On Postcritically Minimal Newton Maps

by

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Statutory Declaration

(on Authorship of a Dissertation)

I, Khudoyor Mamayusupov, hereby declare that I have written this PhD thesis independently, unless where clearly stated otherwise. I have used only the sources, the data and the support that I have clearly mentioned. This PhD thesis has not been submitted for conferral of degree elsewhere.

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Abstract

The Newton map of an entire map f is defined by $N_f(z) := z - \frac{f(z)}{f'(z)}$. For $f(z) = p(z)e^{q(z)}$, where p and q are polynomials, its Newton map $N_{pe^q}(z) = z - \frac{p(z)}{p'(z)+p(z)q'(z)}$ is a rational function. For a Newton map N_{pe^q} the finite fixed points are superattracting and are roots of p. The point at ∞ is a parabolic fixed point with $\deg(q)$ petals for a Newton map $N_{pe^q}(z)$. For fixed integers $d \geq 3$ and $n \geq 1$, let the degrees of the polynomials be $\deg(p) = d - n$ with p having only simple roots and $\deg(q) = n$ then we have $\deg(N_{pe^q}) = d$. The parameter plane (parametrized by coefficients of p and q) of Newton maps satisfying properties above is of complex dimension $d - 2$. Due to existence of the parabolic fixed point for these Newton maps, we can not have post-critically finiteness condition in this family. But there exist analogous notion, that we call *"post-critically minimal"*, to the notion of post-critically finite.

The properties of post-critically minimal Newton maps are analogous to those of post-critically finite Newton maps of polynomials. Using surgery tools developed by Haïssinsky and Cui we give a full classification of *post-critically minimal* Newton maps in terms of Newton maps of polynomials. The latter was recently classified by Y. Mikulich.

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Structure of the thesis

The thesis is organized in six chapters.

• **Chapter 1:** This is an introduction to the subject of the thesis.

• **Chapter 2:** In this chapter we give known results on Newton maps of entire functions. We give a description of a Newton map in terms of partial fraction decomposition of rational functions. We prove that every immediate basin of a rational Newton map has one or several accesses to ∞ improving a known result for Newton maps of polynomials. We define the notion of an "attracting" access, which does not exist in case of Newton maps coming from polynomials. In Section 2.1 we introduce a notion of a marked channel diagram for postcritically finite Newton maps of polynomials. This is the main ingredient of the Haïssinsky surgery, which is covered in Chapter 4. We also give examples of marked channel diagrams for degrees of 2 and 3 of functions.

• **Chapter 3:** In this chapter we introduce a notion of post-critically minimal Newton map, which is the main object of the thesis. We give a full description of a post-critically minimal Newton map on its Fatou components. We define the spaces of function in the thesis. In the Sections 3.2 and 3.3 we do a case study of spaces of lower degree Newton maps.

• **Chapter 4:** This chapter consists of three sections. In the first section, the preliminaries, we define quasiconformal and David maps. We state a surgery method developed by McMullen to change any rational function in the Fatou set to the other function, which we call a general post-critically minimal. As a corollary of it we obtain that the stable components of the parameter space of rational Newton maps contain a unique "center", which is a post-critically minimal Newton map. In Section 4.2 we formulate a surgery developed by Cui to turn rational functions with parabolic cycles into rational functions without parabolic cycles. Section 4.3 is devoted to the Haïssinsky surgery, the main tool of the surgery method we use in the thesis. We prove that it can be applied to post-critically finite Newton maps of polynomials. We improve the result by including the case when a critical point may land at the repelling fixed point that is being converted to the parabolic. The other improvement of the theorem is that we lose the conjugacy in the process of the surgery only in the immediate basins of the marked fixed points that we are working with. This makes our life easy when we are dealing with injectivity of the parabolic surgery. The final remark we give is that the surgery applied to a post-critically finite Newton map of a polynomial gives us a post-critically minimal rational Newton map, which defines a natural mapping from the one space to the other one.

• **Chapter 5:** This chapter provides the main results of the thesis. We prove that the Haïssinsky surgery is injective and surgective between corresponding spaces of Newton maps.

The thesis ends with two appendices.

• **Appendix 1:** In Appendix 1 we generalize the notion of a Newton map allowing the multipliers at the fixed points to be any complex numbers. These type of functions are called Formal Newton maps. In the special cases we prove that the Julia set is connected obtained by a corollary of Shishikura theorem. We give a canonical postcritically minimal Newton map associated to a formal Newton map.

• **Appendix 2:** In Appendix 2 we propose some open problems in the field of Newton maps.

Notations

$\mathbb{C}$	–	The set of complex numbers
$\hat{\mathbb{C}}$	–	The complex Riemann sphere
$\deg(f, z)$	–	The local degree of f at a point z
$K(p)$	–	The filled-in Julia set of a polynomial p
$J(f)$	–	The Julia set of a rational function f
$F(f)$	–	The Fatou set of a rational function f
N_f	–	The Newton map of an entire function f, i.e. $N_f(z) := z - \frac{f(z)}{f'(z)}$
C_f	–	The set of critical points of a function f
P_f	–	The post-critical set of a rational function f
$\mathcal{N}(d-n, n)$	–	The space of degree d normalized Newton maps of pe^q , see Definition 3.1
$\mathcal{N}_{\mathrm{pcm}}(d-n, n)$	–	The space of *post-critically minimal* Newton maps in $\mathcal{N}(d-n, n)$, see Definition 3.2
$\mathcal{N}_{\mathrm{pcf}}(d)$	–	The space of degree d *post-critically finite* Newton maps
$\Delta(f)$	–	The (unmarked) channel diagram of f, see Definition 2.14
$\Delta_n^*(f)$	–	The marked channel diagram of f, see Definition 2.15
$\mathcal{N}_{\mathrm{pcf}}^{+,n}(d)$	–	The space of *post-critically finite* Newton maps in $\mathcal{N}_{\mathrm{pcf}}(d)$ with markings Δ_n^+, see Definition 3.3.

Thesis Summary

1. **The Goal:** The goal of my thesis is to obtain a classification result for the next big class of rational Newton maps, which has the form $z - \frac{p(z)}{p'(z)+p(z)q'(z)}$, where p and q are polynomials. This class consists of all Newton maps coming from transcendental entire maps of the form $p(z)e^{q(z)}$.

2. **Description of rational Newton maps:** The first task was to give a simple description of a function in this space. We solve this task using the partial fractional decomposition of rational functions. We have necessary and sufficient condition for a rational function to be a Newton map.

3. **Understanding of Fatou components:** The next target was to have a basic understanding of Julia set and Fatou components, accesses to ∞ within an immediate basins of attracting and parabolic fixed points. The difference from Newton maps of polynomials (case of $q =$ const.) is that for our functions the point at ∞ is parabolic and so it has parabolic basins. We generalize the notion of access for our setting and prove the result on the number of accesses to ∞.

4. **Marked Channel Diagrams:** We introduce a notion of a marked channel diagram. It is a channel diagram of post-critically Newton map of polynomial (case of $q =$ const.) with marked invariant rays. We mark at most one invariant ray in every immediate basin that we want it to be marked.

5. **Spaces of Newton maps:** The parameter space of degree d Newton maps of polynomials (parametrized by locations of roots of polynomials) is of complex dimension $d-2$. In this space the connected components of hyperbolic functions are called hyperbolic components, and every bounded hyperbolic component contains a unique "center", which is known to be a post-critically finite function. Similarly, we define spaces of rational Newton maps for each degrees of q. In each space we define *stable* functions: these are functions for which all the critical points belong to the basin of ∞ or to the basins of attracting cycles. Since for this family of functions the point at ∞ is persistently parabolic with multiplier $+1$, these stable functions form an open set in the parameter plane. The connected and bounded components contain a *"center"*, which is a post-critical minimal Newton map.

6. **Post-critical minimal Newton maps.** We introduce a notion of post-critical minimal Newton map, for which the critical orbits on the Julia set and on attracting Fatou domains are finite; for critical points in the basin of ∞ there exist minimal critical orbit relations: in every immediate basin of ∞ there exists a unique (possibly with higher multiplicity)

critical point and all other critical points in the basin of ∞ will land to the critical point in one of the immediate basins of ∞ in *minimal iterate*, so that there exist no other types of Fatou components.

7. **A description of post-critical minimal Newton maps.** We obtain the description of a post-critical minimal Newton map in its Fatou components. We construct a normalized Riemann map for every Fatou component with the commutative diagram. The dynamics is conjugated only to the two types model maps.

8. **Haïssinsky surgery.** P. Haïssinsky developed a surgery tool to turn attracting domains of a rational function to parabolic domains of other rational function. This procedure is referred as "Haïssinsky surgery". We extend Haïssinsky surgery construction allowing superattracting domains and also obtain a stronger result. For a Newton map of polynomial the basins of attraction of fixed points have a common boundary point at ∞, the surgery operation changes this point to a parabolic point. The resulting function is in the class of Newton maps of entire functions.

   Haïssinsky surgery is carried out in the immediate basins of attracting fixed points of a rational function. When a root of a polynomial p is simple, it is a superattracting fixed point of the Newton map. In order to be able to successfully apply the surgery we have to change the multipliers to be non-zero at the given basins.

   We define a notion of Haïssinsky surgery equivalence class, which is defined as the class of Newton maps of polynomials with given markings of accesses to ∞ in the immediate basins of superattracting fixed points, where Haïssinsky surgery applied through. Two Newton maps of polynomials belong to the same class if the results of Haïssinsky surgeries give affine conjugate rational functions.

9. **Main result.** We prove that Haïssinsky surgery is an injective and surjective mapping between the corresponding spaces of Newton maps. First, we give a classification of Haïssinsky surgery equivalence classes. Finally, we show shat the surgery induces a natural surjective mapping between corresponding quotient spaces.

10. **Formal Newton maps.** We generalize the definition of a Newton map and obtain a large family of rational functions as: Let $a_i \in \mathbb{C} \setminus \{0\}$, $z_i \in \mathbb{C}$ for $1 \leq i \leq d$ be given. We define $f(z) := z - \frac{1}{\sum_{i=1}^{d} \frac{a_i}{z-z_i}}$ and call it a formal Newton map. We study connectivity of the Julia set for a formal Newton map. We obtain the corresponding post-critical minimal Newton map to the formal Newton map that preserves conjugacy on the Julia sets.

Chapter 1

Introduction

The iteration theory of rational maps on the Riemann sphere was born a century ago after extensive work by P. Fatou and G. Julia. The beauty of this field is that it uses diverse tools coming from topology, geometry, complex analysis, group theory, combinatorics and many other fields.

One of the open problems in the field is to understand and distinguish different possible kinds of dynamics. In recent years, there has been very substantial progress on the understanding of rational functions that arise as Newton maps of polynomials. Dynamical classifications have been given for two large families of rational functions: polynomials of all degrees in terms of "Hubbard trees" (Douady, Hubbard, and Poirier in the 1980's and 1990's), and Newton maps of polynomials in terms of forward invariant connected finite graphs (Head, Tan, Lou, Schleicher, Rückert, Mikulich, Lodge). The result in [LMS2] gives a classification of all post-critically finite Newton maps of polynomials.

An important tool developed by J. Hubbard and D. Schleicher that we shall refer to as "spider theory" uses binary sequences to encode unicritical post-critically finite polynomials by means of a finite graph. In contrast to the rich combinatorial results for polynomials, far less is known about dynamics of rational maps that are not polynomials.

One way better understand dynamical properties of rational functions is to work with less rigid topological models. Mating of two polynomials produces such models [MP12]. In [Tan97] Tan proved that every post-critically finite cubic Newton map is a mating or a capture. Recently, Aspenberg and Roesch extended these results to most cubic Newton maps, towards proving Tan's conjecture: all (not necessarily post-critically finite) cubic Newton maps of polynomials are either matings or captures [AR]. A mating is an operation to obtain a rational function from two polynomials, while capture produces a rational function from only a single polynomial.

This thesis gives a full classification of the next big class of rational maps after the class of Newton maps of polynomials (referred to as polynomial

Newton maps); namely the class of all rational Newton maps coming from transcendental entire functions (exponential Newton maps) of the form pe^q, for polynomials p and q. For the entire function pe^q one can easily compute that its Newton map has a form $id - \frac{p}{p'+pq'}$, which is a rational function. We refer to these functions as exponential Newton maps. The finite fixed points of exponential Newton maps are attracting and these are the roots of a polynomial p. M. Haruta in [Har99] studied these functions and showed that the area of every immediate basin of an attracting fixed point is finite if $\deg q \geq 3$.

P. Haïssinsky in [Ha98] developed a surgery tool to turn attracting domains of a rational function to parabolic domains of other rational function. This procedure is referred to as "Haïssinsky surgery". For a Newton map of polynomial the basins of attraction of fixed points have a common boundary point at ∞, the surgery operation changes this point to a parabolic point. The resulting function is in the class of Newton maps of entire functions.

The other tool that we have in our arsenal is turning parabolic basins to attracting ones. This surgery in full generality was accomplished by G. Cui. Similarly, this procedure is referred to as "Cui (plumbing) surgery". Both tools share similar properties, one of which is a topological conjugacy away from marked basins, in particular on Julia sets. The main result of this thesis is proving that the Haïssinsky surgery gives a natural bijection between the space of polynomial Newton maps and the space of exponential Newton maps.

Haïssinsky surgery is carried out in the immediate basins of attracting fixed points of a rational function. When a root of a polynomial p is simple, it is a superattracting fixed point of the Newton map. In order to be able to successfully apply the surgery we have to change the multipliers to be non-zero at the given basins. This is a standard surgery and it does not change the map away from a neighborhood of the fixed points. We carry out this procedure by choosing the multipliers to be $\frac{1}{2}$, so that we are still in the family of Newton maps of polynomials.

If the orbits of all critical points are finite then this rational function is called post-critically finite. Every Fatou critical point of a post-critically finite rational function eventually terminates at the superattracting periodic points. Now we define the notion of *"post-critical minimality"* for *rational Newton maps* with the parabolic fixed point at ∞. The critical orbits on the Julia set and on attracting Fatou domains are finite, for critical points in the basin of ∞ there exist minimal critical orbit relations: in every immediate basin of ∞ there exists a unique (possibly with higher multiplicity) critical point and all other critical points in the basin of ∞ will land to the critical point in one of the immediate basins of ∞ in *minimal iterate*, so that there exist no other types of Fatou components. The post-critical minimality condition is automatically satisfied for *post-critically finite* Newton maps of polynomials. For rational functions with parabolic fixed point, if a critical point is captured by the critical point in one of the immediate basins of ∞, then its orbit may take some positive number of iterates after visiting the immediate basin before landing at the critical point. The latter behavior is not allowed for post-critically minimal rational functions. This is the difference in the parameter spaces as well. The parameter space of degree d Newton maps of polynomials (parametrized by locations of roots of polynomials) is of complex dimension $d-2$. In this space the connected components of hyperbolic functions are called hyperbolic components, and every bounded hyperbolic component contains a unique "center", which is known to be a post-critically finite function.

Similarly, define stable components of a parameter plane of rational Newton maps of the form $id - \frac{p}{p'+pq'}$, for fixed $\deg(p) = m$ and $\deg(g) = n$, parametrized by coefficients of p and q, to be the connected components consisting of functions where all critical points belong either to attracting basins or parabolic basin of ∞. Since for this family of functions the point at ∞ is persistently parabolic with multiplier $+1$, these stable functions form an open set in the parameter plane. Similar to hyperbolic components, one can observe that by using a suitable surgery, developed by C. McMullen [McM86] (see Theorem 4.8), every function within a bounded stable component can be quasiconformally perturbed to the "center" one, which is a *post-critically minimal* function, while keeping dynamics unchanged on the Julia set. The stable components also contain other types of "rigid" models, called "half-centers" - the functions where some free critical points take some iterates after entering the immediate basin of the parabolic fixed point before landing at the critical point in there. We do not study these type of functions in this thesis.

Relaxing the condition of post-critically finiteness comes with some cost; post-critical minimality is much weaker than post-critically finiteness. However, we can still distinguish post-critically minimal Newton maps by their combinatorics. We do not want to build a parallel theory to the successful theory of classification of post-critically finite Newton maps of polynomials, but we will rather use it to give a full classification of post-critically *minimal* rational Newton maps of transcendental entire maps. We shall prove that under a suitable natural equivalence relation Haïssinsky surgery is an injective and surjective mapping between the corresponding spaces of Newton maps. In order to illustrate how large the family of degree $d \geq 3$ Newton maps we are dealing with is, let us fix the degree of polynomials p and q to be $d-n$ and $1 \leq n \leq d$, respectively. Then the parameter space of degree d rational Newton maps (Newton maps for the entire maps $p(z)e^{q(z)}$) has complex dimension $d-2$. The space of degree d Newton maps of *polynomials* P also has complex dimension $d-2$. When we write d as a sum of two non-negative integers $d-n$ and $n \geq 1$ then it is clear that for every $d \geq 3$ we have $d-1$ distinct spaces of rational Newton maps "parallel" to the space of degree d Newton maps of polynomials.

For every $n \geq 1$ there is a bijection between the space of surgery equivalence classes of degree d post-critically finite Newton maps of polynomials and the space of affine conjugacy classes of degree d post-critically minimal exponential Newton maps, that are of the form $id - \frac{p}{p'+pq'}$ with $\deg p = d - n$ and $\deg q = n$ fixed. A Haïssinsky surgery equivalence class of degree d post-critically finite Newton maps is defined as the class of Newton maps of polynomials with given markings of accesses to ∞ in the immediate basins of superattracting fixed points, where Haïssinsky surgery applied through. Two Newton maps of polynomials belong to the same class if the results of Haïssinsky surgeries give affine conjugate rational functions. Now it is clear that the space we are working with is much larger than the space of Newton maps of polynomials because of different ways of marking of accesses to ∞. Haïssinsky surgery equivalence classes are not exactly a product space, since in order to

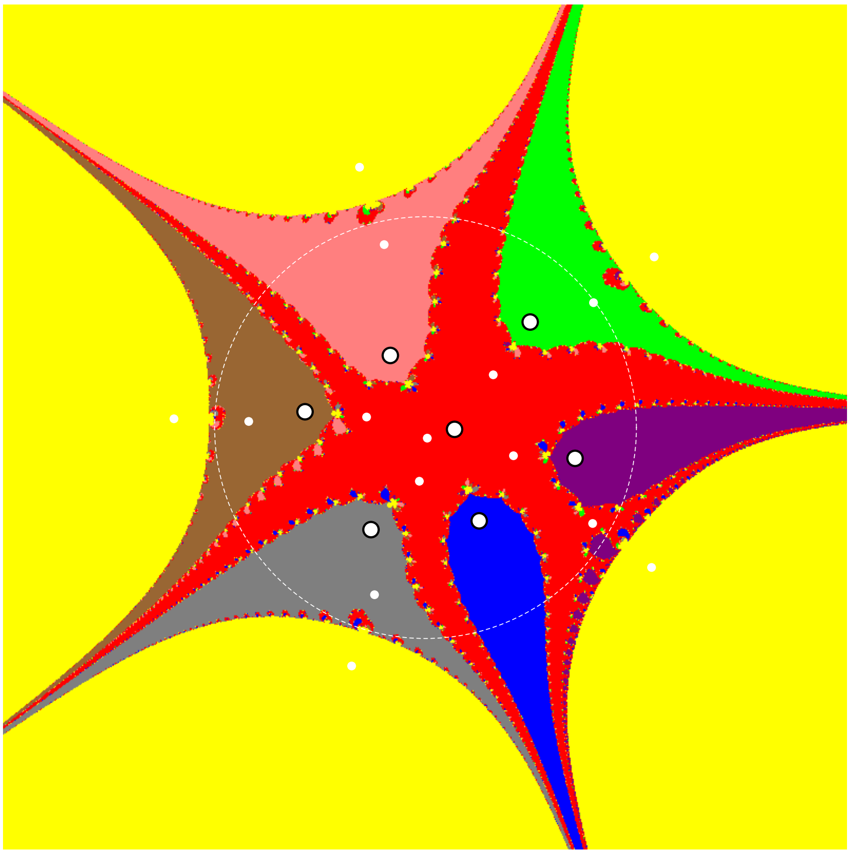


Figure 1.1: The Julia set of the Newton map of degree 12. Yellow is the basin of parabolic fixed point at ∞ with 5 petals, dashed is the unit circle, thick white dots with black circle boundary are fixed points, white dots are critical points.

be able to apply a surgery we need to specify the underlying Newton map of polynomial and then we specify its marking. When $n = d$, the maximal allowed limit, there is a unique fixed point for a Newton map, which necessarily is the parabolic fixed point at ∞. We know that degree d rational function has $2d - 2$ critical points counted with multiplicities. If more than one critical point falls into the same immediate basin, the critical points create more than one access to ∞, through each one of them we can apply a parabolic surgery. If each of fixed points of underlying post-critically finite Newton map has one access to ∞ then, of course, there is only one way to apply Haïssinsky surgery to change all basins at once, the case of $n = d$. In all other cases there are at least two different ways of applying Haïssinsky surgeries.

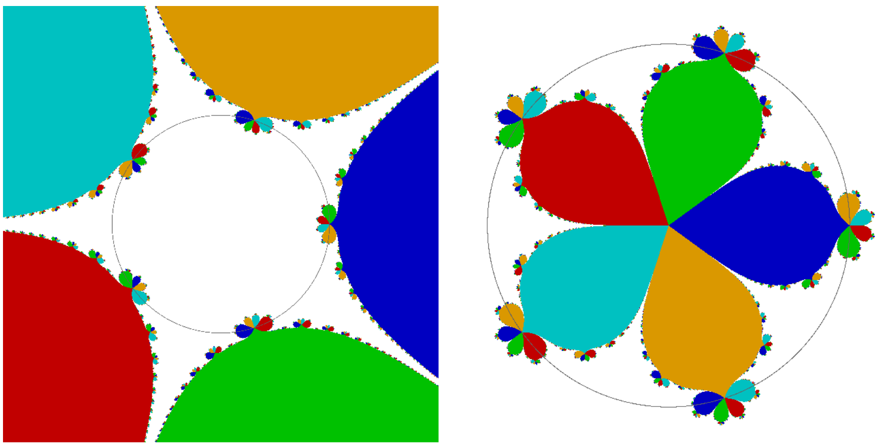


Figure 1.2: Dynamics of the Newton for $f(z) = ze^{-\frac{1}{5}z^5}$. The immediate basin of the root 0 is white, the other colors correspond to immediate basins of ∞ and their backward images. The unit circle $\mathbb{S}^1$ is marked in grey. Right: The same situation in $\zeta = \iota(z) = 1/z$ coordinates: $\iota \circ N_f \circ \iota^{-1}(\zeta) = \zeta - \zeta^6$ is a polynomial, Image courtesy Mayer, Schleicher.

In general for a given function we can apply $m_1 \cdots m_n$ distinct Haïssinsky surgeries to change n basins, where $m_i \geq 1$ is the number of accesses in a marked basin and $\sum_1^n m_i = d$. But in some cases a pair of "different surgeries" result in the same function up to affine conjugation. In special cases this can happen even we apply Haïssinsky surgery through different accesses of a single Newton map of polynomial with an extra symmetry. For instance, we can apply two "different kinds" of Haïssinsky surgeries along two different accesses of $\frac{z^2+1}{2z}$, the quadratic Newton map, but both will produce functions that are affine conjugate to each other. The reason for this is that the two accesses of $\frac{z^2+1}{2z}$ are transformed into each other via an affine map of the form $z \mapsto -z$. We call these kind of surgeries equivalent surgeries. More precisely, Haïssinsky surgeries are said to be *equivalent*, denoted by $\sim_H$, if they produce the same function up to affine conjugation.

Consider a Newton map N_{pe^q} of degree $d \geq 3$ and polynomials p and q of degree $d-n$, $n \geq 0$, respectively. Let us normalize p and q:

case one, $q \not\equiv const.$; we assume that q' is monic,

case two, $q \equiv const.$; we assume that $p(1) = 0$ (i.e. $z = 1$ is a root of p).

Moreover, in both cases we assume that one of the polynomials p, q (or both) with degree at least 2 is centered. Furthermore, we assume that p is monic and has only simple roots.

The set $\mathcal{N}(d-n,n)$ denotes the space of normalized Newton maps N_{pe^q} of degree $d \geq 3$ with n petals at ∞. For instance, $\mathcal{N}(d) := \mathcal{N}(d,0)$ is the space of degree $d \geq 3$ normalized Newton maps for polynomials that are monic, centered, have a root at $z = 1$ and all roots are simple. For every natural number $n \leq d$ denote $\mathcal{N}_{\mathrm{pcm}}(d-n,n)$ the space of post-critically *minimal* Newton maps in $\mathcal{N}(d-n,n)$. In particular, denote $\mathcal{N}_{\mathrm{pcf}}(d) := \mathcal{N}_{\mathrm{pcf}}(d,0)$ the space of degree d post-critically *finite* Newton maps for polynomials that are centered, monic and have a root at $z = 1$, and with only simple roots. Denote $\mathcal{N}^{+,n}_{\mathrm{pcf}}(d)$ the space of post-critically *finite* Newton maps from $\mathcal{N}_{\mathrm{pcf}}(d)$ with markings Δ^+_n.

Theorem 1.1 (Main theorem)**.** *For every pair of non-negative integers $d \geq 3$ and $1 \leq n \leq d$, Haïssinsky surgery is a surjective mapping from $\mathcal{N}^{+,n}_{pcf}(d)$ to $\mathcal{N}(d-n,n)$. Two Haïssinsky surgeries applied to N_{p_1} and N_{p_2} belonging to $\mathcal{N}^{+,n}_{pcf}(d)$ are equivalent if and only if N_{p_1} and N_{p_2} are affine conjugate. The mapping $\mathcal{F}_n$ given by Haïssinsky surgery induces (natural) bijection from $\mathcal{N}^{+,n}_{pcf}(d)/\sim_H$ to $\mathcal{N}_{pcm}(d-n,n)$.*

Chapter 2

Background on Dynamics of Newton maps

In this chapter we summarize known results on Newton maps of entire functions.

Definition 2.1 (Newton map)**.** Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function (polynomial or transcendental entire function). A meromorphic function given by $N_f(z) := z - \frac{f(z)}{f'(z)}$ is called the Newton map of $f(z)$.

The following theorem describes a Newton map in terms of its fixed point multipliers.

Theorem 2.2. *[RS07] Let $N : \mathbb{C} \to \hat{\mathbb{C}}$ be a meromorphic function. It is the Newton map of an entire function $f : \mathbb{C} \to \mathbb{C}$ if and only if for each fixed point ξ, $N(\xi) = \xi$, there is a natural number $m = m_\xi \in \mathbb{N}$ such that $N'(\xi) = (m-1)/m$. In this case there exists a constant $c \in \mathbb{C}\backslash\{0\}$ such that $f = ce^{\int \frac{d\zeta}{\zeta - N(\zeta)}}$. Two entire functions f, g have the same Newton map if and only if $f = c \cdot g$ for some constant $c \in \mathbb{C}\backslash\{0\}$.*

It is a natural question to ask if a Newton map extends to the Riemann sphere as a holomorphic function.

Theorem 2.3 (Rational Newton map)**.** *[RS07] Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. Its Newton map N_f is a rational function if and only if there are polynomials p and q such that f has the form $f = pe^q$. More precisely, let $m,n \geq 0$ be the degrees of p and q, respectively. If $n = 0$ and $m \geq 2$, then ∞ is repelling with multiplier $\frac{m}{m-1}$. If $n = 0$ and $m = 1$, then N_f is constant. If $n > 0$, then ∞ is parabolic with multiplier $+1$ and multiplicity $n + 1 \geq 2$.*

How to decide if a given rational function is a Newton map? It follows from the above theorem that the transcendental entire functions f that give rise to rational Newton maps are exactly those of the form $f(z) = p(z)e^{q(z)}$, where

$p(z)$ and $q(z)$ are polynomials. Note that every rational function of degree at least 2 has a fixed point which is either repelling or parabolic with multiplier $+1$, we call this type of fixed points *weakly repelling.* Here is an easy criterion based on elementary fact on partial fraction decomposition (see Ahlfors [Ahl]) to check whether or not a given rational map is a Newton map.

Theorem 2.4 (Description of rational Newton maps)**.** *Let a rational function $N : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of degree $d \geq 2$ be given. Assume ∞ is a weakly repelling fixed point of N. Let a partial fraction decomposition: $\frac{1}{z-N(z)} = \sum_{i=1}^{k} r_i(\frac{1}{z-z_i}) + s(z)$ be given for unique polynomials r_i for $1 \leq i \leq k$, and s, normalized as $r_i(0) = 0$ for $1 \leq i \leq k$, where z_i runs over all fixed points of N in $\mathbb{C}$. Then N is a Newton map of an entire function if and only if there exist natural numbers $m_i \geq 0$ such that $r_i(z) \equiv m_i \cdot z$. In this case, let $p = p(z) = \prod_{i=1}^{k}(z - z_i)^{m_i}$ (if there is no finite fixed point of N then we let $p(z) = 1$ and $k = 0$) and $q = q(z) = \int s(w)dw$ be polynomials, then $N = N_{pe^q}$ and $d = k + \deg(q)$.*

Proof. Let $N : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational function of degree $d \geq 2$ with $N(\infty) = \infty$. Let the partial fraction decomposition of $\frac{1}{z-N(z)}$ be given: $\frac{1}{z-N(z)} = \sum_{i=1}^{k} r_i(\frac{1}{z-z_i}) + s(z)$ and assume $r_i(z) \equiv m_i \cdot z$. Then we have $\frac{1}{z-N(z)} = \sum_{i=1}^{k} \frac{m_i}{z-z_i} + s(z)$. Elementary algebra shows that $N(z) = z - \frac{p(z)}{p'(z)+p(z)\cdot q'(z)}$, where $p(z) := \prod_{i=1}^{k}(z - z_i)^{m_i}$ and $q = q(z) = \int_0^z s(w)dw$ are polynomials. It follows that N is a Newton map of en entire function pe^q by the uniqueness of Newton maps (Theorem 2.3).

Converse is also true; let N be a Newton map of the entire function $f = pe^q$. Let $p(z) := \prod_{i=1}^{k}(z - z_i)^{m_i}$, where z_i runs over all distinct roots of p, then we obtain $\frac{1}{z-N_f(z)} = \frac{f'}{f} = \frac{p'e^q+pq'e^q}{pe^q} = \frac{p'+pq'}{p} = \frac{p'}{p} + q' = \sum_{i=1}^{k} \frac{m_i}{z-z_i} + q'(z)$. The result follows by uniqueness of a partial fraction decomposition of a rational function.

Now we relate the degree of a Newton map to the number of distinct roots of p and the degree of q. If the ratio $\frac{z(p'(z)+p(z)\cdot q'(z))-p(z)}{p'(z)+p(z)\cdot q'(z)}$ has some cancellation factor in its numerator and denominator then the system of the following polynomial equations;

$$z(p'(z) + p(z)q'(z)) - p(z) = 0 \tag{2.1}$$

$$p'(z) + p(z)q'(z) = 0 \tag{2.2}$$

has a solution for some $z = z_0$. By plugging (2.2) into the equation (2.1) we obtain $p(z_0) = 0$. Combining it with the equation (2.2) we derive to $p'(z_0) = 0$, which means that $z = z_0$ is a multiple root of p. Thus, we have $d = k+deg(q)$, where k is the number of of distinct roots of p. $\square$

Denote the local degree of a function f at a point z by $\deg(f, z)$.

Definition 2.5. Set $C_f = \{\text{critical points of } f\} = \{z | \deg(f, z) > 1\}$ and

$$P_f = \bigcup_{n \geq 1} f^n(C_f).$$

The set P_f is called the post-critical set of f. The function f is called *post-critically finite* (*PCF*) if P_f is finite. The function f is called *geometrically finite* if the intersection $P_f \cap J(f)$ is a finite set.

The next notion is the main object to consider first when one studies a Newton map. Since we are dealing with a general family of rational Newton maps, which are of the form $id - \frac{p}{p' + pq'}$ for polynomials p and q, and so we allow a parabolic fixed point at ∞.

Definition 2.6 (The Basin of Attraction)**.** Let ξ be an attracting or parabolic fixed point of f. The *basin* $\mathcal{A}(\xi)$ of ξ is

$$\operatorname{int}\{z \in \hat{\mathbb{C}} : \lim_{n \to \infty} f^{\circ n}(z) = \xi\},$$

the interior of the set of starting points z which eventually converge to ξ under iteration. The *immediate basin* $\mathcal{A}^\circ(\xi)$ of ξ is the forward invariant connected component of the basin. For a parabolic fixed point there could be more than one immediate basin.

If the fixed point is attracting then in the above definition we do not need to take the interior of the set since it is always an open set, which contains its attracting point with some neighborhood. We only need the interior in the definition when we consider a basin of a parabolic point, since the parabolic point itself does not belong to its basin but is located on the boundary of its immediate basin, thus its grand orbit is dense in the boundary of its basin. For a rational Newton map the basin of a parabolic fixed point at ∞ can be understood as a virtual basin (see [MS06] and [RS07] for the definition of a virtual basin for meromorphic Newton maps).

A basin of an attracting or a parabolic periodic point is defined similarly.

Theorem 2.7 (Przytycki, Mayer-Schleicher)**.** *The immediate basin of a fixed point of a rational Newton map is simply connected and unbounded.*

The above result is a union of two separate works on Newton maps. Przytycki in [Prz89] studies the case of Newton maps of polynomials and Mayer-Schleicher in [MS06] covers the case of Newton maps of entire functions. In the latter case the Newton map does not need to be rational.

Shishikura strengthened the above theorem by proving that not only *immediate* basins are simply connected but *all* components of the Fatou set are simply connected for every rational function with a single weakly repelling

Figure 2.1: The Julia set of the cubic Newton map N_{pe^q}, for $p(z) = z^2 + 2$ and $q(z) = z$. The basins of superattracting fixed points are in green and blue. Its double critical point at $z = -2$ belongs to the immediate basin of parabolic fixed point at ∞ (in yellow).

fixed point. As a corollary of Shishikura's theorem we obtain that the *Julia set* for a rational *Newton map* is *connected*. Recently, Barański, Fagella, Jarque, and Karpińska generalized Shishikura's theorem to the setting of meromorphic Newton maps proving that the Julia set is connected for all Newton maps of entire maps [BFJK].

The following lemma gives the structure of an immediate basin of a post-critically finite Newton map of polynomial.

Lemma 2.8. *[MR09, Lemma 2.2 (Only Critical Point)] Let N_p be a post-critically finite Newton map, $\xi \in \mathbb{C}$ a fixed point of N_p and $\mathcal{A}^\circ(\xi)$ the immediate basin of ξ. Then ξ is a superattracting fixed point of N_p and there is no critical point in $\mathcal{A}(\xi)$ except ξ.*

Besides the connectivity we mostly deal with locally connected Julia sets.

Theorem 2.9 (Tan-Yongcheng)**.** *[TY96] The Julia set of a geometrically finite rational function is locally connected if it is connected.*

In particular, the Julia set of a geometrically finite rational Newton map is locally connected.

The following lemma will be used in the proof of surjectivity of Haïssinsky surgery.

Lemma 2.10 (Most Fatou Components Are Small)**.** *[Mil06] Let f be a rational function with a connected Julia set. The Julia set of f is locally connected if and only if for any $\epsilon > 0$ there are only finitely many Fatou components with diameter greater than ϵ and all of these with locally connected boundaries.*

Definition 2.11 (Access to ∞)**.** Let $\mathcal{A}^\circ$ be the immediate basin of the fixed point $\xi \in \mathbb{C}$ or the parabolic fixed point at ∞ for a rational Newton map f. Consider a curve $\Gamma : [0, \infty) \to \mathcal{A}^\circ$ with $\lim_{t\to\infty} \Gamma(t) = \infty$ (for attracting basins we may further assume that $\Gamma(0) = \xi$, so both ends are fixed under the Newton map f). Its homotopy class within $\mathcal{A}^\circ$ defines an *access to* ∞ for $\mathcal{A}^\circ$, in other words a curve Γ' with the same properties lies in the same access as Γ if the two curves with one endpoint at ∞ fixed and the other end point may vary are homotopic in $\mathcal{A}^\circ$. For a parabolic immediate basin there always exist an access, call it an *attracting access*, through which orbits of points within the basin converge to the parabolic fixed point at ∞.

Every fixed point of a rational Newton map has one or several accesses to infinity.

Remark 2.12. For the case of parabolic immediate basins we can always choose a base point $x_0 \in \mathcal{A}^\circ(\infty)$ so that when we consider a homotopy class of curves $\Gamma : [0, \infty) \to \mathcal{A}^\circ(\infty)$ with $\lim_{t\to\infty} \Gamma(t) = \infty$ then we can assume that $\Gamma(0) = x_0$.

The following is the generalization of the result from [HSS01, Proposition 6] for general rational Newton maps.

Proposition 2.13 (Accesses to ∞)**.** *Let N_{pe^q} be a (rational) Newton map of degree $d \geq 3$ and $\mathcal{A}^\circ$ an immediate basin of a fixed point ξ of N_{pe^q} (an attracting or a parabolic fixed point). Assume that $\mathcal{A}^\circ$ contains k critical points of N_{pe^q} (counting multiplicities), then $N_{pe^q}|_{\mathcal{A}^\circ}$ is a branched covering map of degree $k+1$, and $\mathcal{A}^\circ$ has exactly k different accesses to ∞.*

Proof. Let N_{pe^q} a rational Newton map and $\mathcal{A}^\circ$ its immediate basin be given. There are two cases; case of basin of attracting fixed point or the next case basin of a parabolic fixed point. The case of attracting immediate basins for Newton maps of polynomials is treated in [HSS01, Proposition 6]. Since arguments in the proof use only local dynamics of the function within the basin their result is true for attracting immediate basins of rational Newton maps, too.

It remains to prove the theorem for parabolic immediate basins of N_{pe^q}, it is the case when $\deg(q) > 0$. The proof is essentially the same, for the sake of completeness we give the full proof.

Following [HSS01], let $\mathbb{D}$ denote the unit disk and $\mathcal{A}^\circ$ be one of the immediate basins of ∞ with its Riemann map $\psi : \mathcal{A}^\circ \to \mathbb{D}$, uniquely determined as $\psi(c) = 0$ and $\psi'(c) > 0$, where c is any point in $\mathcal{A}^\circ$. Then the composition map $f = \psi \circ N_{pe^q} \circ \psi^{-1}$ is a proper map of the unit disc $\mathbb{D}$ with a degree which is equal to the degree of $N_{pe^q}|_{\mathcal{A}^\circ}$. The critical points of N_{pe^q} in $\mathcal{A}^\circ$ are mapped preserving multiplicities to the critical points of f. By assumption N_{pe^q} has k critical points in $\mathcal{A}^\circ$, which is simply connected, by the Riemann-Hurwitz formula the degree of $N_{pe^q}|_{\mathcal{A}^\circ}$ in $\mathcal{A}^\circ$ is $k+1$.

Every proper self map of $\mathbb{D}$ is a Blaschke product, thus has an extension to $\hat{\mathbb{C}}$, denote the extension again by f. Both f and the restriction $N_{pe^q}|_{\mathcal{A}^\circ}$ have the same degree. Then f has $k+2$ fixed points, one of which is a double parabolic, since we have a parabolic dynamics in $\mathbb{D}$, and the other $k-1$ fixed points are simple and repelling with real multipliers, and all are located on the unit circle. The unit disk $\mathbb{D}$, the unit circle $\mathbb{S}^1$ and $\hat{\mathbb{C}} \setminus \bar{\mathbb{D}}$ are invariant by f, it follows that f can not have a critical point on $\mathbb{S}^1$, and is a covering map $\mathbb{S}^1 \to \mathbb{S}^1$ of degree $k+1$, and orbit for every $z \in \hat{\mathbb{C}} \setminus \mathbb{S}^1$ converges to the unique parabolic fixed point on $\mathbb{S}^1$. Thus the Julia set is the unit circle $\mathbb{S}^1$. The linearizing coordinates of $k-1$ repelling fixed points define $k-1$ accesses among k accesses, the other one comes from a Fatou coordinate of the parabolic fixed point on $\mathbb{S}^1$. We call the access associated to the parabolic fixed point as an *"attracting"* and all other $k-1$ accesses (if there is any) are called *"repelling"*.

Assume that the boundary of $\mathcal{A}^\circ$ is locally connected, which is true e.g. when a Newton map is geometrically finite. Carathéodory's Theorem assures that the inverse map to $\psi : \mathcal{A}^\circ \to \mathbb{D}$ extends to the closed unit disk as a continuous map. By continuity, $\psi^{-1} \circ f = N_{pe^q} \circ \psi^{-1}$. All of the $k+1$ fixed points of f correspond to $k+1$ fixed points of N_{pe^q} on $\partial\mathcal{A}^\circ$. A fixed point of N_{pe^q} that is on the boundary of an immediate basin is the only parabolic fixed point at ∞, so the domain $\mathcal{A}^\circ$ has accesses to ∞ in $k-1$ different directions.

In the case when $\mathcal{A}^\circ$ is not locally connected, so that the inverse to the Riemann map does not extend continuously to the closed unit disk, the statement still holds true. Consider a Koenigs coordinate of a repelling fixed point ξ_j that conjugates f locally near the point ξ_j to the linear map $z \mapsto f'(\xi_j)z$, we take a segment of a straight-line through the origin, which is invariant. We take an invariant curve in the petal associated to the parabolic fixed point of f. Let γ be the preimage of this curve that lands at ξ_j in the dynamical plane of f. Then we have $\gamma \subset f(\gamma)$. Now we pull the curve γ by the Riemann map ψ to $\mathcal{A}^\circ$. The accumulation set of $\psi^{-1}(\gamma)$ in $\partial\mathcal{A}^\circ$ is connected [Mil06, Section 17] and since γ is invariant we conclude that the accumulation set is pointwise fixed by N_{pe^q}. But ∞ is the only fixed point on the Julia set. This gives us k accesses of $\mathcal{A}^\circ$ to ∞. We need to show that they are all different and the only ones.

It is clear that simple curves within $\mathbb{D}$ converging to a given fixed point of f are homotopic so that every fixed point of f defines a unique access in $\mathcal{A}^\circ$. Different fixed points of f lead to non-homotopic curves in $\mathcal{A}^\circ$ and thus to different accesses. Indeed, let l_i, $l_j \subset \mathbb{D}$ be the radial lines converging to $\xi_i \neq \xi_j$ respectively, parametrized by the radius. Assume by contrary that $\psi^{-1}(l_i)$ and $\psi^{-1}(l_j)$ are homotopic curves in $\mathcal{A}^\circ$ by a homotopy fixing end points; $\psi^{-1}(l_i(1)) = \psi^{-1}(l_j(1)) = \infty$, then one of the components bounded by a simple closed curve $\psi^{-1}(l_i) \cup \psi^{-1}(l_j)$ must be contained in $\mathcal{A}^\circ$. Call this component V; then $\psi(V)$ must be one of the sectors bounded by l_i and l_j; call it S. Both V and S are Jordan domains, so ψ^{-1} extends as a homeomorphism from $\bar{S}$ onto $\bar{V}$, by Carathéodory theorem; but then the extension sends the set $\mathbb{S}^1 \cap S$ nowhere.

Conversely, we show that every access in $\mathcal{A}^\circ$ to ∞ comes from a fixed point of f.

Let $\Gamma : [0,1] \to \mathcal{A}^\circ \cup \infty$ be a curve representing an access. Then $\psi(\Gamma)$ lands at a point $\upsilon \in \mathbb{S}^1$ by [Mil06, Corollary 17.10], define it as the associated point of Γ. Then for every $k \geq 1$, $N_{pe^q}(\Gamma)$ represents an access and thus has its associated point $\upsilon_k \in \mathbb{S}^1$. Since the Newton map N_{pe^q} has a parabolic fixed point at ∞, so it is locally a homeomorphism near ∞ and every fixed point of f gives rise to an access, all υ_k must be contained in the same connected component of $\mathbb{S}^1$ with the fixed points removed; this component is an interval, say I, on which $\{\upsilon_k\}$ must be a monotone sequence converging under f to a fixed point υ of f in $\bar{I}$, i.e. to one of the endpoints. If υ is a one of the repelling fixed points of f then it is impossible. Assume υ is a parabolic fixed point of f then the sequence $\{\upsilon_k\} \subset \mathbb{S}^1$ converges tangentially to the parabolic fixed point, which is also not possible since $\mathbb{S}^1$ is the Julia set of f and every orbit that converges to a parabolic fixed point must follow the attracting direction, which is not tangent to $\mathbb{S}^1$.

$\square$

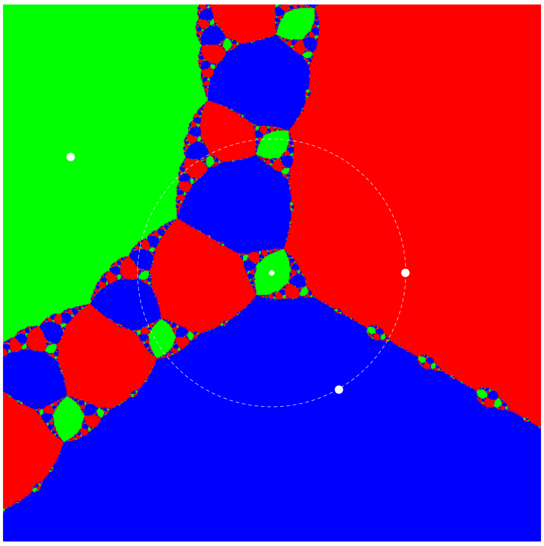


Figure 2.2: The Julia set of the post-critically finite cubic Newton map of the polynomial $p(z) = (z-1)(z-(-1.5+\frac{\sqrt{3}}{2}i))(z-(.5-\frac{\sqrt{3}}{2}i))$. Colors correspond to the basins of three superattracting fixed points. Its unique free critical point at $z = 0$ is fixed under second iterate of N_p.

2.1 Marked Channel Diagrams of Newton map

In this section we give a notion of a channel diagram of a post-critically finite Newton map. But the notion could be defined for a post-critically minimal Newton map, which is defined in Definition 3.6. The channel diagram tells us all about the possible applications of Haïssinsky surgeries on a Newton map. Let the superattracting fixed points of a post-critically finite Newton map N_p be denoted by $a_1, a_2, \ldots, a_d$, and their immediate basins by $\mathcal{A}_i^\circ$. Let $\phi_i : (\mathcal{A}_i^\circ, a_i) \to (\mathbb{D}, 0)$ be a Riemann map (global Böttcher coordinate) with the property that $\phi_i(N_p(z)) = \phi_i^{k_i}$ for each $z \in \mathbb{D}$, where $k_i - 1 \geq 1$ is the multiplicity of a_i as a critical point of N_p. The map $z \mapsto z^{k_i}$ fixes the $k_i - 1$ internal rays in $\mathbb{D}$. Under ϕ_i^{-1} these rays map to the $k_i - 1$ pairwise disjoint (except for endpoints) simple curves $\Gamma_i^1, \Gamma_i^2, \ldots, \Gamma_i^{k_i-1} \subset \mathcal{A}_i^\circ$ that connect a_i to ∞, are pairwise non-homotopic in $\mathcal{A}_i^\circ$ (with homotopies fixing the endpoints) and are invariant under N_p as sets. They represent all accesses to ∞ of $\mathcal{A}_i^\circ$ (see Proposition 2.13).

Definition 2.14 (Channel Diagram Δ)**.** The union

$$\Delta = \bigcup_{i=1}^{d} \bigcup_{j=1}^{k_i-1} \overline{\Gamma_i^j}$$

forms a connected graph in $\hat{\mathbb{C}}$ that is called the *channel diagram.*

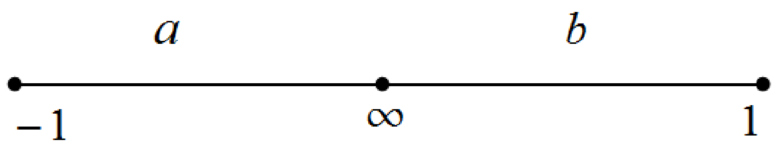
It follows from the definition that $N_p(\Delta) = \Delta$. The channel diagram records the mutual locations of the immediate basins of N_p. The main goal of the thesis is to establish a one to one correspondence between post-critically finite and post-critically minimal families of Newton maps. The correspondence comes from a parabolic surgery, see Theorem 4.15. In order to perform a parabolic surgery along immediate basins of N_p we need to *mark/label* a number of accesses, at most one in every immediate basin of fixed points of N_p.

Definition 2.15 (Marked Channel Diagram Δ_n^+)**.** Let a post-critically finite Newton map N_p with superattracting fixed points $a_1, a_2, \ldots, a_d$ and the channel diagram $\Delta = \bigcup_{i=1}^{d} \bigcup_{j=1}^{k_i-1} \overline{\Gamma_i^j}$ be given. For each $i \in \{1, \ldots, d\}$ we mark at most one fixed ray $\Gamma_i^{j^*}$ in the immediate basin of a_i. If a ray in the immediate basin of a_i is marked then we call the basin of a_i as a *marked basin.* A basin can be marked or unmarked. The marked channel diagram is a channel diagram Δ with marking that is an extra information about which fixed rays are selected/marked. If we have marked $n \leq d$ rays we denote the marked channel diagram by Δ_n^+.

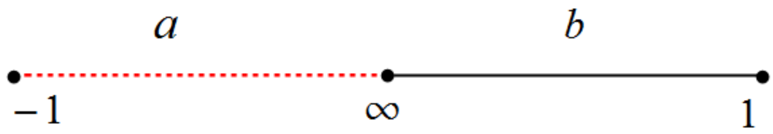
Remark 2.16. We call a channel diagram as *unmarked* channel diagram to distinguish it with the marked channel diagram.

The *marking* defines the single unique access among all accesses within the *marked* immediate basins through which the parabolic surgery is performed.

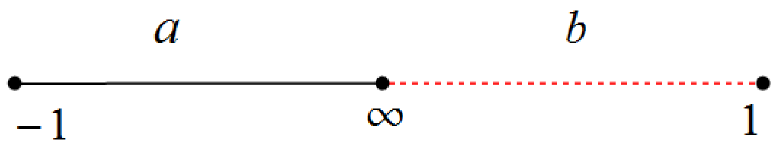
As above, we have $N_p(\Delta_n^+) = \Delta_n^+$. Now we illustrate this notion for the quadratic and cubic Newton maps.



(a) Channel diagram (unmarked)



(b) marked channel diagram



(c) marked channel diagram

Figure 2.3: Unmarked and marked channel diagrams of $\frac{z^2+1}{2z}$, a post-critically finite quadratic Newton map of a polynomial. Top: an unmarked channel diagram. Center and Bottom: 2 marked channel diagrams with one marking.

There exist a quadratic Newton map unique up to affine conjugacy which has two superattracting fixed points. Each of its fixed points has only one access to ∞ in its immediate basin. It is the post-critically finite quadratic Newton map of polynomial and has a form $\frac{z^2+1}{2z}$. Thus in total there are $2 = 1 + 1$ marked Δ_1^+ and one Δ_2^+. An illustration of (unmarked) and marked channel diagrams of the Newton maps are given in Figure 2.3, where -1 and 1 are the only two superattracting points. Labels a and b represent invariant rays in the immediate basins. The figure is a view in $z \mapsto 1/z$ coordinates. Sub-figure **(a)** is an unmarked channel diagram, sub-figures **(b)** and **(c)** are the only possible one marked channel diagrams, where dashed red lines represent the marked rays respectively. The symmetry of the function, which is the rotation by 180 degrees around the origin, interchanges marked channel diagrams of **(b)** and of **(c)**, so we have **(b)**$\sim$**(c)**, means that the marked channel diagrams **(b)** and **(c)** are equivalent, please, refer to Definition 5.1 for the exact formulation of the equivalence of surgeries.

Every cubic Newton map coming from polynomials has three superattracting fixed points, but only one of them may have more than one access in its

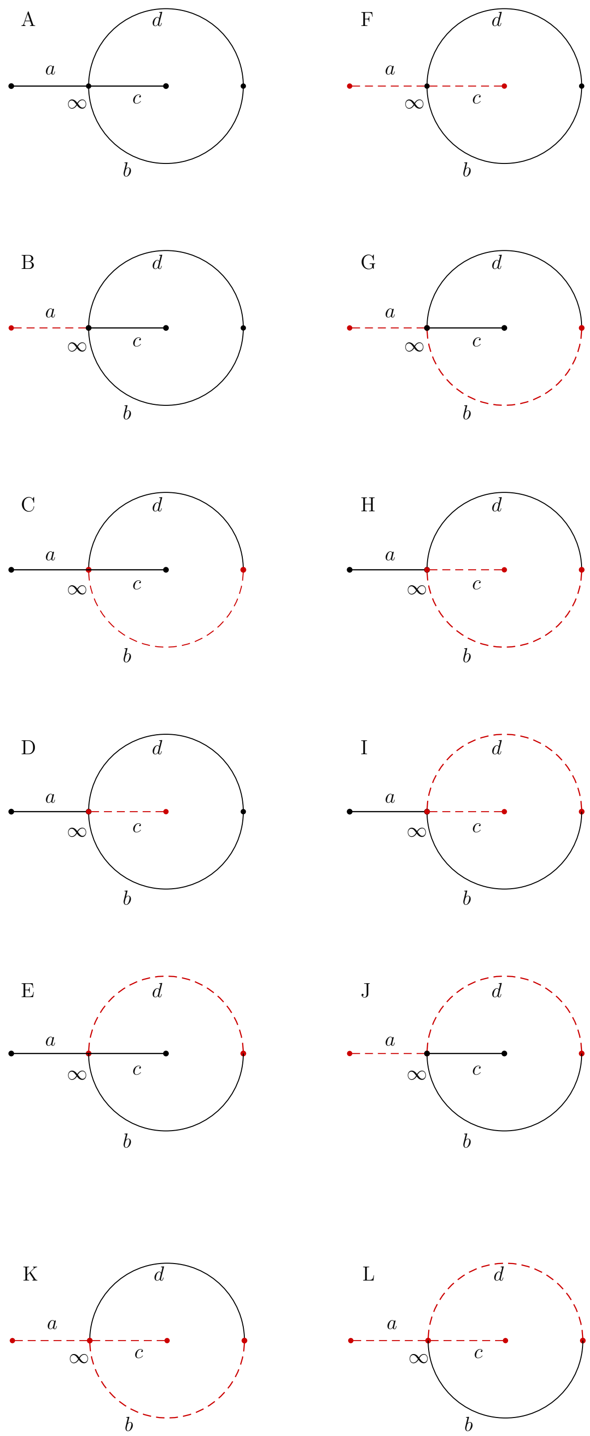


Figure 2.4: An illustration of unmarked and marked channel diagrams of $\frac{2z^3}{3z^2-1}$, the post-critically finite cubic Newton map of polynomial.

immediate basin since we have one free critical point. Thus there exists a post-critically finite cubic Newton map, unique up to affine conjugacy, with two accesses in one of its immediate basins. For $n = 2$: For this extreme case we have two invariant rays (accesses to ∞) within the immediate basin and there is only a single invariant ray (access to ∞) for each of the other immediate basins. Thus there are $4 = 2 + 1 + 1$ marked channel diagrams Δ_1^+ (with one marking). For $n = 2$: there are five marked Δ_2^+ with two markings. Since both marked rays can not be selected from the same immediate basin, thus in total we have $5 = 1 + 2 + 2$ possibilities. For $n = 3$: there are $2 = 1 + 1$ marked channel diagram Δ_3^+ with three marking, since only one immediate basin has two accesses to ∞. In Figure 2.4 an illustration of unmarked and marked channel diagrams of $\frac{2z^3}{3z^2-1}$, a cubic Newton map for the polynomial $z^3 - z$ are shown. This figure is in $z \mapsto 1/z$ coordinates. Dashed red lines represent the marked rays/accesses. There are 4 fixed points at -1, 0 and 1, all are superattracting except ∞ which is repelling. The Julia set of the function is depicted in Figure 2.5. For this function there exist two invariant

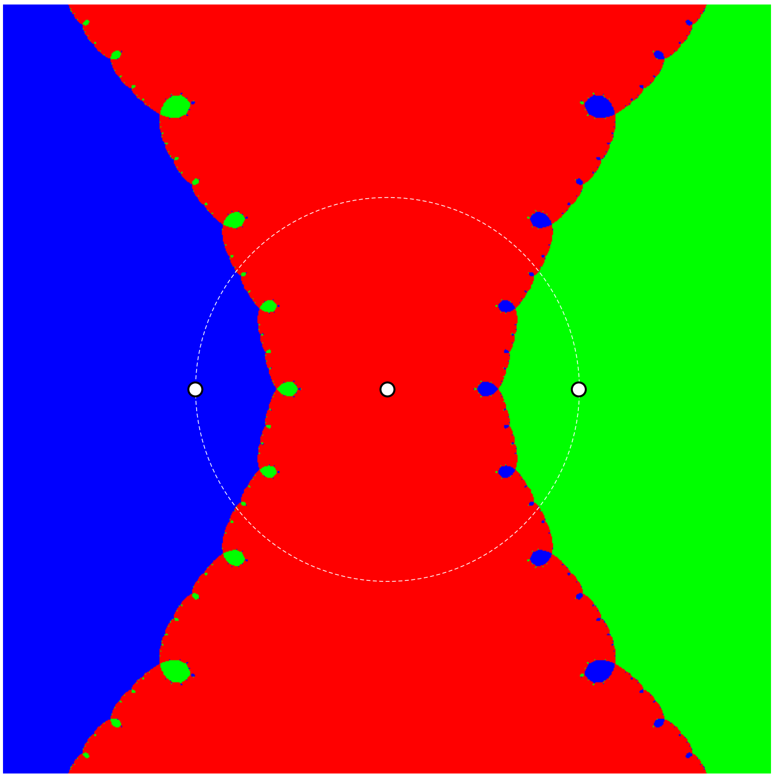


Figure 2.5: The Julia set of $\frac{2z^3}{3z^2-1}$, the cubic Newton map with a double critical point at 0 with the full invariant basin given in red.

rays (accesses) in the immediate basin of 0 since it is a double critical point. We denote these rays by b and d. The other two labels a and c represent fixed rays of -1 and 1 correspondingly. As above, a, b, c and d together with their end points (including ∞) define an unmarked channel diagram, this is shown in **A** in Figure 2.4. Excluding **A**, the unmarked channel diagram, there are in total 11 marked channel diagrams labeled **B** through **L**.

For the cubic Newton map $f(z) = \frac{2z^3}{3z^2-1}$, it is easy to observe that if a Möbius map M is an automorphism of f i.e. $M \circ f = f \circ M$ holds, then we have either $M(z) = z$ or $M(z) = -z$. The map $z \mapsto -z$ interchanges rays a and c, similarly rays b and d are interchanged. From this fact we have 5 pairs of equivalent marked channel diagrams: **B**$\sim$**D**, **C**$\sim$**E**, **G**$\sim$**I**, **H**$\sim$**J** and finally, **K**$\sim$**L**. All other marked channel diagrams are never equivalent to each other. In total we have $11 - 5 = 6$ *different* marked ones, of which two are with one marking and three are with two markings and the last one is with three markings.

For all other post-critically finite cubic Newton maps we have only 3 number of one marked Δ_1^+ and 3 number of two Δ_2^+ and the unique three marked Δ_3^+. In general, we are not interested in counting the number of different possible (non-symmetric) marked channel diagrams that may exist for a given Newton map, this an interesting question in its own right though.

We have fully covered degree 2 and 3 Newton maps to have some sort of intuition about different possibilities that can occur in lower degrees. Later we shall define the equivalence relation on marked channel diagrams (see Definition 5.1), this is one of the reasons why we do not need to count the total number of all possibilities for markings.

Chapter 3

Rational Newton maps

3.1 The Post-Critically Minimal Newton maps

Consider a Newton map $N_{pe^q}(z) = z - \frac{p(z)}{p'(z)+p(z)q'(z)}$ of degree $d \geq 3$, and let $n \leq d$ be the degree of q, then the number of different roots of p is equal $d - n$. It follows from definition of Newton map that the leading coefficient of p cancels, so we can assume that p is monic, and similarly the constant term of q is also not relevant, since it gets canceled under taking the derivative. Any automorphism of $\hat{\mathbb{C}}$ fixing ∞ is an affine transformation of the form $z \mapsto az+b$ $(a \neq 0)$, which is in general a composition of a scaling and a translation, if $a = 1$ then it is just the translation by b.

When $q(z) \not\equiv$ const. by scaling we can change the leading coefficient of q to be any given nonzero number, for instance we can make q' a monic polynomial. Indeed, a scaling by a gives us

$$N_{pe^q}(az)/a = (az - \frac{p(az)}{p'(az) + p(az)q'(az)})/a = z - \frac{p(az)}{ap'(az) + p(az)aq'(az)}.$$

Let $q'(z) = b_{n-1}z^{n-1} + b_{n-2}z^{n-2} + \cdots$ be a derivative of the polynomial q, where $b_{n-1} \neq 0$ is the leading coefficient of $q'(z)$, then we obtain $aq'(az) = b_{n-1}a^n z^{n-1} + b_{n-2}a^{n-1}z^{n-2} + \cdots$. By a choice of a, such that $b_{n-1}a^n = 1$, we can assume that $q'(z)$ is monic. In other words, we let $p_a(z) := p(az)$ and $q_z(z) := q(az)$ then $N_{pe^q}(az)/a = N_{p_a e^{q_a}}(z)$. If we use the translation by b then the conjugacy is

$$N_{pe^q}(z+b) - b = z - \frac{p(z+b)}{p'(z+b) + p(z+b)q'(z+b)}.$$

Now we are only left with one more freedom, essentially a translation. By translation we may further assume that either p or q is centered; all roots sum up to zero.

When $q(z) \equiv$ const. by translation we make p centered; and by scaling we can have $p(1) = 0$. Indeed, let $p_a(z) := p(az)$ then $N_p(az)/a = N_{p_a}(z)$. If we let $p_a(1) = p(a) = 0$ then a is one of the d roots of p.

We can change the multiplier at each finite fixed point of a Newton map by a suitable quasiconformal surgery, therefore, we may further assume that the roots of p are all simple.

Let us normalize polynomials p and q:

case one; $q \not\equiv$ const.: we assume that q' is monic,

case two; $q \equiv$ const.: we assume that $p(1) = 0$ (i.e. $z = 1$ is a root of p).

Moreover, in both cases we assume that either p or q (the one with degree at least 2) is centered, for this we use translation. Furthermore, we assume that p is monic (for this we use the scaling degree of freedom) and has only simple roots.

These lead us to define the following main objects of the thesis.

Definition 3.1. Denote by $\mathcal{N}(d-n, n)$ the space of degree $d \geq 3$ Newton maps N_{pe^q} normalised as above. For instance, $\mathcal{N}(d) := \mathcal{N}(d, 0)$ is the space of degree $d \geq 3$ Newton maps for polynomials, which are monic and centered, have a root at $z = 1$ and all roots are simple.

Definition 3.2. For every natural number $n \leq d$, denote by $\mathcal{N}_{\text{pcm}}(d-n, n)$ the space of post-critically minimal Newton maps in $\mathcal{N}(d-n, n)$. In particular, denote by $\mathcal{N}_{\text{pcf}}(d) := \mathcal{N}_{\text{pcf}}(d, 0)$ the space of degree d *post-critically finite* Newton maps for polynomials that are centered, monic and have a root at $z = 1$.

Definition 3.3. Denote by $\mathcal{N}^{+,n}_{\text{pcf}}(d)$ the space of post-critically finite Newton maps in $\mathcal{N}_{\text{pcf}}(d)$ with markings Δ^+_n (the marked channel diagram) at accesses in the marked immediate basins.

In the above elementary algebra we obtained the following.

Lemma 3.4. *Let two functions f, $\tilde{f} \in \mathcal{N}(d-n, n)$ be conjugate by an affine map ϕ. Then*

if $n = 0$, the case of a Newton map of polynomial, $\phi(z) = az$ where a is a fixed point of f;

if $n \geq 1$, $\phi(z) = az$ where $a^n = 1$.

The affine conjugacy class of a function from $\mathcal{N}(d-n, n)$ within its space consists of d elements if $n = 0$, and n elements if $n \geq 1$.

During normalization process we have a choice for the root at $z = 1$, and thus we don't have a true parameter space: some number of maps are conformally conjugate as in the above lemma. It is now clear that the parameter plane of $\mathcal{N}(d)$ is of complex dimension $d-2$. In this space every bounded hyperbolic component contain a "center": the unique function which is known to be post-critically finite [Mil12]. Every function within the hyperbolic component can be quasiconformally perturbed to the "center" without changing neither topology nor dynamics on Julia sets.

Definition 3.5 (Stable function)**.** We call a function $f \in \mathcal{N}(m,n)$ *stable* if the set of critical points C_f of f belongs to the basin of ∞ or to the basins of attracting cycles of f, in particular all critical points belong to the Fatou set.

The stable functions in the parameter plane of $\mathcal{N}(d-n,n)$ form an open set, because the parabolic point at ∞ is "persistent". Connected components of this set are called *stable components*.

It is also clear now that the parameter plane of $\mathcal{N}(d-n,n)$ is also of complex dimension $d-2$. We consider Newton maps in $\mathcal{N}(d-n,n)$ of degree $d \geq 3$. The following type of functions play a role of "center" in the stable components in $\mathcal{N}(d-n,n)$.

Definition 3.6 (Post-Critically Minimal Newton map)**.** A geometrically finite Newton map $f = N_{pe^q} \in \mathcal{N}(d-n,n)$ with $n \geq 1$ is called post-critically minimal (PCM) if

a) all non-repelling periodic points are superattracting, except ∞ which is the parabolic fixed point with the multiplier $+1$;

b) all critical points in the immediate basin of a superattracting cycle are on the cycle;

c) every immediate basin of the parabolic fixed point at ∞ contains a single critical point (possibly with higher multiplicity);

d) if c is a critical point in a strictly pre-periodic Fatou component and m_c is the *minimal* number for which $f^{\circ m_c}(c)$ is in a periodic component U, then only two possibilities could happen;

   I if U is a superattracting immediate basin, then $f^{\circ m_c+k_c}(c)$ is a critical point, where k_c is the *minimal* number for which $f^{\circ k_c}(U)$ contains a critical point, equivalently the orbit of c is finite; or

   II if U is a parabolic immediate basin of ∞, then $f^{\circ m_c}(c)$ is the critical point in U. In this case U is f invariant.

   In both cases m_c is the pre-period of the Fatou component containing c.

Remark 3.7. The given definition is inspired by the work of McMullen in [McM86] where he develops a surgery method to replace any function on its Fatou components with rigid model maps; obtaining a new function which satisfies conditions b)-d) of above definition, see Definition 4.6 and Theorem 4.8. Condition a) is posed since we are working with Newton maps.

For the space $\mathcal{N}(d)$ by allowing in every immediate attracting basin a unique critical point we could define the notion of PCM maps as defined in $\mathcal{N}(d-n,n)$, basically item (b) is changed from superattracting to attracting. But if a function in $\mathcal{N}(d)$ is PCF then it is also PCM. The only difference in the definitions of PCM and PCF lies on the conditions posed to the critical points in parabolic basin of ∞, whereas for functions in $\mathcal{N}(d)$ the point ∞ is repelling.

Post-critically minimal Newton maps enjoy similar properties as post-critically finite Newton maps of polynomials do.

Proposition 3.8 (Characterization of post-critically minimal Newton maps)**.** *Let $f \in \mathcal{N}(d-n,n)$ be a PCM Newton map and U be any Fatou component of f, let $V = f(U)$. Then U contains the unique "center" ξ_U which is either the critical point or it maps to a point in a superattracting cycle or it maps to the unique critical point in the parabolic immediate basin of ∞ in finite minimal number of iterations under f. Moreover, there exist Riemann maps $\psi_U : U \to \mathbb{D}$ and $\psi_V : V \to \mathbb{D}$ with $\psi_U(\xi_U) = 0$ and $\psi_V(\xi_V) = 0$ such that,*

(a) if U is an immediate basin of the parabolic fixed point at ∞ then $V = U$ and

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "\psi_U"'] & U \arrow[d, "\psi_U"] \\ \mathbb{D} \arrow[r, "P_k"] & \mathbb{D}, \end{tikzcd}$$

where $P_k(z) := \frac{z^k+a}{1+az^k}$ with $a = \frac{k-1}{k+1}$ the parabolic Blaschke product, and $k-1$ is the multiplicity of the single critical point ξ_U in U;

(b) in all other Fatou components we have

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "\psi_U"'] & V \arrow[d, "\psi_V"] \\ \mathbb{D} \arrow[r, "z \mapsto z^k"] & \mathbb{D}, \end{tikzcd}$$

where $k-1$ is the multiplicity of a single critical point in U.

Remark 3.9. In this thesis by a *component* we mean a *connected* component.

Post-critically finite Newton maps can not have a parabolic fixed point, thus the first diagram is the key difference between post-critically *finite* and post-critically *minimal* Newton maps.

Proof. We first show the existence of "centers" in every Fatou component. Since the Julia set $J(f)$ is connected, every Fatou component is simply connected. Let U be a Fatou component of f. We want to show that U contains the unique "center", a point that maps to the critical cycle or to the critical point in a parabolic immediate basin in finite minimal time. If U is an *immediate basin* of a superattracting cycle (fixed points are also counted) that contains a critical point (at least one element of the cycle should be a critical point of f) or U is one of the immediate basins of the parabolic fixed point at ∞, then the existence of the unique "center" follows from Definition 3.6. It is the critical point, denote it by ξ_U. Let T be a component of $f^{-1}(U)$ other than U, if U is fully invariant Fatou component then we are done.

Let C_T be the union of the set of critical points of f in T and the set $f^{-1}(\xi_U) \cap T$. Then by Definition 3.6 we have $f(C_T) = \xi_U$. We want to show that the set C_T consists of the single element. Since $\pi_1(U \setminus \xi_U) = \mathbb{Z}$ and the map $f : V \setminus C_V \to U \setminus \xi_U$ is a covering, the induced map $f_* : \pi_1(V \setminus C_V) \to \pi_1(U \setminus \xi_U)$ is injective, hence C_V is a single point. Denote it by ξ_V and call it the "center" which is unique with the property that it maps to ξ_U under the function f. Induction finishes the argument for all other components of Fatou set.

Now we continue to prove the existence of Riemann maps with commutative diagrams. We have two cases, attracting or parabolic domains. For an attracting component of the Fatou set the existence of the Riemann map satisfying the commutative diagram of the theorem is given in [DH, Proposition 4.2]. The proof was carried out for post-critically finite polynomials but it also works for every attracting component of Fatou set of post-critically minimal Newton maps too, since restricted to these components a rational Newton map behaves like post-critically finite rational function does.

For parabolic domains we need to adapt the proof in the following, since a rational function has an infinite critical orbit in a parabolic basin.

In all other cases (periodic or not), let ψ_V be the Riemann map of a Fatou component V sending its "center" ξ_V to the origin (clearly if U is fixed then we let $V = U$ and $\psi_V = \psi_U$). Then the composition map $F = \psi_V \circ f \circ \psi_U^{-1}$ is a proper map of the unit disc with the only fixed point at the origin, which is also critical. Then $F(z) = uz^k$ and $|u| = 1$. Replace ψ_V and ψ_U by $\mu\psi_V$ and $s\psi_U$ respectively (we denote them again by ψ_V and ψ_U respectively and the composition map by F) then $F(z) = s\mu^{-k}uz^k$, with the choice of $s\mu^{-k_U}u = 1$ we obtain $F(z) = z^{k_U}$, where $k_U - 1$ is the multiplicity of the critical point ξ_U ($k_U = 1$ if the center ξ_U is not a critical point of f). If $U = V$ i.e. U is the superattracting immediate basin then we have $s^{1-k}u = 1$, hence $u = s^{k-1}$ so we have $k - 1$ choices for ψ_U. Now consider the case when U is not an immediate basin (U could be a parabolic component as well) i.e. $U \neq V$ then for a fixed choice of ψ_V we have $\mu^{-k_U}u = 1$, hence $u = \mu^{k_U}$ so we have k_U choices for ψ_U.

In the case when U is a parabolic *immediate basin* of ∞, let ψ_U be the Riemann map sending ξ_U to the origin. Then $F = \psi_U \circ f \circ \psi_U^{-1}$ is the map of the unit disc, which has an extension to the closure, with the unique critical point at the origin. Since the Riemann map ψ_U is uniquely defined up to post-composition by a rotation of the circle, let us post-compose ψ_U by a rotation such that $F(1) = 1$ and it is the parabolic fixed point (we post-compose this rotation to the Riemann map and denote the composition again by ψ_U). Let $F(0) = \psi_U \circ f \circ \psi_U^{-1}(0) = \psi_U(f(\xi_U)) = v \in \mathbb{D}$. Note also that under this normalization, the orbit of 0 under the map F converges to 1 and $F'(1) = 1$. Now let $M(z) = \frac{z-v}{1-\bar{v}z}$ be the Möbius map, an automorphism of the unit disc $\mathbb{D}$, sending v to the origin. The composition $M \circ F$ fixes the origin, which is also its unique critical point. Then clearly, $M \circ F(z) = u(z)z^k$, where u is nonzero conformal function, which is necessarily a constant with $|u| = 1$. Note that, $M \circ F(1) = M(1) = 1$, thus we obtain $u = \frac{1-v}{1-\bar{v}}$. Finally, $F(z) = M^{-1}(z) \circ z^k = \frac{z^k+v}{1+\bar{v}z^k}$. By letting again $F(1) = 1$ we derive after simple algebra to $u = \frac{1-v}{1-\bar{v}}$ is a real number. Since $|u| = 1$ it follows that $u = \pm 1$. Assume $u = -1$ then $1 - v = 1 - \bar{v}$, hence $2 = v + \bar{v}$ and $Re(v) = 1$, which is a contradiction. Thus $u = 1$, finally v is a positive real number, by letting $F'(1) = 1$ we obtain $v = \frac{k-1}{k+1}$, hence $F = P_k(z) := \frac{z^k + \frac{k-1}{k+1}}{1+\frac{k-1}{k+1}z^k}$, where $k - 1$ is a multiplicity of ξ_U as a critical point of f.

□

3.2 The space of quadratic Newton maps

The space of quadratic Newton maps is trivial. We have only three distinct cases depending on the number of petals at ∞.

$\mathcal{N}(2)$ Case of no petals at ∞; $q'(z) \equiv 0$. The space of quadratic Newton maps of polynomials. We have $p = (z - \alpha)(z + \alpha)$, by further scaling we may assume that $\alpha = 1$, then the unique quadratic Newton map is $z - \frac{z^2-1}{2z} = \frac{z^2+1}{2z}$, which can be further conjugated to $z \mapsto z^2$. Thus we have $\mathcal{N}(2) = \mathcal{N}_{\text{pcf}}(2) = \{\frac{z^2+1}{2z}\}$.

$\mathcal{N}(1,1)$ Case of one petal at ∞; $q'(z) \equiv 1$. Since we can assume that the polynomial p is centered, which means its unique root is at the origin, then it is $p(z) \equiv z$, we obtain the quadratic Newton map with one petal $z - \frac{z}{1+z} = \frac{z^2}{z+1}$, which can be further conjugated to $z \mapsto z^2 + z$. Thus we have $\mathcal{N}(1,1) = \mathcal{N}_{\text{pcm}}(1,1) = \{\frac{z^2}{z+1}\}$

$\mathcal{N}(0,2)$ Case of two petals at ∞; $p(z) \equiv 1$ and $q'(z) \equiv z$. A quadratic Newton map with two petals at ∞ is $z - \frac{1}{z} = \frac{z^2-1}{z}$. It follows that $\mathcal{N}(0,2) = \mathcal{N}_{\text{pcm}}(0,2) = \{\frac{z^2-1}{z}\}$.

We mention again that each of $\mathcal{N}_{\text{pcf}}(2)$, $\mathcal{N}_{\text{pcm}}(1,1)$ and $\mathcal{N}_{\text{pcm}}(0,2)$ consists of a single element.

3.3 The spaces of cubic Newton maps

We have four distinct one-parameter families of Newton maps; $\mathcal{N}(3)$, $\mathcal{N}(2,1)$, $\mathcal{N}(1,2)$, and $\mathcal{N}(0,3)$ depending on the number of petals at ∞.

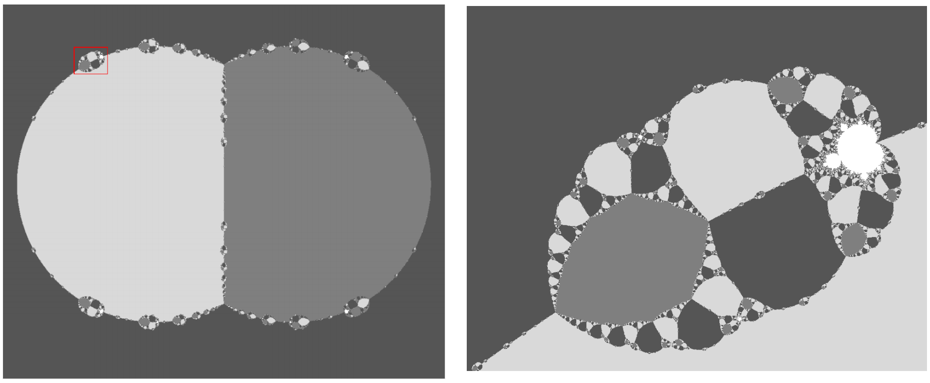


Figure 3.1: Left: Parameter plane of cubic Newton maps of polynomials. Right: Zoom in around a little island in the parameter plane.

The space $\mathcal{N}(3)$, the cubic Newton maps of polynomials, is studied in [Ro08, Tan97], see Figure 3.1 for the parameter plane. Note that the space $\mathcal{N}(3)$ is the space of cubic rational maps with 3 superattracting fixed points. It follows that ∞ is repelling and we obtain the space of cubic Newton maps of polynomials.

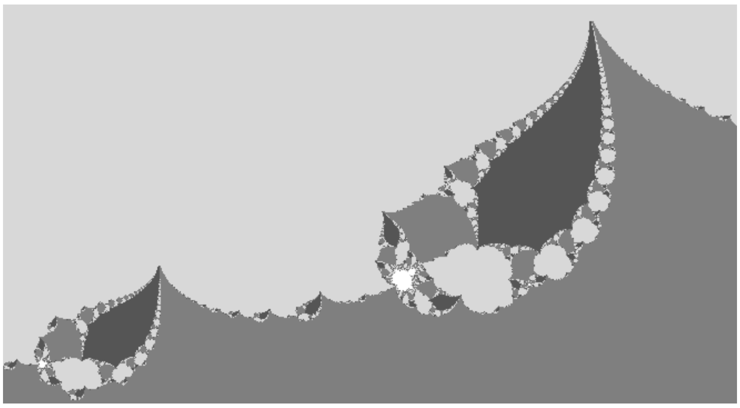


Figure 3.2: A fragment of the parameter plane of the $\mathcal{N}(2,1)$. Light gray, gray, and dark gray colors correspond to the functions for which a free critical point converges to the parabolic fixed point at $z=1$, to the superattracting fixed point at $z=0$, and to the superattracting fixed point at $z=\infty$, respectively.

The space $\mathcal{N}(2,1)$ is one-parameter family of Newton maps of the form

$f_c(z) = z - \frac{z^2+c}{z^2+2z+c}$, $c \neq 0$ (here $p(z) = z^2 + c$ and $q(z) = z$). Note that, f_c and f_c' are affine conjugate if and only if $c' = c$. The parameter plane of this family is depicted in Figure 3.2 with a different parametrization $z^2\frac{z+c-2}{cz-1}$. General family of *all cubic rational functions with two superattracting fixed points* is thoroughly studied by Baranski in [Bar01a, Bar01b]. See [PT09] for the general family of cubic rational maps.

The space $\mathcal{N}(1,2)$ is the family of Newton maps of the form $f_c(z) = z - \frac{z}{z^2+cz+1}$ (here $p(z) = z$ and $q(z) = \frac{z^2}{2} + cz$). Functions f_c and f_c' are Möbius conjugate if and only if ${c'}^2 = c^2$. The parameter plane of $\mathcal{N}(1,2)$ is depicted in Figure 3.3.

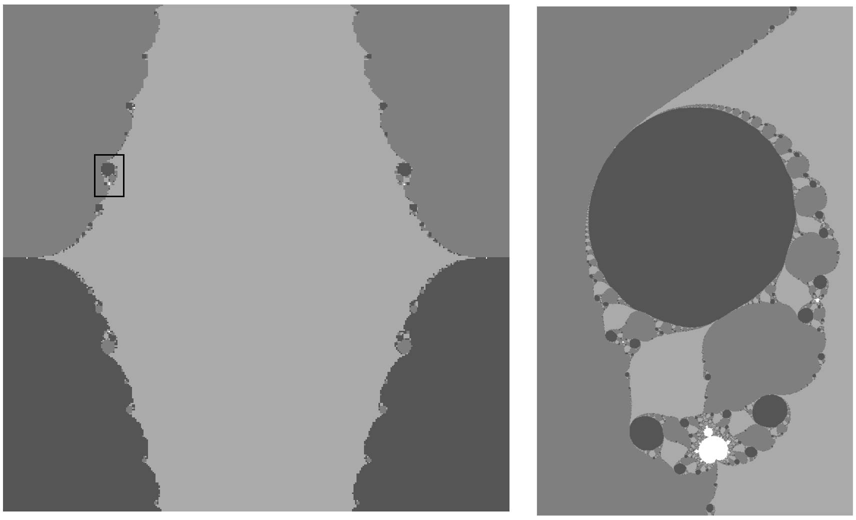


Figure 3.3: The parameter plane of $\mathcal{N}(1,2)$. Left: Parameter plane of cubic Newton maps with 2 petals at a parabolic fixed point. Right: Zoom in around a little island.

The space $\mathcal{N}(0,3)$ is the family of Newton maps of the form $f_c(z) = z - \frac{1}{z^2+c}$, here $p(z) = 1$ and $q(z) = \frac{z^3}{3} + cz$. Functions f_c and f_c' in this family are Möbius conjugate if and only if ${c'}^3 = c^3$.

Chapter 4

Surgery

In this chapter we shall build basis for Main Theorem 5.5. We formulate Cui plumbing surgery (Theorem 4.10) and Haïssinsky surgery (Theorem 4.15) which will enable us to change parabolic points into repelling points, and vice versa, respectively. More precisely, by Cui plumbing surgery for a given post-critically minimal Newton map with a parabolic fixed point at ∞ we change the parabolic domains into attracting, thus producing a Newton map of polynomial. By Haïssinsky surgery we do the reverse of that process, namely for a given post-critically finite Newton map of polynomial we change its repelling fixed point at ∞ into a parabolic fixed point, thus obtaining a rational map, which turns out to be a Newton map in $\mathcal{N}(d-n, n)$ with n the number of marked accesses to ∞, where we are doing surgery. In the next chapter, using the properties of these processes we shall prove in Theorems 5.2–5.3 that the Haïssinsky surgery is well defined and is an injective and a surjective mapping from the marked space of post-critically finite Newton maps of polynomials to the space of post-critically minimal Newton maps.

4.1 Parabolic dynamics and preliminaries

Turning hyperbolics (attracting and repelling fixed points) into parabolic fixed points, or perturbing parabolic fixed points into hyperbolics, is a big issue in complex dynamics. In [GM93, page 16], Goldberg and Milnor formulated the following conjecture: for a polynomial f which has a parabolic cycle, there exists a small perturbation of f such that the immediate basin of the parabolic cycle of f is converted to the basins of some attracting cycles; and the perturbed polynomial on its Julia set is topologically conjugate to f, when restricted to the Julia set. Affirmative answer to the conjecture for geometrically finite functions were given by many, including Haïssinsky, Cui, Tan and Kawahira. We must remark that the local dynamics near repelling and parabolic fixed points are never conjugate to each other. Any quasiconformal conjugacy to a repelling germ is again a repelling germ. Any topological conjugacy to a parabolic germ of the form

$$z \mapsto z + a_{n+1} z^{n+1} + \cdots \tag{4.1}$$

where $n \geq 1$ is an integer and $a_{n+1} \neq 0$ a complex number, is a parabolic germ of the same form. In local conformal coordinates the parabolic germ (4.1) can be written as $z \mapsto z + z^{n+1} + \beta z^{2n+1} + \cdots$, where the fixed point index β is a conformal invariant. In fact, β is the unique formal invariant together with n. There is a formal, not necessarily convergent, power series that formally conjugates (4.1) to $z \mapsto z + z^{n+1} + \beta z^{2n+1}$. It is easy to see that the fixed point index does not depend on the choice of complex coordinates, and is a conformal invariant (see [Mil06, Section 12]).

Now we define quasiconformal and David homeomorphisms and their properties. We state the quasiconformal and David integrability theorems, as well.

Definition 4.1 (K-quasiconformal homeomorphism)**.** Let U and V be domains in $\mathbb{C}$, and let $K \geq 1$ be given. Set $k := \frac{K-1}{K+1}$. Then $\phi : U \to V$ is K-quasiconformal if and only if:

(i) ϕ is a homeomorphism;

(ii) the partial derivatives $\partial\phi$ and $\bar{\partial}\phi$ exist in the sense of distributions and belong to $L^2_{\text{loc}}(U)$ (i.e. are locally square integrable);

(iii) and satisfy $|\bar{\partial}\phi| \leq k|\partial\phi|$ in $L^2_{\text{loc}}(U)$.

The following properties of quasiconformal homeomorphisms are of great importance for our purposes:

1. If ϕ is a K-quasiconformal homeomorphism then the inverse is also K-quasiconformal homeomorphism;
2. Absolute continuity: If ϕ is quasiconformal, then it maps sets of measure zero to sets of measure zero.
3. Quasiconformal removability of quasiarcs: If Γ is a quasiarc (the image of a straight line under a quasiconformal homeomorphism) and $\phi : U \to V$ a homeomorphism that is K-quasiconformal on $U \setminus \Gamma$, then ϕ is K-quasiconformal on U, and hence Γ is quasiconformally removable. In particular, points, lines and smooth arcs are quasiconformally removable.
4. Compactness: The set of K-quasiconformal homeomorphisms $\phi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ fixing three points is compact in the topology of uniform convergence on compact subsets.
5. Weyl's Lemma: If ϕ is 1-quasiconformal, then ϕ is conformal. In other words, if ϕ is a quasiconformal homeomorphism and $\partial_{\bar{z}}\phi = 0$ almost everywhere, then ϕ is conformal.

Theorem 4.2. *[BF14, Theorem 1.28 (Integrability Theorem–global version)] Let S be a simply connected Riemann surface and $\mu = \mu(z)\frac{\partial \bar{z}}{\partial z} : S \to \mathbb{D}$ be a measurable Beltrami form on S. Suppose $K_\mu(z) := \frac{1+|\mu(z)|}{1-|\mu(z)|}$, the dilatation of μ, is uniformly bounded, i.e. $K_\mu < \infty$ or, equivalently, the essential supremum of μ on S satisfies*

$$||\mu||_\infty < 1.$$

Then μ is integrable, i.e. there exists a quasiconformal homeomorphism $\phi : S \to \mathbb{D}$ (respectively onto $\mathbb{C}$ or $\hat{\mathbb{C}}$) which solves the Beltrami equation, i.e. such that

$$\mu(z) = \frac{\partial_{\bar{z}}\phi(z)}{\partial_z \phi(z)}$$

for a.e. $z \in S$.

If S is isomorphic to $\mathbb{D}$ (respectively to $\mathbb{C}$ or $\hat{\mathbb{C}}$) then $\phi : S \to \mathbb{D}$ (respectively onto $\mathbb{C}$ or $\hat{\mathbb{C}}$) is unique up to post-composition with automorphisms of $\mathbb{D}$ (respectively of $\mathbb{C}$ or $\hat{\mathbb{C}}$).

Definition 4.3 (David homeomorphism and David–Beltrami differential)**.** An orientation preserving homeomorphism $\phi : U \to V$ for domains U and V in $\hat{\mathbb{C}}$ is called David homeomorphism (David map or David) if it belongs to the Sobolev class $W^{1,1}_{\text{loc}}(U)$, i.e. has locally integrable distributional partial derivatives in U, and its induced David–Beltrami differential

$$\mu_\phi := \frac{\partial_{\bar{z}}\phi}{\partial_z \phi}\frac{d\bar{z}}{dz}$$

satisfies the following condition; there exist constants $M > 0$, $\alpha > 0$ and $\epsilon_0 > 0$ such that

$$\text{Area}(\{z \in U : |\mu_\phi(z)| > 1 - \epsilon\}) < Me^{-\frac{\alpha}{\epsilon}} \text{ for } \epsilon < \epsilon_0,$$

or, equivalently, if there exist constants $M > 0$, $\alpha > 0$ and $K_0 > 1$ such that

$$\text{Area}(\{z \in U : K_\phi(z) > K\}) < Me^{-\alpha K} \text{ for } K > K_0,$$

where $K_\phi(z) := \frac{1+|\mu_\phi(z)|}{1-|\mu_\phi(z)|}$, the real dilatation of ϕ.

The condition in the definition is referred to as *the area condition.* The area in the definition is a spherical area, for domains that are bounded we use the Euclidean area instead.

Definition 4.4 (ACL, absolute continuity on lines)**.** A continuous function $f : U \to \mathbb{C}$ is said to be absolutely continuous on lines if for any family of parallel lines in any disc D compactly contained in U (that is, $\overline{D} \subset U$), f is absolutely continuous on almost all of them, i.e. for every $\epsilon > 0$ there exists $\delta > 0$ such that $\sum_j |f(b_j) - f(a_j)| < \epsilon$ for every finite sequence of non–intersecting intervals (a_j, b_j) whose closure are contained in a horizontal line I and have a total length $\sum_j |b_j - a_j| < \delta$ for almost all I in D.

We remark that in the definition of David homeomorphism the condition of ϕ being in $W^{1,1}_{\text{loc}}(U)$ can be replaced by the *a priori* weaker requirement of ϕ being *absolutely continuous on lines.*

David homeomorphisms are different from quasiconformal homeomorphisms in many respects:

- The inverse of David homeomorphism may not be David.
- Let $\{\phi_n\}_{n\geq 1}$ be a sequence of David homeomorphisms with the same constants (M, α, K_0) in a domain $U \subset \mathbb{C}$ which fix two given points of U. Then $\{\phi_n\}_{n\geq 1}$ has a subsequence which converges locally uniformly to David homeomorphism in U. The constants of the limit map *a priori* may be different from (M, α, K_0), [Tuk91].

The similar properties of David homeomorphisms to those of quasiconformal homeomorphisms are;

1. David removability of quasiarcs;
2. Absolute continuity: Every David homeomorphism ϕ and its inverse are absolutely continuous, i.e. $\text{Area}(A) = 0$ if and only if $\text{Area}(\phi(A)) = 0$ for a measurable set $A \subset U$.

Moreover, David–Beltrami differentials are always integrated by David homeomorphisms.

Theorem 4.5. *[BF14, David Integrability Theorem] Let μ be David–Beltrami differential on a domain $U \subset \hat{\mathbb{C}}$. Then there exists David homeomorphism $\phi : U \to V$, whose complex dilatation μ_ϕ coincides with μ almost everywhere.*

The integrating map is unique up to post-composition by a conformal map, i.e. if $\widetilde{\phi} : U \to \tilde{V}$ is another David homeomorphism such that $\mu_{\widetilde{\phi}} = \mu$ almost everywhere, then $\widetilde{\phi} \circ \phi^{-1} : V \to \widetilde{V}$ is conformal.

Definition 4.6 (Rigid models)**.** We define the three rigid models for proper self-maps of the unit disk $\mathbb{D}$ as:

(a) the *elliptic model* $z \mapsto e^{2\pi i\theta} z$, for $\theta \in \mathbb{R} \setminus \mathbb{Q}$;

(b) the *hyperbolic model* $z \mapsto z^d$ for some $d > 1$; and

(c) the *parabolic model* $z \mapsto P_d(z)$ for $d > 1$, where

$$P_d(z) := \frac{z^k + a}{1 + az^k}, \text{ for } a = \frac{k-1}{k+1},$$

is the degree d parabolic Blaschke product with a single critical point at $z = 0$ and a parabolic fixed point of multiplicity three at $z = 1$.

Remark 4.7.

1. It is easy to observe that if B is any of the above models, and ϕ is a quasisymmetric automorphism of B (i.e. $B = \phi^{-1} \circ B \circ \phi$) on $\mathbb{D}$, then ϕ must be a rigid rotation $z \mapsto \alpha z$. For the elliptic model, α can be arbitrary; for the hyperbolic model, $\alpha^d = 1$, and in the parabolic model, $\alpha = 1$. This is the reason for calling the models rigid. Note that different types of rigid models are not quasisymmetrically conjugate on $\mathbb{S}^1$.

2. However, if we relax the conjugacy but only ask equivalence then they are no longer rigid. For instance, the hyperbolic and parabolic models are conformally equivalent on D. Set $\phi_1 = id$ and $\phi_2(z) = \frac{z+a}{1+az}$, then $\phi_2(z^d) = P_d(\phi_1)$. Analogously, set $\phi_1 = id$ and $\phi_2(z) = e^{-2\pi i\theta}z$, then for $B(z) = e^{2\pi i\theta}z$ we have $\phi_2 \circ B = id \circ \phi_1$, so the rigid rotation B is conformally equivalent to id.

We end this section with the following theorem of McMullen [McM86, BF14], and its corollary, the proof for which is given in Section 5.3.

Theorem 4.8. *Let f be a rational function and $\Gamma \subset J(f)$ a forward invariant connected component different from a single point. Then there exists a rational function g and a quasiconformal homeomorphism $\phi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that ϕ conjugates $g|_{J(g)}$ to $f|_\Gamma$, in particular, $\phi(\Gamma) = J_g$. Moreover, g has the following properties. The Julia set $J(g)$ is connected and the periodic Fatou components of g are Siegel discs, superattracting basins or parabolic basins. On each periodic Fatou component, the first return map is conjugate to one of the three rigid models. Moreover, if c is a critical point in a strictly preperiodic Fatou component, and n is the smallest number for which $g^{\circ n}(c)$ is in a periodic component U, then:*

(a) if U is a Siegel disc, $g^{\circ n}(c)$ is the indifferent periodic point in U, i.e. the center of the Siegel disc;

(b) if U is a superattracting basin, $g^{\circ n}(c)$ is the superattracting periodic point;

(c) if U is a parabolic basin, $g^{\circ n+k}(c)$ is a critical point, where k is the smallest number for which $g^{\circ k}(U)$ contains a critical point.

Recall from Definition 3.5 that a function $f \in \mathcal{N}(d-n, n)$ is *stable* if C_f, the set of critical points of f, belongs to the basin of ∞ or to the basins of attracting or superattracting cycles of f.

Corollary 4.9 (Case of rational Newton maps)**.** *If $f \in \mathcal{N}(d-n, n)$ is stable, then there exists a post-critically minimal Newton map, unique up to affine conjugacy, satisfying the conclusions of the Theorem 4.8 if we take $\Gamma = J(f)$. In this case only items (b) and (c) of the theorem are possible. Moreover, every bounded stable component in $\mathcal{N}(d-n, n)$ has a unique "center", which is a post-critically minimal Newton map.*

4.2 Cui plumbing surgery

For a geometrically finite rational function g with a parabolic cycle, G. Cui developed a surgery method to turn parabolic domains of g into attracting domains while still maintaining a global semi-conjugacy. As a result he obtained the following as an affirmative answer to the Goldberg-Milnor conjecture in the case of geometrically finite rational functions.

Theorem 4.10. *[CT] Suppose that g is a geometrically finite rational function and X is a parabolic cycle of g. Then there exist a continuous family of geometrically finite rational functions $\{f_t\}$ $(1 \le t < \infty)$ and a continuous family of quasiconformal conjugacies $\{\phi_t\}$ from f_1 to $\{f_t\}$, such that the following conditions hold:*

(1) $\{f_t\}$ uniformly converges to g as $t \to \infty$.

(2) $\{\phi_t\}$ uniformly converges to a quotient map ϕ of $\hat{\mathbb{C}}$ as $t \to \infty$ and $\phi \circ f_1 = g \circ \phi$, i.e. the following diagram is commutative;

$$\begin{tikzcd} \hat{\mathbb{C}} \arrow[r, "f_1"] \arrow[d, "\phi"'] & \hat{\mathbb{C}} \arrow[d, "\phi"] \\ \hat{\mathbb{C}} \arrow[r, "g"] & \hat{\mathbb{C}}. \end{tikzcd}$$

Moreover, ϕ is a homeomorphism from $J(f_1)$ onto $J(g)$.

(3) For every periodic Fatou domain D of g, if D is a parabolic component associated with X, then $\phi^{-1}(D)$ is contained in an attracting domain of f_1 and ϕ is quasiconformal homeomorphism on any domain compactly contained in $\phi^{-1}(D)$. Otherwise, $\phi^{-1}(D)$ is a Fatou domain of f_1 and ϕ is conformal on $\phi^{-1}(D)$.

Remark 4.11. Note that f_1 in the above theorem is sub-hyperbolic geometrically finite function; that is, all of its non-repelling cycles are attracting. For such rational function g, the theorem converts all parabolic domains into attracting domains. Since the semi-conjugacy ϕ is conformal in other Fatou components, the multipliers of attracting cycles of g are preserved. For a post-critically minimal Newton map g, item (3) of the theorem allows us to conclude that f_1 could be further changed by quasiconformal surgery to a post-critically finite Newton map.

The theorem uses the following notion.

Definition 4.12 (Quotient map)**.** Let h be a continuous surjective map on $\hat{\mathbb{C}}$. The map h is called a *quotient* map if $h^{-1}(y)$ is either a singleton or a full continuum for every point $y \in \hat{\mathbb{C}}$, i.e. $\hat{\mathbb{C}} \setminus h^{-1}(y)$ is a simply connected domain.

4.3 Haïssinsky surgery for Newton maps

Haïssinsky in [Ha98, BF14] developed a parabolic surgery that changes a repelling fixed point of a polynomial into a parabolic fixed point of some other polynomial q while preserving the topology and dynamics of the Julia set. This surgery construction is the basis for this thesis.

We write

$$a(t) \asymp b(t)$$

for two positive valued functions $a(t)$ and $b(t)$ if there exists a constant $C > 0$ such that $C^{-1}b(t) \leq a(t) \leq Cb(t)$. In this case, we say that a and b are *comparable*.

Let p be a polynomial of degree $\deg(p) \geq 2$ and let $K(p)$ and $J(p)$ be its filled Julia set and Julia set, respectively. Recall that P_f denotes the post-critical set of f, that is the closure of the critical orbits of f. If α is an attracting or parabolic fixed point, we set $\mathcal{A}(\alpha)$ to be its basin of attraction and $\mathcal{A}^\circ(\alpha)$ its immediate basin.

Theorem 4.13 (P. Haïssinsky)**.** *[Ha98] Let p be a polynomial of degree at least 2 with a (non-super)attracting fixed point α and a repelling fixed point $\beta \in \partial\mathcal{A}^\circ(\alpha)$, $\beta \notin P_p$. Suppose also that β is accessible from the basin $\mathcal{A}^\circ(\alpha)$. Then there exist a polynomial q of the same degree and a David homeomorphism $\phi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, such that:*

1. *$\phi(K(p)) = K(q)$, $\phi(\beta)$ is a parabolic fixed point with $q'(\phi(\beta)) = 1$, and $\phi(\mathcal{A}^\circ(\alpha)) = \mathcal{A}^\circ(\phi(\beta))$;*
2. *for all $z \notin \mathcal{A}(\alpha)$, $\phi \circ p = q \circ \phi$; in particular, $\phi : J(p) \to J(q)$ is a homeomorphism which conjugates f to g on the Julia sets.*
3. *ϕ is conformal on $\hat{\mathbb{C}} \setminus \overline{\mathcal{A}(\alpha)}$.*

Remark 4.14. Note that the construction of surgery is local (along the given access) and can be performed on rational functions that are not necessarily polynomials. By a slight and sensible change of the proof, this theorem can be stated replacing the attracting fixed point by several cycles and the repelling point by cycles such that their periods divide those of the attracting points related to them, provided the repelling points that are to become parabolic are not accumulated by the recurrent critical orbits of p.

Proof is involved, refer to [BF14, Ha98] for the details. We give a sketch of the proof in Theorem 4.15 that is modified for Newton maps. In some places of the proof we give more details and also include the case when the repelling fixed point that is going to become the parabolic is a landing point of a critical orbit.

The following is the special case of the theorem for Newton maps of polynomials, where we have a degree $d \geq 3$ Newton map of polynomial, we select

$n \leq d$ roots ξ_i, select one access for each, then we make the function parabolic where each of these (accesses) immediate basins become attracting parabolic petals.

Theorem 4.15 (Haïssinsky surgery for polynomial Newton map)**.** *Let a post-critically finite Newton map $N_p \in \mathcal{N}_{pcf}^{+,n}(d)$ with Δ_n^+ its marked channel diagram be given. Let $\mathcal{A}(\xi_j)$ be the marked basins of superattracting fixed points ξ_j, for given $1 \leq j \leq n$. Then there exist a David homeomorphism ϕ and $N_{\tilde{p}e^{\tilde{q}}} \in \mathcal{N}_{pcm}(d-n,n)$, such that*

(i) $\phi(\infty) = \infty$ and $\phi(\bigcup_{1\leq j\leq n} \mathcal{A}(\xi_j))$ is the basin of the parabolic fixed point at ∞ of $N_{\tilde{p}e^{\tilde{q}}}$;

(ii) $\phi \circ N_p = N_{\tilde{p}e^{\tilde{q}}} \circ \phi$ for all $z \notin \bigcup_{1\leq j\leq n} \mathcal{A}^\circ(\xi_j)$; in particular, $\phi : J(N_p) \to J(N_{\tilde{p}e^{\tilde{q}}})$ is a homeomorphism which conjugates N_p to $N_{\tilde{p}e^{\tilde{q}}}$;

(iii) ϕ is conformal on the interior of $\hat{\mathbb{C}} \setminus \bigcup_{1\leq j\leq n} \mathcal{A}(\xi_j)$;

(iv) $N_{\tilde{p}e^{\tilde{q}}}$ is post-critically minimal and the marked accesses of the marked channel diagram Δ_n^+ correspond to the attracting accesses of the parabolic basin of ∞ for $N_{\tilde{p}e^{\tilde{q}}}$.

Remark 4.16. Note that in this version of the theorem the conjugacy breaks down only in the marked immediate basins.

Proof. We shall sketch the proof of Theorem 4.13 here as well. The proof is involved, so we divide it into many independent parts.

Preliminaries. We need several results on extensions of maps, lifting property, and Blaschke products that we intensively use in many places of our theorems. We use the following lemma when we construct a local topological conjugacy between Newton maps at their parabolic point at ∞, for the proof of the lemma refer to [CT, Lemma 3.4.]

Lemma 4.17. *[CT] Suppose rational maps f and g with parabolic fixed points z_0 and z_1 respectively are given. Let ϕ_0 be a K-quasiconformal conjugacy between their attracting flowers. Then for any $\epsilon > 0$, there is a local $(K+\epsilon)$-quasiconformal conjugacy ϕ between f and g such that $\phi = \phi_0$ on a smaller attracting flower.*

We extensively use the following theorem on extension of quasisymmetric maps between boundaries of quasidiscs, and quasiannuli. For its proof, please, kindly refer to [BF14, Proposition 2.30].

Theorem 4.18 (Quasiconformal interpolation)**.**

(a) Suppose G_1 and G_2 are quasidiscs bounded by γ_1 and γ_2 and let $f : \gamma_1 \to \gamma_2$ be quasisymmetric. Then f extends to a quasiconformal map $\hat{f} : \overline{G_1} \to \overline{G_2}$.

(b) *For $j = 1, 2$, suppose A_j are open quasiannuli bounded by the quasicircles γ_j^i, γ_j^o. Let $f^i : \gamma_1^i \to \gamma_2^i$ and $f^o : \gamma_1^o \to \gamma_2^o$ be quasisymmetric maps between the inner and outer boundaries respectively. Then there exists a quasiconformal map $f : \overline{A_1} \to \overline{A_2}$ extending the boundary maps f^i and f^o.*

We shall use the following classical result on lifting property of covering maps.

Lemma 4.19. *Let Y, Z and W be path-connected and locally path-connected Hausdorff spaces with base points $y \in Y, z \in Z$ and $w \in W$. Suppose $p : W \to Y$ is an unbranched covering and $f : Z \to Y$ is a continuous map such that $f(z) = y = p(w)$.*

$$\begin{tikzcd} Z, z \arrow[r, "\tilde{f}"] \arrow[dr, "f"'] & W, w \arrow[d, "p"] \\ & Y, y \end{tikzcd}$$

There exists a continuous lift $\tilde{f}$ of f to p with $\tilde{f}(z) = w$ if and only if

$$f_*(\pi_1(Z, z)) \subset p_*(\pi_1(W, w)),$$

where π_1 denotes the fundamental group. This lift is unique if it exists.

We must mention that Haïssinsky surgery is not directly applicable to the superattracting domains. In order to overcome the problem we need to change these domains to (non-super)attracting ones. The next lemma resolves this issue.

Lemma 4.20. *Let N_p be a post-critically finite Newton map of degree $d > 1$ with ξ a superattracting fixed point with its basin $\mathcal{A}(\xi)$ and with a marked access Δ_1^+ to ∞ in the immediate basin $\mathcal{A}^\circ(\xi)$. There exist a Newton map $N_{\tilde{p}}$ of the same degree d and a quasiconformal homeomorphism $\phi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that*

(a) *ϕ is conformal except on $\mathcal{A}(\xi)$;*

(b) *$\phi \circ N_p = N_{\tilde{p}} \circ \phi$ except on a compact set in $\mathcal{A}^\circ(\xi)$, in particular, $\phi(F(N_p)) = F(N_{\tilde{p}})$ and $\phi(J(N)) = J(N_{\tilde{p}})$.*

(c) *$\phi(\xi)$ is an attracting fixed point of $N_{\tilde{p}}$ with the multiplier $N'_{\tilde{p}}(\phi(\xi)) = \frac{1}{2}$, and with the immediate basin $\mathcal{A}^\circ(\phi(\xi)) = \phi(\mathcal{A}^\circ(\xi))$, which contains a single critical point $c \in \mathcal{A}^\circ(\phi(\xi))$.*

(d) *$\phi(\Delta_1^+)$ becomes a marked access in $\mathcal{A}^\circ(\phi(\xi))$.*

(e) *Every Fatou component of $N_{\tilde{p}}$ contains at most one critical point; and if U is a component of $\mathcal{A}(\phi(\xi))$ with a critical point c_U, then there exists a minimal m_c such that $N_{\tilde{p}}^{\circ m_c}(c_U)$ is a critical point in $N_{\tilde{p}}^{\circ m_c}(U)$.*

Remark 4.21. The lemma is still valid if we replace one basin with several basins, we just need to work one by one with all marked basins. We change the multiplier to $1 - \frac{1}{2} = \frac{1}{2}$ with the goal of obtaining a Newton map of some other polynomial $\tilde{p}$ by theorem 2.2. But this choice is purely artificial; actually, it is possible to change the multiplier to any non-zero $\lambda \in \mathbb{D}^*$. The lemma says that not only is the multiplier changed at the superattracting fixed point, but we can also keep the marked access "unchanged". Moreover, if we let $R : \mathcal{A}^\circ(\phi(\xi)) \to \mathbb{D}$ be a Riemann map such that $R(c) = 0$ and let $0 < R(\phi(\xi)) < 1$ (after post-composition by a rotation), then the marked access in $\mathcal{A}^\circ(\xi)$ transports to the marked access containing $R^{-1}([R(\phi(\xi)), 1])$ in $\mathcal{A}^\circ(\phi(\xi))$ of $N_{\tilde{p}}$, where the following holds: $R \circ N_{\tilde{p}} \circ R^{-1} = B(z)$, where $B(z) = \frac{z^k+b}{1+bz^k}$ for $0 < b < a$ and $a = \frac{k-1}{k+1}$ and $k = \deg(N_p, \xi)$ with exact value of $b = b_0$, which is specified later in the proof.

Proof of Lemma 4.20. Let N_p be a post-critically finite Newton map of degree $d > 1$ with ξ a superattracting fixed point with the basin $\mathcal{A}(\xi)$ and with a single marked access Δ_1^+ in the immediate basin $\mathcal{A}^\circ(\xi)$. By lemma 2.8, the point ξ is the only critical point in $\mathcal{A}^\circ(\xi)$. Its local dynamics is conjugate to $z \mapsto z^{d_\xi}$, where $d_\xi > 1$ is the multiplicity of the critical point ξ. Let $\psi : \mathcal{A}^\circ(\xi) \to \mathbb{D}$ be the Riemann map (Böttcher coordinate) such that $\psi(\xi) = 0$. It is unique up to rotation; normalize it so that the fixed ray corresponding to the marked access Δ_1^+ maps to $[0, 1]$, which we call the "zero ray". We want to change the multiplier at ξ to be $1 - \frac{1}{2} = \frac{1}{2}$ so that we obtain a Newton map of some other polynomial $\tilde{p}$ by Theorem 2.2. The proof of this lemma uses standard surgery tools of holomorphic dynamics; compare [BF14, Section 4.2].

The Blaschke product $B(z) = \frac{z^k+b}{1+bz^k}$ with $0 \le b < a$ and $a = \frac{k-1}{k+1}$ has a unique attracting fixed point $\alpha \in [0, 1)$, which attracts every point in $\mathbb{D}$. The multiplier at α depends continuously on b. When $b = 0$ the Blaschke product is $z \mapsto z^k$; denote it by $B_0(z) = z^k$. Then $B_0'(0) = 0$ and when $b \nearrow a$, the multiplier at α converges to 1; in particular α also converges to 1. It can be shown that the multiplier map is an increasing function of b in the interval $[0, 1]$, but this fact is not relevant for our construction. We choose $b = b_0$ such that the multiplier at $\alpha = \alpha_0$ is equal to $1 - \frac{1}{2} = \frac{1}{2}$, i.e $B'(\alpha_0) = \frac{1}{2}$.

For a fixed $\alpha_0 < r < 1$ we set $\mathbb{D}_r = \{|z| < r\}$ and $\mathbb{S}_r^1 = \{|z| = r\}$. Note that $B_0^{-1}(\mathbb{D}_r) = \mathbb{D}_{r^{-k}}$ and $B_0^{-1}(\mathbb{S}_r^1) = \mathbb{S}_{r^{-k}}^1$. Define A_0 to be the closed annulus $\overline{\mathbb{D}_{r^{-k}} \setminus \mathbb{D}_r}$. We have $B(\mathbb{D}_r) \subset \mathbb{D}_r$ and denote by $A_1 = \overline{\mathbb{D}_r \setminus B(\mathbb{D}_r)}$ a closed annulus.

The maps $f^i = B|_{B^{-1}(\mathbb{S}_r^1)} : B^{-1}(\mathbb{S}_r^1) \to \mathbb{S}_r^1$ and $f^o = B|_{\mathbb{S}_{r^{-k}}^1} : \mathbb{S}_{r^{-k}}^1 \to \mathbb{S}_r^1$ are covering maps of degree k. These are the inner and the outer maps between the inner and the outer boundaries of A_0 and A_1 respectively. We can extend these maps to all of A_0 as a quasiregular covering map of degree k, where a quasiregular map is a map that is locally uniformly quasiconformal homeomorphism except at finitely many critical points. To do this we choose a fundamental domain, which is a sector

$$S = \{z : r < |z| < r^{-k},\ 0 < Arg(z) < 2\pi/k\}. \tag{4.2}$$

Its boundary consists of two arcs and two line segments at angles 0 and $2\pi/k$. Note that $B(r) = B(re^{2\pi in/k})$ for all n. First, define it on $\left[r, r^{-k}\right]$ to the interval $[B(r), r]$ as a scaling by the positive real number $0 < B(r)/r < 1$, i.e. $z \mapsto \frac{B(r)}{r}z$. On the other segment $\left[re^{2\pi i/k}, r^{-k}e^{2\pi i/k}\right]$ define it as a lift, or as scaling by $B(r)/r$ and rotation by $e^{2\pi i/k}$, i.e. $z \mapsto \frac{B(r)}{r}e^{-2\pi i/k}z$. If we identify these two line segments $\left[r, r^{-k}\right]$ and $\left[re^{2\pi i/k}, r^{-k}e^{2\pi i/k}\right]$, then the closure of the sector 4.2 maps injectively via $z \mapsto z^k$ to the annulus $\{r^k < |z| < r\}$. Then the inverse of this map is an unbranched covering. Now we lift the inner and the outer maps f^i and f^o to the annulus $\{r^k < |z| < r\}$. Now we are in position to apply part (b) of Theorem 4.18, which produces a quasiconformal homeomorphism as an extension of lifted inner and outer quasisymmetric maps to the interior of the annulus. Now transporting the extension to the sector and extending it to the rest of the annulus A_0 by copying the same map to each of the other fundamental sectors we obtain a degree k quasiregular function between A_0 and A_1. Denote it by $h_0 : A_0 \to A_1$. Define a new quasiregular function $g : \mathbb{D} \to \mathbb{D}$ as following

$$g = \begin{cases} B_0 & \text{on } \mathbb{D} \setminus \overline{\mathbb{D}_r} \\ B & \text{on } \overline{\mathbb{D}_r} \\ h & \text{on } A_0 \end{cases}$$

Note that $g(\alpha_0) = \alpha_0$ and $\deg(g, 0) = k$ and also the interval $[\alpha_0, 1]$ is invariant. Since $z = 1$ is a repelling fixed point for B the interval $[\alpha_0, 1]$ is contained in the marked access coming from B_0. We will define a g-invariant Beltrami form μ. Let μ_0 be a conformal structure, then define μ on $\mathbb{D}_r$ as $\mu = \mu_0$. In the annulus A_0 define it as a pullback by h, i.e. $\mu = h^*\mu_0$. Finally, we extend μ to the rest $\mathbb{D}$ by the dynamics of g, observing that for every $z \in \mathbb{D} \setminus \mathbb{D}_r$, there is a unique $n \geq 0$ such that $g^n(z)$ belongs to the half open annulus $A_0 \setminus \mathbb{S}^1_r$; moreover, $g^n = B_0^n$ at the point z. Hence μ is defined recursively as

$$\mu = \begin{cases} \mu_0 & \text{on } \mathbb{D}_r \\ h^*\mu_0 & \text{on } A_0 \\ (B_0^n)^*\mu & \text{on } B_0^n(A_0) \end{cases}$$

By our construction μ is g-invariant and the maximum dilatation satisfies $||\mu|| = ||h^*\mu_0|| < 1$. Let us transport g to immediate basin $\mathcal{A}^\circ(\xi)$ of N_p and define a topological function $F : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ as

$$F(z) = \begin{cases} \psi^{-1} \circ g \circ \psi(z) & z \in \mathcal{A}^\circ(\xi) \\ N_p & z \notin \mathcal{A}^\circ(\xi) \end{cases}$$

Note that $F(z) = N_p(z)$ for $z \in \psi^{-1}(\mathbb{D} \setminus \mathbb{D}_r)$. We will now define an F-invariant Beltrami form μ_F in $\hat{\mathbb{C}}$. In the immediate basin $\mathcal{A}^\circ(\xi)$, define μ_F as the pull back of μ by the Riemann map $\psi : \mathcal{A}^\circ(\xi) \to \mathbb{D}$; for other components of the basin $\mathcal{A}(\xi)$ we spread it by the dynamics of F. We put the standard complex structure on the complement of the basin $\mathcal{A}(\xi)$, thus obtaining an F-invariant Beltrami form μ_F. We apply the measurable Riemann mapping Theorem (Theorem 4.2) to μ_F deducing a quasiconformal homeomorphism $\phi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, unique up to automorphism of $\hat{\mathbb{C}}$. The conjugation $\phi \circ F \circ \phi^{-1}$ is a rational function on $\hat{\mathbb{C}}$. Let us normalize ϕ as fixing ∞. Then $\phi \circ F \circ \phi^{-1}$ satisfies conditions of the theorem 2.2 and since ∞ is a repelling fixed point, it is a Newton map of some polynomial denote the polynomial by $\tilde{p}$. Then $N_{\tilde{p}}$ is the resulting function (by further normalizing ϕ we can make sure that $\tilde{p}$ is centered, but it is not relevant for us now). We have that N_p and $N_{\tilde{p}}$ are conjugate in some neighborhoods of their Julia sets. The conjugating map transforms accesses of N_p in $\mathcal{A}^\circ(\xi)$ to that of $N_{\tilde{p}}$, so we obtain a marked access of $N_{\tilde{p}}$ corresponding to the Δ_1^+. We still use the same notation Δ_1^+ for the resulting access. By the construction all of the conditions of the lemma are satisfied. □

The next lemma is for the Blaschke products that we use throughout the thesis.

Lemma 4.22. *Let $E : \mathbb{D} \to \mathbb{D}$ be a proper map of degree k with a critical point at $z = 0$ and a maximum multiplicity $k - 1$, and whose Julia set is the unit circle $\mathbb{S}^1$. Assume that either E has a parabolic fixed point at $z = 1$, or an attracting fixed point in the unit disk $\mathbb{D}$ and a repelling fixed point at $z = 1$. Then E has the form (4.3) or (4.4) with $0 < b < a$; in particular in the latter case the attracting fixed point belongs to $(0, 1)$.*

Proof of Lemma 4.22. Indeed, let E be such a map. Let $E(0) = c$, and let $M_c(z) = \frac{z-c}{1-\bar{c}z}$ be an automorphism of $\mathbb{D}$. Then the map $M_c \circ E$ is a proper map of $\mathbb{D}$ such that $M_c \circ E(0) = 0$ and also 0 is the only critical point with the maximum multiplicity $k-1$. Then $M_c \circ B(z) = uz^k$ with constant $|u| = 1$, (compare the proof of the Proposition 3.8). Since 1 is fixed by E, we obtain $\frac{1-c}{1-\bar{c}} = u$. Finally we deduce $E = \frac{z^k+b}{1+\bar{b}z^k}$, where $b = \frac{c(1-\bar{c})}{1-c}$. Since $E(1) = 1$ and $E(0) = c$, hence $c = b = \bar{b}$, so b is real. The non-repelling unique fixed point, if there is one, also belongs the to real line. An easy calculation shows that $E'(z) = \frac{kz^{k-1}(1-b^2)}{(1+bz^k)^2}$, and for odd values of k we have $E'(-1) = \frac{k(1+b)}{1-b} > 1$ and $E(-1) = -1$. If k is even then -1 is never fixed by E. Additionally, $E'(1) = \frac{k(1-b)}{1+b} = 1$ if $b = a = \frac{k-1}{k+1} > 0$, so in this case 1 is a parabolic fixed point of E and we obtain the parabolic Blaschke product B_{par}. Since the Julia set is the unit circle, in all other cases we obtain that $b \leq a$, otherwise 1 becomes attracting and the Julia set becomes a Cantor set. Therefore, there exist an attracting fixed point at the value, which is less than 1. If z

is a fixed point of E and $b < 0$ then from $z^k + b = z + bz^k$, it follows that $b(1 - z^{k+1}) = z(1 - z^{k-1})$, hence $z < 0$. If k is even then $|z| < |b|$, which is a contradiction since the orbit of 0 should converge to z, since E as a function of z in $(-1, 0)$ is decreasing. If k is odd $|z| > |b|$ and $E'(z) > 0$, and a similar argument shows that this case is also not possible. Hence $b > 0$ and the orbit of 0 increasingly converges to an attracting fixed point $\alpha \in (0, 1)$. Lemma is proved. $\square$

Now we start the proof of the theorem. By applying Lemma 4.20 to N_p with its marked basins we obtain a $N_{\tilde{p}}$ such that every marked immediate basin is now (non-super)attracting with a single critical point in it. The compositions $\phi \circ \psi$ and $\psi \circ \phi$ of a quasiconformal homeomorphism ψ with David homeomorphism ϕ are again David homeomorphisms, since a quasiconformal homeomorphism distorts an area in a bounded fashion by Astala theorem [A94]. It is enough to work with $N_{\tilde{p}}$, but we may still denote it by $N_p = N_{\tilde{p}}$. It is easy to carry out the surgery for the Blaschke products in the unit disk and then transfer it to the marked immediate basins via Riemann maps. As it was used in the proof of Lemma 4.20, let $b = b_0$ and

$$B_{\text{par}}(z) = \frac{z^k + a}{1 + az^k}, \quad a = \frac{k-1}{k+1}, \quad \text{and} \tag{4.3}$$

$$B(z) = \frac{z^k + b}{1 + bz^k}, \quad b = b_0 \tag{4.4}$$

be the parabolic and the attracting Blaschke products of degree $k > 1$ with the unique critical point of multiplicity $k - 1$ located at $z = 0$ and the Julia set the unit circle. The map B has an attracting fixed point $\alpha = \alpha_0 \in (0, 1)$ with $B'(\alpha_0) = \frac{1}{2}$, which attracts every point in $\mathbb{D}$, while $z = 1$ is a repelling fixed point with real multiplier $\lambda = \frac{1-b}{1+b}k$. On the other hand, B_{par} has the parabolic fixed point at $z = 1$, which also attracts every point in $\mathbb{D}$. In both cases the interval $[0, 1]$ is invariant, which corresponds to attracting invariant rays, and the critical orbit marches monotonically along this segment towards $z = \alpha$, in the case of B, and towards $z = 1$ in the case of B_{par}.

Now we consider the local dynamics around the repelling fixed point. Let $f(z) = \lambda z$ with $\lambda > 1$ be the repelling model map. We define the sector

$$S := \{z \in \mathbb{C} \mid \theta \le \arg z \le 2\pi - \theta \text{ and } 0 < |z| < 1/\lambda^m\},$$

for $m > m_0$ where m_0 is large and $0 < \theta < \pi$, see Fig. 4.1. We write Q_m^f for the quadrilaterals bounded by the segments $[(1/\lambda^{m+1})e^{\pm i\theta}, (1/\lambda^m)e^{\pm i\theta}]$ and arcs of radii $1/\lambda^{m+1}$ and $1/\lambda^m$ contained in S, see Fig. 4.1. Note that $Q_m^f = f^{-m}(Q_0^f)$. It is easy to observe that the map $z \mapsto \omega(z) = \frac{Log\lambda}{Logz}$ conjugates f on $\mathbb{D}_{\lambda^{-m}} \setminus S$ to the following parabolic model map

$$g : \omega \mapsto \frac{\omega}{\omega + 1}$$

on the cusp $C = g(\mathbb{D}_{\lambda^{-m}} \setminus S)$ with vertex at the origin, i.e. $\omega \circ f = g \circ \omega$ on $\mathbb{D}_{\lambda^{-m}} \setminus S$, see Fig. 4.1.

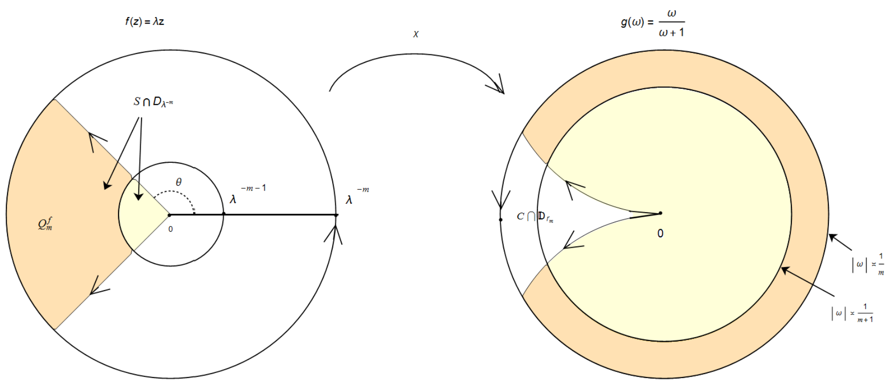


Figure 4.1: Pictorial illustration of the function χ. The total of the shaded region on the left picture corresponds to $S \cap \mathbf{D}_{\lambda^{-m}}$ and $r_m = |\omega(\lambda^{-m}e^{\pm i\theta})| \asymp \frac{1}{m}$. The initial map $z \mapsto \omega(z)$ sends the white region (the complement of S) from the left to the white in the right.

Lemma 4.23. *There is David extension χ of $z \mapsto \omega(z)$ to a neighborhood of the origin, with $K_\chi \asymp m$ on Q^f_m.*

The above lemma is needed to conclude the following lemma. We would like to mention that the inverse of the map χ constructed in the previous lemma is not David.

Using the previous lemma we obtain the following result for Blaschke products, which will be used in Haïssinsky surgery. Basically, we cut two sectors and replace one dynamics with the other and pull back the local dynamics to all of the attracting basin.

Lemma 4.24. *There exist a piecewise C^1 homeomorphism $\phi : \mathbb{D} \to \mathbb{D}$ and a sector $S_B \subset \mathbb{D}$ with vertex at 1, which is a neighborhood of α as in Fig 4.2, such that:*

(i) for all $z \in \mathbb{D} \setminus S_b$, $\phi \circ B(z) = B_{par} \circ \phi$;

(ii) there is a set S'_B which is the intersection of S_B with a neighborhood of 1, such that $\phi : \mathbb{D} \setminus \bigcup_k B^{-k}(S'_B) \to \phi(\mathbb{D} \setminus \bigcup_k B^{-k}(S'_B))$ is quasiconformal homeomorphism;

(iii) on the quadrilaterals Q^B_m in S'_B defined as Q^f_m for Lemma 4.23, we have $K_\phi \asymp m$, for all $m \geq m_0$.

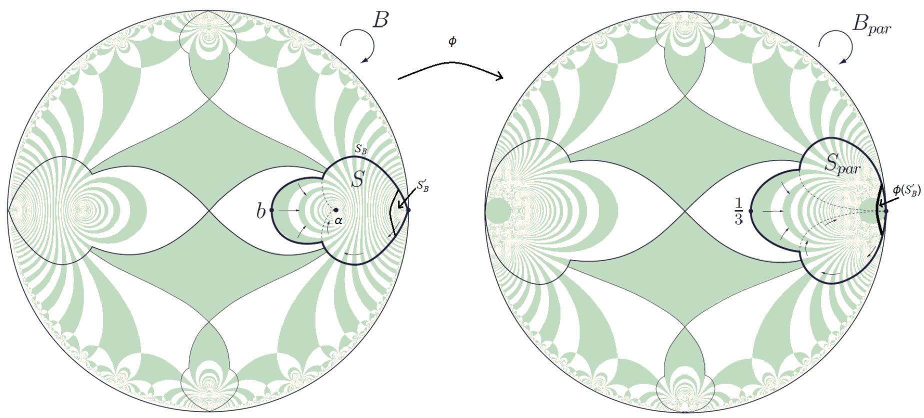


Figure 4.2: An illustration of the construction of ϕ for the Lemma 4.24, for the degree $n = 2$. Left: Quadratic Blaschke product with attracting basin, Right: Quadratic Blaschke product with parabolic basin (Figure courtesy T. Kawahira)

Refer for the proofs of the last two lemmas to [BF14, Lemma 9.20 and Lemma 9.21].

Topological surgery. In order to ease the notations let us do the construction for one basin only and denote by ξ an attracting fixed point and $\mathcal{A}^\circ$ its immediate basin with a single critical point c_ξ of local degree $k = \deg(N_p, c_\xi)$. Consider the Riemann map $R = R_{\mathcal{A}^\circ} : \mathcal{A}^\circ \to \mathbb{D}$ such that $R(c_\xi) = 0$; now we can still post-compose it with a rotation. After the rotation assume that 1 is fixed by the proper map $R \circ N_p \circ R^{-1} : \mathbb{D} \to \mathbb{D}$. Now this choice of the Riemann map exists and is unique. Since there is a unique Blaschke product of degree k defined as in (4.4) with the fixed point $\alpha \in (0, 1)$ and a critical point at $z = 0$, we have $R \circ N_p \circ R^{-1} = B$, and in particular by Lemma 4.22, we have $R(\xi) = \alpha$. Define the map $\phi : \mathbb{D} \to \mathbb{D}$ as a partial conjugacy between B and B_{par}. Let $\hat{B} = \phi^{-1} \circ B_{\text{par}} \circ \phi$ be the conjugation. Hence, $\hat{B} = B$ except on the sector S_B.

We transfer this data to the immediate basin; define on $\mathcal{A}^\circ$ the map $F = R^{-1} \circ \hat{B} \circ R$ which coincides with N_p except on the sector $R^{-1}(S_B)$.

Finally set

$$G(z) = \begin{cases} F(z) & \text{for } z \in \mathcal{A}^\circ \\ N_p & \text{for } z \notin \mathcal{A}^\circ \end{cases}$$

This map G is our topological model: a ramified covering of degree d, piecewise C^1. We have to make sure that G satisfies “post-critical minimality” condition; we know that N_p does by construction. Let $c_U \in U$ be a critical point in a component of the basin $\mathcal{A}$ and let $N_p^{m_U}(U) = \mathcal{A}^\circ$ for minimal $m_U > 0$. Recall that by construction $N_p^{m_U}(c_U) = c_\xi$, with c_U a unique critical point of N_p in

$\mathcal{A}^\circ$. By the definition of the Riemann map $R = R_{\mathcal{A}^\circ} : \mathcal{A}^\circ \to \mathbb{D}$ with $R(\xi) = \alpha$, $R(c_\xi) = 0$, since we have $F = R^{-1} \circ \hat{B} \circ R = R^{-1} \circ \phi^{-1} \circ B_{\text{par}} \circ \phi \circ R$. It is easy to observe that $B^{-1}(b) = B_{\text{par}}^{-1}(a) = \{0\}$, and since $\phi(B(0)) = \phi(b) = a = B_{\text{par}}(0)$, it follows that $\phi(0) = 0$. We have $F(c_\xi) = R^{-1}(b)$ and $F^{-1}(R^{-1}(b)) = \{c_\xi\}$; hence, c_ξ is a critical point of F. This is a crucial property of our model map: after the straightening theorem, we obtain an actual post-critically minimal Newton map.

Straightening of almost complex structure and conclusion of the Haïssinsky theorem (Theorem 4.13). Let us emphasize again that the full strength of David Integrability Theorem is needed: the existence of the solution in order to straighten the almost complex structure, and its uniqueness to conclude that the resulting composition map is indeed a rational function (holomorphic).

We will define a G-invariant almost complex structure μ in $\hat{\mathbb{C}}$. Let $\hat{\mu} = \partial_{\bar{z}}\phi/\partial_z$. By definition this David-Beltrami form is defined in $\mathbb{D}$ and invariant by $\hat{B}$. We transport it to the immediate basin $\mathcal{A}^\circ$ by defining the pull back $\mu = R^*\hat{\mu}$. We have the following commutative diagram:

$$\begin{tikzcd}
(D, \mu_0) \arrow[r, "B_{par}"] & (D, \mu_0) \\
(D, \hat{\mu}) \arrow[r, "\hat{B}"] \arrow[u, "\phi"] & (D, \hat{\mu}) \arrow[u, "\phi"'] \\
(\mathcal{A}^\circ, \mu) \arrow[r, "F"] \arrow[u, "R"] & (\mathcal{A}^\circ, \mu) \arrow[u, "R"']
\end{tikzcd}$$

We extend it recursively by the dynamics of F (which is equal N_p outside the sector $R^{-1}(S_B)$) to the rest of the dynamical plane:

$$\mu = \begin{cases} (N_p^n)^*\mu & \text{on } N_p^{-n}(\mathcal{A}^\circ) \\ \mu_0 & \text{on } C \setminus \bigcup_n N_p^{-n}(\mathcal{A}^\circ) \end{cases}$$

By definition the map G leaves μ invariant.

Compare [BF14, Lemma 9.23] to the following lemma, where we include the case when $\infty \in P_{N_p}$, the post-critical set of N_p.

Lemma 4.25. *G invariant μ, defined as above, satisfies the hypothesis of David Integrability Theorem (4.5).*

Proof of Lemma 4.25. Let V be a simply connected linearizable neighborhood of ∞. Let U be a connected component of $N_p^{-1}(V)$ compactly contained in V. Let $\sum_\infty = R^{-1}(S'_B)$ in U. Set $\rho = 1/N'_p(\infty) = \frac{d}{d-1} > 1$ and let K_μ be the dilatation ratio of μ; by Koebe's Distortion Theorem and Lemma 4.24,

$$\text{Area}\,\{z \in \Sigma_\infty \mid K_\mu > n\} \asymp \left(1/\rho^{2n}\right) \text{Area}\,\{\Sigma_\infty\}$$

If $N_p^k(y) = \infty$ for some $y \in \mathbb{C}$, then let $\sum_y$ be the connected component of $N_p^{-k}(\sum_\infty)$ with vertex y. Here we need to be cautious: if y is a critical point then we end up having more than one component with a shared vertex at y. Note that the local degree $\deg(N_p, y) < d$ is finite. We need to consider all the different components separately; in total, we shall have $\deg(N_p, y)$ number of components. The map $N_p^k : (N_p^{-1}(V), y) \to (V, \infty)$ is a covering, ramified at y if y is a critical point. Since $N_p : U \to V$ fixes ∞ we lift it by N_p^k as a repelling germ of the same multiplier in a **punctured** neighborhood of y as the following: after fixing some base point $w_0 \in U \setminus \{\infty\}$, its its image $N_p(w_o)$ is a base point in $V \setminus \{\infty\}$, let $y_0 \in N_p^{-k}(w_0)$ be a base point in $N_p^{-k}(U) \setminus \{y\}$ and finally, let $z_0 \in N_p^{-k}(N_p(w_0))$ be a base point in $N_p^{-k}(V) \setminus \{y\}$. Now we can apply Lemma 4.19 to $N_p^k \circ N_p$ obtaining a lift F with the following commutative diagram:

$$\begin{tikzcd}
(N_p^{-k}(U) \setminus \{y\}, y_0) \arrow[r, "F"] \arrow[d, "N_p^k"'] & (N_p^{-k}(V) \setminus \{y\}, z_0) \arrow[d, "N_p^k"] \\
(U \setminus \{\infty\}, w_0) \arrow[r, "N_p"] & (V \setminus \{\infty\}, N_p(w_0))
\end{tikzcd}$$

Note that F is a one-to-one conformal map from $N_p^{-k}(U)\setminus\{y\}$ to $N_p^{-k}(V)\setminus\{y\}$ given by $F = N_p^{-k} \circ N_p^{k+1}$ with $F(y_0) = z_0$. We extend it using the Riemann removability theorem to y as a repelling germ. Easy calculation shows that the derivative is $F'(y) = N_p'(\infty) = \rho$. By bounded distortion theorem, there exists a constant $C > 0$ such that

$$\text{Area}\{z \in \Sigma_y \mid K_\mu > n\} \le \left(C/\rho^{2n}\right) \text{Area}\Sigma_y$$

for all preimages of y of ∞. We deduce that for n large enough,

$$\begin{aligned}
\text{Area}\left\{z \in \hat{\mathbb{C}} \mid K_\mu > n\right\} &= \sum_{k\ge 0} \sum_{N_p^k(y)=\infty} \text{Area}\{z \in \Sigma_y,\ K_\mu > n\} \\
&\le \sum_{k\ge 0} \sum_{N_p^k(y)=\infty} \left(C/\rho^{2n}\right) \text{Area } \Sigma_y,
\end{aligned}$$

In the above summands with $N_p^k(y) = \infty$, if y is a critical point then we include all $\deg(N_p, y)$ number of different Σ_y. Hence,

$$\text{Area}\left\{z \in \hat{\mathbb{C}} \mid K_\mu > n\right\} \le \left(C/\rho^{2n}\right) \text{Area } X = C' e^{-2n \ln \rho},$$

where $X = \bigcup_{k\ge 0, N_p^k(y)=\infty} \Sigma_y$. Since the area of X depends on which metric we are using, if we want to use the Euclidean metric then we have to make sure that ∞ belongs to the interior of a complement of X in $\hat{\mathbb{C}}$. This can be done by conjugating N_p by a Mobius map in the beginning of the construction so that ∞ belongs to an immediate basin; otherwise, we can use a spherical metric in $\hat{\mathbb{C}}$ then area of X is bounded anyway. This proves the estimate on K_μ and finishes the proof of the lemma. $\square$

David Integrability Theorem (Theorem 4.5) asserts the existence of a map ϕ which integrates μ. However, to obtain a holomorphic function there is still some work to be done.

Lemma 4.26 (Straightening, Lemma 9.25 [BF14]). *If there is David homeomorphism $\phi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, such that the equation $\partial_{\bar{z}}\psi = \mu_\phi \partial_z \psi$ has a local solution, unique up to post-composition by a conformal map, and such that $G^*\mu_\phi = \mu_\phi$ a.e., then $Q = \phi \circ G \circ \phi^{-1}$ is a rational function..*

Proof of Lemma 4.26. Indeed, on every disk where G is injective, $\phi \circ G$ has the same Beltrami coefficient as ϕ; hence, there exists a conformal map, say Q, such that $\phi \circ G = Q \circ \phi$, which follows from uniqueness of the solution. There is a discrete set of points, which is the set of critical points of G, where G is not locally injective; hence by the Riemann Removability Theorem Q is a holomorphic rational function. $\square$

Here we complete the proof of Theorem 4.15. By construction ∞ is a parabolic fixed point of Q with multiplicity $+1$ and all unmarked fixed points in $\mathbb{C}$ are superattracting, since ϕ is conformal away from marked basins; hence, Q is a Newton map by Theorem 2.3. Since G satisfies "post-critical minimality" condition, hence Q is a post-critically *minimal* Newton map. Let ϕ_0 be the quasiconformal homeomorphism coming from Lemma 4.20, and let ϕ_1 be David homeomorphism and $G = N_{\tilde{p}e^{\tilde{q}}}$ coming from Lemma 4.26. Both ϕ_0 and ϕ_1 are conformal except on the corresponding marked basins. Set $\phi = \phi_1 \circ \phi_0$. Finally, we have $\phi \circ N_p = N_{\tilde{p}e^{\tilde{q}}} \circ \phi$ in the domain away from marked immediate basins of N_p, thus for ϕ and $N_{\tilde{p}e^{\tilde{q}}}$ all the conditions of Theorem 4.15 are automatically fulfilled by the construction. $\square$

Chapter 5

Main Results

This chapter has three sections. In the first two of the sections we prove results of Haïssinsky surgery applied to post-critically finite Newton maps. In the last section we prove the existence of a natural bijection between the space of post-critically finite Newton maps and the space of post-critically minimal Newton maps.

5.1 Injectivity of Haïssinsky surgery

In this section we prove injectivity of Haïssinsky surgery, the Theorem 4.15 for the space of Newton maps. Haïssinsky surgery defines a map from the space of n marked post-critically finite Newton maps of polynomials (denoted $\mathcal{N}^{+,n}_{\mathrm{pcf}}(d)$) to the space of post-critically minimal Newton maps for pe^q with $\deg(q) = n$ (denoted $\mathcal{N}_{\mathrm{pcm}}(d-n, n)$). We consider all possible applications of a Haïssinsky surgery to Newton maps of polynomials of given degree $d \geq 3$ as one object of study. Different surgeries applied to the same Newton map of a polynomial with different accesses may produce the same rational function up to affine conjugacy. For instance, for $n = 1$ we have two ways to apply Haïssinsky surgery to $\frac{2z^3}{3z^2-1} \in \mathcal{N}_{\mathrm{pcf}}(3)$ to the two immediate basins, each with one access to ∞, which is also studied at the end of Chapter 2, where we counted different possible applications of surgery. The third immediate basin has 2 accesses, see Fig. 2.5 for its Julia set. The resulting Newton map is $z - \frac{z^2+c}{z^2+2z+c}$ for $c = -\frac{1}{4}$ and belongs to $\mathcal{N}_{\mathrm{pcm}}(2, 1)$. There exists a single Newton map with this property in $\mathcal{N}_{\mathrm{pcm}}(2, 1)$. Thus the two results of Haïssinsky surgery produce the same map. Similarly, consider applications of Haïssinsky surgery to the above function $\frac{2z^3}{3z^2-1} \in \mathcal{N}_{\mathrm{pcf}}(3)$, this time, to its third immediate basin, which has 2 accesses to ∞. We can perform surgery in two ways. But the results of the two surgeries are the same function in $\mathcal{N}_{\mathrm{pcm}}(2, 1)$. It is $z - \frac{z^2+c}{z^2+2z+c}$ for a concrete value $c = 2$. It is the unique Newton map with two accesses in the parabolic immediate basin of ∞. Since we identify functions that are Möbius conjugate, so we do not distinguish these "different" surgeries, which produce the same result up to affine conjugation. The definition is as following.

Definition 5.1 ($\sim_H$ Haïssinsky equivalence on surgeries)**.** Let F and G be results of application of Haïssinsky surgery to N_{p_1} with marking $\Delta_n^+(N_{p_1})$ and N_{p_2} with marking $\Delta_n'^+(N_{p_2})$, both belonging to $\mathcal{N}_{\text{pcf}}^{+,n}(d)$, respectively. The two surgeries are said to be equivalent if there exists an affine map M such that $M \circ F = G \circ M$. Notation $\sim_H$ is used for equivalent surgeries.

The following theorem characterizes equivalent surgeries, which states that distinct surgeries produce non-conjugate (distinct) functions unless underlying functions are conjugate themselves.

Theorem 5.2 (Injectivity of Haïssinsky surgery)**.** *Haïssinsky surgeries applied to N_{p_1} with marking $\Delta_n^+(N_{p_1})$ and N_{p_2} with marking $\Delta_n'^+(N_{p_2})$, both belonging to $\mathcal{N}_{pcf}^{+,n}(d)$, are equivalent if and only if there exists an affine map L such that*

- $L \circ N_{p_1} = N_{p_2} \circ L$
- $L(\Delta_n^+(N_{p_1})) = \Delta_n'^+(N_{p_2})$

Proof. For one direction: if for an affine map L we have

- $L \circ N_{p_1} = N_{p_2} \circ L$
- $L(\Delta_n^+(N_{p_1})) = \Delta_n'^+(N_{p_2})$,

and we are applying Haïssinsky surgery to N_{p_1} and N_{p_2} through marked channel diagrams $\Delta_n^+(N_{p_1})$ and $\Delta_n'^+(N_{p_2})$ respectively, then the result trivially follows by the construction of Haïssinsky surgery. The converse is the main part of the theorem, which we deal with it now.

For the other direction: Let us use simpler notation for the functions involved in the theorem: $f = N_{p_1}$, $g = N_{p_2}$, and let F and G be the resulting functions of application of Haïssinsky surgery to f with marking $\Delta_n^+(f)$ and g with marking $\Delta_n'^+(g)$, respectively. Let $\mathcal{A}(\xi_j)$ for $1 \leq j \leq n$ denote the marked basins of f. There exists a homeomorphism $\phi_f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that the following diagram commutes:

$$\begin{tikzcd}
\hat{\mathbb{C}} \setminus \cup_{1\leq j\leq n}\mathcal{A}^\circ(\xi_j) \arrow[r, "f"] \arrow[d, "\phi_f"'] & \hat{\mathbb{C}} \setminus \cup_{1\leq j\leq n}\mathcal{A}^\circ(\xi_j) \arrow[d, "\phi_f"] \\
\hat{\mathbb{C}} \setminus \phi_f\big(\cup_{1\leq j\leq n}\mathcal{A}^\circ(\xi_j)\big) \arrow[r, "F"] & \hat{\mathbb{C}} \setminus \phi_f\big(\cup_{1\leq j\leq n}\mathcal{A}^\circ(\xi_j)\big),
\end{tikzcd} \qquad \text{D1}$$

where $\hat{\mathbb{C}} \setminus \cup_{1\leq j\leq n}\mathcal{A}^\circ(\xi_j)$ is the complement of the union of marked *immediate basins* of f. Moreover, $\mathcal{A}_F(\infty) = \phi_f(\cup_{1\leq j\leq n}\mathcal{A}(\xi_j))$ is the parabolic basin of ∞ for F. As above, for $1 \leq j \leq n$ let $\mathcal{A}(\chi_j)$ denote the marked basins of superattracting fixed points χ_j of g.

Similarly, there exists a homeomorphism ϕ_g such that the following diagram commutes:

$$\begin{tikzcd} \hat{\mathbb{C}} \setminus \cup_{1\le j\le n}\mathcal{A}^\circ(\chi_j) \arrow[r, "g"] \arrow[d, "\phi_g"'] & \hat{\mathbb{C}} \setminus \cup_{1\le j\le n}\mathcal{A}^\circ(\chi_j) \arrow[d, "\phi_g"] \\ \hat{\mathbb{C}} \setminus \phi_g\big(\cup_{1\le j\le n}\mathcal{A}^\circ(\chi_j)\big) \arrow[r, "G"] & \hat{\mathbb{C}} \setminus \phi_g\big(\cup_{1\le j\le n}\mathcal{A}^\circ(\chi_j)\big), \end{tikzcd} \tag{D2}$$

where $\hat{\mathbb{C}} \setminus \cup_{1\le j\le n}\mathcal{A}^\circ(\chi_j)$ is the complement of the union of marked *immediate basins* of g. Moreover, $\mathcal{A}_G(\infty) = \phi_g(\cup_{1\le j\le n}\mathcal{A}(\chi_j))$ is the parabolic basin of ∞ for G.

Assume the two surgeries are equivalent; $F \sim_H G$, i.e. there exists an affine map M such that the following diagram commutes

$$\begin{tikzcd} \hat{\mathbb{C}} \arrow[r, "F"] \arrow[d, "M"'] & \hat{\mathbb{C}} \arrow[d, "M"] \\ \hat{\mathbb{C}} \arrow[r, "G"] & \hat{\mathbb{C}}. \end{tikzcd} \tag{D3}$$

It is easy to observe that from diagram D3 we obtain $M(\mathcal{A}_F(\infty)) = \mathcal{A}_G(\infty)$ and $M(\hat{\mathbb{C}} \setminus \mathcal{A}_F(\infty)) = \hat{\mathbb{C}} \setminus \mathcal{A}_G(\infty)$, moreover the attracting accesses of $\mathcal{A}_F(\infty)$ for F transform via M to the attracting accesses of $\mathcal{A}_G(\infty)$ for G. From diagrams D1, D2 and D3 it follows that

$$\phi_g^{-1} \circ M \circ \phi_f \circ f = g \circ \phi_g^{-1} \circ M \circ \phi_f$$

on $\hat{\mathbb{C}} \setminus \cup_{1\le j\le n}\mathcal{A}(\xi_j)$. The homeomorphism

$$\psi^1 = \phi_g^{-1} \circ M \circ \phi_f : \hat{\mathbb{C}} \setminus \cup_{1\le j\le n}\mathcal{A}^\circ(\xi_j) \to \hat{\mathbb{C}} \setminus \cup_{1\le j\le n}\mathcal{A}^\circ(\chi_j)$$

conjugates f to g in the complement of the union of marked immediate basins $\hat{\mathbb{C}} \setminus \cup_{1\le j\le n}\mathcal{A}^\circ(\xi_j)$ of f.

We want to extend ψ^1 to $\hat{\mathbb{C}}$ as a global conjugacy between f and g, and what is missing are the marked immediate basins $\cup_{1\le j\le n}\mathcal{A}^\circ(\xi_j)$ of f. To accomplish this we use normalized Riemann maps (Böttcher coordinates) coming from Proposition 3.8. Let us sort the indices such that $\mathcal{A}^\circ(\xi_j)$ and their counterparts $\mathcal{A}^\circ(\chi_j)$ are cyclically ordered at ∞ for $1 \le j \le n$. Let us pick $\mathcal{A}^\circ(\xi_j)$ an immediate basin for f. By Proposition 3.8 there exists a Riemann map $\psi_{j_f} : (\mathcal{A}^\circ(\xi_j), \xi_j) \to (\mathbb{D}, 0)$ such that $\psi_{j_f} \circ f \circ \psi_{j_f}^{-1}(z) = z^{k_j}$, where $k_j = \deg(f, \xi_j)$. We have $k_j - 1$ choices for ψ_{j_f}.

Let $R(t) = \{re^{2\pi i t}, 0 \le r \le 1\}$ be a *radial line at angle* t in $\mathbb{D}$. We fix some choice of a Riemann map ψ_{j_f} and define $R_j(t) = \psi_{j_f}^{-1}(R(t))$, *a ray of angle* t in $\mathcal{A}(\xi_j)$. The radial lines $R(t)$ at angles $t \in \{0, \frac{1}{k-1}, \dots, \frac{k-2}{k-1}\}$ are fixed by $z \mapsto z^{k_j}$. Hence, the rays in $\mathcal{A}^\circ(\xi_j)$ at those angles are fixed by f define all accesses to ∞ within the immediate basin. Once we label each access, the different choices of ψ_{j_f} does nothing but cyclically permute (a shift) the labels of accesses. Note that accesses do not depend on a choice of a Riemann map. Let us choose the Riemann map ψ_{j_f} such that the rays at angles $0, \frac{1}{k-1}, \dots, \frac{k-2}{k-1}$ in $\mathcal{A}^\circ(\xi_j)$ are ordered in anti-clockwise direction, 0 ray being the one that was marked. By Theorem 2.9, the Julia set of f is locally connected and the boundary of every Fatou component is locally connected, hence, every ray lands by Carathéodory's theorem. Also note that every f-invariant ray lands at $\infty \in \partial\mathcal{A}^\circ(\xi_j)$.

We have the same construction for g: the Riemann maps

$$\phi_{j_g}(\mathcal{A}^\circ(\chi_j), \chi_j) \to (\mathbb{D}, 0)$$

such that $\psi_{j_g} \circ g \circ \psi_{j_g}^{-1}(z) = z^{k_j}$, where $k_j = \deg(f, \xi_j) = \deg(g, \chi_j)$. We normalize these Riemann maps of marked immediate basins of g as well, in the same ordering used for f. We define rays in $\mathcal{A}^\circ(\chi_j)$; and every ray for g also lands.

We construct conjugating maps between corresponding marked immediate basins of f and g. Consider the map

$$\psi_j^2 := \phi_{j_g}^{-1} \circ \phi_{j_f} : \mathcal{A}^\circ(\xi_j) \to \mathcal{A}^\circ(\chi_j)$$

which is conformal. The following diagrams commute;

$$\begin{tikzcd}
\mathcal{A}^\circ(\xi_j) \arrow[r, "f"] \arrow[d, "\psi_{j_f}"'] & \mathcal{A}^\circ(\xi_j) \arrow[d, "\psi_{j_f}"] \\
\mathbb{D} \arrow[r, "z \mapsto z^{k_j}"] & \mathbb{D} \\
\mathcal{A}^\circ(\chi_j) \arrow[r, "g"] \arrow[u, "\psi_{j_g}"] & \mathcal{A}^\circ(\chi_j). \arrow[u, "\psi_{j_g}"']
\end{tikzcd}$$

It is now natural to check if both ψ^1 and ψ_j^2 match up on $\partial\mathcal{A}^\circ(\xi_j)$. For this we define an equivalence relation on $\mathbb{S}^1$ for ψ_{j_f} (and ψ_{j_g}) classes of rays (identified by angles) that land at a common point. Alternatively, since the inverse to ψ_{j_f} (correspondingly the inverse to ψ_{j_g}) has the continuous extension to the closed unit disk by Carathéodory's Theorem, every equivalence class consists of points of $\mathbb{S}^1$ that are mapped to the same point under the continuous extension of the inverse of ψ_{j_f} (correspondingly the continuous extension of the inverse of ψ_{j_g}).

All f-invariant rays land at ∞, and thus these belong to the same class. All iterated pre-fixed (the image is an invariant ray) rays split into distinct equivalent classes. It is clear that our equivalence relation is generated by the closure of the equivalence relation defined by f-invariant rays and their iterated preimages. By the normalized Riemann maps, ψ_{j_f} for f, and ψ_{j_g} for g, we obtain the same equivalence relation for both f and g. Indeed, the map ψ^1 sends bijectively the iterated preimages of ∞ in the f plane to the corresponding iterated preimages of ∞ in the g plane. Thus ψ_j^2 extends continuously to the closure $\overline{\mathcal{A}^\circ(\xi_j)}$. Since $\infty \in J(f)$ therefore iterated preimages of ∞ are dense in $\partial\mathcal{A}^\circ(\xi_j)$, hence for every point $z \in \partial\mathcal{A}^\circ(\xi_j)$ the equivalent class of rays landing at z is a limit of classes of rays landing at iterated preimages of ∞ in $\partial\mathcal{A}^\circ(\xi_j)$. Moreover, the extension (denote again ψ_j^2) coincides with ψ^1 on the iterated preimages of ∞. By construction the maps ψ^1 and ψ_j^2 agree on a dense subset of their common domains of definition; namely, on the point at ∞ and its iterated preimages in $\partial\mathcal{A}^\circ(\xi_j)$. It follows that ψ^1 and ψ_j^2 coincide everywhere on their common domains of definition. Hence the orientation preserving homeomorphism

$$\psi = \begin{cases} \psi^1(z), & z \in \hat{\mathbb{C}} \setminus \cup_{1\le j\le n}\mathcal{A}^\circ(\xi_j) \\ \psi_j^2(z), & z \in \mathcal{A}^\circ(\xi_j), \text{ for } 1 \le j \le n, \end{cases}$$

conjugates f to g in $\hat{\mathbb{C}}$.

Finally, we invoke the rigidity part of Thurston's theorem on characterization of branched coverings [DH93] (actually, we need to apply a result from [BCT14], where we include a point at ∞ as an extra marked point to the postcritical set) to degree $d \ge 3$ functions f and g deducing the existence of L, a conformal conjugacy $L \circ f = g \circ L$. [1] Moreover, L sends the marked fixed critical points of $\Delta_n^+(f)$ to those of $\Delta_n'^+(g)$, hence all of the marked channel diagram: $L(\Delta_n^+(f)) = \Delta_n'^+(g)$. □

[1]Alternatively, by the proof structure of Chapter 6 of [DH], we can construct the conformal conjugacy by hand by keeping the conformal conjugacy at small disc neighborhoods of superattracting periodic points of f compactly contained in their immediate basins and interpolating this conformal map to a quasiconformal map ϕ_0 of the sphere. Next, we keep taking lifts and obtain a sequence of quasiconformal maps with the same complex dilatation. We only need to require for all $m > 0$, $\phi_m \circ f = g \circ \phi_{m+1}$ and $\phi_m = \phi_0$ in a small disc neighborhood of some superattracting fixed point of f so that we fix a base point from this domain to define the lifts. The sequence $\{\phi_m\}_{m\ge0}$ has a convergent subsequence. Let ϕ be its limit. It is clear that ϕ is a conformal map of $\hat{\mathbb{C}}$ since the domain where ϕ_m are not conformal shrinks to the Julia set of f. The claim follows since the Julia set of f has measure zero. In the domain we have $\phi = \phi_0$. We have constructed the initial map to satisfy $\phi_0 \circ f = g \circ \phi_0$ in the domain, hence $\phi \circ f = g \circ \phi$ by the identity principle of holomorphic functions. More details of this procedure are given in the next chapter, where we construct such a sequence for a pair of post-critically minimal Newton maps.

5.2 Surjectivity of Haïssinsky surgery

In this section we prove surjectivity of Haïssinsky surgery in the space of Newton maps. For a given post-critically minimal Newton map we use Cui plumbing surgery to change its parabolic fixed point at ∞ to repelling one. The result is later quasiconformally changed to the post-critically finite Newton map. Moreover, Cui plumbing surgery preserves attracting accesses to ∞ in its parabolic immediate basins of attraction. This gives us for the resulting post-critically finite Newton map *a marking*, to which we apply Haïssinsky surgery. We shall show that the resulting map of Haïssinsky surgery is affine conjugate to the given post-critically minimal Newton map.

We recall from Definitions 3.1–3.3: $\mathcal{N}(d-n,n)$ denotes the space of normalized Newton maps N_{pe^q} of degree $d \geq 3$ with n petals at ∞. For instance, $\mathcal{N}(d) := \mathcal{N}(d,0)$ is the space of degree $d \geq 3$ normalized Newton maps for polynomials that are monic, centered, have a root at $z=1$ and all roots are simple. For every natural number $n \leq d$ denote $\mathcal{N}_{\text{pcm}}(d-n,n)$ the space of post-critically *minimal* Newton maps in $\mathcal{N}(d-n,n)$. In particular, denote $\mathcal{N}_{\text{pcf}}(d) := \mathcal{N}_{\text{pcf}}(d,0)$ the space of degree d post-critically *finite* Newton maps for polynomials that are centered, monic and have a root at $z=1$, and with only simple roots. Denote $\mathcal{N}^{+,n}_{\text{pcf}}(d)$ the space of post-critically *finite* Newton maps from $\mathcal{N}_{\text{pcf}}(d)$ with markings Δ^+_n.

Theorem 5.3 (Surjectivity of Haïssinsky surgery)**.** *For every pair of non-negative integers $d \geq 3$ and $1 \leq n \leq d$, the mapping $\mathcal{F}_n$ given by Haïssinsky surgery induces a (natural) surjective mapping between the quotient space $\mathcal{N}^{+,n}_{pcf}(d)/\sim_H$ and the space of affine conjugacy classes of Newton maps in $\mathcal{N}_{pcm}(d-n,n)$.*

Proof. The proof is involved. For a given function from $\mathcal{N}_{\text{pcm}}(d-n,n)$ we obtain a new rational function by perturbing the parabolic point at ∞, then we apply Haïssinsky surgery to the perturbed function, which is actually a Newton map from $\mathcal{N}^{+,n}_{\text{pcf}}(d)$. We show that the function which we took from $\mathcal{N}_{\text{pcm}}(d-n,n)$ and the result of Haïssinsky surgery are affine conjugate to each other. Thus, proving that Haïssinsky surgery induces a surjective mapping from the space $\mathcal{N}^{+,n}_{\text{pcf}}(d)/\sim_H$ to the space of affine conjugacy classes of functions in $\mathcal{N}_{\text{pcm}}(d-n,n)$. The surgery does not differ distinct elements within the conjugacy class, therefore we must work with the quotient spaces. We split the whole proof into four parts, Part A-Part D, as following.

Part A We apply Cui plumbing surgery (Theorem 4.10) to a given PCM Newton map $N_{p_1e^{q_1}} \in \mathcal{N}_{\text{pcm}}(d-n,n)$ of degree $d \geq 3$. We study properties of the resulting rational function f_1 and the quotient map ϕ such that $\phi \circ f_1 = N_{p_1e^{q_1}} \circ \phi$ and ϕ is not injective only in the Fatou components of f_1 that map to parabolic domains of $N_{p_1e^{q_1}}$, in particular, it is a homeomorphism from $J(f_1)$ onto $J(N_{p_1e^{q_1}})$. Next, we change f_1 in its attracting immediate basins to the one with superattracting cycles such that the result of this intermediate surgery produces a post-critically finite Newton map, denote it by N_p.

Part B We apply Haïssinsky surgery to N_p of Part A, with a corresponding marked channel diagram, which is uniquely obtained from $N_{p_1e^{q_1}}$. Denote by $N_{p_2e^{q_2}}$ the result of the surgery.

Part C We construct a *topological* conjugacy Ψ between the given function $N_{p_1e^{q_1}}$ and $N_{p_2e^{q_2}}$, the resulting map of Part B. We accomplish this by gluing local conjugacy, composing the Riemann maps, at the parabolic basin with the topological conjugacy, which is not changed during the process of three surgeries of Part A and Part B, on the Julia set and superattracting cycles.

Part D Using the topological conjugacy of Part C, which is a conformal map at the petals and superattracting cycles of $N_{p_1e^{q_1}}$, by applying interpolation technique several times we construct a set of quasiconformal homeomorphisms of $\hat{\mathbb{C}}$, which is denoted by $\{\Psi_1, \ldots \Psi_k\}$, where k is a total number of superattracting periodic points of $N_{p_1e^{q_1}}$. Next we work with Ψ_k, from the previous step, and construct, by taking lifts of a local conjugation, $\{\psi_m\}_{m\geq 0}$ a sequence of quasiconformal homeomorphisms of $\hat{\mathbb{C}}$ with bounded complex dilatation. Finally, by extracting a converging sub-sequence of the latter we obtain a *conformal* conjugacy between $N_{p_1e^{q_1}}$ and $N_{p_2e^{q_2}}$, finishing the proof of the theorem.

Part A. Let a post-critically minimal Newton map $N_{p_1e^{q_1}} \in \mathcal{N}_{\mathrm{pcm}}(d-n, n)$ of degree $d \geq 3$ be given. We invoke Cui plumbing surgery (Theorem 4.10) to deduce the existence of a geometrically finite rational function f_1 and a quotient map ϕ such that $\phi \circ f_1 = N_{p_1e^{q_1}} \circ \phi$ and ϕ is a homeomorphism from $J(f_1)$ onto $J(N_{p_1e^{q_1}})$. The following diagram is commutative

$$\begin{tikzcd} \hat{\mathbb{C}} \arrow[r, "f_1"] \arrow[d, "\phi"'] & \hat{\mathbb{C}} \arrow[d, "\phi"] \\ \hat{\mathbb{C}} \arrow[r, "N_{p_1e^{q_1}}"] & \hat{\mathbb{C}}. \end{tikzcd}$$

Now we study properties of the functions f_1 and ϕ. Without loss of generality we can assume that ∞ is a fixed point of f_1, after Möbius conjugation if necessary. We obtain $\phi(\infty) = \infty$ since $\phi(\infty) = N_{p_1e^{q_1}}(\phi(\infty))$, and note that ∞ is the only fixed point of $N_{p_1e^{q_1}}$ on its Julia set. For the Newton map $N_{p_1e^{q_1}}$ the parabolic cycle consists of only a point at ∞. For every immediate basin U of ∞ the map ϕ is quasiconformal on any domain compactly contained in $\phi^{-1}(U)$, in particular ϕ^{-1} sends the unique critical point of $N_{p_1e^{q_1}}$ in U to the critical point of f_1 in $\phi^{-1}(U)$. Let $c \in U$ be the critical point of $N_{p_1 e^{q_1}}$ in U. Since ϕ is a homeomorphism on the Julia set we have $\deg(f_1, \phi^{-1}(c)) = \deg(N_{p_1 e^{q_1}}, c)$, thus there is no other critical point of f_1 in U. Indeed, let K be a neighborhood of $\phi^{-1}(c)$ compactly contained in $\phi^{-1}(U)$, by the theorem we know that ϕ is quasiconformal on K, thus $\phi^{-1}(c)$ is a single point, moreover it is a critical point of f_1. The following diagram commutes

$$\begin{tikzcd} f_1^{-1}(K) \arrow[r, "f_1"] \arrow[d, "\phi"'] & K \arrow[d, "\phi"] \\ \phi(f_1^{-1}(K)) \arrow[r, "N_{p_1 e^{q_1}}"] & \phi(K), \end{tikzcd}$$

hence ϕ is quasiconformal on $f_1^{-1}(K)$. Induction shows that ϕ is quasiconformal in all of iterated preimages of K. Now assume c_1 is a critical point of $N_{p_1 e^{q_1}}$ such that $N^l_{p_1 e^{q_1}}(c_1) = c \in U$ for a minimal $l > 0$. Since ϕ is homeomorphism where the above diagram commutes, it follows that after iteratively applying the conjugacy for iterative preimages of K we obtain that $\phi^{-1}(c_1)$ is a critical point of f_1 and $f_1^l(\phi^{-1}(c_1)) = \phi^{-1}(c)$ for the same minimal $l > 0$, moreover since ϕ is a homeomorphism on the Julia set we have $\deg(f_1, \phi^{-1}(c_1)) = \deg(N_{p_1 e^{q_1}}, c_1)$. Furthermore, there are no other critical points of f_1 in the Fatou component containing $\phi^{-1}(c_1)$ than $\phi^{-1}(c_1)$.

Similarly, by induction we shall show that ϕ is conformal in every $\phi^{-1}(U)$, where U is a Fatou component of $N_{p_1 e^{q_1}}$ that is not a parabolic domain. These types of components could only be basin components of the superattracting periodic points of $N_{p_1 e^{q_1}}$. If U is a superattracting immediate basin of $N_{p_1 e^{q_1}}$ then by Cui plumbing theorem (Theorem 4.10) $\phi^{-1}(U)$ is an immediate basin of the superattracting periodic point of f_1 and ϕ is conformal on $\phi^{-1}(U)$, therefore ϕ^{-1} sends the superattracting periodic points of $N_{p_1 e^{q_1}}$ to those of f_1. Let V be a component of $N^{-1}_{p_1 e^{q_1}}(U)$ other than U. The following diagram commutes

$$\begin{tikzcd} \phi^{-1}(V) \arrow[r, "f_1"] \arrow[d, "\phi"'] & \phi^{-1}(U) \arrow[d, "\phi"] \\ V \arrow[r, "N_{p_1 e^{q_1}}"] & U, \end{tikzcd}$$

hence ϕ is conformal in $\phi^{-1}(V)$. By induction, ϕ is conformal in $\phi^{-1} \circ N^{-l}_{p_1 e^{q_1}}(U)$ for all $l \geq 1$. What we have is that for every component of $F(f_1)$, that is preserved by the conjugacy ϕ, the critical orbits terminate in finite time.

We have to mention that in all immediate basins of f_1 that are counterparts to the parabolic domains of $N_{p_1 e^{q_1}}$ we can change the multipliers to zero, see [BF14, Chapter 4.2] and [CG93, Theorem 5.1], compare with Lemma 4.20. Then the resulting function is a post-critically finite Newton map since it satisfies all the conditions of Theorem 2.2, denote it by N_p. What we have in this process is that the new rational function N_p and the old f_1 are conjugate except in small neighborhoods of attracting fixed points of f_1. This intermediate surgery produces a quasiconformal homeomorphism ϕ_1 such that the following diagram is commutative

$$\begin{tikzcd} \hat{\mathbb{C}} \setminus \phi_1^{-1}(A) \arrow[r, "N_p"] \arrow[d, "\phi_1"'] & \hat{\mathbb{C}} \setminus \phi_1^{-1}(A) \arrow[d, "\phi_1"] \\ \hat{\mathbb{C}} \setminus A \arrow[r, "f_1"] & \hat{\mathbb{C}} \setminus A, \end{tikzcd}$$

where A is the union of all basins that are affected by the intermediate surgery, moreover ϕ_1 is conformal in the interior of $\hat{\mathbb{C}} \setminus \phi_1^{-1}(A)$. *A priori* some of the attracting fixed points of f_1, that are newly created by perturbing a parabolic fixed point of $N_{p_1 e^{q_1}}$, could actually be superattracting, then we could use Haïssinsky surgery to f_1 right away to avoid intermediate surgery at that attracting fixed point, since we need to do the reverse of this process again in Lemma 4.20 during process of Haïssinsky surgery. We only change the multipliers of attracting fixed point into 0 in order to make sure that the resulting rational function is a post-critically *finite* Newton map. Let us summarize what we have.

- The quotient map ϕ restricted to the Julia set of f_1 is a topological conjugacy between the Julia sets of f_1 and $N_{p_1 e^{q_1}}$ and it is a conformal conjugacy on the Fatou components of f_1 that are mapped to the non-parabolic domains of $N_{p_1 e^{q_1}}$ via ϕ.
- The quasiconformal homeomorphism ϕ_1 is a conjugacy between f_1 and N_p on the complement of disk neighborhoods of attracting fixed points of f_1 and it is conformal in all basins of superattracting periodic points of f_1. Thus, a quotient map $\phi \circ \phi_1$ is a topological conjugacy between the Julia sets of N_p and $N_{p_1 e^{q_1}}$, and it is a conformal map where ϕ is conformal.

We *mark* the basins of N_p that are created by Cui plumbing surgery. We also need *marked* accesses in every marked basin. We know that (Proposition 2.13) every parabolic immediate basin of $N_{p_1 e^{q_1}}$ has a unique attracting access along which the orbits starting at the points from its basin converge to its limit, a parabolic fixed point at ∞. Note that, since ϕ is a homeomorphism between the boundary of one basin with some other basin, the accesses of former transform to the accesses of the latter via ϕ. Thus, we have the marked accesses of N_p.

Part B. Now for every $1 \leq j \leq n$ all the marked basins $\mathcal{A}(\xi_j)$ and the marked access in each are given. We apply Haïssinsky surgery (Theorem 4.15) to N_p to those basins with accesses deducing the existence of a David homeomorphism ϕ_2 and a post-critically *minimal* Newton map $N_{p_2 e^{q_2}}$ such that

- ϕ_2 is conformal in every Fatou component of N_p that is not marked,
- $\phi_2 \circ N_p = N_{p_2 e^{q_2}} \circ \phi_2$ for all $z \notin \bigcup_{1 \le j \le n} \mathcal{A}^\circ(\xi_j)$. i.e. the following diagram commutes

$$\begin{tikzcd}
\hat{\mathbb{C}} \setminus \cup_{1 \le j \le n} \mathcal{A}^\circ(\xi_j) \arrow[r, "N_p"] \arrow[d, "\phi_2"'] & \hat{\mathbb{C}} \setminus \cup_{1 \le j \le n} \mathcal{A}^\circ(\xi_j) \arrow[d, "\phi_2"] \\
\hat{\mathbb{C}} \setminus \phi_2(\cup_{1 \le j \le n} \mathcal{A}^\circ(\xi_j)) \arrow[r, "N_{p_2 e^{q_2}}"] & \hat{\mathbb{C}} \setminus \phi_2(\cup_{1 \le j \le n} \mathcal{A}^\circ(\xi_j)).
\end{tikzcd}$$

Part C. We shall construct a topological conjugacy between $N_{p_1 e^{q_1}}$ and $N_{p_2 e^{q_2}}$ that is conformal in the Fatou set of $N_{p_1 e^{q_1}}$.

By construction of ϕ, ϕ_1 and ϕ_2 it follows that the map $\Psi = \phi_2 \circ \phi_1^{-1} \circ \phi^{-1}$ is a conjugacy between $N_{p_1 e^{q_1}}$ and $N_{p_2 e^{q_2}}$ in the complement of parabolic basins of $N_{p_1 e^{q_1}}$. Moreover Ψ is conformal in the basins of superattracting periodic points of $N_{p_1 e^{q_1}}$. We want to extend this conjugacy to the parabolic basin $\mathcal{A}^1(\infty)$ as well. We shall use the full power of Proposition 3.8 to construct the topological conjugacy via gluing the Riemann maps of corresponding parabolic components. Let $\mathcal{A}_j^{\circ 1}$ be an immediate basin of parabolic fixed point of $N_{p_1 e^{q_1}}$ and let c_j^1 be the unique critical point in $\mathcal{A}_j^{\circ 1}$, for $1 \le j \le n$. Note that $\Psi(\partial \mathcal{A}_j^{\circ 1})$ is the boundary of exactly one parabolic component of $N_{p_2 e^{q_2}}$ for $1 \le j \le n$, denote it by $\mathcal{A}_j^{\circ 2}$, since it could only be an immediate basin. Let c_j^2 be the unique critical point in $\mathcal{A}_j^{\circ 2}$. Let $\psi_j^{\circ 1} : \mathcal{A}_j^{\circ 1} \to \mathbb{D}$ and $\psi_j^2 : \mathcal{A}_j^{\circ 2} \to \mathbb{D}$ be the corresponding uniquely defined Riemann maps sending the critical points c_j^1 and c_j^2 to the origin, moreover we have $k_j = \deg(N_{p_1 e^{q_1}}, c_j^1) = \deg(N_{p_2 e^{q_2}}, c_j^2)$ such that the following diagrams commute;

$$\begin{tikzcd}
\mathcal{A}_j^{\circ 1} \arrow[r, "N_{p_1 e^{q_1}}"] \arrow[d, "\psi_j^1"'] & \mathcal{A}_j^{\circ 1} \arrow[d, "\psi_j^1"] \\
\mathbb{D} \arrow[r, "P_{k_j}"] & \mathbb{D} \\
\mathcal{A}_j^{\circ 2} \arrow[r, "N_{p_2 e^{q_2}}"] \arrow[u, "\psi_j^2"] & \mathcal{A}_j^{\circ 2}, \arrow[u, "\psi_j^2"']
\end{tikzcd}$$

where $P_{k_j}(z) = \frac{z^{k_j} + a_j}{1 + a_j z^{k_j}}$ for $a_j = \frac{k_j - 1}{k_j + 1}$, is the parabolic Blaschke product of $\mathbb{D}$. Note that under these normalizations the marked access for both immediate basins are mapped via the Riemann maps to the same access associated to the invariant ray $(0,1)$ for P_{k_j}. For every $1 \le j \le n$, the composition $\psi_j^2 \circ (\psi_j^1)^{-1}$: $\mathcal{A}_j^{\circ 1} \to \mathcal{A}_j^{\circ 2}$ is a conformal conjugacy between $N_{p_1 e^{q_1}}$ and $N_{p_2 e^{q_2}}$ on $\mathcal{A}_j^{\circ 1}$.

By Carathéodory's theorem the inverses to both maps ψ_j^1 and ψ_j^2 extend to the boundary of the unite disk. We define the equivalence relation on the unit circle $\mathbb{S}^1$ induced by the extension; $x \sim y \in \mathbb{S}^1$ if and only if both are mapped to the same point on the boundary by the inverse of ψ_j^1. Similarly we define the equivalence relation for the inverse map of ψ_j^2. We shall show that these two maps define the same equivalence relation on $\mathbb{S}^1$. Indeed, we have k_j+1 fixed points of P_{k_j}, of which $k-2$ are distinct repelling fixed points, and a triple fixed point at 1. In total there are $k-1$ invariant accesses, all of them correspond to the accesses to ∞ in each of the immediate basins $\mathcal{A}_j^{\circ 1}$ and $\mathcal{A}_j^{\circ 2}$. We identify all fixed points P_{k_j} since they all map to ∞ under the inverse map. Now we take preimages of a given fixed point. There are k_j-1 preimages on $\mathbb{S}^1$ of every fixed point other than the fixed point itself. Similarly, in $\mathcal{A}_j^{\circ 1}$ the preimages of ∞ by $N_{p_1e^{q_1}}$, since the Newton map is locally injective away from its critical points, the invariant rays/accesses to ∞ have preimages which land at the poles in $\partial\mathcal{A}_j^{\circ 1}$, one for each non-homotopic rays/accesses to ∞. This is transported by the Riemann map ψ_j^1 to the unit disk and we identify preimages of fixed points according to the rules as in $\mathcal{A}_j^{\circ 1}$. This gives us k_j-1 different classes of identifications on $\mathbb{S}^1$, one for each corresponding pole other than ∞ of $N_{p_1e^{q_1}}$ in $\partial\mathcal{A}_j^{\circ 1}$. Continuing this process we identify iterated preimages of all fixed points of P_{k_j} in $\mathbb{S}^1$ into the equivalence classes corresponding to the iterated preimages of ∞ on $\partial\mathcal{A}_j^{\circ 1}$. Take the closure of this equivalence relation. Since the above diagram commutes we have the same closed equivalent relation on $\mathbb{S}^1$ for ψ_j^1 and ψ_j^2.

Thus, the map $\psi_j^2\circ(\psi_j^1)^{-1}:\mathcal{A}_j^{\circ 1}\to\mathcal{A}_j^{\circ 2}$ extends to the boundary as a continuous map and equals to Ψ on the dense set of points in the common domain of the definition, namely on ∞ and its iterated preimages. Denote the continuous extension by Ψ_j^2, hence $\Psi_j^2=\Psi$ on $\partial\mathcal{A}_j^{\circ 1}$. The conjugacy is now extended to all of immediate basins of the parabolic fixed point.

Now we extend it to all other components of the parabolic basin $\mathcal{A}^1(\infty)$. Let U be the component of $N_{p_1e^{q_1}}^{-1}(\mathcal{A}_j^{\circ 1})$ (iterated preimage of the immediate basin) other than $\mathcal{A}_j^{\circ 1}$, for a fixed $1\le j\le n$. Let c_u be the unique center of U, that is the point which maps to the critical point in $\mathcal{A}_j^{\circ 1}$, and let $k=\deg(N_{p_1e^{q_1}},c_u)$. Then $\Psi(\partial U)$ is the boundary of a unique component of $N_{p_2e^{q_2}}(\mathcal{A}_j^{\circ 2})$, denote it by V, and let c_u denote its unique center. There exist Riemann maps $\psi_U:U\to\mathbb{D}$ and $\psi_V:V\to\mathbb{D}$ such that $\psi_U(c_u)=\psi_V(c_v)=0$ and the diagrams commute

$$\begin{tikzcd}
U \arrow[r, "N_{p_1e^{q_1}}"] \arrow[d, "\psi_U"'] & \mathcal{A}_j^{\circ 1} \arrow[d, "\psi_j^1"] \\
\mathbb{D} \arrow[r, "z\mapsto z^k"] & \mathbb{D} \\
V \arrow[u, "\psi_V"] \arrow[r, "N_{p_2e^{q_2}}"] & \mathcal{A}_j^{\circ 2}, \arrow[u, "\psi_j^2"']
\end{tikzcd}$$

Riemann maps are unique up to post-composing by a rotation of k^{th} root of unity. Since we are interested in the composition ${\psi_V}^{-1} \circ \psi_U$, the choice of Riemann maps for both can be restricted to one. Let us fix any choice for ψ_U. Now we choose the map ψ_V to be compatible with the dynamics of the Newton maps. Observe that preimages of the invariant rays/accesses (e.g. a marked access, which is associated to the interval $(0,1)$, the zero ray) by $N_{p_1 e^{q_1}}$ in ∂U are mapped by ψ_U to the preimages under $z \mapsto z^k$ of the invariant rays landing at fixed points for P_{k_j} (e.g. $(0,1)$), since the above diagram is commutative. Note that the map $z \mapsto z^k$ can not differentiate between different preimages. The map Ψ that is a homeomorphism from ∂U onto ∂V comes in handy. Once ψ_U is chosen we fix ψ_V in such a way that those preimages of $(0,1)$ by $z \mapsto z^k$ are pulled back to U such that they land at the corresponding points dictated by Ψ. There is only one choice of ψ_V for doing this. This is compatible with the dynamics of both $N_{p_1 e^{q_1}}$ and $N_{p_2 e^{q_2}}$ on corresponding boundaries of their Fatou components.

We define equivalence relation on S^1 for both ψ_U and ψ_V as we did above. These equivalence relations are the same since both agree on a dense set of common points. Hence ${\psi_V}^{-1} \circ \psi_U$ extends to the closure of U and coincides with Ψ on a dense set of points, thus both are equal on the common domain of definition. This way we extend Ψ to all (first level) components of preimages of immediate parabolic basins.

We can continue in the same way to extend it to all (iterated preimages) of the parabolic components, since the diagrams are commutative with the same type of model maps $z \mapsto z^k$, where k is a common local degree of the Newton maps at the centers of components.

Let us summarize what we have proved so far and give (remind) the definition of Ψ, which we spent the whole Part C.

$$\Psi = \begin{cases} \phi_2 \circ \phi_1^{-1} \circ \phi^{-1}, & z \in \hat{\mathbb{C}} \setminus \mathcal{A}^1(\infty) \\ {\psi_V}^{-1} \circ \psi_U, & z \in U, \end{cases}$$

where, U and V are component of $\mathcal{A}^1(\infty)$ and $\mathcal{A}^2(\infty)$, correspondingly and all the involved maps in the definition of Ψ are defined in this and previous parts. Thus, Ψ is a conjugacy on $\hat{\mathbb{C}}$ between Newton maps $N_{p_1 e^{q_1}}$ and $N_{p_2 e^{q_2}}$, it is conformal in every component of Fatou set of $N_{p_1 e^{q_1}}$. We still have to show that Ψ is globally continuous.

Claim 5.4. *The map Ψ, defined in Part C, is a homeomorphism of $\hat{\mathbb{C}}$.*

Proof of the Claim. It suffices to prove continuity of Ψ on $J(N_{p_1 e^{q_1}})$. Let us fix $\epsilon > 0$, and a sequence of positive numbers $\epsilon_k \to 0$ as $k \to \infty$. Consider a sequence of points $\{w_s\}_{s \geq 1} \subset \hat{\mathbb{C}}$ such that $w_s \to w \in J(N_{p_1 e^{q_1}})$ as $s \to \infty$. We shall prove that

$$\Psi(w_s) \to \Psi(w) \text{ as } s \to \infty. \tag{5.1}$$

If for some subsequence of w_s we have an inclusion $\{w_{s_k}\}_{k \geq 1} \subset J(N_{p_1 e^{q_1}})$, then $\{\Psi(w_{s_k})\}_{k \geq 1} \subset J(N_{p_1 e^{q_1}})$; hence, the limit (5.1) holds over the subsequence

$\{w_{s_k}\}_{k\geq 1}$, since Ψ is a homeomorphism on the Julia set of $N_{p_1e^{q_1}}$. Moreover, if some subsequence $\{w_{n_k}\}_{k\geq 1}$ is contained in one of the Fatou components (marked or unmarked) of $N_{p_1e^{q_1}}$ (i.e. $\{w_{n_k}\}_{k\geq 1} \subset U_{N_{p_1e^{q_1}}}$) then along this subsequence the limit (5.1) holds true since the restriction $\Psi|_{\overline{U}}$ is continuous. Therefore, without loss of generality we assume that $\{w_s\}_{s\geq 1} \subset F(N_{p_1e^{q_1}})$ and no subsequence of $\{w_s\}_{s\geq 1}$ is contained completely in only one of the components of $F(N_{p_1e^{q_1}})$. As a result of this assumption the sequence $\{w_s\}_{s\geq 1}$ leaves any given component of $F(N_{p_1e^{q_1}})$ in finite time. By Lemma 2.9 there are only finitely many components of $F(N_{p_1e^{q_1}})$ with spherical diameter less than the given $\epsilon > 0$. Now we fix any k. Sooner or later the points of $\{w_s\}_{s\geq 1}$ leave any Fatou component of $N_{p_1e^{q_1}}$ with spherical diameter $\geq \epsilon_k$. Note that the spherical distance between $\psi(w_s)$ and $\psi(w_s')$ is less then ϵ_k for all large enough s, where w_s' is any point on the boundary of the component where w_s is located, in particular, w_s' is located on the Julia set, $J(N_{p_1e^{q_1}})$. Clearly along the same ideas, $w_s' \to w$ as $s \to \infty$. Note that w_s' converges to the same w, since Ψ is continuous on $J(N_{p_1e^{q_1}})$. The claim is now proved. □

Part D. [2] Using the topological conjugacy of Part C, which is, in particular, a conformal map at the petals and superattracting cycles of $N_{p_1e^{q_1}}$, by applying interpolation technique several times we construct a set of quasiconformal homeomorphisms of $\hat{\mathbb{C}}$, which is denoted by $\{\Psi_1, \dots \Psi_k\}$, where k is a total number of superattracting periodic points of $N_{p_1e^{q_1}}$. Next we work with Ψ_k, from the previous step, and construct, by taking lifts of a local conjugation, $\{\psi_m\}_{m\geq 0}$ a sequence of quasiconformal homeomorphisms of $\hat{\mathbb{C}}$ with bounded complex dilatation. Finally, by extracting a converging sub-sequence of the latter we obtain a *conformal* conjugacy between $N_{p_1e^{q_1}}$ and $N_{p_2e^{q_2}}$, finishing the proof of the theorem.

We divide the dynamical plane of $N_{p_1e^{q_1}}$ into two parts: some Jordan neighborhood of infinity and the complement of it, which is bounded. We use Ψ as an initial partial conjugacy between petals of $N_{p_1e^{q_1}}$ and $N_{p_2e^{q_2}}$ at infinity. Note that Ψ is a conformal conjugacy restricted on the immediate basins of ∞. Let us fix an $\epsilon = 1$ (the exact value of ϵ is not relevant). Since Ψ is conformal in petal, thus 1-quasiconformal homeomorphism, by Lemma 4.17 we obtain a $1 + \epsilon = 2$-quasiconformal homeomorphism ϕ defined locally at ∞, that is a conjugacy between $N_{p_1e^{q_1}}$ and $N_{p_2e^{q_2}}$ such that $\phi = \Psi$ on a smaller attracting flower bounded by curves $l_1, \dots, l_n$, see Figure 5.1. In the figure, $l_1, \dots, l_n$ denote the boundaries of small petals of ∞. We fix some quasicircle L_1 in the domain of definition of ϕ such that L_1 separates all superattracting periodic points of $N_{p_1e^{q_1}}$ from the smaller attracting flower where we had the equality $\phi = \Psi$. Denote by L_1^+ the unbounded component of the complement

[2]Note that the topological conjugacy Ψ of Part C is not a c-equivalence between $N_{p_1e^{q_1}}$ and $N_{p_2e^{q_2}}$ according to the generalization of Thurston's topological characterization of post-critically finite covering maps to the setting of geometrically finite covering maps with parabolic cycles (refer to [CT]).

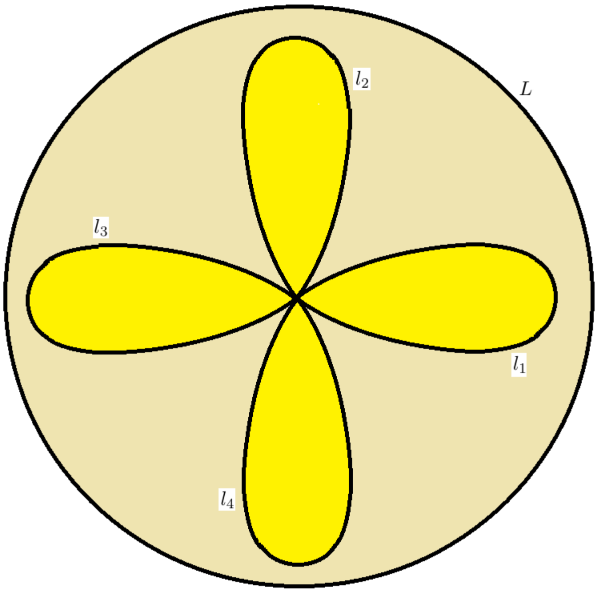


Figure 5.1: A schematic picture of a neighborhood of ∞ with a flower in yellow, for $n = 4$.

of L_1, and by L_1^- the bounded component. Consider $L_2 = \phi(L_1)$ the corresponding quasicircle in the dynamical plane of $N_{p_2e^{q_2}}$. Moreover, L_2 separates the attracting flower from all superattracting periodic points of $N_{p_2e^{q_2}}$. Similarly, denote by L_2^+ the unbounded component of the complement of L_2, and by L_2^- the bounded component.

We shall extend ϕ to the bounded domain L_1^- as a quasiconformal homeomorphism that is conformal on disk neighborhoods of superattracting cycles of $N_{p_1e^{q_1}}$.

We want to construct a quasiconformal homeomorphism that is equal to the map Ψ defined above on neighborhoods of superattracting periodic points, where it is a local conformal conjugacy between $N_{p_1e^{q_1}}$ and $N_{p_2e^{q_2}}$.

In case when there exist no superattracting periodic points of $N_{p_1e^{q_1}}$ we extend ϕ using Theorem 4.18 part (a) to L_1^-. In case when there exist one or more superattracting periodic points of $N_{p_1e^{q_1}}$, we extend ϕ using Theorem 4.18 part (b) to L_1^- sequentially in small disk about every periodic point as following. Let $C_1, C_2, \ldots, C_k$ be a list of disjoint simple closed analytic curves contained in L_1^-, one for each element of superattracting cycles (the critical points and their orbits) that bound the element in its immediate basin for $N_{p_1e^{q_1}}$. For every $i \leq k$ let Ω_i^1 be the closed disk bounded by C_i, i.e. $\Omega_i^1 \Subset L_1^-$.

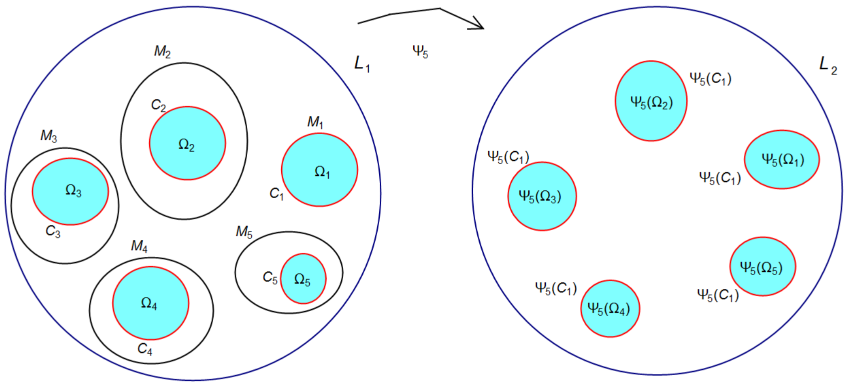


Figure 5.2: A schematic illustration of the construction of the interpolation. Left: The analytic disks in cyan are Ω_i^1 in the $N_{p_1 e^{q_1}}$ plane. Right: The corresponding image for the $N_{p_2 e^{q_2}}$ plane.

Note that, the images $\Psi(\Omega_1^1), \Psi(\Omega_2^1), \ldots, \Psi(\Omega_k^1)$ are closed disks in L_2^- bounded by analytic curves $\Psi(C_1), \Psi(C_2), \ldots, \Psi(C_k)$, each of which surrounds the corresponding superattracting periodic point of $N_{p_2 e^{q_2}}$ in its immediate basin.

We are in position to apply Theorem 4.18 part (b). First, we pick C_1 as the internal boundary and L^1 as the external boundary of a quasiannulus. Theorem 4.18 part (b) produces a quasiconformal homeomorphism of $\hat{\mathbb{C}}$, denote it by Ψ_1, that interpolates inner and outer maps $\Psi|_{C_1}$ and $\Psi|_{L_1}$. Second, we continue the application of Theorem 4.18 part (b) with the next analytic curve C_2 and the map Ψ_1, which is obtained in the first step. We need to specify the boundary maps, too. One way to define the boundaries is shrinking the curve C_2, while keeping the center unchanged, which is a superattracting periodic point of $N_{p_1 e^{q_1}}$. By shrinking we mean that we take an analytic curve $\tilde{C_2}$ within Ω_2^1. Another way is taking some quasicircle located within L_1^-, denote it by M_2, which bounds the curve C_2 and separates it with C_1. In the latter case, we have $\Psi_1|_{M_2}$ and $\Psi_1|_{C_2}$ as external and internal maps, respectively. Interpolation gives us a quasiconformal homeomorphism of $\hat{\mathbb{C}}$, denote this map by Ψ_2. Note that, Ψ_2 is conformal on the union of Ω_1^1 and Ω_2^1. Finally, we take some quasicircle located within L_1^-, denote it by M_k, that bounds the curve C_k and separates it from all other curves $C_1, C_2, \ldots, C_{k-1}$. We consider $\Psi_{k-1}|_{M_k}$ and $\Psi_{k-1}|_{C_k}$ as external and internal maps respectively for the interpolation. We obtain a quasiconformal homeomorphism of $\hat{\mathbb{C}}$, denote it by Ψ_k. Note that Ψ_k is conformal on all of Ω_i^1 for $i \leq k$.

To ease the notation let us denote the last interpolating map by ψ_0 i.e. $\psi_0 = \Psi_k$ and denote by Ω^1 the union of Ω_i^1 for $1 \leq i \leq k$ and the open parabolic flower bounded by $l_1, \ldots, l_n$ and let $\psi_0(\Omega^1) = \Omega^2$. By definition

$\psi_0 = \Psi$ and $\psi_0 \circ N_{p_1e^{q_1}} = N_{p_2e^{q_2}} \circ \psi_0$ on Ω^1. The following diagram commutes

$$\begin{tikzcd} \Omega^1 \arrow[r, "N_{p_1e^{q_1}}"] \arrow[d, "\phi_0"'] & N_{p_1e^{q_1}}(\Omega^1) \arrow[d, "\phi_0"] \\ \Omega^2 \arrow[r, "N_{p_2e^{q_2}}"] & N_{p_2e^{q_2}}(\Omega^2). \end{tikzcd}$$

Lifting. Recall that C_f denotes the set of critical points of f. Let us define sets: $V^i = N_{p_ie^{q_i}}(C_{N_{p_ie^{q_i}}})$ the set of critical values of $N_{p_ie^{q_i}}$, and for $i \in \{1,2\}$ let $T^i = N^{-1}_{p_ie^{q_i}}(V^i)$ be the full preimage of $C_{N_{p_ie^{q_i}}}$ under $N_{p_ie^{q_i}}$.

We have $N_{p_1e^{q_1}} : \hat{\mathbb{C}} \setminus T^i \to \hat{\mathbb{C}} \setminus V^i$ unbranched covering maps for $i \in \{1,2\}$. Note that if $\phi_0(V^1) \neq V^2$ then we include to V^1 and V^2 the sets $\phi_0^{-1}(V^2)$ and $\phi_0^{-1}(V^1)$, correspondingly. We define T^i's accordingly. The maps $\psi_0 \circ N_{p_1e^{q_1}} : \hat{\mathbb{C}} \setminus T^1 \to \hat{\mathbb{C}} \setminus V^2$ and $\psi_0^{-1} \circ N_{p_2e^{q_2}} : \hat{\mathbb{C}} \setminus T^2 \to \hat{\mathbb{C}} \setminus V^1$ are unbranched covering maps. We fix any component of Ω^1 and denote it by Ω^0. Newton map has at least one petal, it is better to take a petal as Ω^0. Let us fix a base point $x_0 \in \Omega^0 \setminus O(C_{N_{p_2e^{q_2}}})$ for the domain $\hat{\mathbb{C}} \setminus V^2$, where $O(C_{N_{p_2e^{q_2}}})$ denotes the union of grand orbits of critical points of $N_{p_2e^{q_2}}$, note that $V^2 \cup T^2 \subset O(C_{N_{p_2e^{q_2}}})$. Actually more is true; the grand total orbit of critical points $O(C_{N_{p_ie^{q_i}}})$ is generated by V^i and also by T^i for $i \in \{1,2\}$. Note that $N^{-1}_{p_2e^{q_2}}(\Omega^0)$ has many components in the immediate basin associated to the petal Ω^0, since $\Omega^0 \subset N^{-1}_{p_2e^{q_2}}(\Omega^0)$ we fix a preimage $N^{-1}_{p_2e^{q_2}}(x_0)$ in $N^{-1}_{p_2e^{q_2}}(\Omega^0)$ denoted by y_0 as a base point for the domain $\hat{\mathbb{C}} \setminus T^2$. Since ψ_0 is bijection, the preimages $\psi_0^{-1}(x_0)$ and $\psi_0^{-1}(y_0)$ are base points for the domains associated to $N_{p_1e^{q_1}}$. We can invoke Lemma 4.19 since the map ψ is a homeomorphism, therefore the induced maps on fundamental groups of involved domains are isomorphisms. The unique lift ψ_1 of $\psi_0 \circ N_{p_1e^{q_1}}$ is a map from $\hat{\mathbb{C}} \setminus T^1$ onto $\hat{\mathbb{C}} \setminus V^2$ such that $\psi_1(\psi_0^{-1}(y_0)) = y_0$ and $\psi_0 \circ N_{p_1e^{q_1}} = N_{p_2e^{q_2}} \circ \psi_1$ on $\hat{\mathbb{C}} \setminus T^1$;

$$\begin{tikzcd} \hat{\mathbb{C}} \setminus T^1 \arrow[r, "\psi_1"] \arrow[dr, "\psi_0 \circ N_{p_1e^{q_1}}"'] & \hat{\mathbb{C}} \setminus T^2 \arrow[d, "N_{p_2e^{q_2}}"] \\ & \hat{\mathbb{C}} \setminus V^2. \end{tikzcd}$$

We extend ψ_1 to the finite set T^1 as a continuous map. Observe that
$\psi_1 = \psi_0 = \Psi$ on Ω^0. The unique lift $\tilde{\psi}_1$ of $\psi_0^{-1} \circ N_{p_2e^{q_2}}$ is a map from $\hat{\mathbb{C}} \setminus T^2$ onto $\hat{\mathbb{C}} \setminus V^1$ such that $\tilde{\psi}_1(y_0) = \psi_0^{-1}(y_0)$ and $\psi_0^{-1} \circ N_{p_2e^{q_2}} = N_{p_1e^{q_1}} \circ \tilde{\psi}_0$ on $\hat{\mathbb{C}} \setminus T^2$;

$$\begin{tikzcd} \hat{\mathbb{C}} \setminus T^1 \arrow[d, "N_{p_1e^{q_1}}"] & \hat{\mathbb{C}} \setminus T^2 \arrow[l, "\psi_1"'] \arrow[dl, "\psi_0^{-1} \circ N_{p_2e^{q_2}}"] \\ \hat{\mathbb{C}} \setminus V^1. & \end{tikzcd}$$

Similarly, we extend $\tilde{\psi}_1$ to the finite set T^2 as a continuous map. It is easy to observe that ψ_1 and $\tilde{\psi}_1$ are inverses to each other on $\hat{\mathbb{C}}$. Moreover ψ_1 is a quasiconformal homeomorphism with the same complex dilatation as ψ_0. By continuing this lifting process we obtain a sequence of quasiconformal maps $\{\psi_m\}_{m\geq 0}$, with the same bound on complex dilatation, such that $\psi_{m+1} = \psi_0 = \Psi$ on Ω^0 and $\psi_m \circ N_{p_1 e^{q_1}} = N_{p_2 e^{q_2}} \circ \psi_{m+1}$ on $\hat{\mathbb{C}}$. Note that $\psi_m = \psi_0$ and is conformal on $N^{-m}_{p_1 e^{q_1}}(\Omega^1)$. The sequence $\{\psi_m\}_{m\geq 0}$ is a normal family, so it has a converging subsequence $\{\psi_{m_k}\}_{k\geq 0}$; let ψ_∞ be the limiting map. Because the space of quasiconformal homeomorphisms with uniformly bounded dilatations is compact the homeomorphism ψ_∞ is quasiconformal. Note that, as constructed by lifts the map ψ_∞ is conformal on $\cup_{m=0}^{\infty} N^{-m}_{p_1 e^{q_1}}(\Omega^1)$, the complement of which is the Julia set of $N_{p_1 e^{q_1}}$, which has a measure zero. Therefore, the map ψ_∞ is conformal on $\hat{\mathbb{C}}$ and we have $\psi_\infty \circ N_{p_1 e^{q_1}} = N_{p_2 e^{q_2}} \circ \psi_\infty$ on Ω^0.

Consider a rational function $R = \psi_\infty^{-1} \circ N_{p_2 e^{q_2}} \circ \psi_\infty$ and note that $R = N_{p_1 e^{q_1}}$ on Ω^0. By the identity principle of holomorphic functions we obtain $R = N_{p_1 e^{q_1}}$ on $\hat{\mathbb{C}}$, i.e. $\psi_\infty \circ N_{p_1 e^{q_1}} = N_{p_2 e^{q_2}} \circ \psi_\infty$ on $\hat{\mathbb{C}}$. The proof of the surjectivity is finished here. $\square$

5.3 Proof of the Main Theorem

In this section we shall prove the Main Theorem. We recall from Definitions 3.1–3.3: $\mathcal{N}(d-n,n)$ denotes the space of normalized Newton maps N_{pe^q} of degree $d \geq 3$ with n petals at ∞. For instance, $\mathcal{N}(d) := \mathcal{N}(d,0)$ is the space of degree $d \geq 3$ normalized Newton maps for polynomials that are monic, centered, have a root at $z = 1$ and all roots are simple. For every natural number $n \leq d$ denote $\mathcal{N}_{\mathrm{pcm}}(d-n,n)$ the space of post-critically *minimal* Newton maps in $\mathcal{N}(d-n,n)$. In particular, denote $\mathcal{N}_{\mathrm{pcf}}(d) := \mathcal{N}_{\mathrm{pcf}}(d,0)$ the space of degree d post-critically *finite* Newton maps for polynomials that are centered, monic and have a root at $z = 1$, and with only simple roots. Denote $\mathcal{N}_{\mathrm{pcf}}^{+,n}(d)$ the space of post-critically *finite* Newton maps from $\mathcal{N}_{\mathrm{pcf}}(d)$ with markings Δ_n^+.

Theorem 5.5 (Main theorem)**.** *Two Haïssinsky surgeries applied to N_{p_1} and N_{p_2} belonging to $\mathcal{N}_{pcf}^{+,n}(d)$ are equivalent if and only if N_{p_1} and N_{p_2} are affine conjugate. The mapping $\mathcal{F}_n$ given by Haïssinsky surgery induces (natural) bijection between the quotient space $\mathcal{N}_{pcf}^{+,n}(d)/\sim_H$ and the space of affine conjugacy classes of Newton maps in $\mathcal{N}_{pcm}(d-n,n)$.*

Proof. Fix a pair of non-negative integers $d \geq 3$ and $1 \leq n \leq d$. Define a map $\mathcal{F}_n : \mathcal{N}_{\mathrm{pcf}}^{+,n}(d) \to \mathcal{N}_{\mathrm{pcm}}(d-n,n)$ given by Haïssinsky surgery. From Theorems 5.2 and 5.3 it follows that Haïssinsky surgeries applied to f with its marking $\Delta_n^+(f))$ and g with its marking $\Delta_n^+(g))$ are affine conjugate if and only if f and g are affine conjugate, that is the two Haïssinsky surgeries are Haïssinsky equivalent, i.e. $f \sim_H g$ (definition 5.1). Going to the quotient by Haïssinsky equivalence $\sim_H$ for $\mathcal{N}_{\mathrm{pcf}}^{+,n}(d)$ and affine conjugation for $\mathcal{N}(d-n,n)$ the mapping $\mathcal{F}_n$ induces bijection. □

Proof of Corollary 4.9. We shall sketch the proof. Let a stable Newton map $f \in \mathcal{N}(d-n,n)$ be given. An application of the Theorem 4.8 to f with $\Gamma = J(f)$ produces a rational function g. It excluded the item (a), because f can not have any Siegel discs, therefore g also can not have any Siegel disc. The resulting function g satisfies all the conditions of the Definition 3.6. The uniqueness of g, which we call a "center" of the stable component of f in $\mathcal{N}(d-n,n)$, follows from the main theorem of the thesis (Theorem 5.5). □

Appendix A

Formal Newton maps

One can generalize the definition of a Newton map and obtain a large family of rational functions in the following way: Let $a_i \in \mathbb{C} \setminus \{0\}$, $z_i \in \mathbb{C}$, for $1 \le i \le d$ be given. Consider a formal Newton map as follows

$$f(z) := z - \frac{1}{\sum_{i=1}^{d} \frac{a_i}{z - z_i}}. \tag{A.1}$$

If z_i are all different from each other, then the degree of the rational function f of the form (A.1) is equal to d, and the number of "free" complex parameters is $2d$. In this way we obtain all rational functions of degree d with $d+1$ distinct fixed points. The points z_i are fixed with multipliers $1 - \frac{1}{a_i}$ and are attracting if $|1 - \frac{1}{a_i}| < 1$ for $1 \le i \le d$. By changing the multipliers at the finite fixed points through quasiconformal surgery [CG93, Theorem 5.1], we can convert all of them to superattracting. The resulting rational function is clearly a Newton map of a polynomial, implying that in this special case the Formal Newton map has a simple fixed point, which is necessarily repelling, other than z_i for $1 \le i \le d$, and its Julia set is connected. If all of $a_i = 1$, we are in the case of Newton maps of polynomials.
We have the following:

Theorem A.1 (Connectivity of the Julia set of a formal Newton map)**.** *Let complex numbers a_i and z_i be given and satisfy $|1 - \frac{1}{a_i}| < 1$ for $1 \le i \le d$, and $z_i \ne z_j$ for $i \ne j$, with $d \ge 2$. Then the Julia set of a formal Newton map $f(z) = z - \frac{1}{\sum_{i=1}^{d} \frac{a_i}{z - z_i}}$ is connected. There is a canonical post-critically finite Newton map of a polynomial N_p corresponding to f, provided the Julia critical points of f have finite orbits, i.e. when f is geometrically finite. This correspondence is quasiconformal and conjugates the dynamics on some neighborhood of the Julia set of f.*

Proof. Assume that $|1 - \frac{1}{a_i}| < 1$ for $1 \le i \le d$, and $z_i \ne z_j$ for $i \ne j$. Then f has attracting fixed points at $z = z_i$ for $1 \le i \le d$, the multipliers of which can be changed to zero by quasiconformal surgery [CG93, Theorem 5.1]. The resulting function has the same degree and is the Newton map of polynomial since its all but one fixed points are superattracting, hence $J(f)$ is connected by Shishikura's theorem [Shi09]. Let $z \in \mathbb{C}$ be given, if $z \neq z_i$ the ratio $\frac{a_i}{z-z_i}$ is never zero or infinite, thus $|\frac{1}{\sum_{i=1}^{d} \frac{a_i}{z-z_i}}|$ is never zero, hence z is not a fixed point of f. Since every rational function has a weakly repelling fixed point then ∞ is the one for f, which necessarily is repelling. Since the Newton map with d superattracting fixed points has degree d so does the function f.

When f is geometrically finite and satisfies conditions of the theorem then quasiconformal surgery, which makes attracting fixed points superattracting, produces the post-critically finite Newton map. In particular, the dynamics on some neighborhood of $J(f)$ is conjugate to that of the the post-critically finite Newton map. □

Theorem A.2 (Rational function with a single fixed point is a rational Newton map)**.** *If there is a single fixed point, let it be at ∞, of a rational function F of degree at least 2 then it has a normal form $F(z) = z - \frac{1}{q(z)}$ for some polynomial q. Moreover, $F = N_f$ the Newton map of an exponential function $exp\left(\int q(w)dw\right)$. In particular, the fixed point at ∞ is necessarily parabolic with the multiplier $+1$.*

Proof. Assume a rational function F of degree ≥ 2 is given. Let its single fixed point be at ∞, otherwise we conjugate by a Möbius map and send its fixed point to ∞. Then the rational function $\frac{1}{z-F(z)}$ has a single pole at ∞. Thus, it is a polynomial, denote by q, then $F(z) = z - \frac{1}{q(z)}$. Observe that F is a Newton map of an entire function $e^{\int q(w)dw}$. Then ∞ is necessarily a parabolic fixed point with $\deg(\int q(w)dw) = \deg(q) + 1$ equal to the number of petals followed by Theorem 2.3. □

Formal Newton maps form a reasonably large family of rational functions. Moreover, all rational function of degree d with $d+1$ different fixed points (all those where all fixed points are simple) can be obtained as a formal Newton map. The value of a_i is uniquely defined by the desired multiplier at the fixed point $z = z_i$.

We can further generalize the formula for the Formal Newton maps by taking the Newton map of formal transformation of the form: For an integer $d \geq 3$ consider

$$\prod_{i=1}^{d-n} (z - z_i)^{a_i} e^{Q(z)},$$

where $a_i \in \mathbb{C} \setminus \{0\}$, $z_i \in \mathbb{C}$ for $1 \leq i \leq d - n$ and Q a polynomial of degree $n \geq 1$. Its formal "Newton map" has the following form:

$$F(z) := z - \frac{1}{\sum_{i=1}^{d-n} \frac{a_i}{z-z_i} + Q'(z)} \tag{A.2}$$

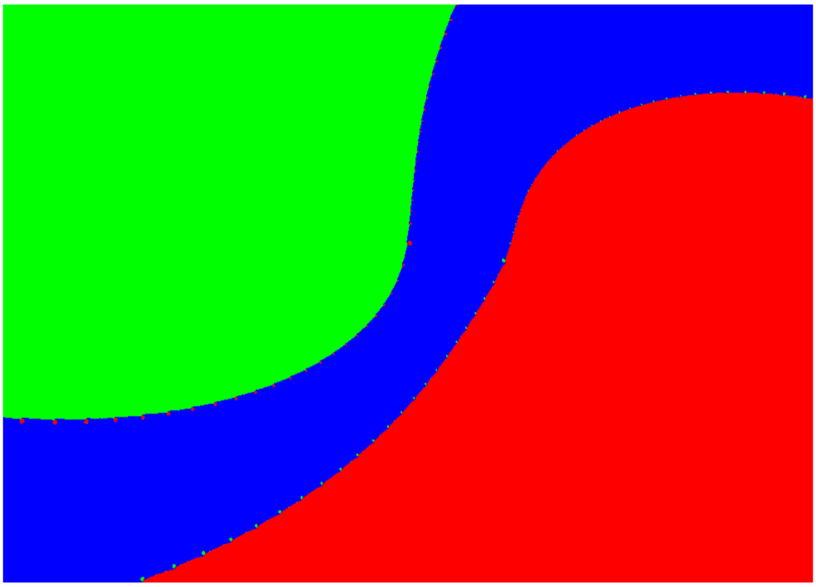


Figure 1.1: The Julia set of a cubic Formal Newton map with attracting fixed points at $z = -1$, $z = 0$, and $z = 1$. Moreover, at the attracting fixed points the multipliers are complex numbers.

which is well defined rational function of degree $d \geq 3$. In the case when $|1 - \frac{1}{a_i}| < 1$ for $1 \leq i \leq d - n$, as above by changing the multipliers at the fixed points $z = z_i$ we can derive that the function A.2 has a fixed point at ∞, which is necessarily parabolic with the multiplier $+1$, and its Julia set is connected. In this general case we summarize the result as following:

Theorem A.3 (Connectivity of the Julia set, general case)**.** *Let for a pair of integers $d \geq 2$ and $1 \leq n \leq d$, complex numbers a_i and z_i for $1 \leq i \leq d - n$, and a polynomial Q of degree n be given. Assume $|1 - \frac{1}{a_i}| < 1$ for all $1 \leq i \leq d - n$, and $z_i \neq z_j$ for $i \neq j$. Then the Julia set of the formal Newton maps $F(z) := z - \frac{1}{\sum_{i=1}^{d-n} \frac{a_i}{z - z_i} + Q'(z)}$ is connected. There exist a post-critically minimal Newton map N_{pe^q} and a quasiconformal map ϕ such that $\phi \circ F = N_{pe^q} \circ \phi$ on $J(f)$ provided the Julia critical points of F have finite orbits, i.e. when F is geometrically finite.*

Proof. A similar argument as in the proof of Theorem A.1 shows that the point ∞ is a weakly repelling fixed point of F, which necessarily is parabolic with the multiplier $+1$. Since ∞ is the only fixed point with this property $J(f)$ is connected by Shishikura's theorem [Shi09]. To construct a rational Newton map we apply a surgery tool [CG93, Theorem 5.1]; for $1 \leq i \leq n$, we convert the attracting fixed points $z = z_i$ to superattracting. The resulting rational function is a rational Newton map with a parabolic fixed point by Theorem 2.2.

Both rational functions are quasiconformally conjugate to each other away from the compact sets containing attracting fixed points, in particular the conjugacy holds in the neighborhood of the Julia set. In order to obtain a post-critically minimal Newton map we can use the theorem of McMullen (Theorem 4.8) provided all critical points on $J(F)$ have finite orbits. $\square$

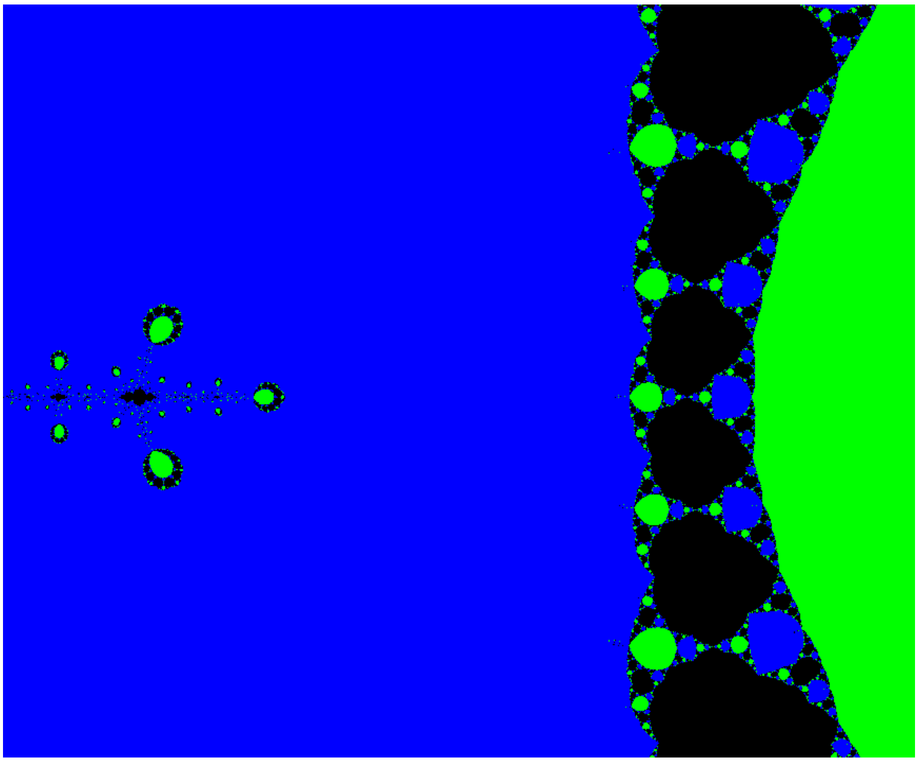


Figure 1.2: A degree 4 formal Newton map with superattracting fixed points at $z = 0$ and $z = 1$, and repelling fixed points with real multipliers at $z = .5$, $z = -1$, and $z = \infty$. It has a two cycle with the basin in black. The basin of $z = 0$ is in blue and the basin of $z = 1$ is in green.

Appendix B

Open Problems

A Dictionary.

There are similarities between the parameter planes of cubic Newton maps with parabolic fixed point at infinity with different number of petals and also with Newton maps of polynomials, see Section 3.3. Every hyperbolic component in $\mathcal{N}(3)$, the space of Newton maps of polynomials, contains a unique "center". The same is true for the spaces $\mathcal{N}(2,1)$, $\mathcal{N}(1,2)$, and $\mathcal{N}(0,3)$. It is interesting to obtain a complete *"dictionary"* between these spaces. It is also not known if all stable components are simply connected in in the parameter planes of $\mathcal{N}(2,1)$, $\mathcal{N}(1,2)$, and $\mathcal{N}(0,3)$. But in $\mathcal{N}(3)$, the hyperbolic components are simply connected [Tan97].

Local connectivity.

The understanding of local connectivity of Julia sets of Newton maps N_{pe^q} is of great importance. In case of post-critically minimal Newton maps N_{pe^q}, when we are able to apply Haïssinsky surgery the local connectivity property is preserved. It remains to investigate local connectivity in the case when there is a Julia critical point with infinite orbit i.e for geometrically **infinite** rational functions.

Regarding local connectivity it is expected that the results corresponding to those for cubic Newton maps of polynomials hold for Newton maps N_{pe^q} of any degrees.

Newton maps as matings of polynomials.

Haïssinsky used techniques from his parabolic surgery to show that a pair of geometrically finite quadratic polynomials $z^2 + c_1$ and $z^2 + c_2$ are matable if and only if $\overline{c_1}$ and c_2 do not belong to the same limb of the Mandelbrot set, generalizing Tan's result for post-critically finite quadratic polynomials [BF14]. It is still open whether or not the Newton maps N_p for polynomials p with degree bigger than 3 are matable. For case of the family $\mathcal{N}(3)$, it is a result of Tan, see [Tan97]

Formal Newton maps.

Not much is known for connectivity of the Julia sets of rational functions with at least 2 weakly repelling fixed points. We would like to have a framework which enables us to study this question within the space of formal Newton maps. One way is approaching to some of these rational functions as a limiting function of formal Newton maps by letting absolute values of multipliers go to 1 from below, in this case the Julia sets are connected by Theorems A.1-A.3. It is plausible that the Julia set in this process depends continuously to the multipliers and is connected.

Post-critically minimal functions in any spaces of rational functions.

There are four one-parameter families of cubic Newton maps $\mathcal{N}(3), \mathcal{N}(2,1)$, $\mathcal{N}(1,2)$, and $\mathcal{N}(0,3)$, see Section 3.3. It would be interesting to construct a dynamics-preserving projection from one to another. The results of the thesis give a partial answer to this question. We have a correspondence between the "centers" of "hyperbolic" components of these families, see Main Theorem (Theorem 5.5) and Corollary 4.9. In this regard we would like to mention that Barański in his thesis [Bar01a, Bar01b] investigated the existence of Mandelbrot-like sets and the bifurcations of those from a parameter plane of cubic Newton maps of polynomials. Post-critically minimal functions are models in the space of Newton maps. One can make use of the definition in general family of rational functions. To obtain model functions an appropriate "surgery" in the basins needs to be done.

"Tuning" of basins of Newton maps.

Post-critically finite Newton maps of polynomials are conformally conjugate to the map $z \mapsto z^d$, in the immediate basins of superattracting fixed points, where $d-1$ is the multiplicity of a critical fixed point. There is a way, similar to matings of polynomials, to glue a new filled-in Julia set of a post-critically finite polynomial in the immediate basins and then pull it back to other components in the basins. Intuitively we are pinching the basins through rational laminations. The possible obstructions are topological, that is a Moore type, or Thurston type. To overcome Thurston type obstruction in most cases we will be able to use the same tool, arcs intersecting obstructions from [PT09] that is used in the classification of post-critically finite Newton maps of polynomials in [LMS2]. The conjecture is that one can "tune" "star like" filled in Julia sets, these are the unicritical polynomials in the hyperbolic components attached to the main cardioid in the Mandelbrot set or in the Multibrot sets. The resulting post-critically finite rational function still has a connected Julia set and the set of these functions could be classified using post-critically finite Newton maps, obtaining a huge space of post-critically finite rational functions.

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