associated to the Young function $\Phi$ satisfying the $\Delta_2$-condition (1.2.1), with $\varrho_n \geq 0$ for all $n \in \mathbb{N}$. Suppose also that*

$$\{\mathbf{u}_n\}_{n\in\mathbb{N}} \text{ be uniformly bounded in } L^q_{\text{loc}}(0,\infty; W^{1,q}(\Omega;\mathbb{R}^d)),\ q > 1. \tag{5.4.22}$$

*Moreover, let the sequence $\{\varrho_n \mathbf{u}_n\}_{n\in\mathbb{N}}$ be equi–integrable in $L^1_{\text{loc}}(0,\infty; L^1(\Omega;\mathbb{R}^d))$. Then, if*

$$\varrho_n \to \varrho \quad \text{in } C_{\text{weak, loc}}([0,\infty); L^1(\Omega)), \tag{5.4.23}$$

$$\mathbf{u}_n \rightharpoonup \mathbf{u} \quad \text{in } L^q_{\text{loc}}(0,\infty; W^{1,q}(\Omega;\mathbb{R}^d)), \tag{5.4.24}$$

$$\varrho_n\mathbf{u}_n \rightharpoonup \mathbf{m} \quad \text{in } L^1_{\text{loc}}(0,\infty; L^1(\Omega;\mathbb{R}^d)) \tag{5.4.25}$$

*as $n \to \infty$, we have*

$$\mathbf{m} = \varrho\mathbf{u} \quad \text{a.e. in } (0,\infty)\times\Omega. \tag{5.4.26}$$

*Proof.* 1. ***Truncation.*** Following the same idea developed by Abbatiello, Feireisl and Novotný in [2], Lemma 8.1, it is enough to suppose that

$$\{\mathbf{u}_n\}_{n\in\mathbb{N}} \text{ is uniformly bounded in } L^\infty_{\text{loc}}(0,\infty; L^\infty(\Omega;\mathbb{R}^d)). \tag{5.4.27}$$

Indeed, let us consider a bounded cut-off function $T : \mathbb{R}^d \to \mathbb{R}^d$ such that

$$T(-\mathbf{z}) = T(\mathbf{z}), \quad T(\mathbf{z}) = \mathbf{z} \text{ if } |\mathbf{z}| \leq 1 \quad \text{for any } \mathbf{z} \in \mathbb{R}^d,$$

and the family of truncation functions $\{T_k\}_{k\in\mathbb{N}}$ such that

$$T_k(\mathbf{z}) = k\, T\left(\frac{\mathbf{z}}{k}\right) \quad \text{for any } \mathbf{z} \in \mathbb{R}^d.$$

Writing

$$\mathbf{u}_n = T_k(\mathbf{u}_n) + \big(\mathbf{u}_n - T_k(\mathbf{u}_n)\big),$$
$$\varrho_n\mathbf{u}_n = \varrho_n T_k(\mathbf{u}_n) + \varrho_n\big(\mathbf{u}_n - T_k(\mathbf{u}_n)\big),$$

on one side, for any $k \in \mathbb{N}$ fixed we have

$$T_k(\mathbf{u}_n) \overset{*}{\rightharpoonup} \overline{T_k(\mathbf{u})} \quad \text{in } L^\infty_{\text{loc}}(0,\infty; L^\infty(\Omega;\mathbb{R}^d)),$$
$$\varrho_n T_k(\mathbf{u}_n) \rightharpoonup \mathbf{m}_k \quad \text{in } L^1_{\text{loc}}(0,\infty; L^1(\Omega;\mathbb{R}^d))$$

as $n \to \infty$; on the other side, for every $[a,b] \subset [0,\infty)$, using Hôlder's inequality (1.1.1) with $r = 1$, $p = q$ and $q = q'$,

$$\int_a^b \int_\Omega |\mathbf{u}_n - T_k(\mathbf{u}_n)|\, \mathrm{d}x\mathrm{d}t \leq 2 \int\int_{\{|\mathbf{u}_n|\geq k\}} |\mathbf{u}_n|\, \mathrm{d}x\mathrm{d}t$$
$$\leq 2\, |\{|\mathbf{u}_n| \geq k\}|^{\frac{1}{q'}} \|\mathbf{u}_n\|_{L^q((a,b)\times\Omega;\mathbb{R}^d)} \to 0$$

as $k \to \infty$, uniformly in $n \in \mathbb{N}$, and, in view of the equi–integrability of $\{\varrho_n\mathbf{u}_n\}_{n\in\mathbb{N}}$, cf. condition (ii) of Theorem 1.5.4,

$$\int_a^b \int_\Omega \varrho_n|\mathbf{u}_n - T_k(\mathbf{u}_n)|\, \mathrm{d}x\mathrm{d}t \leq 2 \int\int_{\{|\mathbf{u}_n|\geq k\}} \varrho_n|\mathbf{u}_n|\, \mathrm{d}x\mathrm{d}t \to 0,$$

as $k \to \infty$, uniformly in $n \in \mathbb{N}$. Finally, noticing that for every $[a,b] \subset [0,\infty)$

$$\|\mathbf{u} - \overline{T_k(\mathbf{u})}\|_{L^1((a,b)\times\Omega;\mathbb{R}^d)} \le \liminf_{n\to\infty} \|\mathbf{u}_n - T_k(\mathbf{u}_n)\|_{L^1((a,b)\times\Omega;\mathbb{R}^d)} \to 0$$

as $k \to \infty$, (5.4.26) reduces to prove that for every $k \in \mathbb{N}$ fixed

$$\mathbf{m}_k = \varrho\overline{T_k(\mathbf{u})} \quad \text{a.e. in } (0,\infty)\times\Omega.$$

2. ***Regularization.*** We claim that it is sufficient to suppose

$$\{\mathbf{u}_n\}_{n\in\mathbb{N}} \text{ to be uniformly bounded in } L^q_{\mathrm{loc}}(0,\infty; W^{m,r}(\Omega;\mathbb{R}^d)) \tag{5.4.28}$$

with $q > 1$ and $m, r$ arbitrarily large. Seeing all the quantities involved as embedded in $\mathbb{R}^d$ with compact support, we consider regularization in the spatial variable by convolution with a family of regularizing kernels $\{\theta_\delta\}_{\delta>0}$, cf. Definition 1.1.1. As in the previous step, writing

$$\varrho_n\mathbf{u}_n = \varrho_n\, \theta_\delta * \mathbf{u}_n + \varrho_n(\mathbf{u}_n - \theta_\delta * \mathbf{u}_n),$$

our goal is to show that for every $[a,b] \subset (0,\infty)$

$$\int_a^b \int_\Omega \varrho_n|\mathbf{u}_n - \theta_\delta * \mathbf{u}_n| \,\mathrm{d}x\mathrm{d}t \to 0$$

as $\delta \to 0$, uniformly in $n \in \mathbb{N}$. To this end, we introduce the Banach space

$$X = W^{1,q}_0 \cap L^\infty(\Omega;\mathbb{R}^d)$$

and observe that, in view of (5.4.22) and (5.4.27),

$$\{\|\mathbf{u}_n\|_X\}_{n\in\mathbb{N}} \text{ is uniformly bounded in } L^q_{\mathrm{loc}}(0,\infty). \tag{5.4.29}$$

Consequently, we may write

$$\int_a^b \int_\Omega \varrho_n|\mathbf{u}_n - \theta_\delta * \mathbf{u}_n| \,\mathrm{d}x\mathrm{d}t = I_1^M + I_2^M$$

with

$$I_1^M = \int_{\{\|\mathbf{u}_n(t,\cdot)\|_X \le M\}} \int_\Omega \varrho_n|\mathbf{u}_n - \theta_\delta * \mathbf{u}_n| \,\mathrm{d}x\mathrm{d}t,$$
$$I_2^M = \int_{\{\|\mathbf{u}_n(t,\cdot)\|_X > M\}} \int_\Omega \varrho_n|\mathbf{u}_n - \theta_\delta * \mathbf{u}_n| \,\mathrm{d}x\mathrm{d}t,$$

where, in view of (5.4.29) - recall that the functions $\varrho_n$ are weakly continuous in time

$$I_2^M \le c \sup_{t\in[a,b]} \|\varrho_n(t,\cdot)\|_{L^1(\Omega)} \|\mathbf{u}_n\|_{L^\infty((a,b)\times\Omega;\mathbb{R}^d)} \,|\{\|\mathbf{u}_n(t,\cdot)\|_X > M\}| \to 0$$

as $M \to \infty$, uniformly in $n \in \mathbb{N}$ and independently of $\delta > 0$. It remains to show smallness of the first integral for fixed $M$. To this end, denoting with $\Psi$ the complementary Young function of $\Phi$, we consider the Orlicz space $L_\Psi(\Omega)$ that can be identified with the dual of $L_\Phi(\Omega)$ as $\Phi$ satisfies the $\Delta_2$–condition 1.2.1. By Proposition 1.2.9, we recover the compact embedding

$$X \hookrightarrow\hookrightarrow L_\Psi(\Omega;\mathbb{R}^d)$$

which, combined with boundedness of convolution on $E_\Psi(\Omega) \subset L_\Psi(\Omega)$, cf. Lemma 1.2.7, gives

$$I_1^M \leq \sup_{t\in[a,b]} \|\varrho_n(t,\cdot)\|_{L_\Phi(\Omega)} \sup_{\|\mathbf{u}_n(t,\cdot)\|_X\leq M} \|\mathbf{u}_n - \theta_\delta * \mathbf{u}_n\|_{E_\Psi(\Omega;\mathbb{R}^d)} \to 0$$

as $\delta \to 0$, uniformly in $n \in \mathbb{N}$.

3. ***Conclusion.*** Using the compact Sobolev embedding (1.1.10) with $k = 1$ and $p = s > d$, from (5.4.23) we get that

$$\varrho_n \to \varrho \quad \text{in } C_{\text{loc}}([0,\infty); W^{-1,s'}(\Omega)) \quad \text{for any } s > d$$

as $n \to \infty$, and thus, to conclude the proof of the Lemma it is sufficient to choose $m = 1$ and $r = s$ in (5.4.28).

□

### 5.4.4 *Limit of the energies*

From (5.2.3) we can notice that the energies $E_n(\tau)$ are non–increasing and for $\gamma > 1$ they are also non–negative, while for $\gamma = 1$ we have

$$\begin{aligned} E_n(\tau) &\geq \int_\Omega \left[\frac{1}{2}\frac{|\mathbf{m}_n|^2}{\varrho_n} + \mathbb{1}_{\varrho_n\geq 1}\, \varrho\log\varrho\right](\tau,\cdot)\, \mathrm{d}x + \frac{1}{\lambda_n}\int_{\overline{\Omega}} \mathrm{d}\,\mathrm{Tr}[\mathfrak{R}_n(\tau)] + \int_{\{0\leq\varrho_n<1\}} \varrho_n \log\varrho_n\, \mathrm{d}x \\ &\geq \text{“non-negative term”} - \frac{|\Omega|}{e}. \end{aligned}$$

for a.e. $\tau > 0$. Hence, for every $[a,b] \subset (0,\infty)$ and every $n \in \mathbb{N}$

$$\|E_n\|_{L^1[a,b]} \leq \int_0^b |E_n(t)|\, \mathrm{d}t \leq b \sup_{t\in[0,b]} |E_n(t)| \leq b\, E_{0,n} \leq c(\overline{E}),$$

and, since the functions $E_n$ are non-increasing, the total variation on $[a,b]$ is given by

$$V_a^b(E_n) = E_n(a) - E_n(b) \leq \begin{cases} E_{0,n} + \frac{|\Omega|}{e} & \text{if } \gamma = 1 \\ E_{0,n} & \text{if } \gamma > 1 \end{cases} \leq c(\overline{E}).$$

We recover that $\{E_n\}_{n\in\mathbb{N}}$ is locally of bounded variation and by Helly's theorem 1.1.8, passing to a suitable subsequence as the case may be, we obtain

$$E_n(t) \to E(t) \quad \text{for every } t \in [0,\infty) \tag{5.4.30}$$

as $n \to \infty$, which in particular implies

$$E_n \to E \quad \text{in } \mathfrak{D}([0,\infty);\mathbb{R}) \tag{5.4.31}$$

as $n \to \infty$, since $E_n : [0,\infty) \to \mathbb{R}$ is a monotone function for all $n \in \mathbb{N}$ and thus, by Proposition 1.3.9, showing (5.4.31) is equivalent to showing almost everywhere convergence.

On the other side, from (5.4.9), (5.4.10) and (5.4.11) we get

$$\begin{aligned} \frac{|\mathbf{m}_n|^2}{\varrho_n} &\overset{*}{\rightharpoonup} \overline{\frac{|\mathbf{m}|^2}{\varrho}} \quad \text{in } L^\infty(0,\infty;\mathcal{M}(\overline{\Omega})) \\ P(\varrho_n) &\overset{*}{\rightharpoonup} \overline{P(\varrho)} \quad \text{in } L^\infty(0,\infty;\mathcal{M}(\overline{\Omega})) \\ \frac{1}{\lambda_n}\operatorname{Tr}[\mathfrak{R}_n] &\overset{*}{\rightharpoonup} \widetilde{\mathfrak{E}} \quad \text{in } L^\infty(0,\infty;\mathcal{M}^+(\overline{\Omega})) \end{aligned}$$

as $n \to \infty$. We can then write

$$E(\tau) = \int_\Omega \left[\frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho)\right](\tau,\cdot)\,\mathrm{d}x + \int_{\overline{\Omega}} \mathrm{d}\mathfrak{E}(\tau) \tag{5.4.32}$$

for a.e. $\tau > 0$, with

$$\mathrm{d}\mathfrak{E} = \mathrm{d}\widetilde{\mathfrak{E}} + \frac{1}{2}\left(\overline{\frac{|\mathbf{m}|^2}{\varrho}} - \frac{|\mathbf{m}|^2}{\varrho}\right)\mathrm{d}x + \left(\overline{P(\varrho)} - P(\varrho)\right)\mathrm{d}x$$

where, once again, from the convexity of the function $P$ and of the superposition $[\varrho, \mathbf{m}] \mapsto \frac{|\mathbf{m}|^2}{\varrho}$, we get

$$\mathfrak{E} \in L^\infty_{\text{weak}}(0,\infty;\mathcal{M}^+(\overline{\Omega})).$$

As pointed out in Section 5.2, we can choose constant $\lambda > 0$ such that

$$\operatorname{Tr}[\check{\mathfrak{R}}(\tau)] \leq \lambda \mathfrak{E}(\tau) \tag{5.4.33}$$

for a.e. $\tau \in (0,T)$; however, with this choice we only get

$$\int_\Omega \left[\frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho)\right](\tau,\cdot)\,\mathrm{d}x + \frac{1}{\lambda}\int_{\overline{\Omega}} \mathrm{d}\operatorname{Tr}[\check{\mathfrak{R}}(\tau)] \leq E(\tau)$$

for a.e. $\tau \in (0,T)$. To obtain (5.2.3), it is sufficient to define a new defect

$$\mathfrak{R} = \check{\mathfrak{R}} + \psi(t)\mathbb{I},$$

where the function $\psi \geq 0$ of time only can be chosen in such a way that

$$\int_\Omega \left[\frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho)\right](\tau,\cdot)\,\mathrm{d}x + \frac{1}{\lambda}\int_{\overline{\Omega}} \mathrm{d}\operatorname{Tr}[\mathfrak{R}(\tau)] = E(\tau)$$

for a.e. $\tau \in (0,T)$. Clearly,

$$\int_{\overline{\Omega}} \nabla_x \boldsymbol{\varphi} : \mathrm{d}\mathfrak{R} = \int_{\overline{\Omega}} \nabla_x \boldsymbol{\varphi} : \mathrm{d}\check{\mathfrak{R}}$$

for any $\boldsymbol{\varphi} \in C^\infty_c([0,\infty) \times \overline{\Omega}; \mathbb{R}^d)$, $\boldsymbol{\varphi}|_{\partial\Omega} = 0$, and therefore, the weak formulation of the balance of momentum (5.2.2) remains valid.

Finally, notice that the couple $[\varrho, \mathbf{m}]$ satisfies the energy inequality (5.2.4) due to lower semi–continuity of the functions $F$ and $F^*$: for a.e. $\tau > 0$

$$\int_0^\tau \int_\Omega [F(\mathbb{D}_x \mathbf{u}) + F^*(\mathbb{S})]\,\mathrm{d}x\mathrm{d}t \leq \liminf_{n\to\infty} \int_0^\tau \int_\Omega [F(\mathbb{D}_x \mathbf{u}_n) + F^*(\mathbb{S}_n)]\,\mathrm{d}x\mathrm{d}t;$$

in particular, $[\varrho, \mathbf{m}]$ satisfies condition (iv) of Definition 5.2.1.

## 5.5 EXISTENCE FOR LINEAR PRESSURE

As in [2] Abbatiello, Feireisl and Novotný proved the existence of dissipative solutions of system (5.1.1)–(5.1.10) when $\gamma > 1$ in (5.1.6), in this section we aim to show existence for $\gamma = 1$ in (5.1.6). We employ an approximation scheme based on

(i) addition of an artificial viscosity term of the type $\varepsilon\Delta_x\varrho$ in the continuity equation (5.1.1) in order to convert the hyperbolic equation into a parabolic one and thus recover better regularity properties of $\varrho$;

(ii) addition of an extra term of the type $\varepsilon\nabla_x\mathbf{u}\cdot\nabla_x\varrho$ in the balance of momentum (5.1.2) in order to eliminate the extra terms arising in the energy inequality to save the a priori estimates;

(iii) regularization of the convex potential $F$ through convolution with a family of regularizing kernels to make it continuously differentiable.

More precisely, we will study the following system:

- **continuity equation**
$$\partial_t\varrho + \mathrm{div}_x(\varrho\mathbf{u}) = \varepsilon\Delta_x\varrho, \tag{5.5.1}$$
on $(0,T)\times\Omega$, with $\varepsilon > 0$, the homogeneous Neumann boundary condition
$$\nabla_x\varrho\cdot\mathbf{n} = 0 \quad \text{on } \partial\Omega, \tag{5.5.2}$$
and the initial condition
$$\varrho(0,\cdot) = \varrho_{0,n} \quad \text{on } \Omega, \quad \varrho_{0,n} \to \varrho_0 \text{ in } L^1(\Omega) \text{ as } n\to\infty, \tag{5.5.3}$$
with $\varrho_{0,n}\in C(\overline{\Omega})$, $\varrho_{0,n} > 0$ for all $n\in\mathbb{N}$.

- **momentum equation**
$$\partial_t(\varrho\mathbf{u}) + \mathrm{div}_x(\varrho\mathbf{u}\otimes\mathbf{u}) + a\nabla_x\varrho + \varepsilon\nabla_x\mathbf{u}\cdot\nabla_x\varrho = \mathrm{div}_x\,\mathbb{S} \tag{5.5.4}$$
on $(0,T)\times\Omega$, with $\varepsilon > 0$, the no-slip boundary condition
$$\mathbf{u}|_{\partial\Omega} = 0 \quad \text{on } \partial\Omega, \tag{5.5.5}$$
and the initial condition
$$(\varrho\mathbf{u})(0,\cdot) = \mathbf{m}_0 \quad \text{on } \Omega. \tag{5.5.6}$$

- **convex potential**
$$F_\delta(\mathbb{D}) = (\xi_\delta * F)(\mathbb{D}) - \inf_{\mathbb{D}\in\mathbb{R}^{d\times d}_{\mathrm{sym}}}(\xi_\delta * F) \tag{5.5.7}$$
for any $\mathbb{D}\in\mathbb{R}^{d\times d}_{\mathrm{sym}}$, with $\{\xi_\delta\}_{\delta>0}$ a family of regularizing kernels in $\mathbb{R}^{d\times d}_{\mathrm{sym}}$ as in Definition 1.1.1, the function $F$ satisfying (5.1.4)–(5.1.5), and such that
$$\mathbb{S} : \mathbb{D}_x\mathbf{u} = F_\delta(\mathbb{D}_x\mathbf{u}) + F^*_\delta(\mathbb{S}). \tag{5.5.8}$$

Even if system (5.5.1)–(5.5.8) is of parabolic type, we are forced to perform a further approximation known as *Faedo-Galerkin technique*. The reason is that the unknown state variable $\mathbf{u}$ appears multiplied by $\varrho$ in (5.5.4), which prevents us from applying the already existing results for parabolic systems that can be found in literature. The idea is to consider a family $\{X_n\}_{n\in\mathbb{N}}$ of finite-dimensional spaces $X_n \subset L^2(\Omega;\mathbb{R}^d)$, such that

$$X_n := \operatorname{span}\{\mathbf{w}_i |\ \mathbf{w}_i \in C_c^\infty(\Omega;\mathbb{R}^d),\ i = 1,\dots,n\},$$

where $\mathbf{w}_i$ are orthonormal with respect to the standard scalar product in $L^2(\Omega;\mathbb{R}^d)$, and to look for approximated velocities

$$\mathbf{u}_n \in C([0,T];X_n).$$

Solvability of the approximated problem will be discussed in the following sections.

### 5.5.1 *On the approximated continuity equation*

Given $\mathbf{u} \in C([0,T];X_n)$, let us focus on identifying that unique solution

$$\varrho = \varrho[\mathbf{u}]$$

of system (5.5.1)–(5.5.3). As our domain $\Omega$ is merely Lipschitz, we cannot simply repeat the same passages performed for instance by Feireisl [33] in the context of the compressible Navier-Stokes system since better regularity for the domain would be required. However, since $X_n$ is finite-dimensional, all the norms on $X_n$ induced by $W^{k,p}$-norms, with $k \in \mathbb{N}$ and $1 \le p \le \infty$, are equivalent and thus, we deduce that

$$\mathbf{u} \in L^\infty(0,T;W^{1,\infty}(\Omega;\mathbb{R}^d)),$$

and there exist two constants $0 < \underline{n} < \overline{n} < \infty$, depending solely on the dimension $n$ of $X_n$, such that for any $t \in [0,T]$

$$\underline{n}\|\mathbf{u}(t,\cdot)\|_{W^{1,\infty}(\Omega)} \le \|\mathbf{u}(t,\cdot)\|_{X_n} \le \overline{n}\|\mathbf{u}(t,\cdot)\|_{W^{1,\infty}(\Omega)}. \tag{5.5.9}$$

It is now enough to apply the following result by Crippa, Donadello and Spinolo [27], Lemma 3.2, to get the existence of weak solutions.

**Lemma 5.5.1.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. For any given $\mathbf{u} \in C([0,T];X_n)$ and $\varepsilon > 0$, there exists a unique weak solution*

$$\varrho = \varrho_{\varepsilon,n} \in L^2((0,T);W^{1,2}(\Omega)) \cap C([0,T];L^2(\Omega))$$

*of system* (5.5.1)–(5.5.3) *in the sense that the integral identity*

$$\left[\int_\Omega \varrho\varphi(t,\cdot)\,\mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega (\varrho\partial_t\varphi + \varrho\mathbf{u}\cdot\nabla_x\varphi - \varepsilon\nabla_x\varrho\cdot\nabla_x\varphi)\,\mathrm{d}x,$$

*holds for any $\tau \in [0,T]$ and any $\varphi \in C^1([0,T]\times\overline{\Omega})$, with $\varrho(0,\cdot) = \varrho_{0,n}$. Moreover, the norm in the aforementioned spaces is bounded only in terms of $\varrho_{0,n}$ and*

$$\sup_{t\in[0,T]} \|\mathbf{u}(t,\cdot)\|_{X_n}.$$

Moreover, applying again a result from Crippa, Donadello and Spinolo [27], Lemma 3.4, we recover the *maximum principle*.

**Lemma 5.5.2.** *Under the same hypothesis of Lemma 5.5.1, the weak solution $\varrho$ satisfies*

$$\|\varrho\|_{L^\infty((0,\tau)\times\Omega)} \leq \overline{\varrho} \exp\left(\tau \| \operatorname{div}_x \mathbf{u}\|_{L^\infty((0,T)\times\Omega)}\right), \tag{5.5.10}$$

*for any $\tau \in [0,T]$, with*

$$\overline{\varrho} := \max_{\Omega} \varrho_{0,n}. \tag{5.5.11}$$

With the previous lemma, we got a bound from above of the weak solution $\varrho = \varrho_{\varepsilon,n}$ of the approximated problem (5.5.1)–(5.5.3); however, in order to recover the existence of the corresponded $\mathbf{u}$, we need also a bound from below. We can then apply the following result by Abbatiello, Feireisl and Novotný [2], Corollary 3.4.

**Lemma 5.5.3.** *Under the same hypothesis of Lemma 5.5.1, the weak solution $\varrho$ satisfies*

$$\operatorname*{ess\,inf}_{(0,\tau)\times\Omega} \varrho(t,x) \geq \underline{\varrho} \exp\left(-\tau \| \operatorname{div}_x \mathbf{u}\|_{L^\infty((0,T)\times\Omega)}\right), \tag{5.5.12}$$

*for any $\tau \in [0,T]$, with*

$$\underline{\varrho} := \min_{\Omega} \varrho_{0,n}. \tag{5.5.13}$$

We conclude this first part with one last result that will be useful in the next section.

**Lemma 5.5.4.** *Let $\mathbf{u}_1, \mathbf{u}_2 \in C([0,T]; X_n)$ be such that*

$$\max_{i=1,2} \|\mathbf{u}_i\|_{L^\infty(0,T;W^{1,\infty}(\Omega;\mathbb{R}^d))} \leq K,$$

*and let $\varrho_i = \varrho[\mathbf{u}_i]$, $i = 1,2$ be the weak solutions of the approximated problem* (5.5.1)–(5.5.3) *sharing the same initial data $\varrho_{0,n}$ in* (5.5.3)*. Then, for any $\tau \in [0,T]$ we have*

$$\|(\varrho_1 - \varrho_2)(\tau,\cdot)\|_{L^2(\Omega)} \leq c_1 \|\mathbf{u}_1 - \mathbf{u}_2\|_{L^\infty(0,\tau;W^{1,\infty}(\Omega;\mathbb{R}^d))} \tag{5.5.14}$$

*with $c_1 = c_1(\varepsilon, \varrho_0, T, K)$.*

*Proof.* The difference $\eta = \varrho_1 - \varrho_2$ verifies equations

$$\partial_t \eta - \varepsilon \Delta_x \eta = F \quad \text{in } (0,T)\times\Omega \tag{5.5.15}$$

$$\nabla_x \eta \cdot \mathbf{n} = 0 \quad \text{in } [0,T]\times\partial\Omega \tag{5.5.16}$$

$$\eta(0) = 0 \tag{5.5.17}$$

where

$$\begin{aligned}
-F = \operatorname{div}_x(\varrho_1\mathbf{u}_1 - \varrho_2\mathbf{u}_2) &= \operatorname{div}_x[\varrho_1(\mathbf{u}_1 - \mathbf{u}_2) + (\varrho_1 - \varrho_2)\mathbf{u}_2] \\
&= \operatorname{div}_x[\varrho_1(\mathbf{u}_1 - \mathbf{u}_2)] + \operatorname{div}_x(\eta\mathbf{u}_2) \\
&= \varrho_1 \operatorname{div}_x(\mathbf{u}_1 - \mathbf{u}_2) + (\mathbf{u}_1 - \mathbf{u}_2)\cdot\nabla_x\varrho_1 + \eta \operatorname{div}_x \mathbf{u}_2 + \mathbf{u}_2 \cdot \nabla_x \eta.
\end{aligned}$$

Testing equation (5.5.15) by $\eta$, noticing that

$$\begin{aligned}
\eta\partial_t\eta &= \partial_t\left(\frac{1}{2}\eta^2\right), \\
\eta\Delta_x\eta &= \eta \operatorname{div}_x \nabla_x \eta = \operatorname{div}_x(\eta\nabla_x\eta) - |\nabla_x\eta|^2,
\end{aligned}$$

and integrating over $(0,\tau)\times\Omega$, we obtain

$$\frac{1}{2}\|\eta(\tau,\cdot)\|^2_{L^2(\Omega)}+\varepsilon\int_0^\tau\|\nabla_x\eta\|^2_{L^2(\Omega;\mathbb{R}^d)}\mathrm{d}t=\int_0^\tau\int_\Omega F\eta\ \mathrm{d}x\mathrm{d}t.$$

Applying Hölder's (1.1.1), Young's (1.1.4) and Poincaré (1.1.6) inequalities, we have

$$\begin{aligned}\left|\int_\Omega F\eta\ \mathrm{d}x\right| &\leq \int_\Omega|\operatorname{div}_x(\mathbf{u}_1-\mathbf{u}_2)\varrho_1\eta|\ \mathrm{d}x+\int_\Omega|(\mathbf{u}_1-\mathbf{u}_2)\cdot\nabla_x\varrho_1\eta|\ \mathrm{d}x\\ &\quad+\int_\Omega|\eta|^2|\operatorname{div}_x\mathbf{u}_2|\ \mathrm{d}x+\int_\Omega|\eta\mathbf{u}_2\cdot\nabla_x\eta|\ \mathrm{d}x\\ &\leq\|\mathbf{u}_1-\mathbf{u}_2\|_{W^{1,\infty}(\Omega;\mathbb{R}^d)}\|\varrho_1\|_{W^{1,2}(\Omega)}\|\eta\|_{L^2(\Omega)}+\|\operatorname{div}_x\mathbf{u}_2\|_{L^\infty(\Omega)}\|\eta\|^2_{L^2(\Omega)}\\ &\quad+\frac{1}{2\varepsilon}\|\mathbf{u}_2\|^2_{L^\infty(\Omega;\mathbb{R}^d)}\|\eta\|^2_{L^2(\Omega)}+\frac{\varepsilon}{2}\|\nabla_x\eta\|^2_{L^2(\Omega;\mathbb{R}^d)}\\ &\leq\frac{1}{2}\|\mathbf{u}_1-\mathbf{u}_2\|^2_{W^{1,\infty}(\Omega;\mathbb{R}^d)}+\frac{1}{2}\|\varrho_1\|^2_{W^{1,2}(\Omega)}\|\eta\|^2_{L^2(\Omega)}+\|\operatorname{div}_x\mathbf{u}_2\|_{L^\infty(\Omega)}\|\eta\|^2_{L^2(\Omega)}\\ &\quad+\frac{1}{2\varepsilon}\|\mathbf{u}_2\|^2_{L^\infty(\Omega;\mathbb{R}^d)}\|\eta\|^2_{L^2(\Omega)}+\frac{\varepsilon}{2}\|\nabla_x\eta\|^2_{L^2(\Omega;\mathbb{R}^d)}\\ &\leq\frac{1}{2}\|\mathbf{u}_1-\mathbf{u}_2\|^2_{W^{1,\infty}(\Omega;\mathbb{R}^d)}+\left(\frac{1}{2}\|\varrho_1\|^2_{W^{1,2}(\Omega)}+K+\frac{1}{2\varepsilon}K^2\right)\|\eta\|^2_{L^2(\Omega)}\\ &\quad+\frac{\varepsilon}{2}\|\nabla_x\eta\|^2_{L^2(\Omega;\mathbb{R}^d)}.\end{aligned}$$

Therefore, for any $\tau\in[0,T]$ we get

$$\begin{aligned}\|\eta(\tau,\cdot)\|^2_{L^2(\Omega)}+\varepsilon\int_0^\tau\|\nabla_x\eta\|^2_{L^2(\Omega;\mathbb{R}^d)}\ \mathrm{d}t&\leq\tau\|\mathbf{u}_1-\mathbf{u}_2\|^2_{L^\infty(0,\tau;W^{1,\infty}(\Omega;\mathbb{R}^d))}\\ &\quad+\int_0^\tau\left(\|\varrho_1\|^2_{W^{1,2}(\Omega)}+2K+\frac{1}{\varepsilon}K^2\right)\|\eta\|^2_{L^2(\Omega)}\mathrm{d}t.\end{aligned}$$

Noticing that from Lemma 5.5.1 we have that

$$\sup_{t\in[0,T]}\|\varrho_1\|_{L^2(\Omega)}\leq c(\varrho_0,K),$$

the previous inequality can be rewritten as

$$f(\tau)\leq\alpha(\tau)+\int_0^\tau c(\varepsilon,\varrho_0,K)f(t)\ \mathrm{d}t$$

with

$$f(\tau)=\|\eta(\tau,\cdot)\|^2_{L^2(\Omega)},\quad\alpha(\tau)=T\|\mathbf{u}_1-\mathbf{u}_2\|^2_{L^\infty(0,\tau;W^{1,\infty}(\Omega;\mathbb{R}^d))}.$$

Since $\alpha=\alpha(\tau)$ is non-decreasing, we can apply Gronwall Lemma 1.5.2 to get that for any $\tau\in[0,T]$

$$f(\tau)\leq\alpha(\tau)e^{c(\varepsilon,\varrho_0,K)\tau}\leq\alpha(\tau)e^{c(\varepsilon,\varrho_0,K)T},$$

which in particular implies (5.5.14). □

### 5.5.2 *On the approximated balance of momentum*

Let us now turn our attention to the approximated problem (5.5.4)–(5.5.8). Following the same approach performed by Feireisl [33], we will first solve the problem on a time interval $[0, T(n)]$ via a fixed point argument, where $T(n)$ depends on the dimension $n$ of the finite-dimensional space $X_n$. Subsequently we will establish estimates independent of time and iterate the same procedure to finally obtain, after a finite number of steps, our solution $\mathbf{u}$ on the whole time interval $[0, T]$.

*Technical preliminaries*

For any $\varrho \in L^1(\Omega)$, consider the operator $\mathcal{M}[\varrho] : X_n \to X_n^*$ such that

$$\langle \mathcal{M}[\varrho]\mathbf{v}, \mathbf{w} \rangle \equiv \int_\Omega \varrho \mathbf{v} \cdot \mathbf{w} \, \mathrm{d}x, \tag{5.5.18}$$

with $\langle \cdot, \cdot \rangle$ the $L^2$-standard scalar product. In particular, we have

$$\|\mathcal{M}[\varrho]\|_{\mathcal{L}(X_n, X_n^*)} = \sup_{\|\mathbf{v}\|_{X_n}, \|\mathbf{w}\|_{X_n} \leq 1} |\langle \mathcal{M}[\varrho]\mathbf{v}, \mathbf{w} \rangle| \leq c(n) \|\varrho\|_{L^1(\Omega)}, \tag{5.5.19}$$

It is easy to see that the operator $\mathcal{M}$ is invertible provided $\varrho$ is strictly positive on $\Omega$, and in particular we have

$$\|\mathcal{M}^{-1}[\varrho]\|_{\mathcal{L}(X_n^*; X_n)} = \frac{1}{\inf\{\|\mathcal{M}[\varrho]\mathbf{v}\|_{X_n^*} : \mathbf{v} \in X_n, \ \|\mathbf{v}\|_{X_n} = 1\}} \leq \frac{c(n)}{\inf_\Omega \varrho}.$$

Moreover, the identity

$$\mathcal{M}^{-1}[\varrho_1] - \mathcal{M}^{-1}[\varrho_2] = \mathcal{M}^{-1}[\varrho_2] \left( \mathcal{M}[\varrho_2] - \mathcal{M}[\varrho_1] \right) \mathcal{M}^{-1}[\varrho_1]$$

can be used to obtain

$$\left\| \mathcal{M}^{-1}[\varrho_1] - \mathcal{M}^{-1}[\varrho_2] \right\|_{\mathcal{L}(X_n^*; X_n)} \leq c \left( n, \inf_\Omega \varrho_1, \inf_\Omega \varrho_2 \right) \|\varrho_1 - \varrho_2\|_{L^1(\Omega)} \tag{5.5.20}$$

for any $\varrho_1, \varrho_2 > 0$.

*Fixed point argument*

The approximate velocities $\mathbf{u} \in C([0, T]; X_n)$ are looked for to satisfy the integral identity

$$\begin{aligned} \left[ \int_\Omega \varrho \mathbf{u}(t, \cdot) \cdot \boldsymbol{\psi} \, \mathrm{d}x \right]_{t=0}^{t=\tau} &= \int_0^\tau \int_\Omega [(\varrho \mathbf{u} \otimes \mathbf{u}) : \nabla_x \boldsymbol{\psi} + a\varrho \operatorname{div}_x \boldsymbol{\psi}] \, \mathrm{d}x\mathrm{d}t \\ &- \int_0^\tau \int_\Omega [\partial F_\delta(\mathbb{D}_x \mathbf{u}) : \nabla_x \boldsymbol{\psi} + \varepsilon \nabla_x \varrho \cdot \nabla_x \mathbf{u} \cdot \boldsymbol{\psi}] \, \mathrm{d}x\mathrm{d}t \end{aligned} \tag{5.5.21}$$

for any test function $\boldsymbol{\psi} \in X_n$ and all $\tau \in [0, T]$. Now, the integral identity (5.5.21) can be rephrased for any $\tau \in [0, T]$ as

$$\langle \mathcal{M}[\varrho(\tau, \cdot)](\mathbf{u}(\tau, \cdot)), \boldsymbol{\psi} \rangle = \langle \mathbf{m}_0^*, \boldsymbol{\psi} \rangle + \langle \int_0^\tau \mathcal{N}[\varrho(s, \cdot), \mathbf{u}(s, \cdot)] \, \mathrm{d}s, \boldsymbol{\psi} \rangle$$

with $\mathcal{M}[\varrho] : X_n \to X_n^*$ defined as in (5.5.18), $\mathbf{m}_0^* \in X_n^*$ such that

$$\langle \mathbf{m}_0^*, \boldsymbol{\psi} \rangle := \int_\Omega \mathbf{m}_0 \cdot \boldsymbol{\psi} \, \mathrm{d}x$$

and $\mathcal{N}[\varrho(s,\cdot), \mathbf{u}(s,\cdot)] \in X_n^*$ such that

$$\begin{aligned} \langle \mathcal{N}[\varrho(s,\cdot), \mathbf{u}(s,\cdot)], \boldsymbol{\psi} \rangle := & \int_\Omega \left[ (\varrho \mathbf{u} \otimes \mathbf{u} - \partial F_\delta(\mathbb{D}_x \mathbf{u})) : \nabla_x \boldsymbol{\psi} + a\varrho \operatorname{div}_x \boldsymbol{\psi} \right] (s,\cdot) \, \mathrm{d}x \\ & - \varepsilon \int_\Omega \nabla_x \varrho \cdot \nabla_x \mathbf{u} \cdot \boldsymbol{\psi}(s,\cdot) \, \mathrm{d}x. \end{aligned}$$

Here, $\varrho = \varrho[\mathbf{u}]$ is the weak solution uniquely determined by $\mathbf{u}$ and thus by Lemmas 5.5.2 and 5.5.3, for any $t \in [0,T]$ we have

$$0 < \underline{\varrho} \exp\left( -t \| \operatorname{div}_x \mathbf{u} \|_{L^\infty((0,T)\times\Omega)} \right) \le \varrho(t,x) \le \overline{\varrho} \exp\left( t \| \operatorname{div}_x \mathbf{u} \|_{L^\infty((0,T)\times\Omega)} \right), \tag{5.5.22}$$

where $\overline{\varrho}$, $\underline{\varrho}$ are defined as in (5.5.11), (5.5.13) respectively. In particular, the operator $\mathcal{M}$ is invertible and hence, for any $\tau \in [0,T]$, we can write

$$\mathbf{u}(\tau,\cdot) = \mathcal{M}^{-1}[\varrho(\tau,\cdot)] \left( \mathbf{m}_0^* + \int_0^\tau \mathcal{N}[\varrho(s,\cdot), \mathbf{u}(s,\cdot)] \, \mathrm{d}s \right).$$

For $K$ and $T(n)$ to be fixed, consider a bounded ball $\mathcal{B}(0, \underline{n}K)$ in the space $C([0,T(n)]; X_n)$, with $\underline{n}$ defined as in (5.5.9),

$$\mathcal{B}(0,\underline{n}K) := \left\{ \mathbf{v} \in C([0,T(n)]; X_n) \Big| \sup_{t\in[0,T(n)]} \| \mathbf{v}(t,\cdot) \|_{X_n} \le \underline{n}K \right\},$$

and define a mapping

$$\mathcal{F} : \mathcal{B}(0,\underline{n}K) \to C([0,T(n)]; X_n)$$

such that for all $\tau \in [0,T(n)]$

$$\mathcal{F}[\mathbf{u}](\tau,\cdot) := \mathcal{M}^{-1}[\varrho(\tau,\cdot)] \left( \mathbf{m}_0^* + \int_0^\tau \mathcal{N}[\varrho(s,\cdot), \mathbf{u}(s,\cdot)] \, \mathrm{d}s \right).$$

Notice that for every $\mathbf{u} \in \mathcal{B}(0,\underline{n}K)$, from (5.5.9) we obtain in particular that for all $t \in [0,T(n)]$

$$\| \mathbf{u}(t,\cdot) \|_{W^{1,\infty}(\Omega;\mathbb{R}^d)} \le K$$

and thus, from (5.5.22) we obtain that for all $t \in [0,T(n)]$

$$\underline{\varrho} e^{-Kt} \le \varrho(t,x) \le \overline{\varrho} e^{Kt}.$$

Moreover, it is easy to deduce that for every $\mathbf{u} \in \mathcal{B}(0,\underline{n}K)$, $\varrho = \varrho[\mathbf{u}]$ and every $t \in [0,T(n)]$

$$\| \mathcal{N}(\varrho(t,\cdot), \mathbf{u}(t,\cdot)) \|_{X_n^*} \le c_2(\overline{\varrho}, K, T),$$

and for every $\mathbf{u}_1, \mathbf{u}_2 \in \mathcal{B}(0,\underline{n}K)$, $\varrho_i = \varrho[\mathbf{u}_i]$, $i = 1,2$ and $t \in [0,T(n)]$, making use of (5.5.14),

$$\| \mathcal{N}(\varrho_1(t,\cdot), \mathbf{u}_1(t,\cdot)) - \mathcal{N}(\varrho_2(t,\cdot), \mathbf{u}_2(t,\cdot)) \|_{X_n^*} \le c_3(\overline{\varrho}, K, T) \| \mathbf{u}_1(t,\cdot) - \mathbf{u}_2(t,\cdot) \|_{W^{1,\infty}(\Omega;\mathbb{R}^d)}.$$

Then, for every $\mathbf{u} \in \mathcal{B}(0, \underline{n}K)$, $\varrho = \varrho[\mathbf{u}]$ and every $t \in [0, T(n)]$

$$\begin{aligned}\|\mathcal{F}(\mathbf{u})(t,\cdot)\|_{X_n} &\leq \|\mathcal{M}^{-1}[\varrho(t,\cdot)]\|_{\mathcal{L}(X_n^*;X_n)}(\|\mathbf{m}_0^*\|_{X_n^*} + \|\mathcal{N}(\varrho(t,\cdot),\mathbf{u}(t,\cdot))\|_{X_n^*}\ t) \\ &\leq \frac{c(n)}{\underline{\varrho}}\ e^{KT(n)}\ \big(\|\mathbf{m}_0^*\|_{X_n^*} + c_2\ T(n)\big),\end{aligned}$$

and for every $\mathbf{u}_1, \mathbf{u}_2 \in \mathcal{B}(0, \underline{n}K)$, $\varrho_i = \varrho[\mathbf{u}_i]$, $i = 1, 2$ and $t \in [0, T(n)]$,

$$\begin{aligned}&\|\mathcal{F}(\mathbf{u}_1)(t,\cdot) - \mathcal{F}(\mathbf{u}_2)(t,\cdot)\|_{X_n} \\ &\quad\leq \left\|\Big(\mathcal{M}^{-1}[\varrho_1(t,\cdot)] - \mathcal{M}^{-1}[\varrho_2(t,\cdot)]\Big)\left[\int_0^t \mathcal{N}(\varrho_1(s,\cdot),\mathbf{u}_1(s,\cdot))\ \mathrm{d}s\right]\right\|_{X_n} \\ &\quad\quad+ \left\|\mathcal{M}^{-1}[\varrho_2(t,\cdot)]\left[\int_0^t [\mathcal{N}(\varrho_1(s,\cdot),\mathbf{u}_1(s,\cdot)) - \mathcal{N}(\varrho_2(s,\cdot),\mathbf{u}_2(s,\cdot))]\ \mathrm{d}s\right]\right\|_{X_n} \\ &\quad\leq t\left\|\mathcal{M}^{-1}[\varrho_1(t,\cdot)] - \mathcal{M}^{-1}[\varrho_2(t,\cdot)]\right\|_{\mathcal{L}(X_n^*;X_n)}\|\mathcal{N}(\varrho_1(t,\cdot),\mathbf{u}_1(t,\cdot))\|_{X_n^*} \\ &\quad\quad+ t\left\|\mathcal{M}^{-1}[\varrho_2(t,\cdot)]\right\|_{\mathcal{L}(X_n^*,X_n)}\|\mathcal{N}(\varrho_1(t,\cdot),\mathbf{u}_1(t,\cdot)) - \mathcal{N}(\varrho_2(t,\cdot),\mathbf{u}_2(t,\cdot))\|_{X_n^*} \\ &\quad\leq c(n)\ \frac{c_2}{(\underline{\varrho})^2}\ e^{2Kt}t\ \|\varrho_1(t,\cdot) - \varrho_2(t,\cdot)\|_{L^1(\Omega)} + c(n)\ \frac{c_3}{\underline{\varrho}}\ e^{Kt}t\ \|\mathbf{u}_1(t,\cdot) - \mathbf{u}_2(t,\cdot)\|_{W^{1,\infty}(\Omega;\mathbb{R}^d)} \\ &\quad\leq c(n)\left(\frac{c_1c_2}{(\underline{\varrho})^2} + \frac{c_3}{\underline{\varrho}}\right)e^{2Kt}t\ \|\mathbf{u}_1(t,\cdot) - \mathbf{u}_2(t,\cdot)\|_{W^{1,\infty}(\Omega;\mathbb{R}^d)} \\ &\quad\leq T(n)\ \frac{c(n)}{\underline{n}}\left(\frac{c_1c_2}{(\underline{\varrho})^2} + \frac{c_3}{\underline{\varrho}}\right)e^{2KT(n)}\ \|\mathbf{u}_1(t,\cdot) - \mathbf{u}_2(t,\cdot)\|_{X_n}.\end{aligned}$$

Now, taking $K > 0$ sufficiently large and $T(n)$ sufficiently small, so that

$$\frac{c(n)}{\underline{\varrho}}\ e^{KT(n)}\ \big(\|\mathbf{m}_0^*\|_{X_n^*} + c_2\ T(n)\big) \leq \underline{n}K,$$

and

$$T(n)\ \frac{c(n)}{\underline{n}}\ \left(\frac{c_1c_2}{(\underline{\varrho})^2} + \frac{c_3}{\underline{\varrho}}\right)e^{2KT(n)} < 1,$$

we obtain that $\mathcal{F}$ is a contraction mapping from the closed ball $\mathcal{B}(0, \underline{n}K)$ into itself. From the Banach-Caccioppoli fixed point theorem 1.5.8, we recover that $\mathcal{F}$ admits a unique fixed point $\mathbf{u} \in C([0, T(n)]; X_n)$, which in particular solves the integral identity (5.5.21).

This procedure can be repeated a finite number of times until we reach $T = T(n)$, as long as we have a bound on $\mathbf{u}$ independent of $T(n)$. the next section will be dedicated to establish all the necessary estimates.

*Estimates independent of time*

We start with the *energy estimates*. It follows from (5.5.21) that $\mathbf{u}$ is continuously differentiable and, consequently, the integral identity

$$\begin{aligned}\int_\Omega \partial_t(\varrho\mathbf{u})\cdot\boldsymbol{\psi}\ \mathrm{d}x &= \int_\Omega [\varrho\mathbf{u}\otimes\mathbf{u} : \nabla_x\boldsymbol{\psi} + a\varrho\,\mathrm{div}_x\,\boldsymbol{\psi} - \partial F_\delta(\mathbb{D}_x\mathbf{u}) : \nabla_x\boldsymbol{\psi}]\ \mathrm{d}x \\ &\quad- \varepsilon\int_\Omega [\nabla_x\varrho\cdot\nabla_x\mathbf{u}\cdot\boldsymbol{\psi}]\ \mathrm{d}x\end{aligned}\tag{5.5.23}$$

holds on $(0, T(n))$ for any $\boldsymbol{\psi} \in X_n$, with $\varrho = \varrho[\mathbf{u}]$. We recall that in this context the pressure potential $P = P(\varrho)$ is such that

$$P(\varrho) = a\ \varrho \log \varrho,$$

and it satisfies the following identity

$$a\varrho \operatorname{div}_x \mathbf{u} = -\partial_t P(\varrho) - \operatorname{div}_x(P(\varrho)\mathbf{u}) + \varepsilon\ a(\log \varrho + 1)\Delta_x \varrho.$$

Now, taking $\boldsymbol{\psi} = \mathbf{u}$ in (5.5.23) and noticing that

$$\begin{aligned}\int_\Omega [\partial_t(\varrho\mathbf{u}) \cdot \mathbf{u} - \varrho\mathbf{u} \otimes \mathbf{u} : \nabla_x \mathbf{u}]\ \mathrm{d}x &= \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{1}{2}\varrho|\mathbf{u}|^2\ \mathrm{d}x + \frac{1}{2}\int_\Omega (\partial_t\varrho + \operatorname{div}_x(\varrho\mathbf{u}))|\mathbf{u}|^2\ \mathrm{d}x \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{1}{2}\varrho|\mathbf{u}|^2 \mathrm{d}x + \frac{\varepsilon}{2}\int_\Omega \Delta_x\varrho|\mathbf{u}|^2\ \mathrm{d}x,\end{aligned}$$

where, using the boundary condition (5.5.2),

$$\begin{aligned}\frac{\varepsilon}{2}\int_\Omega |\mathbf{u}|^2\Delta_x\varrho\ \mathrm{d}x &= \frac{\varepsilon}{2}\int_\Omega |\mathbf{u}|^2 \operatorname{div}_x \nabla_x\varrho\ \mathrm{d}x \\ &= \frac{\varepsilon}{2}\int_\Omega \operatorname{div}_x(|\mathbf{u}|^2\nabla_x\varrho)\ \mathrm{d}x - \frac{\varepsilon}{2}\int_\Omega \nabla_x\varrho \cdot \nabla_x\mathbf{u} \cdot 2\mathbf{u}\ \mathrm{d}x \\ &= \frac{\varepsilon}{2}\int_{\partial\Omega} |\mathbf{u}|^2\nabla_x\varrho \cdot \mathbf{n}\ \mathrm{d}S_x - \varepsilon\int_\Omega \nabla_x\varrho \cdot \nabla_x\mathbf{u} \cdot \mathbf{u}\ \mathrm{d}x \\ &= -\varepsilon\int_\Omega \nabla_x\varrho \cdot \nabla_x\mathbf{u} \cdot \mathbf{u}\ \mathrm{d}x,\end{aligned}$$

and

$$\begin{aligned}\int_\Omega (\log\varrho + 1)\Delta_x\varrho\ \mathrm{d}x &= \int_\Omega (\log\varrho + 1) \operatorname{div}_x \nabla_x\varrho\ \mathrm{d}x \\ &= \int_\Omega \operatorname{div}_x [(\log\varrho + 1)\nabla_x\varrho]\ \mathrm{d}x - \int_\Omega \nabla_x(\log\varrho + 1) \cdot \nabla_x\varrho\ \mathrm{d}x \\ &= \int_{\partial\Omega} (\log\varrho + 1)\nabla_x\varrho \cdot \mathbf{n}\ \mathrm{d}S_x - \int_\Omega \frac{d}{d\varrho}(\log\varrho + 1)|\nabla_x\varrho|^2\ \mathrm{d}x \\ &= -\int_\Omega \frac{1}{\varrho}|\nabla_x\varrho|^2\ \mathrm{d}x = -\int_\Omega P''(\varrho)|\nabla_x\varrho|^2\ \mathrm{d}x,\end{aligned}$$

we finally obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho)\right] \mathrm{d}x = -\varepsilon\int_\Omega P''(\varrho)|\nabla_x\varrho|^2\ \mathrm{d}x - \int_\Omega \partial F_\delta(\mathbb{D}_x\mathbf{u}) : \nabla_x\mathbf{u}\ \mathrm{d}x. \tag{5.5.24}$$

Note that we got rid of the integral $\frac{\varepsilon}{2}\int_\Omega |\mathbf{u}_n|^2\Delta_x\varrho_n \mathrm{d}x$ thanks to the extra term $\varepsilon\nabla_x\varrho_n \cdot \nabla_x\mathbf{u}_n$ in (5.5.4). Since all the quantities involved are at least continuous in time, we may integrate (5.5.24) over $(0, \tau)$ in order to get the following energy equality

$$\begin{aligned}\int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho)\right](\tau, \cdot)\ \mathrm{d}x + \int_0^\tau \int_\Omega \left[\partial F_\delta(\mathbb{D}_x\mathbf{u}) : \nabla_x\mathbf{u} + \varepsilon P''(\varrho)|\nabla_x\varrho|^2\right] \mathrm{d}x\mathrm{d}t \\ = \int_\Omega \left[\frac{1}{2}\frac{|\mathbf{m}_0|^2}{\varrho_{0,n}} + P(\varrho_{0,n})\right] \mathrm{d}x,\end{aligned} \tag{5.5.25}$$

for any time $\tau \in [0, T(n)]$. In particular, if we suppose the initial value of the (modified) total energy

$$E(0) = E_n(0) := \int_\Omega \left[ \frac{1}{2} \frac{|\mathbf{m}_0|^2}{\varrho_{0,n}} + P(\varrho_{0,n}) \right] \mathrm{d}x$$

to be bounded by a constant independent of $n > 0$, the term on the right-hand side of (5.5.25) is bounded.

Now, the following result, collecting all the significant properties of the regularized potential $F_\delta$, is needed.

**Proposition 5.5.5.** *For every fixed $\delta > 0$ and $F$ satisfying hypothesis* (5.1.4)–(5.1.5)*, the function $F_\delta$ defined in* (5.5.7) *is convex, non-negative, infinitely differentiable and such that*

$$F_\delta(\mathbb{D}) \geq \nu \left| \mathbb{D} - \frac{1}{d} \operatorname{Tr}[\mathbb{D}]\mathbb{I} \right|^q - c \quad \text{for all } \mathbb{D} \in \mathbb{R}^{d\times d}_{\text{sym}} \tag{5.5.26}$$

*with $\nu > 0$, $c > 0$, $q > 1$ independent of $\delta$.*

*Proof.* For every fixed $\delta > 0$, the non-negativity of $F_\delta$ is trivial while smoothness follows from the fact that each derivative can be transferred to the mollifiers $\xi_\delta$, cf. Theorem 1.1.2, condition (i).

Moreover, for every $\mathbb{A}, \mathbb{B} \in \mathbb{R}^{d\times d}_{\text{sym}}$ and every $t \in [0,1]$, denoting

$$C_1 := \inf_{\mathbb{D} \in \mathbb{R}^{d\times d}_{\text{sym}}} \int_{\mathbb{R}^{d\times d}_{\text{sym}}} \xi_\delta(|\mathbb{D} - \mathbb{Z}|) F(\mathbb{Z}) \, \mathrm{d}\mathbb{Z}$$

we have

$$\begin{aligned}
&F_\delta(t\mathbb{A} + (1-t)\mathbb{B}) \\
&= \int_{\mathbb{R}^{d\times d}_{\text{sym}}} F\big(t(\mathbb{A}+\mathbb{Z}) + (1-t)(\mathbb{B}+\mathbb{Z})\big) \, \xi_\delta(|\mathbb{Z}|) \, \mathrm{d}\mathbb{Z} + tC_1 - (1-t)C_1 \\
&\leq t \left( \int_{\mathbb{R}^{d\times d}_{\text{sym}}} F(\mathbb{A}+\mathbb{Z}) \, \xi_\delta(|\mathbb{Z}|) \, \mathrm{d}\mathbb{Z} + C_1 \right) + (1-t) \left( \int_{\mathbb{R}^{d\times d}_{\text{sym}}} F(\mathbb{B}+\mathbb{Z}) \, \xi_\delta(|\mathbb{Z}|) \, \mathrm{d}\mathbb{Z} + C_1 \right) \\
&= tF_\delta(\mathbb{A}) + (1-t)F_\delta(\mathbb{B}),
\end{aligned}$$

where we have simply summed and subtracted terms $t\mathbb{Z}$, $tC_1$ in the second line and used the convexity of $F$ in the third line. In particular, we get that for every fixed $\delta > 0$, $F_\delta : \mathbb{R}^{d\times d}_{\text{sym}} \to [0, \infty)$ is convex.

Let now $\mathbb{D} \in \mathbb{R}^{d\times d}_{\text{sym}}$ be fixed. From (5.1.5), we have

$$\begin{aligned}
F_\delta(\mathbb{D}) &= \int_{\mathbb{R}^{d\times d}_{\text{sym}}} F(\mathbb{D} - \mathbb{Z}) \xi_\delta(|\mathbb{Z}|) \, \mathrm{d}\mathbb{Z} - C_1 \\
&\geq \mu \int_{\mathbb{R}^{d\times d}_{\text{sym}}} \left| \left( \mathbb{D} - \frac{1}{d} \operatorname{Tr}[\mathbb{D}]\mathbb{I} \right) - \left( \mathbb{Z} - \frac{1}{d} \operatorname{Tr}[\mathbb{Z}]\mathbb{I} \right) \right|^q \xi_\delta(|\mathbb{Z}|) \, \mathrm{d}\mathbb{Z} - C_1.
\end{aligned}$$

Applying Minkowski's inequality (1.1.2) with

$$\begin{aligned}
f(\mathbb{Z}) &:= \left[ \left( \mathbb{D} - \frac{1}{d} \operatorname{Tr}[\mathbb{D}]\mathbb{I} \right) - \left( \mathbb{Z} - \frac{1}{d} \operatorname{Tr}[\mathbb{Z}]\mathbb{I} \right) \right] \xi_\delta^{1/q}(|\mathbb{Z}|), \\
g(\mathbb{Z}) &:= \left( \mathbb{Z} - \frac{1}{d} \operatorname{Tr}[\mathbb{Z}]\mathbb{I} \right) \xi_\delta^{1/q}(|\mathbb{Z}|),
\end{aligned}$$

we get

$$\begin{aligned}
&\int_{\mathbb{R}^{d\times d}_{\mathrm{sym}}} \left| \left(\mathbb{D} - \frac{1}{d}\,\mathrm{Tr}[\mathbb{D}]\mathbb{I}\right) - \left(\mathbb{Z} - \frac{1}{d}\,\mathrm{Tr}[\mathbb{Z}]\mathbb{I}\right)\right|^q \xi_\delta(|\mathbb{Z}|)\,\mathrm{d}\mathbb{Z} \\
&\quad \geq \left[ \left( \int_{\mathbb{R}^{d\times d}_{\mathrm{sym}}} \left| \mathbb{D} - \frac{1}{d}\,\mathrm{Tr}[\mathbb{D}]\mathbb{I}\right|^q \xi_\delta(|\mathbb{Z}|)\mathrm{d}\mathbb{Z} \right)^{\frac{1}{q}} - \left( \int_{\mathbb{R}^{d\times d}_{\mathrm{sym}}} \left| \mathbb{Z} - \frac{1}{d}\,\mathrm{Tr}[\mathbb{Z}]\mathbb{I}\right|^q \xi_\delta(|\mathbb{Z}|)\mathrm{d}\mathbb{Z} \right)^{\frac{1}{q}} \right]^q ;
\end{aligned}$$

recalling that for any $\delta > 0$ sufficiently small $\operatorname{supp} \xi_\delta \subset K$ with $K \subset \mathbb{R}^{d\times d}_{\mathrm{sym}}$ a compact set and that for any $\delta > 0$

$$\int_{\mathbb{R}^{d\times d}_{\mathrm{sym}}} \xi_\delta(|\mathbb{Z}|)\,\mathrm{d}\mathbb{Z} = \frac{1}{\delta^d} \int_{\mathbb{R}^{d\times d}_{\mathrm{sym}}} \xi\left(\frac{|\mathbb{Z}|}{\delta}\right) \mathrm{d}\mathbb{Z} = \int_{\mathbb{R}^{d\times d}_{\mathrm{sym}}} \xi(|\mathbb{Z}|)\,\mathrm{d}\mathbb{Z} = 1,$$

we obtain the following inequality

$$F_\delta(\mathbb{D}) \geq \mu \left[ \left| \mathbb{D} - \frac{1}{d}\,\mathrm{Tr}[\mathbb{D}]\mathbb{I}\right| - \left( \sup_{\mathbb{Z}\in K} \left| \mathbb{Z} - \frac{1}{d}\,\mathrm{Tr}[\mathbb{Z}]\mathbb{I}\right|^q \right)^{\frac{1}{q}} \right]^q - C_1.$$

Now, for every fixed $q > 1$ and constant $c_1 > 0$, there exist $\alpha = \alpha(q, c_1) \in (0,1)$ and $c_2 = c_2(q, c_1) > 0$ such that

$$(y - c_1)^q \geq \alpha y^q - c_2 \quad \text{for any } y \geq 0;$$

in particular, we get that for all $\mathbb{D} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$

$$F_\delta(\mathbb{D}) \geq \mu\alpha \left| \mathbb{D} - \frac{1}{d}\,\mathrm{Tr}[\mathbb{D}]\mathbb{I}\right|^q - (C_1 + C_2)$$

and thus (5.5.26) holds choosing $\nu = \mu\alpha$ and $c = C_1 + C_2$. $\square$

Repeating the procedure that led to (5.4.18), from (5.5.26) and the $L^q$-version of the trace-free Korn's inequality (1.1.7), we can deduce that

$$\mathbf{u} \text{ is bounded in } L^q(0, T(n); W_0^{1,q}(\Omega; \mathbb{R}^d))$$

by a constant which is independent of $n$ and $T(n) \leq T$. Since all norms are equivalent in $X_n$, this implies that

$$\mathbf{u} \text{ is bounded in } L^q(0, T(n); W^{1,\infty}(\Omega; \mathbb{R}^d));$$

in particular, by virtue of (5.5.10) and (5.5.12), the density $\varrho = \varrho[\mathbf{u}]$ is bounded from below and above by constants independent of $T(n) \leq T$. Since $\varrho$ is bounded from below, one can use (5.5.25) to easily deduce uniform boundedness in $t$ of $\mathbf{u}$ in the space $L^2(\Omega; \mathbb{R}^d)$. Consequently, the functions $\mathbf{u}(t, \cdot)$ remain bounded in $X_n$ for any $t$ independently of $T(n) \leq T$. Thus we are allowed to iterate the previous local existence result to construct a solution defined on the whole time interval $[0, T]$.

Summarizing, so far we proved the following result.

**Lemma 5.5.6.** *For every fixed $\delta > 0$, $\varepsilon > 0$, $n \in \mathbb{N}$, and any $\varrho_{0,n} \in C(\overline{\Omega})$ such that*

$$E_{0,n} := \int_\Omega \left[ \frac{1}{2} \frac{|\mathbf{m}_0|^2}{\varrho_{0,n}} + P(\varrho_{0,n}) \right] \mathrm{d}x$$

*is bounded by a constant independent of $n$, there exist*

$$\begin{aligned} \varrho &= \varrho_{\delta,\varepsilon,n} \in L^2((0,T); W^{1,2}(\Omega)) \cap C([0,T]; L^2(\Omega)), \\ \mathbf{u} &= \mathbf{u}_{\delta,\varepsilon,n} \in C([0,T]; X_n), \end{aligned}$$

*such that*

*(i) the integral identity*

$$\left[ \int_\Omega \varrho\varphi(t,\cdot)\, \mathrm{d}x \right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega (\varrho\partial_t\varphi + \varrho\mathbf{u}\cdot\nabla_x\varphi - \varepsilon\nabla_x\varrho\cdot\nabla_x\varphi)\, \mathrm{d}x$$

*holds for any $\tau \in [0,T]$ and any $\varphi \in C^1([0,T]\times\overline{\Omega})$, with $\varrho(0,\cdot) = \varrho_{0,n}$;*

*(ii) the integral identity*

$$\begin{aligned} \left[ \int_\Omega \varrho\mathbf{u}\cdot\boldsymbol{\varphi}(t,\cdot)\, \mathrm{d}x \right]_{t=0}^{t=\tau} &= \int_0^\tau \int_\Omega [\varrho\mathbf{u}\cdot\partial_t\boldsymbol{\varphi} + (\varrho\mathbf{u}\otimes\mathbf{u}) : \nabla_x\boldsymbol{\varphi} + a\varrho\,\mathrm{div}_x\,\boldsymbol{\varphi}]\, \mathrm{d}x\mathrm{d}t \\ &\quad - \int_0^\tau \int_\Omega \partial F_\delta(\mathbb{D}_x\mathbf{u}) : \nabla_x\boldsymbol{\varphi}\, \mathrm{d}x\mathrm{d}t - \varepsilon\int_0^\tau \int_\Omega \nabla_x\varrho\cdot\nabla_x\mathbf{u}\cdot\boldsymbol{\varphi}\, \mathrm{d}x\mathrm{d}t \end{aligned}$$

*holds for any $\tau \in [0,T]$ and any $\boldsymbol{\varphi} \in C^1([0,T]; X_n)$, with $(\varrho\mathbf{u})(0,\cdot) = \mathbf{m}_0$;*

*(iii) introducing*

$$E(\tau) = E_{\delta,\varepsilon,n}(\tau) = \int_\Omega \left[ \frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho) \right](\tau,\cdot)\, \mathrm{d}x$$

*for any $t \in [0,T]$, the integral equality*

$$[E(t)]_{t=0}^{t=\tau} + \int_0^\tau \int_\Omega \partial F_\delta(\mathbb{D}_x\mathbf{u}) : \nabla_x\mathbf{u}\, \mathrm{d}x\mathrm{d}t + \varepsilon\int_0^\tau \int_\Omega P''(\varrho)|\nabla_x\varrho|^2\, \mathrm{d}x\mathrm{d}t = 0$$

*holds for any time $\tau \in [0,T]$, with $E(0) = E_{0,n}$.*

### 5.5.3 *Limit $\delta \to 0$*

Let now $\varepsilon > 0$ and $n \in \mathbb{N}$ be fixed, and let $\{\varrho_\delta, \mathbf{u}_\delta\}_{\delta>0}$ be the family of weak solutions to problem (5.5.1)–(5.5.8) as in Lemma 5.5.6. Proceeding as before, we can deduce that

$$\{\mathbf{u}_\delta\}_{\delta>0} \text{ is unifrmly bounded in } L^q(0,T; W_0^{1,q}(\Omega;\mathbb{R}^d)).$$

As $n$ is fixed and all norms are equivalent on the finite-dimensional space $X_n$, we get that

$$\{\nabla_x\mathbf{u}_\delta\}_{\delta>0} \text{ is unifrmly bounded in } L^\infty((0,T)\times\Omega;\mathbb{R}^{d\times d}),$$

and therefore, we are ready to perform the limit $\delta \to 0$. Accordingly, we obtain the following result.

**Lemma 5.5.7.** *For every fixed $\varepsilon > 0$, $n \in \mathbb{N}$, and any $\varrho_{0,n} \in C(\overline{\Omega})$ such that*

$$E_{0,n} := \int_\Omega \left[ \frac{1}{2} \frac{|\mathbf{m}_0|^2}{\varrho_{0,n}} + P(\varrho_{0,n}) \right] \mathrm{d}x$$

*is bounded by a constant independent of $n$, there exist*

$$\begin{aligned} \varrho = \varrho_{\varepsilon,n} &\in L^2((0,T); W^{1,2}(\Omega)) \cap C([0,T]; L^2(\Omega)), \\ \mathbf{u} = \mathbf{u}_{\varepsilon,n} &\in C([0,T]; X_n), \end{aligned}$$

*such that*

*(i) the integral identity*

$$\left[ \int_\Omega \varrho\varphi(t,\cdot)\,\mathrm{d}x \right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega (\varrho\partial_t\varphi + \varrho\mathbf{u}\cdot\nabla_x\varphi - \varepsilon\nabla_x\varrho\cdot\nabla_x\varphi)\,\mathrm{d}x \tag{5.5.27}$$

*holds for any $\tau \in [0,T]$ and any $\varphi \in C^1([0,T]\times\overline{\Omega})$, with $\varrho(0,\cdot) = \varrho_{0,n}$;*

*(ii) there exists*

$$\mathbb{S} = \mathbb{S}_{\varepsilon,n} \in L^\infty((0,T)\times\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}})$$

*such that the integral identity*

$$\begin{aligned} \left[ \int_\Omega \varrho\mathbf{u}\cdot\boldsymbol{\varphi}(t,\cdot)\,\mathrm{d}x \right]_{t=0}^{t=\tau} &= \int_0^\tau \int_\Omega [\varrho\mathbf{u}\cdot\partial_t\boldsymbol{\varphi} + (\varrho\mathbf{u}\otimes\mathbf{u}) : \nabla_x\boldsymbol{\varphi} + a\varrho\,\mathrm{div}_x\,\boldsymbol{\varphi}]\,\mathrm{d}x\mathrm{d}t \\ &\quad - \int_0^\tau \int_\Omega \mathbb{S} : \nabla_x\boldsymbol{\varphi}\,\mathrm{d}x\mathrm{d}t - \varepsilon \int_0^\tau \int_\Omega \nabla_x\varrho\cdot\nabla_x\mathbf{u}\cdot\boldsymbol{\varphi}\,\mathrm{d}x\mathrm{d}t \end{aligned} \tag{5.5.28}$$

*holds for any $\tau \in [0,T]$ and any $\boldsymbol{\varphi} \in C^1([0,T]; X_n)$, with $(\varrho\mathbf{u})(0,\cdot) = \mathbf{m}_0$;*

*(iii) introducing*

$$E(\tau) = E_{\varepsilon,n}(\tau) = \int_\Omega \left[ \frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho) \right](\tau,\cdot)\,\mathrm{d}x$$

*for any $t \in [0,T]$, the integral inequality*

$$[E(t)]_{t=0}^{t=\tau} + \int_0^\tau \int_\Omega [F(\mathbb{D}_x\mathbf{u}) + F^*(\mathbb{S})]\,\mathrm{d}x\mathrm{d}t + \varepsilon \int_0^\tau \int_\Omega P''(\varrho)|\nabla_x\varrho|^2\,\mathrm{d}x\mathrm{d}t \le 0 \tag{5.5.29}$$

*holds for any time $\tau \in [0,T]$, with $E(0) = E_{0,n}$.*

### 5.5.4 *Limit $\varepsilon \to 0$*

In order to perform the limit $\varepsilon \to 0$, we need the following result.

**Lemma 5.5.8.** *Let $n \in \mathbb{N}$ be fixed and let $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \mathbb{S}_\varepsilon\}_{\varepsilon>0}$ be as in Lemma 5.5.7. Moreover, let*

$$\begin{aligned} \boldsymbol{f}(\varrho_\varepsilon) &:= \sqrt{\varepsilon}\,\nabla_x\varrho_\varepsilon \\ \boldsymbol{g}(\varrho_\varepsilon, \mathbf{u}_\varepsilon) &:= \sqrt{\varepsilon}\,\nabla_x\varrho_\varepsilon\cdot\nabla_x\mathbf{u}_\varepsilon. \end{aligned}$$

*Then, passing to a suitable subsequences as the case may be, the following convergences hold as* $\varepsilon \to 0$.

$$\varrho_\varepsilon \overset{*}{\rightharpoonup} \varrho \quad \text{in } L^\infty((0,T)\times\Omega), \tag{5.5.30}$$

$$\mathbf{u}_\varepsilon \overset{*}{\rightharpoonup} \mathbf{u} \quad \text{in } L^\infty(0,T;W^{1,\infty}(\Omega;\mathbb{R}^d)), \tag{5.5.31}$$

$$\varrho_\varepsilon\mathbf{u}_\varepsilon \overset{*}{\rightharpoonup} \varrho\mathbf{u} \quad \text{in } L^\infty((0,T)\times\Omega;\mathbb{R}^d), \tag{5.5.32}$$

$$\varrho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon \overset{*}{\rightharpoonup} \varrho\mathbf{u}\otimes\mathbf{u} \quad \text{in } L^\infty((0,T)\times\Omega;\mathbb{R}^{d\times d}), \tag{5.5.33}$$

$$\mathbb{S}_\varepsilon \rightharpoonup \mathbb{S} \quad \text{in } L^1((0,T)\times\Omega;\mathbb{R}^{d\times d}), \tag{5.5.34}$$

$$f(\varrho_\varepsilon) \rightharpoonup \overline{f(\varrho)} \quad \text{in } L^2((0,T)\times\Omega;\mathbb{R}^d), \tag{5.5.35}$$

$$g(\varrho_\varepsilon,\mathbf{u}_\varepsilon) \rightharpoonup \overline{g(\varrho,\mathbf{u})} \quad \text{in } L^2((0,T)\times\Omega;\mathbb{R}^d). \tag{5.5.36}$$

*Proof.* Similarly to the preceding section, we can deduce

$$\|\mathbf{u}_\varepsilon\|_{L^q(0,T;W^{1,q}(\Omega;\mathbb{R}^d))} \leq c_1$$

for some $q > 1$ and a positive constant $c_1$ independent of $\varepsilon > 0$, yielding, in view of Lemmas 5.5.2 and 5.5.3,

$$e^{-c_1T}\underline{\varrho} \leq \varrho_\varepsilon(t,x) \leq e^{c_1T}\overline{\varrho}, \quad \text{for all } (t,x)\in[0,T]\times\overline{\Omega}. \tag{5.5.37}$$

We recover convergence (5.5.30). From the energy inequality (5.5.29), it is easy to deduce

$$\sup_{t\in[0,T]} \|\mathbf{u}_\varepsilon(t,\cdot)\|_{W^{1,\infty}(\Omega;\mathbb{R}^d)} \leq c_2, \tag{5.5.38}$$

from which convergence (5.5.31) follows. Combining (5.5.37) and (5.5.38), we can recover

$$\varrho_\varepsilon\mathbf{u}_\varepsilon \overset{*}{\rightharpoonup} \mathbf{m} \quad \text{in } L^\infty((0,T)\times\Omega;\mathbb{R}^d).$$

Now, notice that (5.5.30) can be strengthened to

$$\varrho_\varepsilon \to \varrho \quad \text{in } C_{\text{weak}}([0,T];L^p(\Omega)) \quad \text{for all } 1<p<\infty$$

as $\varepsilon \to 0$, so that, relaying on the compact Sobolev embedding (1.1.10) with $k=1$ and $p=\infty$, we obtain

$$\varrho_\varepsilon \to \varrho \quad \text{in } C([0,T];W^{-1,1}(\Omega))$$

as $\varepsilon \to 0$. The last convergence combined with (5.5.31), implies

$$\mathbf{m} = \varrho\mathbf{u} \quad \text{a.e. in } (0,T)\times\Omega,$$

and thus, we get (5.5.32). Similarly, from (5.5.31) and (5.5.32) we can deduce (5.5.33). Convergence (5.5.34) can be deduced repeating the same passages performed for (5.4.5), manly using the superlinearity of $F^*$ (5.1.11) combined with the De la Vallée–Poussin criterion 1.2.10 and the Dunford–Pettis theorem 1.5.4. Finally, from (5.5.37) we have in particular that

$$\frac{e^{c_1T}\overline{\varrho}}{\varrho(t,x)} \geq 1, \quad \text{for all } (t,x)\in[0,T]\times\overline{\Omega},$$

and thus, from the energy inequality (5.5.29),

$$\varepsilon\int_0^\tau\int_\Omega |\nabla_x\varrho|^2 \,\mathrm{d}x\mathrm{d}t \leq \varepsilon\, e^{c_1T}\overline{\varrho}\int_0^\tau\int_\Omega P''(\varrho)|\nabla_x\varrho|^2\,\mathrm{d}x\mathrm{d}t \leq c(\overline{\varrho},T).$$

In this way we get (5.5.35) and, in view of (5.5.38), (5.5.36). $\square$

*Remark* 5.5.9. It is worth noticing that the limit density $\varrho$ admits the same upper and lower bounds as in (5.5.37):

$$e^{-c_1 T}\underline{\varrho} \le \varrho(t,x) \le e^{c_1 T}\overline{\varrho}, \quad \text{for all } (t,x) \in [0,T] \times \overline{\Omega}.$$

We are now ready to let $\varepsilon \to 0$ in the weak formulations (5.5.27), (5.5.28); notice in particular that, in view of (5.5.36), for any $\tau \in [0,T]$ and any $\boldsymbol{\varphi} \in C^1([0,T]; X_n)$

$$\varepsilon \int_0^\tau \int_\Omega \nabla_x \varrho \cdot \nabla_x \mathbf{u} \cdot \boldsymbol{\varphi} \, \mathrm{d}x\mathrm{d}t = \sqrt{\varepsilon} \int_0^\tau \int_\Omega \sqrt{\varepsilon}\, \nabla_x \varrho \cdot \nabla_x \mathbf{u} \cdot \boldsymbol{\varphi} \, \mathrm{d}x\mathrm{d}t \to 0$$

as $\varepsilon \to 0$.

**Lemma 5.5.10.** *For every fixed $n \in \mathbb{N}$, and any $\varrho_{0,n} \in C(\overline{\Omega})$ such that*

$$E_{0,n} := \int_\Omega \left[ \frac{1}{2} \frac{|\mathbf{m}_0|^2}{\varrho_{0,n}} + P(\varrho_{0,n}) \right] \mathrm{d}x$$

*is bounded by a constant independent of $n$, there exist*

$$\begin{aligned} \varrho &= \varrho_n \in L^\infty((0,T) \times \Omega), \\ \mathbf{u} &= \mathbf{u}_n \in C([0,T]; X_n), \end{aligned}$$

*with*

$$e^{-cT}\underline{\varrho} \le \varrho(t,x) \le e^{cT}\overline{\varrho}, \quad \textit{for all } (t,x) \in [0,T] \times \overline{\Omega},$$

*for a positive constant $c$, such that*

*(i) the integral identity*

$$\left[ \int_\Omega \varrho \varphi(t,\cdot)\, \mathrm{d}x \right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega (\varrho \partial_t \varphi + \varrho \mathbf{u} \cdot \nabla_x \varphi)\, \mathrm{d}x \tag{5.5.39}$$

*holds for any $\tau \in [0,T]$ and any $\varphi \in C^1([0,T] \times \overline{\Omega})$, with $\varrho(0,\cdot) = \varrho_{0,n}$;*

*(ii) there exists*

$$\mathbb{S} = \mathbb{S}_n \in L^1((0,T) \times \Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}})$$

*such that the integral identity*

$$\begin{aligned} \left[ \int_\Omega \varrho \mathbf{u} \cdot \boldsymbol{\varphi}(t,\cdot)\, \mathrm{d}x \right]_{t=0}^{t=\tau} &= \int_0^\tau \int_\Omega [\varrho \mathbf{u} \cdot \partial_t \boldsymbol{\varphi} + (\varrho \mathbf{u} \otimes \mathbf{u}) : \nabla_x \boldsymbol{\varphi} + a\varrho \operatorname{div}_x \boldsymbol{\varphi}]\, \mathrm{d}x\mathrm{d}t \\ &\quad - \int_0^\tau \int_\Omega \mathbb{S} : \nabla_x \boldsymbol{\varphi}\, \mathrm{d}x\mathrm{d}t \end{aligned} \tag{5.5.40}$$

*holds for any $\tau \in [0,T]$ and any $\boldsymbol{\varphi} \in C^1([0,T]; X_n)$, with $(\varrho \mathbf{u})(0,\cdot) = \mathbf{m}_0$;*

*(iii) introducing for a.e. $t \in [0,T]$*

$$E(\tau) = E_n(\tau) = \int_\Omega \left[ \frac{1}{2} \varrho |\mathbf{u}|^2 + P(\varrho) \right] (\tau,\cdot)\, \mathrm{d}x, \tag{5.5.41}$$

*the integral inequality*

$$[E(t)]_{t=0}^{t=\tau} + \int_0^\tau \int_\Omega [F(\mathbb{D}_x \mathbf{u}) + F^*(\mathbb{S})]\, \mathrm{d}x\mathrm{d}t \le 0$$

*holds for any time $\tau \in [0,T]$, with $E(0-) = E_{0,n} \ge E(0+)$.*

*Remark* 5.5.11. In the energy inequality (5.5.41) we used the weak lower semi-continuity in $L^1$ of the functions $E$, $F$ and $F^*$, and thus for a.e. $\tau > 0$

$$\int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho)\right](\tau,\cdot)\,\mathrm{d}x \le \liminf_{\varepsilon\to\infty} \int_\Omega \left[\frac{1}{2}\varrho_\varepsilon|\mathbf{u}_\varepsilon|^2 + P(\varrho)\right](\tau,\cdot)\,\mathrm{d}x,$$

$$\int_0^\tau \int_\Omega [F(\mathbb{D}_x\mathbf{u}) + F^*(\mathbb{S})]\,\mathrm{d}x\mathrm{d}t \le \liminf_{\varepsilon\to 0} \int_0^\tau \int_\Omega [F(\mathbb{D}_x\mathbf{u}_\varepsilon) + F^*(\mathbb{S}_\varepsilon)]\,\mathrm{d}x\mathrm{d}t.$$

### 5.5.5 *Limit $n \to \infty$*

Let $\{\varrho_n, \mathbf{m}_n = \varrho_n\mathbf{u}_n\}_{n\in\mathbb{N}}$ be the family of approximate solutions obtained in Lemma 5.5.10, with correspondent viscous stress tensor $\mathbb{S}_n$ and energy $E_n$. At this stage, as the initial energies are uniformly bounded by a constant independent of $n$, we can recycle the same procedure performed in Section 5.4 to get, passing to suitable subsequences as the case may be, the following family of convergences as $n \to \infty$:

$$\varrho_n \to \varrho \quad \text{in } C_{\text{weak}}([0,T]; L^1(\Omega)), \tag{5.5.42}$$

$$\mathbf{m}_n \to \mathbf{m} \quad \text{in } C_{\text{weak}}([0,T]; L^1(\Omega;\mathbb{R}^d)), \tag{5.5.43}$$

$$\mathbf{u}_n \rightharpoonup \mathbf{u} \quad \text{in } L^q(0,T; W_0^{1,q}(\Omega;\mathbb{R}^d)) \tag{5.5.44}$$

$$\mathbb{S}_n \rightharpoonup \mathbb{S} \quad \text{in } L^1(0,T; L^1(\Omega;\mathbb{R}^{d\times d})), \tag{5.5.45}$$

$$\mathbb{1}_{\varrho_n>0}\frac{\mathbf{m}_n\otimes\mathbf{m}_n}{\varrho_n} \overset{*}{\rightharpoonup} \overline{\mathbb{1}_{\varrho>0}\frac{\mathbf{m}\otimes\mathbf{m}}{\varrho}} \quad \text{in } L^\infty(0,T; \mathcal{M}(\overline{\Omega};\mathbb{R}^{d\times d}_{\text{sym}})), \tag{5.5.46}$$

$$E_n(t) \to E(t) \quad \text{for every } t\in[0,\infty) \text{ and in } L^1(0,T). \tag{5.5.47}$$

with

$$\mathbf{m} = \varrho\mathbf{u} \quad \text{a.e. in } (0,T)\times\Omega,$$

as a consequence of Lemma 5.4.2.

We are now ready to let $n\to\infty$ in the weak formulation of the continuity equation (5.5.39) and the balance of momentum (5.5.40), obtaining that

$$\left[\int_\Omega \varrho\varphi(t,\cdot)\,\mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau\int_\Omega [\varrho\partial_t\varphi + \mathbf{m}\cdot\nabla_x\varphi]\,\mathrm{d}x\mathrm{d}t$$

holds for any $\tau\in[0,T]$ and any $\varphi\in C^1([0,T]\times\overline{\Omega})$, with $\varrho(0,\cdot) = \varrho_0$, and

$$\begin{aligned}\left[\int_\Omega \mathbf{m}\cdot\boldsymbol{\varphi}(t,\cdot)\,\mathrm{d}x\right]_{t=0}^{t=\tau} &= \int_0^\tau\int_\Omega \left[\mathbf{m}\cdot\partial_t\boldsymbol{\varphi} + \mathbb{1}_{\varrho>0}\frac{\mathbf{m}\otimes\mathbf{m}}{\varrho} : \nabla_x\boldsymbol{\varphi} + a\varrho\,\mathrm{div}_x\,\boldsymbol{\varphi}\right]\mathrm{d}x\mathrm{d}t \\ &\quad - \int_0^\tau\int_\Omega \mathbb{S} : \nabla_x\boldsymbol{\varphi}\,\mathrm{d}x\mathrm{d}t + \int_0^\tau\int_{\overline{\Omega}} \nabla_x\boldsymbol{\varphi} : \mathrm{d}\mathfrak{R}\,\mathrm{d}t\end{aligned} \tag{5.5.48}$$

holds for any $\tau\in[0,T]$ and any $\boldsymbol{\varphi}\in C^1([0,T];X_n)$, with $n$ arbitrary. As clearly explained by Abbatiello, Feireisl and Novotný [2], Section 3.4, by a density argument it is possible to extend the validity of the integral identity (5.5.48) for any $\boldsymbol{\varphi}\in C^1([0,T]\times\overline{\Omega})$, $\boldsymbol{\varphi}|_{\partial\Omega} = 0$. Finally, notice that

$$\mathfrak{R}\in L^\infty_{\text{weak}}(0,T;\mathcal{M}^+(\overline{\Omega};\mathbb{R}^{d\times d}_{\text{sym}}))$$

appearing in (5.5.48) has been chosen in such a way that

$$\mathrm{d}\mathfrak{R} = \left( \overline{\mathbb{1}_{\varrho>0} \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho}} - \mathbb{1}_{\varrho>0} \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) \mathrm{d}x + \psi(t)\mathbb{I},$$

where the time-dependent function $\psi$ is chosen in such a way to guarantee

$$\int_\Omega \left[ \frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \right] (\tau, \cdot)\, \mathrm{d}x + \frac{1}{\lambda} \int_{\overline{\Omega}} \mathrm{d}\, \mathrm{Tr}[\mathfrak{R}(\tau)] = E(\tau)$$

for a.e. $\tau \in (0, T)$.

We proved the following result.

**Theorem 5.5.12.** *Let $\gamma = 1$ in (5.1.6). For every fixed initial data*

$$[\varrho_0, \mathbf{m}_0, E_0] \in L^1(\Omega) \times L^1(\Omega; \mathbb{R}^d) \times [0, \infty),$$

*with*

$$\int_\Omega \left[ \frac{1}{2} \frac{|\mathbf{m}_0|^2}{\varrho_0} + \varrho_0 \log \varrho_0 \right] \mathrm{d}x \leq E_0, \tag{5.5.49}$$

*problem (5.1.1)–(5.1.10) admits a dissipative solution in the sense of Definition 5.2.1.*

## 5.6 EXISTENCE OF WEAK SOLUTIONS

Choosing $q > d$ in (5.1.5) and $\gamma = 1$ in (5.1.6), we get the existence of dissipative weak solutions to models describing a general viscous compressible fluid (5.1.1)–(5.1.10), or equivalently, the Reynold stress $\mathfrak{R}$ appearing in Definition 5.2.1 is identically zero. In particular, we improve the work by Matušů-Nečasová and Novotný [62], where existence was achieved in the framework of measure-valued solutions.

For an arbitrary $T > 0$, we can repeat the same procedure performed in Section (5.5) until we get to Lemma 5.5.10. We can now prove the following crucial result.

**Lemma 5.6.1.** *Let $q > d$ in (5.1.5), $\gamma = 1$ in (5.1.6) and let $\{\varrho_n, \mathbf{m}_n = \varrho_n \mathbf{u}_n\}_{n \in \mathbb{N}}$ be the family of approximate solutions obtained in Lemma 5.5.10. Then, passing to a suitable subsequence as the case may be,*

$$\varrho_n \mathbf{u}_n \otimes \mathbf{u}_n \rightharpoonup \varrho \mathbf{u} \otimes \mathbf{u} \quad \text{in } L^1((0,T) \times \Omega; \mathbb{R}^{d \times d}) \tag{5.6.1}$$

*as $n \to \infty$.*

*Proof.* Proceeding as in Section 5.4.1 and due to Lemma 5.4.2, we have

$$\begin{aligned} \varrho_n &\to \varrho \quad \text{in } C_{\mathrm{weak}}([0,T]; L^1(\Omega)), \\ \varrho_n \mathbf{u}_n &\to \varrho \mathbf{u} \quad \text{in } C_{\mathrm{weak}}([0,T]; L^1(\Omega; \mathbb{R}^d)) \end{aligned}$$

as $n \to \infty$, where the sequence $\{\varrho_n \mathbf{u}_n(t, \cdot)\}_{n \in \mathbb{N}}$ is equi-integrable in $L^1(\Omega; \mathbb{R}^d)$ for a.e. $t \in (0, T)$. Thanks to the De la Vallée–Poussin criterion 1.2.10, there exists a Young function $\Psi$ satisfies the $\Delta_2$-condition (1.2.1) such that

$$\varrho_n \mathbf{u}_n \overset{*}{\rightharpoonup} \varrho \mathbf{u} \quad \text{in } L^\infty(0, T; L_\Psi(\Omega; \mathbb{R}^d)),$$

Moreover, due to the compact Sobolev embedding (1.1.10) with $k = 1$ and $p = q > d$, we can prove that the sequence $\{\varrho_n \mathbf{u}_n \otimes \mathbf{u}_n\}_{n\in\mathbb{N}}$ is equi-integrable in $L^1((0,T) \times \Omega; \mathbb{R}^{d\times d})$. Indeed, let $\varepsilon > 0$ be fixed and let the constant $c > 0$ be such that

$$\|\mathbf{u}_n\|_{L^q(0,T;W^{1,q}(\Omega;\mathbb{R}^d))} \leq c,$$

uniformly in $n$. Let $\widetilde{\varepsilon} = \widetilde{\varepsilon}(\varepsilon) > 0$ be chosen in such a way that

$$\widetilde{\varepsilon} < \left(c\, T^{\frac{1}{q'}}\right)^{-1} \varepsilon.$$

From the equi-integrability of the sequence $\{\varrho_n \mathbf{u}_n\}_{n\in\mathbb{N}}$, there exists $\delta = \delta(\widetilde{\varepsilon}) > 0$ such that

$$\int_M |\varrho_n \mathbf{u}_n|(t)\, \mathrm{d}x < \widetilde{\varepsilon}, \quad \text{for every } M \subset \Omega \text{ s.t. } |M| < \delta,$$

for every $n \in \mathbb{N}$. Let $(t_1, t_2) \times M \subset [0,T] \times \Omega$ such that

$$|(t_1, t_2) \times M| < \delta.$$

Then, for every $n \in \mathbb{N}$,

$$\begin{aligned}
\int_{t_1}^{t_2} \int_M |\varrho_n \mathbf{u}_n \otimes \mathbf{u}_n|\, \mathrm{d}x\mathrm{d}t &\leq \int_0^T \int_M |\varrho_n \mathbf{u}_n \otimes \mathbf{u}_n|\, \mathrm{d}x\mathrm{d}t \\
&\leq \|\varrho_n \mathbf{u}_n\|_{L^{q'}(0,T;L^1(M))} \|\mathbf{u}_n\|_{L^q(0,T;W^{1,q}(M))} \\
&\leq c \left[\int_0^T \left(\int_M |\varrho_n \mathbf{u}_n|(t)\, \mathrm{d}x\right)^{q'} \mathrm{d}t\right]^{\frac{1}{q'}} \\
&\leq c\, \widetilde{\varepsilon}\, T^{\frac{1}{q'}} \\
&< \varepsilon.
\end{aligned}$$

Consequently, we can adapt Lemma 5.4.2 replacing the sequence of densities $\{\varrho_n\}_{n\in\mathbb{N}}$ with the sequence of momenta $\{\varrho_n \mathbf{u}_n\}_{n\in\mathbb{N}}$ to obtain (5.6.1). □

Letting $n \to \infty$ in the weak formulation of the continuity equation (5.5.39) and the balance of momentum (5.5.40), we obtain the following result.

**Theorem 5.6.2.** *Let $q > d$ in (5.1.5) and $\gamma = 1$ in (5.1.6). For every fixed initial data*

$$[\varrho_0, \mathbf{m}_0, E_0] \in L^1(\Omega) \times L^1(\Omega; \mathbb{R}^d) \times [0, \infty),$$

*with*

$$\int_\Omega \left[\frac{1}{2} \frac{|\mathbf{m}_0|^2}{\varrho_0} + \varrho_0 \log \varrho_0\right] \mathrm{d}x \leq E_0, \tag{5.6.2}$$

*problem (5.1.1)–(5.1.10) admits a dissipative weak solution*

$$[\varrho, \varrho\mathbf{u}, E] \in C_{\text{weak,loc}}([0,\infty); L^1(\Omega)) \times C_{\text{weak,loc}}([0,\infty); L^1(\Omega; \mathbb{R}^d)) \times \mathfrak{D}([0,\infty)),$$

*meaning that the following holds.*

*(i)* $\varrho \geq 0$ *in* $(0,\infty) \times \Omega$.

*(i) The integral identity*

$$\left[\int_{\Omega} \varrho\varphi(t,\cdot)\ \mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega [\varrho\partial_t\varphi + \varrho\mathbf{u}\cdot\nabla_x\varphi]\ \mathrm{d}x\mathrm{d}t$$

*holds for any* $\tau > 0$ *and any* $\varphi \in C_c^1([0,\infty)\times\overline{\Omega})$*, with* $\varrho(0,\cdot) = \varrho_0$.

*(iii) There exists*

$$\mathbb{S} \in L^1_{\mathrm{loc}}(0,\infty; L^1(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}}))$$

*such that the integral identity*

$$\begin{aligned}\left[\int_{\Omega} \varrho\mathbf{u}\cdot\boldsymbol{\varphi}(t,\cdot)\ \mathrm{d}x\right]_{t=0}^{t=\tau} &= \int_0^\tau \int_\Omega \left[\varrho\mathbf{u}\cdot\partial_t\boldsymbol{\varphi} + \varrho\mathbf{u}\otimes\mathbf{u} : \nabla_x\boldsymbol{\varphi} + a\varrho\,\mathrm{div}_x\,\boldsymbol{\varphi}\right]\ \mathrm{d}x\mathrm{d}t \\ &\quad - \int_0^\tau \int_\Omega \mathbb{S} : \nabla_x\boldsymbol{\varphi}\ \mathrm{d}x\mathrm{d}t\end{aligned}$$

*holds for any* $\tau > 0$ *and any* $\boldsymbol{\varphi} \in C_c^1([0,\infty)\times\overline{\Omega};\mathbb{R}^d)$, $\boldsymbol{\varphi}|_{\partial\Omega} = 0$*, with* $(\varrho\mathbf{u})(0,\cdot) = \mathbf{m}_0$.

*(iv) There exists a càglàd function* $E$*, non–increasing in* $[0,\infty)$*, satisfying*

$$\int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho)\right](\tau,\cdot)\ \mathrm{d}x = E(\tau)$$

*for a.e.* $\tau > 0$*, such that the energy inequality*

$$[E(t)\psi(t)]_{t=\tau_1^-}^{t=\tau_2^+} - \int_{\tau_1}^{\tau_2} E\ \psi'\ \mathrm{d}t + \int_{\tau_1}^{\tau_2} \psi \int_\Omega \left[F(\mathbb{D}_x\mathbf{u}) + F^*(\mathbb{S})\right]\ \mathrm{d}x\mathrm{d}t \le 0$$

*holds for any* $0 \le \tau_1 \le \tau_2$ *and any* $\psi \in C_c^1[0,\infty)$, $\psi \ge 0$*, with* $E(0-) = E_0 \ge E(0+)$.

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