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Research Article

NONLINEAR
Analysis and Application

Metrizability of Cone Metric Spaces Via Renorming the Banach Spaces

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Abstract

In this paper we show that by renorming an ordered Banach space, every cone P can be converted to a normal cone with constant $K = 1$ and consequently due to this approach every cone metric space is really a metric one and every theorem in metric space is valid for cone metric space automatically.

Keywords: Cone metric space; Fixed point.

1 Introduction and Preliminary

In 2007 H. Long-Guang and Z. Xian [4], generalized the concept of a metric space, by introducing cone metric space, and obtained some fixed point theorems for mappings satisfying certain contractive conditions. The study of fixed point theorems in such spaces known as cone metric spaces was taken up by some other mathematicians. But a basic question remained unanswered: *"Are those spaces a real generalization of metric spaces:"* Recently this question has been investigated in the author's paper [2] and in other papers [1, 3, 5, 8, 9]. The authors showed that cone metric spaces are metrizable and defined the equivalent metric using a variety of approaches. However there was another question *"do maps satisfy an analogous contractive condition in the equivalent metric to that satisfied for the cone metric?."* Various authors answered this affirmatively for a number of contractive conditions but it is impossible to answer the question in general. And for recently published article see also [10, 11, 12].

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In this paper we show that by renorming the Banach space which has been partially ordered by a cone, we can obtain a new norm which converts it to normal cone of constant 1, so every cone metric space is metrizable.

Let E be a real Banach space. A nonempty convex subset $P \subset E$ is called a cone in E if it satisfies:

(i) P is closed, nonempty and $P \neq \{0\}$,

(ii) $a, b \in \mathbb{R}$, $a, b \geq 0$ and $x, y \in P$ imply that $ax + by \in P$,

(iii) $x \in P$ and $-x \in P$ imply that $x = 0$.

The space E can be partially ordered by the cone $P \subset E$; by $x \preceq y$ if and only if $y - x \in P$. Also we write $x \prec\prec y$ if $y - x \in P^o$, where P^o denotes the interior of P.
A cone P is called normal if there exists a constant $K > 0$ such that $0 \preceq x \preceq y$ implies $\|x\| \leq K\|y\|$.
In the sequel we suppose that E is a real Banach space, P is a cone in E with nonempty interior i.e. $P^o \neq \emptyset$ and $\preceq$ is the partial ordering with respect to P.

Definition 1.1. ([4]) Let X be a nonempty set. Assume that the mapping $D : X \times X \to E$ satisfies

(i) $0 \preceq D(x, y)$ for all $x, y \in X$ and $D(x, y) = 0$ iff $x = y$

(ii) $D(x, y) = D(y, x)$ for all $x, y \in X$

(iii) $D(x, y) \preceq D(x, z) + D(z, y)$ for all $x, y, z \in X$.

Then D is called a cone metric on X, and (X, D) is called a cone metric space.

2 Main results

Theorem 2.1. *Let $(E, \|.\|)$ a real Banach space with a positive cone P. There exists a norm on E such that P is a normal cone with constant $K = 1$, with respect to this norm.*

Proof. Define $|||.||| : E \to [0, \infty)$ by

$$|||x||| := \inf\{\|u\| : x \preceq u\} + \inf\{\|v\| : v \preceq x\} + \|x\|,$$

for all $x \in E$. We will begin by showing that $|||.|||$ is a norm on E. Firstly, by definition of $|||.|||$ it is clear that, $|||x||| = 0$ if and only if $x = 0$ for all $x \in E$. Also

$$\begin{aligned}
|||-x||| &= \inf\{\|u\| : -x \preceq u\} + \inf\{\|v\| : v \preceq -x\} + \|-x\| \\
&= \inf\{\|u\| : -u \preceq x\} + \inf\{\|v\| : x \preceq -v\} + \|x\| \\
&= \inf\{\|v'\| : v' \preceq x\} + \inf\{\|u'\| : x \preceq u'\} + \|x\| \\
&= |||x|||.
\end{aligned}$$

For $\lambda > 0$,

$$\begin{aligned}
|||\lambda x||| &= \inf\{\|u\| : \lambda x \preceq u\} + \inf\{\|v\| : v \preceq \lambda x\} + \|\lambda x\| \\
&= \inf\left\{\lambda\|\frac{1}{\lambda}u\| : x \preceq \frac{1}{\lambda}u\right\} + \inf\left\{\lambda\|\frac{1}{\lambda}v\| : \frac{1}{\lambda}v \preceq x\right\} + \lambda\|x\| \\
&= \lambda|||x|||.
\end{aligned}$$

Therefore $|||\lambda x||| = |\lambda| |||x|||$ for all $x \in E$ and $\lambda \in \mathbb{R}$.

To prove the triangle inequality of $|||.|||$, let $x, y \in E$

$$\forall \epsilon > 0 \ \exists u_1, v_1 \quad s.t. \quad v_1 \preceq x \preceq u_1, \quad \|u_1\| + \|v_1\| + \|x\| - \epsilon < |||x|||,$$

$$\forall \epsilon > 0 \ \exists u_2, v_2 \quad s.t. \quad v_2 \preceq y \preceq u_2, \quad \|u_2\| + \|v_2\| + \|y\| - \epsilon < |||y|||.$$

Therefore $v_1 + v_2 \preceq x + y \preceq u_1 + u_2$, hence

$$|||x + y||| \leq \|v_1 + v_2\| + \|u_1 + u_2\| + \|x + y\| \leq |||x||| + |||y||| + 2\epsilon.$$

Since $\epsilon > 0$ is arbitrary we obtain

$$|||x + y||| \leq |||x||| + |||y|||.$$

So $|||.|||$ is a norm on E.

Now we shall show that, with the norm $|||.|||$, P is a normal cone with constant $K = 1$; that is for all $x, y \in E$,

$$0 \preceq x \preceq y \Rightarrow |||x||| \leq |||y|||.$$

Suppose that $0 \preceq x \preceq y$. Then

$$0 \leq |||x||| \leq \|0\| + \|y\| + \|x\| = \|y\| + \|x\|. \tag{2.1}$$

If we put $A := \{\|v\| : v \preceq y\}$, then, by (2.1), $|||x|||$ is a lower bound for $A + \|y\|$. So

$$|||x||| \leq \inf(A + \|y\|) = \inf A + \|y\| \leq |||y|||.\square$$

Remark 2.1. We note that the cone P is also closed with respect to $|||.|||$. Let $\{x_n\}$ be sequence in P which is convergent to x in $|||.|||$, by definition of $|||.|||$ we have $\|x_n - x\| \leq |||x_n - x|||$ so if $x_n \xrightarrow{|||.|||} x$, then $x_n \xrightarrow{\|.\|} x$, since $(P, \|.\|)$ is closed therefore $x \in P$.

Corollary 2.1. *Every cone metric space (X, D) is metrizable, with metric defined by $d(x, y) = |||D(x, y)|||$.*

Let $D : X \times X \to E$ be a cone metric, d its equivalent metric, T a self map on X and $\varphi : P \to P$ a map. For all $x, y \in X$, define

$$C(x, y, T, D) := \{D(x, y), D(Tx, Ty), D(Tx, x), D(Ty, y), D(Tx, y), D(x, Ty)\}.$$

If $D(Tx, Ty) \preceq \varphi(U)$ for some $U \in \text{span}(C(x, y, T, D))$ whose coefficients are positive in the linear span, then $d(Tx, Ty) \leq \psi(u)$, for $u \in \text{span}(C(x, y, T, d))$ that corresponds to U, where $\psi(u) = |||\varphi(U)|||$. Example,

$$D(Tx, Ty) \preceq \varphi(D(x, y)) \Rightarrow d(Tx, Ty) \leq \psi(d(x, y)),$$

for all $x, y \in X$.

Therefore, the topology on cone metric spaces is the same as metric spaces topology, and fixed point theorems in cone metric spaces can be considered as the corollaries of corresponding theorems in metric spaces, for example, see the following corollaries.

Example 2.1. *Let $E := \mathbb{R}$, $P := \mathbb{R}^+$ and $D : X \times X \to E$ be a cone metric, $d : X \times X \to \mathbb{R}^+$ its equivalent metric, $T : X \to X$ a self map and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $\varphi(x) = \frac{x}{1+x}$. If $D^* := \varphi(D)$, then its equivalent metric is $d^* = \varphi(d)$, and if,*

$$D(Tx, Ty) \leq \varphi(D(x, y)) = \frac{D(x, y)}{1 + D(x, y)},$$

then $d(Tx, Ty) \leq \varphi(d(x, y)) = \frac{d(x,y)}{1+d(x,y)}$.

Corollary 2.2. ([7, Theorem 2.6]) *Let (X, D) be a complete cone metric space, $P \subset E$ a cone, and T a self-map of X satisfying*

$$D(Tx, Ty) \preceq k[D(Tx, x) + D(Ty, y)]$$

for all $x, y \in X$, where k is a constant, $0 \leq k < 1/2$. Then T has a unique fixed point in X and, for each x in X, the iterative sequence $\{T^n x\}_{n\geq 1}$ converges to the fixed point.

Corollary 2.3. ([6, Corollary 3.4]) *Let (X, D) be a metric cone over the Banach space E with cone P, which is normal with normal constant K. Consider $d : X \times X \to [0, \infty)$ defined by $d(x, y) = \|D(x, y)\|$. Let $T : X \to X$ be a contraction with constant $k < 1$. Then*

$$d(T^n x, T^n y) \leq K k^n d(x, y)$$

for any $x, y \in X$ and $n \geq 0$. Hence $Lip(T^n) \leq K k^n$, for any $n \geq 0$. Therefore $\sum_{n\geq 0} Lip(T^n)$ is convergent, which implies T has a unique fixed point ω, and any orbit converges to ω.

3 Conclusion

Let P be a cone in a Banach space E. By renorming the Banach space E, P is a normal cone with constant $K = 1$. So every cone metric $D : X \times X \to E$ is equivalent to the metric defined by $d(x, y) = |||D(x, y)|||$. Therefore every cone metric defined on a Banach space is really equivalent to a metric. According to this fact, every theorem about Banach spaces is automatically true for the corresponding cone metric spaces, so it is redundant to prove results in cone metric spaces, where the underlying space is a Banach space.

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