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Regularity Properties of Sectorial Operators: Extrapolation, Counterexamples and Generic Classes

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Introduction

The study of partial differential equations is a central topic in modern mathematics. While Picard's existence theorem and Peano's existence theorem give very satisfying answers to the problem of existence and uniqueness of ordinary differential equations, such a complete theory does not exist for partial differential equations. In fact, for typical partial differential equations one has to deal with unbounded operators on infinite-dimensional spaces which causes fundamental new mathematical difficulties which are usually studied with sophisticated functional analytic methods.

In this thesis we consider fundamental problems related to methods that are suited for the study of non-linear parabolic partial differential equations. Very prominent examples of this type are among many others the famous Navier–Stokes equations in fluid dynamics and in differential geometry the mean curvature flow for a surface given as the graph of a function $u\colon \mathbb{R}^n \to \mathbb{R}$ which is described by the evolution equation

$$\text{(MCF)} \qquad \begin{cases} \partial_t u - \Delta u = -\displaystyle\sum_{i,j=1}^{n} \frac{\partial_i u \partial_j u}{1+|\nabla u|^2} \partial_i \partial_j u & \text{in } (0,T)\times\mathbb{R}^n \\ u|_{t=0} = u_0 & \text{in } \mathbb{R}^n. \end{cases}$$

The existence of a unique local strong solution of such a non-linear parabolic equation can be established as follows. After linearizing the non-linear equation one obtains an equation of the form

$$\begin{cases} \dot{u}(t) + A(u(t)) = 0 \\ u(0) = u_0, \end{cases}$$

i.e. the abstract first order Cauchy problem for an (unbounded) closed operator A on some function space X. For example, for (MCF) a natural choice is the realization of $A = -\Delta$ on $L_p(\mathbb{R}^n)$ for some $p \in (1,\infty)$. Such abstract Cauchy problems which are fundamental for the understanding of the behaviour of linear parabolic problems are naturally studied in the theory of strongly continuous one-parameter semigroups of linear operators on Banach spaces, or more shortly C_0-semigroups. This theory is by now very well-understood. Hence, it is very natural to take this theory as a starting point for the understanding of non-linear problems. Indeed, if one sees the right hand side of (MCF) in the abstract setting as a function $u \mapsto G(u)$ the non-linear equation becomes

$$\begin{cases} \dot{u} + Au = G(u) \\ u(0) = u_0. \end{cases}$$

Suppose that the solution operator $L\colon u \mapsto (\dot{u} + Au, u_0)$ is invertible on an appropriate X-valued function space Z. Then one can take its inverse in order to rewrite the equation into a fixed point problem which one may hope to solve with one of the well-known fixed point theorems, e.g. the Banach fixed point theorem or a variant of the Schauder fixed point theorem. One sees that for this at least the linear problem should be well-posed which is equivalent for $-A$ to be the generator of strongly continuous semigroup.

It is then most convenient to see the first component of the solution operator u as the sum of two closed operators $\mathcal{B} + \mathcal{A}$. Here $\mathcal{B}$ is the derivation operator on X-valued functions and $\mathcal{A}$ is the pointwise application of A. For L to be invertible one then needs that for each $f \in Z$ there exists a unique $u \in D(\mathcal{A}) \cap D(\mathcal{B})$ with

$$\mathcal{B}u + \mathcal{A}u = \dot{u} + Au = f.$$

In applications, this property together with some regularity of the given non-linearity G then guarantees the self-mapping property of the respective maps needed for the fixed point theorems. We are now interested in the case where $Z = L_p([0;T];X)$ is some vector-valued L_p-space which is a natural choice in view of Sobolev methods. In this case the above requirement then translates to the fact that for a given $f \in L_p([0,T];X)$ and for the unique solution u of $\dot{u} + Au = f$ together with the initial condition both $\dot{u}$ and Au lie in $L_p([0,T];X)$. In other words, this means that both summands have the same regularity as the right hand side f. Since this is the best regularity one can hope for, one therefore speaks of *maximal regularity* or more precisely of *maximal L_p-regularity*. One can show that for this property to hold for the most general initial values u_0 possible, namely the traces of functions in $W_p^1([0,T];X) \cap L_p([0,T];D(A))$ it suffices to verify it for the initial condition $u_0 = 0$. We are therefore interested in the following property which we now formulate precisely.

Let X be a Banach space and $-A$ the generator of a C_0-semigroup on X. We say that A has *maximal L_p-regularity* for $p \in (1,\infty)$ and $T > 0$ if for all $f \in L_p([0,T];X)$ the unique (mild) solution u of the abstract inhomogeneous Cauchy problem

$$\begin{cases} \dot{u} + Au = f \\ u(0) = 0 \end{cases}$$

satisfies $u \in W_p^1([0,T];X) \cap L_p([0,T];D(A))$, where $D(A)$ is the domain of the closed operator A endowed with its graph norm. In fact, as a working example, one can show maximal regularity for the realization of $-\Delta$ on $L_p(\mathbb{R}^n)$ for all $n \in \mathbb{N}$ and all $p \in (1,\infty)$. Together with some routine applications of the Sobolev embedding theorems and a routine application of the Banach fixed point theorem this yields the existence of a unique local strong solution of

(MCF) and many other important non-linear parabolic partial differential equations.

In this thesis we answer some fundamental structural questions concerning maximal regularity and other closely related regularity properties, thereby making use of the geometric theory of Banach spaces and operator space theory. We now motivate the considered problems in more detail.

The Maximal Regularity Problem Now that we have seen the importance of maximal regularity for non-linear equations, the most natural question to ask is which (negative) semigroup generators have maximal regularity. In fact, maximal regularity can be verified for a broad class of realizations of various differential operators. Moreover, it is a classical result that a necessary condition for A to have maximal regularity is that $-A$ generates an analytic semigroup, i.e. the semigroup mapping $t \mapsto T(t)$ can be extended to an analytic mapping on some sector around the positive real line. Moreover, by an old result of L. de Simon the converse holds on Hilbert spaces. More shortly, one says that Hilbert spaces have the *maximal regularity property*. However, it has been a long open problem whether L_p-spaces for $p \in (1,\infty)$ – or more generally UMD-spaces – have the maximal regularity property as well. In particular, for L_p-spaces this question is of central importance for applications as the Sobolev methods in conjunction with the fixed point theorems may only work for sufficiently large $p > 2$, so the Hilbert space case is not sufficient for many important applications. Here one technical restriction is the validity of various Sobolev embeddings, for example in the example (MCF) one has to work with $p > n+2$.

The maximal regularity problem goes back to H. Brézis and remained open for a long time until it was solved in the seminal work of N.J. Kalton and G. Lancien [KL00]. They showed that the maximal regularity property characterizes Hilbert spaces in the class of all Banach spaces that admit an unconditional Schauder basis and in the class of all separable Banach lattices. In particular, it follows that an L_p-space for $p \in (1,\infty) \setminus \{2\}$ does not have the maximal regularity property. Their approach makes heavy use of abstract methods from the geometric theory of Banach spaces and curiously enough, up to now, no explicit example of a negative generator of a bounded analytic semigroup on L_p for $p \in (1,\infty)$ is known. In fact, the authors of the monograph [DHP03] on maximal regularity write in their introduction:

> So far no specific example of an operator $-A$ in $X = L_p(G)$ is known which generates a bounded analytic C_0-semigroup but $A \notin \mathcal{M}_p(X)$.

The first central goal of this thesis is to develop a general method to construct such explicit counterexamples to the maximal regularity problem. In fact, we will show how one can do this in an arbitrary Banach space not isomorphic to a Hilbert space that admits an unconditional Schauder basis, thereby giving a new more explicit proof of the Kalton–Lancien result (Theorem 2.1.42). In particular, we construct explicit examples on L_p-spaces.

Furthermore, our methods allow us to construct negative generators of *positive* analytic C_0-semigroups on UMD-Banach lattices, e.g. $\ell_p(\ell_q)$ for $p \neq q \notin (1,\infty)$, without maximal regularity. Such counterexamples with additional regularity properties have been out of reach with the old methods. In particular, as there exist positive results for contractive positive semigroups on L_p-spaces (Theorem 4.2.21), this is a first step to close the large gap between positive and negative results.

Furthermore, we apply our new techniques and results to the theory of Schauder bases and to the closedness of the sum problem for sectorial operators.

The Maximal Regularity Extrapolation Problem Although the negative solution of the maximal regularity problem shows that not every negative generator of an analytic C_0-semigroup on L_p for $p \in (1,\infty) \setminus \{2\}$ has maximal regularity, the following important question for applications has remained open.

> Suppose one has given a consistent family of C_0-semigroups $(T_p(t))_{t\geq 0}$ on L_p for $p \in (1,\infty)$; that is, one has $T_{p_1}(t)f = T_{p_2}(t)f$ for all $p_1, p_2 \in (1,\infty)$, all $f \in L_{p_1} \cap L_{p_2}$ and all $t \geq 0$. Further suppose that $(T_2(t))_{t\geq 0}$ is analytic and therefore has maximal regularity. Does it follow that $(T_p(t))_{t\geq 0}$ have maximal regularity for all $p \in (1,\infty)$?

A positive answer to this question would be of fundamental importance for concrete applications as maximal regularity on L_p-spaces can be difficult to verify in practice, whereas analyticity on Hilbert spaces can be established rather easily, for example by form methods. In the positive direction it is known that under the above assumptions the semigroups $(T_p(t))_{t\geq 0}$ are analytic for all $p \in (1,\infty)$. This is classically shown using Stein's interpolation theorem, whereas we will give a new proof that generalizes this extrapolation result to a very broad class of interpolation spaces (Theorem 3.1.10). In fact, the same methods allow us to prove an extrapolation result for maximal regularity for rather general interpolation functors (Theorem 3.2.5), however only under some additional assumption that is for example satisfied when the semigroups satisfy Gaussian or Poisson estimates.

We then show that the general maximal regularity extrapolation problem has a negative answer. In fact, the extrapolation problem behaves in the worst possible way (Theorem 3.2.22): for every interval $I \subset (1,\infty)$ with $2 \in I$ there exists a family of consistent (analytic) C_0-semigroups $(T_p(z))_{z\in\Sigma_{\frac{\pi}{2}}}$ such that $(T_p(z))_{z\in\Sigma_{\frac{\pi}{2}}}$ has maximal regularity if and only if $p \in I$. If we take $I = \{2\}$, then we obtain an example where maximal regularity is satisfied on L_p if and only if $p = 2$.

Generic H^∞-Calculus An important problem is to give a structural description of those semigroups that have maximal regularity on L_p-spaces or even more general Banach spaces as it was done in the Hilbert space case. For example, N.J. Kalton writes in his overview article [Kal01]:

> There are many problems left to resolve. Let us mention just two intriguing questions. [...] The second question is more vague. The maximal regularity conjecture was made because all natural examples on say the spaces L_p have maximal regularity. The problem is to explain why this phenomenon occurs. This would require isolating the properties of a sectorial operator induced by some differential operator which force it to be $\mathcal{R}$-sectorial.

At the moment such a characterization seems to be out of reach even for L_p-spaces. However, there is a powerful positive result on L_p-spaces for $p \in (1,\infty)$ which goes back to L. Weis: the negative generator of a bounded analytic semigroup that is positive and contractive on the real line has maximal regularity. In this thesis we will show the strongest possible variant of this result which is possible to prove using the known techniques (Theorem 4.2.21). This class cannot characterize maximal regularity as one can actually show that they satisfy a strictly stronger property, namely that of a bounded holomorphic functional calculus for the negative semigroup generator. However, we show in our third main result (Corollary 5.5.16) that this class completely characterizes the boundedness of holomorphic functional calculus on L_p-spaces for $p \in (1,\infty)$ modulo the operations of passing through invariant subspace-quotients and applying similarity transforms. So far, there has not been any characterization whatsoever of the boundedness of the holomorphic functional calculus on L_p-spaces. Moreover, our approach seems to be the first application of Pisier's factorization theory for p-completely bounded maps outside the natural habitat of operator space theory and its close connections with abstract harmonic analysis.

Organization of the Thesis The thesis is organized as follows.

Chapter 1 In the first chapter we introduce the necessary mathematical background. It is most convenient to describe maximal regularity as a regularity property of sectorial operators. We start by recalling the definition of sectorial operators and then introduce $\mathcal{R}$-sectorial operators and describe their equivalence with maximal regularity. Furthermore, we introduce other regularity properties of sectorial operators: bounded imaginary powers, bounded H^∞-calculus and the dilation property. Next, we present the various implications between these regularity properties.

Chapter 2 The main goal of this chapter is to give the first explicit counterexamples to the maximal regularity problem on UMD-spaces and to give a new proof of the Kalton–Lancien result (Theorem 2.1.42). In order to motivate our approach, before, we give various other counterexamples between the different regularity properties that are easier to establish. We then construct step by step more general counterexamples to the maximal regularity problem. Each step needs more sophisticated tools from the geometric theory of Banach spaces – of which some are not covered in the literature – and we will develop all results needed along the way. At the end of the chapter we construct positive bounded analytic semigroups on UMD-Banach lattices, for example on $\ell_p(\ell_q)$ for $p \neq q \in (1,\infty)$, without maximal regularity (Theorem 2.1.46). Furthermore, we present in Section 2.1.6 explicit counterexamples to all possible variants of the closedness of the sum problem for sectorial operators, among them one which could now be constructed for the first time explicitly thanks to our new methods. Furthermore, we obtain the new result that on $L_p([0,1])$ there exist Schauder bases that are not $\mathcal{R}$-bases (Theorem 2.1.59).

Chapter 3 In this chapter we give a detailed study of the extrapolation problem for maximal regularity. We first show an extrapolation result for the analyticity of C_0-semigroups (Theorem 3.1.10) for rather general interpolation spaces which is not restricted to the complex interpolation method as the approach using Stein's interpolation theorem. The proof uses a characterization of analytic semigroups which goes back to T. Kato and A. Beurling (Theorem 3.1.6) and for which we give a very elementary proof. Along the way we also obtain a zero-two law for strongly continuous cosine families on UMD-spaces (Theorem 3.1.15), answering positively a conjecture of my supervisor W. Arendt. The used techniques then naturally generalize to the setting of maximal regularity and we obtain an extrapolation theorem for rather general interpolation functors (Theorem 3.2.5).

After that we extend the techniques of Chapter 2 to give counterexamples to the extrapolation problem for maximal regularity in the form described in the first part of the introduction (Theorem 3.2.22).

Chapter 4 In this chapter we give a detailed proof of Weis' result for the boundedness of the H^∞-calculus for the negative generators of bounded analytic C_0-semigroups on L_p-spaces that are positive and contractive on the real line (Theorem 4.2.21). The main tool in the proof is Fendler's dilation theorem (Theorem 4.2.11) for semigroups on L_p-spaces. We extend this theorem to r-contractive C_0-semigroups on closed subspaces of L_p-spaces and show that the r-contractive semigroups are exactly those semigroups that have such a dilation (Theorem 4.2.13). This allows us to prove Weis' result for r-contractive C_0-semigroups on L_p-spaces, a result which is known to experts (see for example [LMX12, Proposition 2.2]). Furthermore, with the extension of Fendler's dilation theorem we obtain a new pointwise ergodic theorem for r-contractive semigroups on closed subspaces of L_p-spaces (Theorem 4.2.20).

Chapter 5 The last chapter is devoted to the proof of the result that the positive contractive analytic C_0-semigroups are generic for the boundedness of the H^∞-calculus which holds for all UMD-Banach lattices (Theorem 5.5.11). In particular, this gives a complete characterization of those sectorial operators on L_p for $p \in (1, \infty)$ that have a bounded H^∞-calculus of angle smaller than $\frac{\pi}{2}$ (Corollary 5.5.16). The proof uses the not broadly known theory of p-matrix normed spaces and p-completely bounded maps which we present along the way in the depth needed for our results. Furthermore, as an application we obtain a renorming result for semigroups on UMD-spaces with a bounded H^∞-calculus (Theorem 5.5.14).

Further Comments At the end of the chapters the reader can find a supplementary section which contains further historical information on the topics covered in the main body of the thesis, further related results with references to the literature and the discussion of some open questions. Moreover, we assume that the reader has some basic knowledge in functional analysis and the theory of partial differential equations, in particular in the theory of C_0-semigroups. Further used concepts and results which are beyond these prerequisites are summarized in the appendices.

This thesis contains material from the published articles [Fac13a], [Fac13b] and [Fac14] and the accepted manuscripts [Faca] and [Facb]. Moreover, the author had the pleasure to work on the article [FN14] in collaboration with T. Nau and on a not yet finished manuscript in collaboration with C. Arhancet and C. Le Merdy. Both works are not part of this thesis.

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Part I

Regularity Properties of Sectorial Operators

1

Sectorial Operators and Their Regularity Properties

In this chapter we introduce sectorial operators as the main objects of our studies. The main interest for us lies in their strong connection with strongly continuous (analytic) semigroups and maximal regularity. We also introduce and present the main known results concerning the regularity properties of being $\mathcal{R}$-sectorial, having a bounded H^∞-calculus, having bounded imaginary powers (BIP) and having a dilation whose structural properties we will investigate in much more detail in the following chapters.

Most of the topics covered in this chapter can be found in more detail in the references [KW04], [DHP03] and [Haa06].

1.1 Sectorial Operators

We make the convention that all Banach spaces considered here and in the rest of the thesis are assumed to be complex unless stated otherwise. For $\omega \in (0, \pi)$ we denote with

$$\Sigma_\omega := \{z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \omega\}$$

an open sector in the complex plane with opening angle ω, where our convention is that $\arg z \in (-\pi, \pi]$.

Definition 1.1.1 (Sectorial Operator). A closed densely defined operator A on a Banach space X is called *sectorial* if there exists an $\omega \in (0, \pi)$ such that

$$\sigma(A) \subset \overline{\Sigma_\omega} \qquad \text{and} \qquad \sup_{\lambda \notin \overline{\Sigma_{\omega+\varepsilon}}} \|\lambda R(\lambda, A)\| < \infty \quad \forall \varepsilon > 0. \tag{S_ω}$$

One defines the *sectorial angle of* A as $\omega(A) := \inf\{\omega : (S_\omega) \text{ holds}\}$.

Remark 1.1.2. The definition of sectorial operators is not universal in the literature. Some authors require a sectorial operator to be injective and to have dense range as well. We will omit these condition from our definition and add explicitly one or both to our assumptions when necessary. Notice that for a sectorial operator A on a Banach space one always has $N(A) \cap \overline{R(A)} = 0$. In particular, if A has dense range, A is injective as well.

Let A be a densely defined operator on some Banach space X. Then it is well-known that $-A$ generates an analytic C_0-semigroup if and only if A is sectorial with $\omega(A) < \frac{\pi}{2}$. Moreover, if $-A$ is the generator of a C_0-semigroup, then A is sectorial with $\omega(A) \le \frac{\pi}{2}$. However, there exist sectorial operators with sectorial angle equal to $\frac{\pi}{2}$ that do not generate C_0-semigroups.

1.2 $\mathcal{R}$-Sectorial Operators and Maximal Regularity

In the study of maximal L_p-regularity a stronger condition for sectorial operators plays a central role. This condition is nowadays called $\mathcal{R}$-sectoriality. We now give the necessary definitions and shortly explain the connection with maximal L_p-regularity. We start with $\mathcal{R}$-boundedness. This notion can already be implicitly found in the work [Bou83] and is for the first time explicitly defined in [BG94, Definition 2.4].

Let $r_k(t) := \operatorname{sign}\sin(2^k\pi t)$ be the k-th *Rademacher function*. Then on the probability space $([0,1], \mathcal{B}([0,1]), \lambda)$, where $\mathcal{B}([0,1])$ is the Borel σ-algebra on $[0,1]$ and λ denotes the Lebesgue measure, the Rademacher functions form an independent identically distributed family of random variables satisfying $\mathbb{P}(r_k = \pm 1) = \frac{1}{2}$.

Definition 1.2.1 ($\mathcal{R}$-Boundedness). A family of operators $\mathcal{T} \subset \mathcal{B}(X)$ on a Banach space X is called $\mathcal{R}$-*bounded* if there exists $p \in [1,\infty)$ and a finite constant $C_p \geq 0$ such that for each finite subset $\{T_1, \ldots, T_n\}$ of $\mathcal{T}$ and arbitrary $x_1, \ldots, x_n \in X$ one has

$$\left\| \sum_{k=1}^{n} r_k T_k x_k \right\|_{L_p([0,1];X)} \leq C_p \left\| \sum_{k=1}^{n} r_k x_k \right\|_{L_p([0,1];X)}. \tag{1.1}$$

The best constant C_p such that (1.1) holds is denoted by $\mathcal{R}_p(\mathcal{T})$.

The property of being $\mathcal{R}$-bounded (but not the constant C_p) is independent of p by the Kahane–Khintchine inequality (Theorem A.3.2). The $\mathcal{R}$-bound behaves in many ways similar to a classical norm. For example, if $\mathcal{S}$ is a second family of operators, one sees that (if the operations make sense)

$$\mathcal{R}_p(\mathcal{T} + \mathcal{S}) \leq \mathcal{R}_p(\mathcal{T}) + \mathcal{R}_p(\mathcal{S}), \qquad \mathcal{R}_p(\mathcal{T}\mathcal{S}) \leq \mathcal{R}_p(\mathcal{T})\mathcal{R}_p(\mathcal{S}).$$

Note that by the orthogonality of the Rademacher functions a family $\mathcal{T} \subset \mathcal{B}(H)$ for some Hilbert space H is $\mathcal{R}$-bounded if and only if $\mathcal{T}$ is bounded in operator norm. In fact, an $\mathcal{R}$-bounded subset $\mathcal{T} \subset \mathcal{B}(X)$ for a Banach space X is always clearly norm-bounded and one can show that the converse holds if and only if X is isomorphic to a Hilbert space [AB02, Proposition 1.13].

Kahane's contraction principle is a basic tool for $\mathcal{R}$-boundedness [KW04, Proposition 2.5].

Proposition 1.2.2 (Kahane's Contraction Principle). *Let X be a Banach space and $x_1, \ldots, x_n \in X$ for $n \in \mathbb{N}$. Then for arbitrary complex numbers $a_1, \ldots, a_n$ one has*

$$\left\| \sum_{k=1}^{n} r_k a_k x_k \right\| \leq 2 \sup_{k=1,\ldots,n} |x_k| \left\| \sum_{k=1}^{n} r_k x_k \right\|.$$

We now list some basic criteria for $\mathcal{R}$-boundedness. For their proofs see [KW04, Example 2.16] and [KKW06, Proposition 3.5].

Proposition 1.2.3. *Let X, Y be two Banach spaces. Then the following hold.*

(a) *Let $N\colon \Sigma_{\theta'} \to \mathcal{B}(X,Y)$ be analytic and $\{N(\lambda) : \lambda \in \partial\Sigma_\theta, \lambda \neq 0\}$ be $\mathcal{R}$-bounded for some $\theta \in (0,\theta')$. Then for every $\theta_1 \in (0,\theta)$ the sets*

$$\{N(\lambda) : \lambda \in \Sigma_\theta\} \qquad \text{and} \qquad \{\lambda N'(\lambda) : \lambda \in \Sigma_{\theta_1}\}$$

are $\mathcal{R}$-bounded.

(b) *Let $\mathcal{T} \subset \mathcal{B}(X,Y)$ be $\mathcal{R}$-bounded. If X has non-trivial type, then $\mathcal{T}^* = \{T^* : T \in \mathcal{T}\} \subset \mathcal{B}(Y^*,X^*)$ is $\mathcal{R}$-bounded.*

Now, if one replaces norm-boundedness by $\mathcal{R}$-boundedness, one obtains the definition of an $\mathcal{R}$-sectorial operator.

Definition 1.2.4 ($\mathcal{R}$-Sectorial Operator). A sectorial operator A on a Banach space X is called *$\mathcal{R}$-sectorial* if for some $\omega > \omega(A)$ one has

$$\mathcal{R}\{\lambda R(\lambda, A) : \lambda \notin \overline{\Sigma_\omega}\} < \infty. \tag{$\mathcal{R}_\omega$}$$

One defines the *$\mathcal{R}$-sectorial angle* as $\omega_R(A) := \inf\{\omega : (\mathcal{R}_\omega) \text{ holds}\}$. If A is not $\mathcal{R}$-sectorial, we set $\omega_R(A) := \infty$.

By definition, one has $\omega(A) \le \omega_R(A)$. However, there are examples of sectorial operators A on closed subspaces of L_p for $p \in (1,2)$ for which the strict inequality $\omega(A) < \omega_R(A) < \infty$ holds (such an example was found by N.J. Kalton and is apparently contained in the unpublished manuscript [KWa]). In Hilbert spaces an operator is sectorial if and only if it is $\mathcal{R}$-sectorial and the equality $\omega(A) = \omega_R(A)$ does always hold. In general Banach spaces $\mathcal{R}$-sectorial operators clearly are sectorial, the converse question whether every sectorial operator is $\mathcal{R}$-sectorial will be addressed in the following.

One can show with the usual techniques that a sectorial operator A on some Banach space is $\mathcal{R}$-sectorial with $\omega_R(A) < \frac{\pi}{2}$ if and only if $-A$ generates an analytic C_0-semigroup which is $\mathcal{R}$-bounded on some sector Σ_δ ($\delta > 0$). We therefore use the following terminology in analogy to analytic semigroups.

Definition 1.2.5 ($\mathcal{R}$-Analytic Semigroup). An analytic semigroup $(T(z))_{z\in\Sigma_{\tilde{\delta}}}$ on a Banach space X is called *$\mathcal{R}$-analytic* of angle δ for some $\delta \in (0,\tilde{\delta})$ if

$$\mathcal{R}\{T(z) : z \in \Sigma_\delta, |z| \le 1\} < \infty.$$

More precisely, one has the following equivalence between $\mathcal{R}$-analytic semigroups and $\mathcal{R}$-sectorial operators. For proofs we refer to [KW04, Theorem 1.11 and Remarks 1.12] and [KW04, Remark 2.22a)].

Proposition 1.2.6. *Let A be a linear operator on some Banach space X. Then the following assertions are equivalent.*

(i) *For some $\omega \geq 0$ the operator $A + \omega$ is $\mathcal{R}$-sectorial with $\omega_R(A) < \frac{\pi}{2}$.*

(ii) *There exists an $\alpha_0 > 0$ such that $\mathcal{R}\{\alpha R(i\alpha) : |\alpha| > \alpha_0\} < \infty$.*

(iii) *$-A$ is the generator of an $\mathcal{R}$-analytic semigroup.*

In this case the supremum of those δ for which $(T(z))_{z\in\Sigma_\delta}$ is $\mathcal{R}$-analytic is given by $\frac{\pi}{2} - \omega_R(A)$.

In order to give a first impression, we now give a first very elementary example of a contractive C_0-semigroup which is not $\mathcal{R}$-bounded on the real line. Notice, however, that the example is not analytic.

Example 1.2.7. Let $(U(t))_{t\in\mathbb{R}}$ be the shift $U(t)f = f(\cdot + t)$ on $L_p(\mathbb{R})$ for $p \in [1, \infty)$. Then $(U(t))_{t\in\mathbb{R}}$ is a C_0-group of positive isometries. Observe that on the one hand for the functions $f_k = \mathbb{1}_{[k,k+1]}$ for $k \in \mathbb{N}$ one has for all $n \in \mathbb{N}$

$$\left\| \sum_{k=1}^{n} r_k U(k) f_k \right\| = \left\| \sum_{k=1}^{n} r_k \, \mathbb{1}_{[0,1]} \right\| \simeq \int_0^1 \left| \sum_{k=1}^{n} r_k(t) \right| dt \simeq n^{1/2}$$

by the Khintchine inequality (Theorem A.3.1). On the other hand, one has for all $n \in \mathbb{N}$

$$\left\| \sum_{k=1}^{n} r_k f_k \right\| = \left\| \mathbb{1}_{[1,n+1]} \right\|_{L_p(\mathbb{R})} = n^{1/p}.$$

This shows that $\{U(t) : t \in \mathbb{R}\}$ is not $\mathcal{R}$-bounded for $p > 2$. Analogous calculations for $f_n = \mathbb{1}_{[0,1]}$ show that $\{U(t) : t \in \mathbb{R}\}$ is not $\mathcal{R}$-bounded for $p < 2$ either.

Maximal L_p-Regularity The main interest for $\mathcal{R}$-sectorial operators comes from their close connection with the concept of maximal regularity.

Definition 1.2.8 (Maximal Regularity). The generator $-A$ of a C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X has (p,T)-*maximal regularity* (for $T > 0$ and $p \in (1,\infty)$) if for all $f \in L_p([0,T];X)$ the mild solution $u(t) = \int_0^t T(t-s)f(s)\,ds$ of the inhomogeneous abstract Cauchy problem

$$\begin{cases} \dot{u}(t) + A(u(t)) = f(t) \\ \qquad\qquad u(0) = 0 \end{cases}$$

satisfies $u \in W_p^1([0,T];X) \cap L_p([0,T];D(A))$. As a shorthand we will sometimes also say that the semigroup $(T(t))_{t\geq 0}$ has maximal regularity.

One can show that this property is independent of $p \in (1,\infty)$ and $T \in (0,\infty)$ (see [Dor93, Theorem 4.2]). Therefore one simply speaks of *maximal regularity*. As explained in the introduction maximal regularity has become an important tool in the study of non-linear partial differential equations in the past decade. The study of maximal regularity can at least be traced back to the fundamental work of G. Da Prato and P. Grisvard [DPG75]. Both from the point of view of applications and the abstract theory the following equivalent characterization by L. Weis [Wei01b, Theorem 4.2] which was independently also found by N.J. Kalton [Wei01b, p. 737] (and is based on the preliminary works [CdPSW00] and [CP01]) is extremely useful. Its proof relies on an operator-valued variant of the Mikhlin multiplier theorem. For the definition and main results on UMD-spaces we refer to Appendix A.3.4.

Theorem 1.2.9. *Let X be a Banach space and $-A$ the generator of an analytic C_0-semigroup $(T(z))_{z\in\Sigma}$ on X. Then the following hold:*

(a) *If $-A$ has maximal regularity, then A is $\mathcal{R}$-sectorial with $\omega_R(A) < \frac{\pi}{2}$.*

(b) *Conversely, on a UMD-space X, $-A$ has maximal regularity if A is an $\mathcal{R}$-sectorial operator with $\omega_R(A) < \frac{\pi}{2}$.*

Later we will need the following weaker condition.

Remark 1.2.10. It is shown in the original proof of Weis that condition (b) of Theorem 1.2.9 can be replaced by the following weaker condition: There exists a $C \geq 0$ such that for all $a \in \mathbb{R}$ with $|a| \in [1,2]$ one has

$$\mathcal{R}\{ia2^n R(ia2^n, A) : n \in \mathbb{Z}\} \leq C.$$

More generally, suppose that for a sectorial operator A there exists a $C \geq 0$ and a $\theta \in (0,\pi)$ such that for all $a \in \mathbb{R}$ with $|a| \in [1,2]$ one has

$$\mathcal{R}\{ae^{i\theta}2^n R(ae^{i\theta}2^n, A) : n \in \mathbb{Z}\} \leq C.$$

Then A is already $\mathcal{R}$-sectorial with $\omega_R(A) \leq \theta$. This follows from the proofs of [Wei01b, Theorem 4.2] and [Wei01b, Corollary 3.7].

Note that in particular $\mathcal{R}$-analyticity is always necessary for maximal regularity and that on a Hilbert space a sectorial operator A has maximal regularity if and only if $\omega(A) < \frac{\pi}{2}$. The last point gives a very satisfying L_2-theory. From the point of view of concrete applications the L_2-theory is not always sufficient as non-linearties may only be treated for sufficiently large $p > 2$, e.g. for the equation governing the mean curvature flow (MCF) from the introduction one needs $p > n+2$. It is therefore very desirable to develop a general L_p-theory. A question attributed to H. Brézis and presented in [CL86] asks whether the Hilbert space result generalizes to the L_p-case. This is the so-called *maximal regularity problem*. We formalize this problem by introducing the following terminology.

Definition 1.2.11. A Banach space X has the *maximal regularity property* or shortly *(MRP)* if every negative generator of an analytic C_0-semigroup on X has maximal regularity.

Using this terminology, we have just seen the following positive result.

Theorem 1.2.12. *Every Hilbert space has the maximal regularity property (MRP).*

Now, the general maximal regularity problem reads as follows.

Problem 1.2.13. Which Banach spaces do have the maximal regularity property (MRP)? In particular, do the reflexive L_p-spaces have (MRP)?

So the maximal regularity problem asks on UMD-spaces whether there exists a sectorial operator A with $\omega(A) < \frac{\pi}{2}$ which is not $\mathcal{R}$-sectorial or which is $\mathcal{R}$-sectorial with $\omega_R(A) > \frac{\pi}{2}$. In [KL00], N.J. Kalton and G. Lancien give a very satisfying answer to the maximal regularity problem. Namely, in the class of all Banach spaces admitting an unconditional Schauder basis only Hilbert spaces have the maximal regularity property (MRP). In particular, a reflexive L_p-space has (MRP) if and only if $p = 2$. Therefore for $p \in (1, \infty) \setminus \{2\}$ there exists a sectorial operator on L_p which does not have maximal regularity. However, Kalton and Lancien's approach only shows the pure existence of such an operator without giving an explicit example. Later, we will present a new approach to Kalton and Lancien's result which is explicit enough to give concrete counterexamples.

Notice, however, that the maximal regularity property (MRP) does not characterize Hilbert spaces as one has the following result.

Theorem 1.2.14. *The spaces ℓ_∞ and $L_\infty[0,1]$ have the maximal regularity property (MRP).*

Proof. By a celebrated result of H.P. Lotz [Lot85, Theorem 3] the generator of every C_0-semigroup on these spaces is bounded. In particular, every strongly continuous analytic semigroup on these spaces has maximal regularity. □

1.3 Bounded H^∞-Calculus for Sectorial Operators

One can use the integral representation of analytic functions to associate, given a sectorial operator, bounded operators to certain bounded analytic functions. This gives a basic functional calculus which for some sectorial operators can be extended to the space of all bounded analytic functions on some sector. The boundedness of this functional calculus has become an important tool in the study of sectorial operators and its study was initiated by the pioneering works [McI86] and [CDMY96]. We now give the definitions and present its main properties.

Definition 1.3.1. For $\theta \in (0,\pi)$ we define

$$\begin{aligned} H_0^\infty(\Sigma_\theta) &:= \left\{ f\colon \Sigma_\theta \to \mathbb{C} \text{ analytic} : |f(\lambda)| \le C \frac{|\lambda|^\varepsilon}{(1+|\lambda|)^{2\varepsilon}} \text{ on } \Sigma_\theta \text{ for } C, \varepsilon > 0 \right\}, \\ H^\infty(\Sigma_\theta) &:= \{ f\colon \Sigma_\theta \to \mathbb{C} \text{ analytic and bounded} \}, \end{aligned}$$

Moreover, we set $H^\infty(\Sigma_{\theta+}) := \cup_{\theta' \in (\theta,\pi)} H^\infty(\Sigma_{\theta'})$. Then endowed with the norm $\|f\|_{H^\infty(\Sigma_\theta)} := \sup_{z\in\Sigma_\theta} |f(z)|$ the algebras $H_0^\infty(\Sigma_\theta)$ and $H^\infty(\Sigma_\theta)$ are normed respectively Banach algebras.

Now, let A be a sectorial operator on some Banach space X and let $\theta > \omega(A)$. Then for $f \in H_0^\infty(\Sigma_\theta)$ one defines

$$f(A) = \frac{1}{2\pi i} \int_{\partial\Sigma_{\theta'}} f(\lambda) R(\lambda, A)\, d\lambda \qquad (\omega(A) < \theta' < \theta).$$

This is well-defined by the growth estimate on f and by the invariance of the contour integral and induces an algebra homomorphism $H_0^\infty(\Sigma_\theta) \to \mathcal{B}(X)$.

Let us now additionally assume for the rest of the paragraph that A is injective and has dense range. For $n \ge 2$ let $\rho_n(\lambda) := \frac{n}{n+\lambda} - \frac{1}{1+\lambda n} \in H_0^\infty(\Sigma_\theta)$. Then one has $\rho_n(A) = n(n+A)^{-1} - \frac{1}{n}(\frac{1}{n}+A)^{-1}$ for $n \in \mathbb{N}$, which is a uniformly bounded family because of the sectoriality of A. Moreover, one has $(f\rho_n)(\lambda) \to f(\lambda)$ for all $\lambda \in \Sigma_\theta$ and $\rho_n(A)x \to x$ for all $x \in X$. Then one can extend the functional calculus by letting

$$f(A)x := \lim_{n\to\infty} (f\rho_n)(A)x,$$

provided the above limit exists for all $x \in X$. A particular interesting case is given when the above procedure works for all $f \in H^\infty(\Sigma_\theta)$. One can show that this is exactly the case when the homomorphism $H_0^\infty(\Sigma_\theta) \to \mathcal{B}(X)$ is bounded. This leads to the next definition which can be made independently of any additional assumptions on A.

Definition 1.3.2 (Bounded H^∞-Calculus). A sectorial operator A is said to have a *bounded $H^\infty(\Sigma_\theta)$-calculus* for some $\theta \in (\omega(A),\pi)$ if the homomorphism $f \mapsto f(A)$ from $H_0^\infty(\Sigma_\theta)$ to $\mathcal{B}(X)$ is bounded. The infimum of the θ for which these homomorphisms are bounded is denoted by $\omega_{H^\infty}(A)$. We say that A has a *bounded H^∞-calculus* if A has a bounded $H^\infty(\Sigma_\theta)$-calculus for some $\theta \in (0,\pi)$. If A does not have a bounded H^∞-calculus, we let $\omega_{H^\infty}(A) := \infty$.

Let A be a sectorial operator with a bounded $H^\infty(\Sigma_\theta)$-calculus which has dense range. We have just explained that then the holomorphic functional calculus defined above can be extended to a Banach algebra homomorphism $u\colon H^\infty(\Sigma_\theta) \to \mathcal{B}(X)$. Although $H_0^\infty(\Sigma_\theta)$ is not dense in $H^\infty(\Sigma_\theta)$, one can show that the above extension $u\colon H^\infty(\Sigma_\theta) \to \mathcal{B}(X)$ is unique under the following mild assumptions.

(a) $u\colon H^\infty(\Sigma_\theta) \to \mathcal{B}(X)$ is linear and multiplicative.

(b) One has $u((\lambda - \cdot)^{-1}) = R(\lambda, A)$ for all $\lambda \notin \overline{\Sigma_\theta}$.

(c) For functions $f_n, f \in H^\infty(\Sigma_\theta)$ that satisfy $f_n(\lambda) \to f(\lambda)$ for all $\lambda \in \Sigma_\theta$ and $\sup_{n\in\mathbb{N}} \|f_n\|_{H^\infty(\Sigma_\theta)} < \infty$ one has $f_n(A)x \to f(A)x$ for all $x \in X$.

If $\omega(A) < \frac{\pi}{2}$, one can plug $f_z\colon \lambda \mapsto e^{-\lambda z} - 1/(1+\lambda)$ into the functional calculus for $z \in \Sigma_{\pi/2-\omega(A)}$. For such z one has $f_z(A) = T(z) - (1+A)^{-1}$, which means that the functional calculus reproduces the bounded analytic C_0-semigroup generated by $-A$. Although the extension of the functional calculus is only defined if A has dense range, one can reduce to this case if X is reflexive. Indeed, in this case one can always decompose a sectorial operator as

$$A = \begin{pmatrix} A_{00} & 0 \\ 0 & 0 \end{pmatrix}$$

with respect to the decomposition $X = \overline{R(A)} \oplus N(A)$ such that A_{00} is an injective sectorial operator with dense range on $\overline{R(A)}$. In this case $x \mapsto \lim_{n\to\infty} -\frac{1}{n} R(-\frac{1}{n}, A)x$ is the projection onto the null space $N(A)$.

Note that it follows directly from the definition of the functional calculus that one always has $\omega(A) \le \omega_{H^\infty}(A)$ for a sectorial operator A. Moreover, Kalton's example for the inequality $\omega(A) < \omega_R(A)$ on subspaces of L_p in its original formulation even shows that there exist sectorial operators A for which the strict inequalities $\omega(A) < \omega_{H^\infty}(A) < \infty$ hold. For a published example of this strict inequality see also [Kal03].

There is a close connection to $\mathcal{R}$-boundedness and $\mathcal{R}$-sectorial operators as well. For the geometric properties of Banach spaces used in the following theorems we refer to Appendix A.3. A proof of the following theorem can be found in [KW04, Theorem 12.8].

Theorem 1.3.3. *Let X be a Banach space with Pisier's property (α) and A a sectorial operator on X with a bounded $H^\infty(\Sigma_\theta)$-calculus for some $\theta \in (0, \pi)$. Then for all $\theta' \in (\theta, \pi)$ and all $C \ge 0$ the set*

$$\{f(A) : \|f\|_{H^\infty(\Sigma_{\theta'})} \le C\}$$

is $\mathcal{R}$-bounded.

Note that this also implies under the above assumptions that a sectorial operator with a bounded H^∞-calculus is $\mathcal{R}$-sectorial. More generally, the following holds [KW01, Theorem 5.3].

Theorem 1.3.4. *Let X be a Banach space with property (Δ). Let A be a sectorial operator on X with a bounded H^∞-calculus. Then A is $\mathcal{R}$-sectorial and one has $\omega_R(A) = \omega_{H^\infty}(A)$.*

In particular it follows that a sectorial operator on a Hilbert space with a bounded H^∞-calculus satisfies $\omega(A) = \omega_{H^\infty}(A)$.

We now give an elementary example of sectorial operators with a bounded H^∞-calculus and an important criterion for the boundedness of the functional calculus.

Example 1.3.5. Let A be a bounded invertible sectorial operator. Then A has a bounded H^∞-calculus with $\omega_{H^\infty}(A) = \omega(A)$. Indeed, let $\theta > \omega(A)$ and $f \in H_0^\infty(\Sigma_\theta)$. Since the spectrum of A is bounded and A is invertible, there exist $\delta > 0$ and $R > 0$ such that $\sigma(A) \subset \Omega := \overline{\Sigma_\theta} \cap B(0,R) \cap \overline{B}(0,\delta)^c$. Then, by Cauchy's integral theorem we have

$$f(A) = \frac{1}{2\pi i} \int_{\partial\Sigma_\theta} f(\lambda) R(\lambda, A)\, d\lambda = \frac{1}{2\pi i} \int_{\partial\Omega} f(\lambda) R(\lambda, A)\, d\lambda.$$

Now, the norm of the resolvent $\|R(\lambda, A)\|$ is bounded by a constant $M \geq 0$ on the compact set $\partial\Omega$. Therefore one has

$$\|f(A)\| \leq M\ell(\partial\Omega) \sup_{\lambda\in\partial\Omega} |f(\lambda)| \leq M\ell(\partial\Omega) \|f\|_{H^\infty(\Sigma_\theta)}.$$

For angles bigger than $\frac{\pi}{2}$ one can rewrite the functional calculus with the help of the Laplace transform (denoted by $\mathcal{L}$) in terms of the semigroup. This alternative representation is useful for the transference techniques to be applied later in this section. We follow the presentation in [LM99b].

Lemma 1.3.6. *Let $-A$ be the generator of a bounded C_0-semigroup $(T(t))_{t\geq 0}$ on some Banach space X. Let $\theta \in (\frac{\pi}{2}, \pi)$ and $f \in H_0^\infty(\Sigma_\theta)$. Then there exists a unique $b \in L_1(\mathbb{R}_+)$ such that*

$$f = \mathcal{L}b \qquad \textit{and} \qquad f(A)x = \int_0^\infty b(t) T(t)x\, dt \quad \forall x \in X.$$

Proof. Let $\theta' \in (\frac{\pi}{2}, \theta)$. We define $b(t) = -\int_{\partial\Sigma_{\sigma'}} f(\lambda) e^{\lambda t}\, d\lambda$, which converges for all $t > 0$. Moreover, one has by Tonelli's theorem and the fact that $\left|\frac{\lambda}{\operatorname{Re}\lambda}\right|$ is bounded by some constant $M > 0$ on $\partial\Sigma_{\theta'}$ that

$$\int_0^\infty |b(t)|\, dt \leq \int_0^\infty \int_{\partial\Sigma_{\theta'}} \left|f(\lambda) e^{\lambda t}\right| |d\lambda|\, dt = \int_{\partial\Sigma_{\theta'}} |f(\lambda)| \int_0^\infty e^{\operatorname{Re}\lambda t}\, dt\, |d\lambda|$$

$$= \int_{\partial\Sigma_{\theta'}} \left|\frac{f(\lambda)}{\lambda}\right| \left|\frac{\lambda}{\operatorname{Re}\lambda}\right| |d\lambda| \leq M \int_{\partial\Sigma_{\theta'}} \left|\frac{f(\lambda)}{\lambda}\right| |d\lambda| < \infty.$$

From this it follows directly that b is measurable and that $b \in L_1(\mathbb{R}_+)$. Moreover, for any $z \in \overline{\Sigma_{\frac{\pi}{2}}}$ we have by Cauchy's integral formula and Fubini's theorem

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{\partial\Sigma_{\theta'}} \frac{f(\lambda)}{\lambda - z} \, d\lambda = -\frac{1}{2\pi i} \int_{\partial\Sigma_{\theta'}} f(\lambda) \int_0^\infty e^{(\lambda - z)t} \, dt \, d\lambda \\ &= -\int_0^\infty e^{-zt} \frac{1}{2\pi i} \int_{\partial\Sigma_{\theta'}} f(\lambda) e^{\lambda t} \, d\lambda \, dt = \int_0^\infty b(t) e^{-zt} \, dt = (\mathcal{L}b)(z). \end{aligned}$$

This shows that f is the Laplace transform of b. In a similar fashion one obtains for all $x \in X$ by the resolvent formula

$$\begin{aligned} f(A)x &= \frac{1}{2\pi i} \int_{\partial\Sigma'_\theta} f(\lambda) R(\lambda, A)x \, d\lambda = -\frac{1}{2\pi i} \int_{\partial\Sigma'_\theta} f(\lambda) \int_0^\infty e^{\lambda t} T(t)x \, dt \, d\lambda \\ &= -\int_0^\infty T(t)x \frac{1}{2\pi i} \int_{\partial\Sigma_{\theta'}} e^{\lambda t} f(\lambda) \, d\lambda \, dt = \int_0^\infty b(t) T(t)x \, dt. \end{aligned}$$ □

With the above description of the functional calculus we obtain the following useful criterion.

Proposition 1.3.7. *Let $-A$ be the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on some Banach space X. Then for $\theta \in (\frac{\pi}{2}, \pi)$ the following are equivalent:*

(i) *A has a bounded $H^\infty(\Sigma_\theta)$-calculus.*

(ii) *There is a constant $C \geq 0$ such that for any $b \in L_1(\mathbb{R}_+)$ whose Laplace transform is in $H_0^\infty(\Sigma_\theta)$ one has*

$$\left\| \int_0^\infty b(t) T(t) \, dt \right\|_{\mathcal{B}(X)} \leq C \, \|\mathcal{L}b\|_{H^\infty(\Sigma_\theta)}.$$

Proof. This follows directly from Lemma 1.3.6. □

As a first application of the criterion we obtain the boundedness of the H^∞-calculus for the negative generators of (vector)-valued shift semigroups.

Example 1.3.8. Let X be a UMD-space and $p \in (1, \infty)$. We consider the shift group $(V(t))_{t\in\mathbb{R}}$ on $L_p(\mathbb{R}; X)$ defined by $(V(t)f)(s) = f(s-t)$. Now, for $\theta \in (\frac{\pi}{2}, \pi)$ choose $b \in L_1(\mathbb{R}_+)$ such that its Laplace transform $\mathcal{L}b$ lies in $H_0^\infty(\Sigma_\theta)$. Then one has for $f \in L_p(\mathbb{R}; X)$

$$\int_0^\infty b(t) V(t) f \, dt = \int_0^\infty b(t) f(\cdot - t) \, dt = b * f,$$

the convolution of b with f. It follows that the left hand side is a Fourier multiplier operator for the multiplier

$$m(t) = (\mathcal{F}b)(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} b(x) e^{-itx}\, dx = \frac{1}{\sqrt{2\pi}} (\mathcal{L}b)(it).$$

We now verify that m is indeed a bounded multiplier, i.e. the Fourier multiplier operator can be extended to a bounded operator on $L_p(\mathbb{R}; X)$. Observe that one clearly has $\sup_{t\in\mathbb{R}} |m(t)| \le (2\pi)^{-1/2} \|\mathcal{L}b\|_{H^\infty(\Sigma_\theta)}$. Further, it follows from Cauchy's integral formula that for $\theta' \in (\frac{\pi}{2}, \theta)$ one has for $t \neq 0$

$$\begin{aligned} \left|tm'(t)\right| &= \frac{1}{\sqrt{2\pi}} \left|it(\mathcal{L}b)'(it)\right| \le \frac{t}{(2\pi)^{3/2}} \int_{\partial\Sigma_{\theta'}} \left|\frac{(\mathcal{L}b)(\lambda)}{(\lambda - it)^2}\right| d\,|\lambda| \\ &\le \frac{t}{(2\pi)^{3/2}} \|\mathcal{L}b\|_{H^\infty(\Sigma_\theta)} \int_{\partial\Sigma_{\theta'}} |\lambda - it|^{-2}\, d\,|\lambda|. \end{aligned}$$

Let $t_0 = 2t\sin^{-1}(\theta')$. Note that the last integral can be roughly estimated as

$$\begin{aligned} \int_{\partial\Sigma_{\theta'}} |\lambda - it|^{-2}\, d\,|\lambda| &\le 2\int_0^{t_0} (\sin(\theta' - \frac{\pi}{2})t)^{-2}\, ds + 2\int_{t_0}^{\infty} (\cos(\theta' - \frac{\pi}{2})s - t)^{-2}\, ds \\ &= 4t^{-1}\sin^{-1}(\theta')\cos^{-2}(\theta') + 2\sin^{-1}(\theta') \int_{2t}^{\infty} (s-t)^{-2}\, ds \\ &= 4t^{-1}\sin^{-1}(\theta')\cos^{-2}(\theta') + 2\sin^{-1}(\theta')t^{-1}, \end{aligned}$$

where the integrand in the first sum is estimated by the minimal distance to the point it and the integrand in the second sum by the imaginary part.

Altogether we have shown that there exists a constant $C_\theta \ge 0$ such that

$$\sup_{t\in\mathbb{R}} |m(t)| + |tm'(t)| \le C \|\mathcal{L}b\|_{H^\infty(\Sigma_\theta)}.$$

Hence, it follows from Zimmermann's extension of the classical Mikhlin multiplier theorem to UMD-spaces [Zim89, Proposition 3] that m is a bounded Fourier multiplier and that for some constant $D_\theta \ge 0$ one has

$$\left\| \int_0^\infty b(t) V(t)\, dt \right\|_{\mathcal{B}(X)} \le D_\theta \|\mathcal{L}b\|_{H^\infty(\Sigma_\theta)}.$$

By Proposition 1.3.7 and by the fact $\omega_{H^\infty}(C) \ge \omega(C)$, the negative generator $C = \frac{d}{dt} \otimes \mathrm{Id}_X$ of $(V(t))_{t\in\mathbb{R}}$ has a bounded H^∞-calculus with $\omega_{H^\infty}(C) = \frac{\pi}{2}$.

The criterion can also be used to show the boundedness of the functional calculus for bounded C_0-groups on UMD-spaces. The proof uses the following transference result, a technique which goes back to R.R. Coifman and G. Weiss [CW76]. We do not prove this result for the moment as we will obtain a more general transference result later on in Theorem 5.3.1. Again, $(V(t))_{t\in\mathbb{R}}$ is the vector-valued shift group defined by $(V(t)f)(s) = f(s-t)$.

Theorem 1.3.9 (Transference Principle). *Let $(U(t))_{t\in\mathbb{R}}$ be a C_0-group on a Banach space X such that $M = \sup_{t\in\mathbb{R}} \|U(t)\| < \infty$. Then for all $p \in [1,\infty)$ and all $b \in L_1(\mathbb{R})$ one has*

$$\left\| \int_{\mathbb{R}} b(t)U(t)\,dt \right\|_{\mathcal{B}(X)} \le M^2 \left\| \int_{\mathbb{R}} b(t)V(t)\,dt \right\|_{\mathcal{B}(L_p(\mathbb{R};X))}.$$

Now, the boundedness of the functional calculus is a direct consequence. This application of the transference principle to the boundedness of the H^∞-calculus goes back to [Cow83] in the scalar case and to [HP98] in the vector-valued case.

Corollary 1.3.10. *Let $(U(t))_{t\in\mathbb{R}}$ be a bounded C_0-group on a UMD-space. Then its negative infinitesimal generator A has a bounded H^∞-calculus with $\omega_{H^\infty}(A) \le \frac{\pi}{2}$.*

Proof. By the transference principle (Theorem 1.3.9), the problem can be reduced to the case of the vector-valued shift group on $L_2(\mathbb{R};X)$ which has been treated in Example 1.3.8. □

Extended Functional Calculus and Fractional Powers We now shortly explain how the functional calculus can be extended from bounded analytic functions to polynomially bounded analytic functions. This extension is necessary if one wants to work with fractional powers of logarithms of sectorial operators. Of course, one can not expect to obtain bounded operators.

Let $\rho(\lambda) = \frac{\lambda}{(1+\lambda)^2}$. We consider functions whose growth at 0 and ∞ can be estimated by multiples of $|\lambda|^{-\alpha}$ and $|\lambda|^{\alpha}$ respectively.

Definition 1.3.11. Let $\alpha \ge 0$ and $\theta \in (0,\pi)$. Denote by $H_\alpha(\Sigma_\theta)$ the space of all analytic functions $f : \Sigma_\theta \to \mathbb{C}$ for which

$$\sup\{|\rho(\lambda)|^{\alpha}\,|f(\lambda)| : \lambda \in \Sigma_\theta\} < \infty.$$

Note that if $f \in H_\alpha(\Sigma_\theta)$ and $k \in \mathbb{N}$ with $k > \alpha$, one can write $f(\lambda) = \rho(\lambda)^{-k}(\rho^k f)(\lambda)$, where the second factor lies in $H_0^\infty(\Sigma_\theta)$. For such functions we can extend the functional calculus for sectorial operators in the following way.

Definition 1.3.12. Let A be a sectorial operator with dense range and $\theta \in (\omega(A),\pi)$. For $f \in H_\alpha(\Sigma_\theta)$ choose $k \in \mathbb{N}$ with $k > \alpha$ and let

$$\begin{aligned} f(A) &= \rho(A)^{-k}(\rho^k f)(A) \\ D(f(A)) &= \{x \in X : (\rho^k f)(A)x \in D(\rho(A)^{-k}).\end{aligned}$$

One then shows that A is a well-defined operator. For this and further details see [KW04, Appendix B].

In particular, one can define fractional powers of sectorial operators with dense range. These have the following natural properties. Proofs of the first two assertions can be found in [KW04, Theorem 15.16] and [KKW06, Proposition 3.4], whereas the third follows from the composition formula proved in [KW04, Proposition 15.11].

Proposition 1.3.13. *Let A be a sectorial operator with dense range on a Banach space X. Then the following hold.*

(a) *For $\alpha \in (0, \frac{2\pi}{\omega(A)})$ the operator A^α is sectorial with $\omega(A^\alpha) = \alpha\omega(A)$. Further, one has $(A^\alpha)^z = A^{\alpha z}$ for all $z \in \mathbb{C}$.*

(b) *If A is $\mathcal{R}$-sectorial, for $\alpha \in (0, \frac{2\pi}{\omega_R(A)})$ the operator A^α is $\mathcal{R}$-sectorial with $\omega_R(A^\alpha) = \alpha\omega_R(A)$.*

(c) *If A has a bounded H^∞-calculus, for $\alpha \in (0, \frac{2\pi}{\omega_{H^\infty}(A)})$ the operator A^α has a bounded H^∞-calculus with $\omega_{H^\infty}(A) = \alpha\omega_{H^\infty}(A)$.*

1.4 Sectorial Operators which Have a Dilation

A further regularity property which is not so inherent to sectorial operators but nevertheless very important for their study is the existence of group dilations. The introduction of this powerful concept goes back to B. Sz.-Nagy [SN53]. Although the concept of dilations is intuitively clear once one has seen some basic examples, it is important for the treatment in this thesis to give precise definitions of semigroup dilations on general Banach spaces. We follow the terminology used in [AM14].

Definition 1.4.1. Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on some Banach space X. Further let $\mathcal{X}$ denote a class of Banach spaces. We say that

(i) $(T(t))_{t\geq 0}$ has a *strict dilation* in $\mathcal{X}$ if for some Y in $\mathcal{X}$ there are contractive linear operators $J\colon X \to Y$ and $Q\colon Y \to X$ and a C_0-group $(U(t))_{t\in\mathbb{R}}$ of isometries on Y such that

$$T(t) = QU(t)J \qquad \text{for all } t \geq 0.$$

(ii) $(T(t))_{t\geq 0}$ has a *loose dilation* in $\mathcal{X}$ if for some Y in $\mathcal{X}$ there are bounded linear operators $J\colon X \to Y$ and $Q\colon Y \to X$ and a bounded C_0-group $(U(t))_{t\in\mathbb{R}}$ on Y such that

$$T(t) = QU(t)J \qquad \text{for all } t \geq 0.$$

Later in Chapter 4, we will obtain complete characterizations of those semigroups on (subspaces) of Hilbert and L_p-spaces which admit a strict dilation in the class of Hilbert spaces and in the class of L_p-spaces for fixed $p \in (1,\infty)$ respectively. The main connection with the other regularity properties is the following observation.

Proposition 1.4.2. *Let A be a sectorial operator on a Banach space X such that $-A$ generates a C_0-semigroup which has a loose dilation in the class of all UMD-Banach spaces. Then A has a bounded H^∞-calculus with $\omega_{H^\infty}(A) \le \frac{\pi}{2}$.*

Proof. The boundedness of the H^∞-calculus clearly passes through dilations. Hence, it suffices to consider the case of a bounded C_0-group on a UMD-space for which the assertion was shown in Corollary 1.3.10. □

The following theorem by A. Fröhlich and L. Weis [FW06, Corollary 5.4] is a partial converse to Proposition 1.4.2. Its proof uses square function techniques which we do not cover here, for an overview on this topic we refer to [LM07].

Theorem 1.4.3. *Let A be a sectorial operator on some UMD-space with $\omega_{H^\infty}(A) < \frac{\pi}{2}$. Then the semigroup $(T(t))_{t\ge 0}$ generated by $-A$ has a loose dilation to the space $L_2([0,1];X)$.*

Together with Theorem 1.3.4 this shows that on UMD-spaces the existence of loose dilations and of a bounded H^∞-calculus are equivalent under the restriction $\omega_R(A) < \frac{\pi}{2}$. However, we will see in Section 2.2 that this characterization does not extend to the case $\omega_R(A) = \omega(A) = \frac{\pi}{2}$.

1.5 Bounded Imaginary Powers (BIP)

A third regularity property of sectorial operators which is also of historical importance is that of having bounded imaginary powers. Although we are mainly interested in H^∞-calculus and maximal regularity, bounded imaginary powers can be seen as somewhere between the stronger property of having a bounded H^∞-calculus and the weaker property of having maximal regularity (at least for sufficiently nice Banach spaces). This intermediate role will be useful in our later studies. Since we are not interested in bounded imaginary powers for its own sake, we only give the definition and present the connections with the other regularity properties. In particular, we do not present the fundamental applications to the study of evolution equations, which has been the key motivation for the development of this subject in the first place. Let us start with the definition of bounded imaginary powers.

Definition 1.5.1 (Bounded Imaginary Powers (BIP)). A sectorial operator with dense range on a Banach space X is said to have *bounded imaginary powers (BIP)* if for all $t \in \mathbb{R}$ the operator A^{it} associated to the functions $\lambda \mapsto \lambda^{it}$ via the holomorphic functional calculus is bounded.

In this case $(A^{it})_{t\in\mathbb{R}}$ is already a C_0-group on X. This follows from the following useful characterization (see for example [Haa06, Corollary 3.5.7].

Proposition 1.5.2. *Let A be a sectorial operator with dense range on a Banach space X. The following assertions are equivalent.*

(i) *A^{it} is a bounded operator for all $t \in \mathbb{R}$, i.e. A has bounded imaginary powers.*

(ii) *The operators A^{it} for $t \in \mathbb{R}$ form a C_0-group of bounded operators on X.*

(iii) *The operator $i \log A$ generates a C_0-group $(U(t))_{t\in\mathbb{R}}$ of bounded operators on X.*

In this case we have $U(t) = A^{it}$ for all $t \in \mathbb{R}$.

The growth of the C_0-group $(A^{it})_{t\in\mathbb{R}}$ is used to define the BIP-angle.

Definition 1.5.3. For a sectorial operator A with dense range and bounded imaginary powers on some Banach space one defines

$$\omega_{\mathrm{BIP}}(A) := \inf\{\omega \geq 0 : \|A^{it}\| \leq Me^{\omega|t|} \text{ for all } t \in \mathbb{R} \text{ and some } M \geq 0\}.$$

If A does not have bounded imaginary powers, we set $\omega_{BIP}(A) := \infty$.

Notice that if a sectorial operator A with dense range has a bounded $H^\infty(\Sigma_\theta)$-calculus for some $\theta \in (0,\pi)$, then it follows from the estimate

$$|\lambda^{it}| \leq \exp(\mathrm{Re}(it \log \lambda)) \leq \exp(|t|\theta)$$

for all $\lambda \in \Sigma_\theta$ that A has bounded imaginary powers with $\omega_{\mathrm{BIP}}(A) \leq \omega_{H^\infty}(A)$. A less obvious fact is that BIP implies $\mathcal{R}$-sectoriality on UMD-spaces [DHP03, Theorem 4.5].

Theorem 1.5.4. *Let A be a sectorial operator with dense range and bounded imaginary powers on a UMD-space. Then A is $\mathcal{R}$-sectorial with $\omega_R(A) \leq \omega_{\mathrm{BIP}}(A)$.*

2 Counterexamples

In the previous chapter we have introduced the regularity properties having a dilation, having a bounded H^∞-calculus, having bounded imaginary powers and being $\mathcal{R}$-sectorial. We have seen that on L_p for $p \in (1,\infty)$ and even on more general Banach spaces the following implications hold:

$$\text{dilation} \quad \Rightarrow \quad H^\infty \quad \Rightarrow \quad \text{BIP} \quad \Rightarrow \quad \mathcal{R}\text{-sectorial} \quad \Rightarrow \quad \text{sectorial}.$$

The goal of this chapter is to give explicit counterexamples which show that none of the converse implications $\Leftarrow$ holds. We present two different approaches to construct such counterexamples. The first one uses Schauder multipliers. We develop this approach further to give the first explicit example of a sectorial operator on L_p for $p \in (1,\infty)\setminus\{2\}$ (even on UMD-spaces) which is not $\mathcal{R}$-sectorial. As the main result of this chapter we then give a fundamentally new proof of the Kalton–Lancien Theorem (Theorem 2.1.42). Further, we construct positive analytic bounded semigroups on $\ell_p(\ell_q)$ for $p \neq q \in (1,\infty)$ without maximal regularity. This is the first example of a positive analytic semigroup on a UMD-Banach lattice which does not have maximal regularity. Furthermore we give explicit counterexamples to the closedness of the sum problem for two sectorial operators (Corollary 2.1.63), results which until now have been out of reach with the known methods.

The second approach uses a theorem of S. Monniaux to give examples of sectorial operators with bounded imaginary powers which do not have a bounded H^∞-calculus. Although the approach is rather straightforward, this method seems to be conceptually new. We conclude this chapter by constructing an example of a generator $-A$ of a C_0-semigroup on a Hilbert space without any loose dilation in the class of Hilbert spaces such that A has a bounded H^∞-calculus with $\omega_{H^\infty}(A) = \frac{\pi}{2}$. Further, we show how some of the regularity properties behave in exotic Banach spaces.

This chapter contains material from the published articles [Fac13a] and [Fac14] and from the accepted manuscript [Facb].

2.1 The Schauder Multiplier Method

In this section we develop the most fruitful method to construct systematically counterexamples which is known at the moment: the Schauder multiplier method. This method goes back to the pioneering works [BC91] and [Ven93]. We will develop this method from scratch starting with the counterexamples which are already known and which we think are easier to understand. Step by step we will then use more sophisticated and deep theorems from the geometric theory of Banach spaces. Whereas the basic theory of Schauder bases is summarized in Appendix A.1, the deeper and not so well-known results are presented in the main body of the text. After dealing with H^∞-calculus and bounded imaginary powers, we first present a self-contained example of a sectorial operator on L_p which is not $\mathcal{R}$-sectorial. On the one hand this is the most interesting case from the point of view of applications and on the other hand we hope that this makes the proof of the Kalton–Lancien Theorem in the general case more transparent.

2.1.1 Schauder Multipliers

We start our journey by giving the definition of Schauder multipliers and by studying its fundamental properties. After that we show how Schauder multipliers can be used to construct (analytic) semigroups. This technique goes back to the works of J. Baillon and P. Clément [BC91] and A. Venni [Ven93]. We follow the presentation in [Ven93].

From now on we need the theory of Schauder bases (and Schauder decompositions). All necessary definitions and results can be found in Appendix A.1. We will almost exclusively work with Schauder bases, however at some places we will need the results for more general Schauder decompositions. Nevertheless, it suffices for our proposes to deal with scalar-valued multipliers throughout. For the treatment of operator-valued multipliers we refer to [CdPSW00] and [Wit00]. The definition of a Schauder multiplier is then – at least for Schauder bases – very natural.

Definition 2.1.1 (Schauder Multiplier). Let $(\Delta_m)_{m\in\mathbb{N}}$ be a Schauder decomposition for a Banach space X. For a sequence $(\gamma_m)_{m\in\mathbb{N}} \subset \mathbb{C}$ the operator A defined by

$$D(A) = \left\{ x = \sum_{m=1}^{\infty} \Delta_m x : \sum_{m=1}^{\infty} \gamma_m \Delta_m x \text{ exists} \right\}$$

$$A\left(\sum_{m=1}^{\infty} \Delta_m x\right) = \sum_{m=1}^{\infty} \gamma_m \Delta_m x$$

is called the *Schauder multiplier* associated to $(\gamma_m)_{m\in\mathbb{N}}$.

2.1.1.1 Basic Properties of Schauder Multipliers

We now discuss some elementary properties of Schauder multipliers.

Proposition 2.1.2. *The Schauder multiplier A associated to a sequence $(\gamma_m)_{m\in\mathbb{N}}$ is a densely defined closed linear operator.*

Proof. It is clear that A is linear. Note that the domain $D(A)$ contains all finite sums of the form $\sum_{m=1}^{N} \Delta_m x$ for $x \in X$ and $N \in \mathbb{N}$ which form a dense subspace of X. It remains to show that A is closed. For this let $(x_n)_{n\in\mathbb{N}} \subset X$ with $x_n \to x$ and $Ax_n \to y$. Applying the projections Δ_m for $m \in \mathbb{N}$ we obtain that

$$\Delta_m(y) \xleftarrow[n\to\infty]{} \Delta_m(Ax_n) = \gamma_m \Delta_m(x_n) \xrightarrow[n\to\infty]{} \gamma_m \Delta_m(x).$$

This shows that $y = \sum_{m=1}^{\infty} \gamma_m \Delta_m(x)$. Hence, $x \in D(A)$ and $Ax = y$. $\square$

A central problem in the theory of Schauder multipliers is to determine for a given Schauder basis $(e_m)_{m\in\mathbb{N}}$ (or more generally for a Schauder decomposition $(\Delta_m)_{m\in\mathbb{N}}$) for a Banach space X the set of all sequences $(\gamma_m)_{m\in\mathbb{N}}$ for which the associated Schauder multiplier is bounded, or equivalently by the closed graph theorem for which the domain is the whole space. Obviously, these sequences form a vector space. In general, it is an extremely difficult problem to determine this space exactly. For example, the trigonometric basis $(e^{imz})_{m\in\mathbb{Z}}$ with respect to the enumeration $(0,-1,1,-2,2,\ldots)$ is a Schauder basis for $L_p([0,1])$ for $p \in (1,\infty)$. In this particular case the above problem asks for a characterization of all bounded Fourier multipliers in L_p which for $p \neq 2$ seems to be intractable at the moment.

However, some elementary general properties of this sequence space can be obtained easily. In particular, in the case of an unconditional Schauder decomposition most of the difficulties vanish.

Proposition 2.1.3. *Let $(\Delta_m)_{m\in\mathbb{N}}$ be a Schauder decomposition for X.*

(a) *Let $(\gamma_m)_{m\in\mathbb{N}}$ be the sequence associated to a bounded Schauder multiplier. Then $(\gamma_m)_{m\in\mathbb{N}} \in \ell_\infty$.*

(b) *The associated multipliers are bounded for all $(\gamma_m)_{m\in\mathbb{N}} \in \ell_\infty$ if and only if $(\Delta_m)_{m\in\mathbb{N}}$ is unconditional.*

(c) *Let $(\gamma_m)_{m\in\mathbb{N}} \in BV$, the space of all sequences with bounded variation. Then the Schauder multiplier associated to $(\gamma_m)_{m\in\mathbb{N}}$ is bounded.*

Proof. (a) Let A be the bounded Schauder multiplier associated to $(\gamma_m)_{m\in\mathbb{N}}$. Then for $x \in X$ one has

$$|\gamma_m| \|\Delta_m x\| = \|A\Delta_m x\| \le \|A\| \|\Delta_m x\|$$

for all $m \in \mathbb{N}$, which shows $(\gamma_m)_{m\in\mathbb{N}} \in \ell_\infty$ with $\|(\gamma_m)\|_\infty \le \|A\|$.

(b) If $(\Delta_m)_{m\in\mathbb{N}}$ is unconditional, then the operator associated to a bounded sequence $(\gamma_m)_{m\in\mathbb{N}}$ is bounded with operator norm smaller than $K\|(\gamma_m)\|_\infty$, where K denotes the unconditional constant of $(\Delta_m)_{m\in\mathbb{N}}$. Conversely, for an arbitrary sequence $(\varepsilon_m)_{m\in\mathbb{N}}$ in $\{-1,1\}^{\mathbb{N}}$ the boundedness of the associated

Schauder multiplier implies that the series $\sum_{m=1}^{\infty} \varepsilon_m \Delta_m x$ converges for all $x \in X$. Hence, $(\Delta_m)_{m\in\mathbb{N}}$ is unconditional.

(c) Let $(\gamma_m)_{m\in\mathbb{N}} \in BV$. For $x = \sum_{m=1}^{\infty} \Delta_m x$ we can rewrite the partial sums of Ax by using summation by parts for $N \in \mathbb{N}$ as

$$\begin{aligned}\sum_{m=1}^{N} \gamma_m \Delta_m x &= \left(\sum_{m=1}^{N} \Delta_m x\right)\gamma_{N+1} + \sum_{m=1}^{N} (\gamma_m - \gamma_{m+1}) \sum_{n=1}^{m} \Delta_n x \\ &= \gamma_{N+1} P_N x + \sum_{m=1}^{N} (\gamma_m - \gamma_{m+1}) P_m x.\end{aligned}$$

As every sequence in BV converges, the first term converges. The second series converges absolutely as

$$\sum_{m=1}^{\infty} \|P_m x\| \, |\gamma_m - \gamma_{m+1}| \le K \|x\| \, \|(\gamma_m)\|_{BV},$$

where K denotes the decomposition constant of $(\Delta_m)_{m\in\mathbb{N}}$. Hence, A is a bounded operator with

$$\|A\| \le K \left(\|(\gamma_m)\|_\infty + \|(\gamma_m)\|_{BV}\right). \qquad \square$$

Remark 2.1.4. In general, condition (c) of Proposition 2.1.3 is optimal. For if $X = BV$, then the sequence $(e_m)_{m\in\mathbb{N}_0}$ defined by e_0 as the constant sequence $\mathbb{1}$ and $e_m = (\delta_{mn})_{n\in\mathbb{N}}$ is a conditional Schauder basis for BV and the multiplier associated to a sequence $(\gamma_m)_{m\in\mathbb{N}_0}$ is bounded if and only if $(\gamma_m)_{m\in\mathbb{N}_0} \in BV$. Let us shortly explain why this is true. Recall that the norm of a sequence $(x_n)_{n\in\mathbb{N}} \in BV$ is given by

$$\|(x_n)\|_{BV} = |x_1| + \sum_{k=1}^{\infty} |x_{k+1} - x_k|.$$

So an element $(x_n)_{n\in\mathbb{N}}$ in BV is a Cauchy sequence and therefore one has $\lim_{n\to\infty} x_n = x$ for some $x \in \mathbb{C}$. We now claim that

$$(x_n)_{n\in\mathbb{N}} = x e_0 + \sum_{k=1}^{\infty} (x_k - x) e_k.$$

Indeed, for $N \in \mathbb{N}$ one has

$$\left\|(x_n) - x e_0 - \sum_{k=1}^{N} (x_k - x) e_k\right\|_{BV} = |x_{N+1} - x| - \sum_{k=N+1}^{\infty} |x_{k+1} - x_k| \xrightarrow[n\to\infty]{} 0$$

and it is easy to see that the expansion is unique. This shows that $(e_m)_{m\in\mathbb{N}}$ is a Schauder basis for BV. Now, let T be a bounded Schauder multiplier associated to a sequence $(\gamma_m)_{m\in\mathbb{N}_0}$ with respect to this basis. Then for all $N \in \mathbb{N}$ one has

$$\left\| T\left(\sum_{k=1}^{N} e_k\right)\right\|_{BV} = \left\|\sum_{k=1}^{N} \gamma_k e_k\right\|_{BV} = |\gamma_1| + |\gamma_N| + \sum_{k=1}^{N-1} |\gamma_{k+1} - \gamma_k| \leq 2\|T\|$$

Hence, we have shown $(\gamma_m)_{m\in\mathbb{N}_0} \in BV$ as asserted.

2.1.1.2 Schauder Multipliers as Generators of Analytic Semigroups

Given an arbitrary Banach space X, it is difficult to guarantee, roughly spoken, the existence of non-trivial strongly continuous semigroups on this space. Of course, every bounded operator generates such a semigroup by means of exponentiation. Such an argument does in general not work to show the existence of C_0-semigroups with an unbounded generator. Indeed, on $L_\infty([0,1])$ a result by H.P. Lotz [Lot85, Theorem 3] shows that every generator of a strongly continuous semigroup is already bounded.

One therefore has to make additional assumptions on the Banach space. A very convenient and rather general assumption for separable Banach spaces is to require the existence of a Schauder basis or a Schauder decomposition for that space. Indeed, all classical separable Banach spaces have a Schauder basis. Moreover, for a long time it had been an open problem whether all separable Banach spaces have a Schauder basis. Indeed, the existence of a Schauder basis for a given Banach space shows that this space has the approximation property. In a landmarking paper P. Enflo [Enf73] showed that there indeed exist separable Banach spaces without the approximation property. The analogous question whether every separable Banach space has a Schauder decomposition has also a negative answer [AKP99]. The solution of this problem relies on the existence of separable so-called hereditarily indecomposable Banach spaces which we will meet in Section 2.3.1.

The next proposition shows that Schauder decompositions allow us to construct systematically strongly continuous semigroups (with unbounded generators) on the underlying Banach spaces.

Proposition 2.1.5. *Let $(\Delta_m)_{m\in\mathbb{N}}$ be a Schauder decomposition for some Banach space X and $(\gamma_m)_{m\in\mathbb{N}}$ be a positive non-decreasing sequence of real numbers. Then the Schauder multiplier associated to $(\gamma_m)_{m\in\mathbb{N}}$ is an injective sectorial operator with dense range and $\omega(A) = 0$. In particular, $-A$ generates an analytic C_0-semigroup $(T(z))_{z\in\Sigma_{\frac{\pi}{2}}}$.*

Proof. For every $t \geq 0$ let $T(t)$ be the Schauder multiplier associated to the sequence $(e^{-\gamma_m t})_{m\in\mathbb{N}}$. The sequence $(e^{-\gamma_m t})_{m\in\mathbb{N}}$ is decreasing and therefore a fortiori of bounded variation. By Proposition 2.1.3(c) the operator $T(t)$ is bounded. More precisely, one has for $\gamma_\infty = \lim_{m\to\infty} \gamma_m \in (0, \infty]$

$$\|(e^{-\gamma_m t})\|_{\mathrm{Var}} = \sum_{m=1}^{\infty} e^{-\gamma_m t} - e^{-\gamma_{m+1} t} = e^{-\gamma_1 t} - e^{-\gamma_\infty t}.$$

It is clear that the family $(T(t))_{t\geq 0}$ satisfies the semigroup law. Therefore $(T(t))_{t\geq 0}$ is a bounded semigroup. Notice that $(T(t))_{t\geq 0}$ is also strongly continuous. Indeed, the strong continuity is clear for all elements in the dense subspace of all $x \in X$ which possess a finite expansion with respect to the decomposition $(\Delta_m)_{m\in\mathbb{N}}$. The general case then follows from the local boundedness of the semigroup and an approximation argument. Notice that it is easy to check that $[0, \infty) \subset \rho(-A)$. In order to show that the generator B of the semigroup coincides with $-A$, it therefore suffices to show that $B \subset -A$. For this notice that for $x \in D(B)$ one has

$$\Delta_m(Bx) = \lim_{t\downarrow 0} \frac{e^{-\gamma_m t}\Delta_m(x) - \Delta_m(x)}{t} = -\gamma_m \Delta_m(x).$$

Hence, $x \in D(A)$ and $Bx = -Ax$.

It is clear that A is injective and has dense range. Notice that for $\alpha > 0$ the fractional power A^α is the Schauder multiplier associated to the sequence $(\gamma_m^\alpha)_{m\in\mathbb{N}}$. Hence, the same reasoning as above shows that $-A^\alpha$ generates a bounded C_0-semigroup for all $\alpha > 0$. Hence, by Proposition 1.3.13 one has $\alpha\omega(A) = \omega(A^\alpha) \leq \frac{\pi}{2}$. As α can be chosen arbitrarily large, this shows that $\omega(A) = 0$. It is well-known that this implies that the semigroup $(T(t))_{t\geq 0}$ generated by $-A$ extends to a bounded analytic semigroup of angle $\frac{\pi}{2}$. □

2.1.2 Sectorial Operators without a Bounded H^∞-Calculus

In this subsection we apply the so far developed methods for Schauder multipliers to give examples of sectorial operators without a bounded H^∞-calculus. The elegant approach used in this section is already known and goes back to [Lan98] and [LM99b]. Nevertheless we give full details for the sake of completeness and in order to familiarize the reader to the key arguments in an easier case. Before giving counterexamples, we start with a positive result in order to show a way we cannot go.

Proposition 2.1.6. *For an unconditional basis $(e_m)_{m\in\mathbb{N}}$ of a Banach space X let A be the Schauder multiplier associated to a sequence $(\gamma_m)_{m\in\mathbb{N}}$ such that $\theta := \sup_{m\in\mathbb{N}} |\arg(\gamma_m)| < \pi$. Then A has a bounded H^∞-calculus with $\omega_{H^\infty}(A) = \theta$.*

Proof. Notice that for $\lambda \in \rho(A)$ one has $e_m^*(R(\lambda, A)x) = (\lambda - \gamma_m)^{-1} e_m^*(x)$. Therefore for $f \in H_0^\infty(\Sigma_\psi)$ with $\psi \in (\theta, \pi)$ we have

$$e_m^*(f(A)x) = \int_{\partial \Sigma_\psi} \frac{f(\lambda)}{\lambda - \gamma_m} e_m^*(x)\, d\lambda = f(\gamma_m) e_m^*(x).$$

This shows that $f(A)$ is the Schauder multiplier associated to the sequence $(f(\gamma_m))_{m \in \mathbb{N}}$. Since by assumption the basis $(e_m)_{m \in \mathbb{N}}$ is unconditional, Proposition 2.1.3(b) shows that

$$\|f(A)\| \le K \|f(\gamma_m)\|_\infty \le K \|f\|_{H^\infty(\Sigma_\psi)}.$$

Hence, A has a bounded H^∞-calculus with $\omega(A) \le \theta$. As one clearly has $\omega(A) = \theta$, we obtain $\omega_{H^\infty}(A) = \theta$. □

In particular, on sufficiently nice Banach spaces such Schauder multipliers are $\mathcal{R}$-sectorial.

Corollary 2.1.7. *For an unconditional basis $(e_m)_{m \in \mathbb{N}}$ of a Banach space X with property (Δ) let A be the Schauder multiplier associated to a sequence $(\gamma_m)_{m \in \mathbb{N}}$ with $\theta := \sup_{m \in \mathbb{N}} |\arg(\gamma_m)| < \pi$. Then A is $\mathcal{R}$-sectorial with $\omega_{\mathcal{R}}(A) = \theta$.*

Proof. This follows from Proposition 2.1.6 and Theorem 1.3.4. □

The above result shows that one cannot obtain examples of sectorial operators with dense range without having a bounded H^∞-calculus by using Schauder multipliers with respect to an unconditional basis. However, one can produce counterexamples from Schauder multipliers with respect to a conditional basis. The following method goes back to [Lan98] and [LM99b, Theorem 4.1].

Theorem 2.1.8. *Let $(e_m)_{m \in \mathbb{N}}$ be a conditional Schauder basis for a Banach space X. Then the Schauder multiplier A associated to the sequence $(2^m)_{m \in \mathbb{N}}$ is a sectorial operator with dense range and $\omega(A) = 0$ which does not have a bounded H^∞-calculus.*

Proof. By Proposition 2.1.5 everything is already shown except for the fact that A does not have a bounded H^∞-calculus. For this notice that for each $f \in H^\infty(\Sigma_\theta)$ for some $\theta \in (0, \pi)$ the operator $f(A)$ is given by the Schauder multiplier associated to the sequence $(f(2^m))_{m \in \mathbb{N}}$. Now, assume that A has a bounded $H^\infty(\Sigma_\theta)$-calculus for some $\theta \in (0, \pi)$. By [Haa06, Section 9.1.2] on the interpolation of sequences by analytic functions, for every element in ℓ_∞ there exists an $f \in H^\infty(\Sigma_\theta)$ such that $(f(2^m))_{m \in \mathbb{N}}$ is the given sequence. This means that every element in ℓ_∞ defines a bounded Schauder multiplier. Then Proposition 2.1.3(b) implies that $(e_m)_{m \in \mathbb{N}}$ is unconditional in contradiction to our assumption. □

Notice that the following corollary even includes (separable) Hilbert spaces.

Corollary 2.1.9. *Let X be a Banach space that admits a Schauder basis. Then there exists a sectorial operator A with dense range and $\omega(A) = 0$ that does not have a bounded H^∞-calculus.*

Proof. We will later see in Remark 2.1.41 that every Banach space that admits a Schauder basis does also admit a conditional Schauder basis. Then the result follows directly from Theorem 2.1.8. □

Corollary 2.1.10. *Let $(e_m)_{m\in\mathbb{N}}$ be a Schauder basis for a Banach space X. Then the Schauder multiplier associated to the sequence $(2^m)_{m\in\mathbb{N}}$ is a sectorial operator with dense range and has a bounded H^∞-calculus if and only if $(e_m)_{m\in\mathbb{N}}$ is unconditional.*

Proof. Apply Proposition 2.1.6 and Theorem 2.1.8. □

Next we give a concrete example of a sectorial operator of the above form which has bounded imaginary powers but no bounded H^∞-calculus by Theorem 2.1.8. This example is due to [LM99b, p. 15] and [Lan98].

Example 2.1.11. We consider the trigonometric system $(e^{imz})_{m\in\mathbb{Z}}$ which is with respect to the enumeration $(0,-1,1,-2,2,\ldots)$ of $\mathbb{Z}$ a conditional basis of $L_p([0,2\pi])$ for $p \in (1,\infty)\setminus\{2\}$. Let A be the Schauder multiplier associated to the sequence $(2^m)_{m\in\mathbb{Z}}$. As a consequence of the boundedness of the Hilbert transform on L_p for $p \in (1,\infty)$, it suffices to consider separately the operator on the two complemented parts with respect to the decomposition

$$L_p[0,2\pi] = \overline{\operatorname{span}}\{e^{imz} : m < 0\} \oplus \overline{\operatorname{span}}\{e^{imz} : m \geq 0\}.$$

Observe that A has a bounded H^∞-calculus if and only if both parts have a bounded H^∞-calculus. It then follows from the proof of Proposition 2.1.5 that A is a sectorial operator with $\omega(A) = 0$ which does not have a bounded H^∞-calculus. We now show that A has bounded imaginary powers with $\omega_{\mathrm{BIP}}(A) = 0$. For this we observe that

$$\begin{aligned} A^{it}\left(\sum_{m\in\mathbb{Z}} a_m e^{imz}\right) &= \sum_{m\in\mathbb{Z}} (2^m)^{it} a_m e^{imz} = \sum_{m\in\mathbb{Z}} a_m \exp(imt\log 2) e^{imz} \\ &= \sum_{m\in\mathbb{Z}} a_m \exp(im(t\log 2 + z)) = S(t\log 2)\left(\sum_{m\in\mathbb{Z}} a_m e^{imz}\right), \end{aligned}$$

where $(S(t))_{t\in\mathbb{R}}$ is a periodic shift group on $L_p([0,2\pi])$.

We will study examples in the spirit of Example 2.1.11 more systematically in Section 2.3.

2.1.3 Sectorial Operators without Bounded Imaginary Powers

Similarly as in the case of the H^∞-calculus one can use Schauder multipliers to construct sectorial operators that do not have bounded imaginary powers. We start with a weighted version of Example 2.1.11 which gives an example of an $\mathcal{R}$-sectorial operator without bounded imaginary powers. However, before we need to state some facts from harmonic analysis.

It is a natural question to ask for which weights w the trigonometric system is a Schauder basis for the space $L_p([0,2\pi],w)$. Indeed, a complete characterization of these weights is known. Moreover, these weights play an important role in harmonic analysis. We will identify the torus $\mathbb{T}$ with the interval $[0,2\pi)$ on the real line and functions in some L_p-space over $[0,2\pi]$ with their periodic extensions or with L_p-functions on the torus. The following class of weights was first introduced in the work [HMW73].

Definition 2.1.12 (A_p-weight). Let $p \in (1,\infty)$. A function $w\colon \mathbb{R} \to [0,\infty]$ with $w(t) \in (0,\infty)$ almost everywhere is called an *A_p-weight* if there exists a constant $K \geq 0$ such that for every compact interval $I \subset \mathbb{R}$ with positive length one has

$$\left(\frac{1}{|I|}\int_I w(t)\,dt\right)\left(\frac{1}{|I|}\int_I w(t)^{-1/(p-1)}\,dt\right)^{p-1} \leq K.$$

We denote the set of all A_p-weights with $\mathcal{A}_p(\mathbb{R})$. The smallest constant such that the above inequality holds is called the *$\mathcal{A}_p(\mathbb{R})$-weight constant*. Moreover, we set in the periodic case

$$\mathcal{A}_p(\mathbb{T}) \coloneqq \{w \in \mathcal{A}_p(\mathbb{R}) : w \text{ is } 2\pi\text{-periodic}\}.$$

As an example, the 2π-periodic extension of the function $t \mapsto |t|^\alpha$ for $\alpha \in \mathbb{R}$ lies in $\mathcal{A}_p(\mathbb{T})$ if and only if $\alpha \in (-1,p-1)$ [BG03, Example 2.4]. The characterization below can be found in [Nie09, Proposition 2.3] and essentially goes back to methods developed by R. Hunt, B. Muckenhoupt and R. Wheeden in [HMW73].

Theorem 2.1.13. *Let $w\colon \mathbb{R} \to [0,\infty]$ with $w(t) \in (0,\infty)$ almost everywhere be a 2π-periodic weight and $p \in (1,\infty)$. Then the trigonometric system is a Schauder basis for $L_p([0,2\pi],w)$ with respect to the enumeration $(0,-1,1,-2,2,\ldots)$ of $\mathbb{Z}$ if and only if $w \in \mathcal{A}_p(\mathbb{T})$.*

It is also possible to extend the classical multiplier theorems to L_p-spaces weighted with A_p-weights. We need the following weighted periodic version of the Marcinkiewicz multiplier theorem. It is proved in [BG03, Theorem 4.4] via transference methods from its continuous analogue in [Kur80, Theorem 2].

Let $I = [n_1, n_2]$ be an interval in $\mathbb{Z}$. For a function $\psi\colon \mathbb{Z} \to \mathbb{C}$ we define the *variation of ψ on I* as

$$\operatorname{var}(\psi, I) := \sum_{m=n_1}^{n_2-1} \left|\psi(m+1) - \psi(m)\right|.$$

Moreover, we consider the following dyadic decomposition of the integers.

$$\Lambda_n = \begin{cases} [2^{n-1}, 2^n] \cap \mathbb{Z} & \text{for } n \geq 1 \\ [-1, 1] \cap \mathbb{Z} & \text{for } n = 0 \\ [-2^{-n}, -2^{-n-1}] \cap \mathbb{Z} & \text{for } n \leq -1. \end{cases}$$

The Marcinkiewicz multiplier theorem then gives the following sufficient criterion for a multiplier to be bounded on L_p.

Theorem 2.1.14 (Marcinkiewicz Multiplier Theorem). *Let $p \in (1, \infty)$ and $w \in \mathcal{A}_p(\mathbb{T})$ with $\mathcal{A}_p(\mathbb{R})$-constant C. Further let $\psi\colon \mathbb{Z} \to \mathbb{C}$ with*

$$\|\psi\|_{\mathfrak{M}_1(\mathbb{Z})} := \sup_{n \in \mathbb{Z}} |\psi(n)| + \sup_{n \in \mathbb{Z}} \operatorname{var}(\psi, \Lambda_n) < \infty.$$

Then ψ defines a bounded Fourier multiplier on $L_p([0, 2\pi], w)$ and the norm of the induced Fourier multiplier operator can be estimated by $K_{p,C} \|\psi\|_{\mathfrak{M}_1(\mathbb{Z})}$, where $K_{p,C}$ is a constant that only depends on p and C.

Notice that this in particular implies that the periodic Hilbert transform is bounded on $L_p([0, 2\pi], w)$ for every A_p-weight $w \in \mathcal{A}_p(\mathbb{T})$. Now we are ready to give a new example of an $\mathcal{R}$-sectorial operator on some L_p-space which does not have bounded imaginary powers, a discrete variant of the example given in [KW04, Example 10.17]. This example is of particular interest because it shows that the classical Dore–Venni theorem does not cover the complete spectrum of sectorial operators with maximal regularity. Observe that the use of weights allows us to construct unbounded imaginary powers. In particular, in the Hilbert space case we recover an explicit example of a sectorial operator without bounded imaginary powers.

Example 2.1.15. Let $p \in (1, \infty)$ and $w \in \mathcal{A}_p(\mathbb{T})$ be an A_p-weight. Then the trigonometric system $(e^{imz})_{m \in \mathbb{Z}}$ is a Schauder basis for $L_p([0, 2\pi], w)$ by Theorem 2.1.13. Let A again be the Schauder multiplier associated to the sequence $(2^m)_{m \in \mathbb{Z}}$. One sees as in Example 2.1.11 that A is a sectorial operator. It remains to show that A is $\mathcal{R}$-sectorial. Notice that for $\lambda = a2^l e^{i\theta} \in \mathbb{C} \setminus [0, \infty)$ with $|a| \in [1, 2]$ one has

$$\lambda R(\lambda, A)\left(\sum_{m \in \mathbb{Z}} a_m e^{imz}\right) = \sum_{m \in \mathbb{Z}} \frac{\lambda}{\lambda - 2^m} a_m e^{imz} = \sum_{m \in \mathbb{Z}} \frac{a e^{i\theta}}{a e^{i\theta} - 2^{m-l}} a_m e^{imz}$$

$$= \sum_{m\in\mathbb{Z}} \frac{ae^{i\theta}}{ae^{i\theta} - 2^m} a_m e^{i(m+l)z}.$$

Consequently, for $\lambda_k = a2^{l_k}e^{i\theta}$ with $k \in \{1,\dots,n\}$ and $x_1,\dots,x_n \in L_p([0,2\pi],w)$ one has by Kahane's contraction principle (Proposition 1.2.2)

$$\begin{aligned}\left\|\sum_{k=1}^n r_k\lambda_k R(\lambda_k,A)x_k\right\| &= \left\|\sum_{k=1}^n r_k ae^{i\theta}R(ae^{i\theta},A)e^{il_kz}x_k\right\| \\ &\le |a|\left\|R(ae^{i\theta},A)\right\|\left\|\sum_{k=1}^n r_k e^{il_kz}x_k\right\| \le C_p\left\|R(ae^{i\theta},A)\right\|\left\|\sum_{k=1}^n r_k x_k\right\|\end{aligned}$$

for some universal constant $C_p > 0$. Now it is straightforward to check that for every $\theta_0 > 0$ the sequence $(\frac{ae^{i\theta}}{ae^{i\theta}-2^m})_{m\in\mathbb{Z}}$ satisfies the assumptions of the Marcinkiewicz multiplier theorem (Theorem 2.1.14) or alternatively of Proposition 2.1.3(c) uniformly in $\theta \in [\theta_0, 2\pi)$ and in $|a| \in [1,2]$. By Remark 1.2.10 this shows that A is $\mathcal{R}$-sectorial with $\omega_R(A) = 0$.

By the same calculation as in Example 2.1.11 on the dense set of trigonometric polynomials the operator A^{it} for $t \in \mathbb{R}$ is given by $S(t\log 2)$, where $(S(t))_{t\in\mathbb{R}}$ is a periodic shift group. Notice however, that for example for $w(t) = |t|^\alpha$ for a suitable chosen $\alpha \in \mathbb{R}$ such that $w \in \mathcal{A}_p(\mathbb{T})$ this shift obviously does not leave $L_p([0,2\pi],w)$ invariant. Hence, A does not have bounded imaginary powers.

2.1.4 Sectorial Operators without Maximal Regularity: The Maximal Regularity Problem

After the preparatory sections, in this section we use semigroups generated by Schauder multipliers to give a negative answer to the maximal regularity problem. Our examples of sectorial operators without maximal regularity will be explicit, namely Schauder multipliers associated to the sequence $(2^m)_{m\in\mathbb{N}}$ with respect to some Schauder basis. For this we need to develop a completely new approach to the Kalton–Lancien Theorem (Theorem 2.1.42). This was done by the author in the articles [Fac13a] and [Fac14] on which this presentation is based.

The key idea is to associate to a Schauder multiplier operators on $\operatorname{Rad}(X)$ which are again of diagonal form. The $\mathcal{R}$-boundedness will then be equivalent to the boundedness of the introduced diagonal operators which can be investigated by similar methods as before. As appetizers we give easy and explicit counterexamples for the spaces ℓ_1 and c_0. After that we focus on the L_p-case. Here the arguments are getting more involved. Nevertheless, in order to stay as concrete as possible we will directly exploit the structure of the Haar basis of $L_p[0,1]$. After that we extract the key concept which allows us to give counterexamples: non-symmetric Schauder bases. The use of non-symmetric bases lies at the heart of our completely new approach. We then develop the necessary concepts and tools for symmetric Schauder bases from scratch. At the end of this study we will prove a deep structure theorem for Banach spaces with an unconditional basis originally due to J. Lindenstrauss and M. Zippin (Theorem 2.1.39). After all this preliminary work the general case of the Kalton–Lancien Theorem (Theorem 2.1.42) will follow rather directly. Moreover, we construct negative generators of positive bounded analytic C_0-semigroups on some UMD-Banach lattices without maximal regularity (Theorem 2.1.46).

2.1.4.1 Associated Semigroups on Rad(X)

In this subsection we explain how we can describe the $\mathcal{R}$-analyticity of a semigroup in terms of the analyticity of an associated semigroup on Rad(X). The idea to study $\mathcal{R}$-analyticity with the help of associated semigroups on Rad(X) goes back to W. Arendt and S. Bu [AB03]. In our counterexamples we will always show that the analyticity of the associated semigroup is violated instead of working directly with $\mathcal{R}$-analyticity or maximal regularity. Our object of interest is the following.

Definition 2.1.16 (Associated Semigroup on Rad(X)). Let $(T(z))_{z\in\Sigma}$ be an analytic C_0-semigroup on a Banach space X. Given a sequence $(q_n)_{n\in\mathbb{N}} \subset (0,1)$, one defines the *associated semigroup* $(\mathcal{T}(z))_{z\in\Sigma}$ on the finite Rademacher sums by

$$\mathcal{T}(z)\left(\sum_{n=1}^{N} r_n x_n\right) := \sum_{n=1}^{N} r_n T(q_n z) x_n \qquad \text{for } z \in \Sigma \text{ and } N \in \mathbb{N}.$$

For $x \in X$ one often uses the notation $r_n \otimes x$ for the function $\omega \mapsto r_n(\omega)x$ in Rad(X). If the semigroup is generated by a Schauder multiplier, the associated semigroup on Rad(X) takes the form of a multiplier as well.

Remark 2.1.17. Let $(T(z))_{z\in\Sigma}$ be the analytic C_0-semigroup generated by a Schauder multiplier $-A$ associated to a sequence $(-\gamma_m)_{m\in\mathbb{N}}$ as studied in Proposition 2.1.5. Then $\mathcal{T}(z)$ acts on finite Rademacher sums as

$$\begin{aligned}\mathcal{T}(z)\left(\sum_{m,n=1}^{N} a_{nm} r_n \otimes e_m\right) &= \sum_{n=1}^{N} r_n T(q_n z)\left(\sum_{m=1}^{N} a_{nm} e_m\right) \\ &= \sum_{n,m=1}^{N} e^{-q_n \gamma_m z} a_{nm} r_n \otimes e_m.\end{aligned} \tag{2.1}$$

The following theorem allows us to study the $\mathcal{R}$-analyticity of a semigroup with the help of the associated analytic semigroup on Rad(X). In some sense the $\mathcal{R}$-boundedness is hidden within the associated semigroup. The next proof goes back to W. Arendt & S. Bu [AB03, Theorem 3.6].

Theorem 2.1.18. *Let $(T(z))$ be an analytic C_0-semigroup on a Banach space X. Then the following hold:*

(a) *If $(T(t))_{t\geq 0}$ is locally $\mathcal{R}$-bounded, then $(\mathcal{T}(t))_{t\geq 0}$ extends to a C_0-semigroup on* $\operatorname{Rad}(X)$.

Moreover, if $(T(z))_{z\in\Sigma}$ is $\mathcal{R}$-analytic, then the associated semigroup extends to an analytic C_0-semigroup $(\mathcal{T}(z))_{z\in\Sigma}$ on $\operatorname{Rad}(X)$.

Further, in both cases there exist $M,\omega > 0$ such that

$$\|\mathcal{T}(t)\| \leq Me^{wt} \qquad \textit{respectively} \qquad \|\mathcal{T}(z)\| \leq Me^{\omega|z|}$$

holds independently of the chosen sequence $(q_n)_{n\in\mathbb{N}} \subset (0,1)$.

(b) *Conversely, if the associated semigroup $(\mathcal{T}(z))_{z\in\Sigma}$ is strongly continuous and analytic for some $(q_n)_{n\in\mathbb{N}}$ being dense in $(0,1)$, then $(T(z))$ is $\mathcal{R}$-analytic.*

Proof. We start by proving (a) in the $\mathcal{R}$-analytic case, the first case is of course completely analogous. For z in some sector Σ one has

$$\left\|\mathcal{T}(z)\left(\sum_{n=1}^{N} r_n x_n\right)\right\| = \left\|\sum_{n=1}^{N} r_n T(q_n z) x_n\right\| \leq \mathcal{R}\{T(\lambda z) : \lambda \in (0,1)\} \left\|\sum_{n=1}^{N} r_n x_n\right\|.$$

Since the finite Rademacher sums are dense in $\operatorname{Rad}(X)$, $\mathcal{T}(z)$ extends to a bounded linear operator on $\operatorname{Rad}(X)$. Now let $z \in \Sigma$ be arbitrary and M be given by $M := \mathcal{R}\{T(z) : z \in \Sigma, |z| \leq 1\}$. There exist unique $n \in \mathbb{N}$, $s \in [0,1)$ such that $z = (n+s)\frac{z}{|z|}$. Then for $\omega = \log M$ one has

$$\|\mathcal{T}(z)\| \leq \left\|\mathcal{T}\left(\frac{z}{|z|}\right)\right\|^n \left\|\mathcal{T}\left(s\frac{z}{|z|}\right)\right\| \leq Me^{n\log M} \leq Me^{\omega|z|}.$$

The strong continuity can easily be checked for finite Rademacher sums and can then be extended to arbitrary elements of $\operatorname{Rad}(X)$ by the local boundedness of $z \mapsto \mathcal{T}(z)$ in operator norm.

We now turn to the proof of (b). Assume that the associated semigroup $(\mathcal{T}(z))_{z\in\Sigma}$ is an analytic C_0-semigroup. Let $z_1,\ldots,z_N \in \Sigma$ with $\arg z = \theta$ for some $\theta \in (0,\frac{\pi}{2})$ with $e^{i\theta} \in \Sigma$ and $|z| \leq 1$. Then by the density of $(q_m)_{m\in\mathbb{N}}$ in $(0,1)$ one can choose for all $n = 1,\ldots,N$ disjoint subsequences $(q_{m_l}^{(n)})_{l\in\mathbb{N}}$ with $q_{m_l}^{(n)} e^{i\theta} \to z_n$ for $l \to \infty$. Now let $M := \sup\{\|\mathcal{T}(z)\| : z \in \Sigma, |z| \leq 1\}$. Then one has for all $x_1,\ldots,x_N \in X$

$$\left\|\sum_{n=1}^{N} r_n T(z_n) x_n\right\| = \lim_{l\to\infty} \left\|\sum_{n=1}^{N} r_n T(q_{m_l}^{(n)} e^{i\theta}) x_n\right\| \leq \left\|\mathcal{T}(e^{i\theta})\right\| \left\|\sum_{n=1}^{N} r_n x_n\right\|$$

$$\leq M \left\| \sum_{n=1}^{N} r_n x_n \right\|.$$

Here we have used that the norm of the Rademacher average is independent of the concrete choice of the random variables. This shows that the set $\{T(te^{i\theta}) : t \in (0,1)\}$ is $\mathcal{R}$-bounded. Of course, by an analogous argument one obtains that the set $\{T(te^{-i\theta}) : t \in (0,1)\}$ is $\mathcal{R}$-bounded. It now follows from the analyticity of the semigroup and Proposition 1.2.3 together with a rescaling argument that the semigroup $(T(z))$ is $\mathcal{R}$-analytic. □

2.1.4.2 Warm-up: Counterexamples on c_0 and ℓ_1

Before we consider counterexamples on general Banach spaces admitting an unconditional basis following [Fac14], we construct counterexamples for the concrete Banach spaces c_0 and ℓ_1. They also illustrate our approach. The following elementary lemma will be useful in the future and throws light on the special role played by the sequence $(2^m)_{m\in\mathbb{N}}$ when used as a multiplier sequence. As the proof only involves a direct computation, we omit it.

Lemma 2.1.19. *The function $d(t) = e^{-2^m t} - e^{-2^{m+1} t}$ $(m \in \mathbb{N})$ possesses a unique maximum in $[0,1]$ at $t_0 = \frac{\log 2}{2^m}$. Moreover, the maximum value $d(t_0) = \frac{1}{4}$ is independent of m.*

The Space c_0 Let $(e_m)_{m\in\mathbb{N}}$ be the standard unit vector basis of c_0. Then the *summing basis* $(s_m)_{m\in\mathbb{N}}$ given by $s_m := \sum_{k=1}^{m} e_k$ is a conditional basis of c_0 [AK06, Example 3.1.2].

Proposition 2.1.20. *Let $(s_m)_{m\in\mathbb{N}}$ be the summing basis of c_0. Then $-A$ given by*

$$D(A) = \left\{ x = \sum_{m=1}^{\infty} a_m s_m : \sum_{m=1}^{\infty} 2^m a_m s_m \quad \text{exists} \right\}$$

$$A\left(\sum_{m=1}^{\infty} a_m s_m \right) = \sum_{m=1}^{\infty} 2^m a_m s_m$$

generates an analytic C_0-semigroup $(T(z))_{z\in\Sigma_{\pi/2}}$ that is not $\mathcal{R}$-bounded on $[0,1]$.

Proof. Assume that $\mathcal{R}\{T(t) : 0 < t \leq 1\} < \infty$. Then $(\mathcal{T}(t))_{t\geq 0}$ is a C_0-semigroup on $\operatorname{Rad}(c_0)$ by Theorem 2.1.18. We now consider $x_N := \sum_{m=1}^{N} (s_{2m} - s_{2m-1}) \otimes r_m$ for $N \in \mathbb{N}$. Its norm in $\operatorname{Rad}(c_0)$ is

$$\left\| \sum_{m=1}^{N} r_m (s_{2m} - s_{2m-1}) \right\| = \int_0^1 \left\| \sum_{m=1}^{N} r_m(\omega) e_{2m} \right\|_\infty d\omega = 1.$$

One has by (2.1)

$$\mathcal{T}(1)(x_N) = \sum_{m=1}^{N} (e^{-2^{2m} q_m} s_{2m} - e^{-2^{2m-1} q_m} s_{2m-1}) r_m.$$

In particular for the choice $q_m = \frac{\log 2}{2^{2m-1}}$ the first coordinate of the sequence given by the above expression evaluated in ω is $-\frac{1}{4}\sum_{m=1}^{N} r_m(\omega)$. Contradictory to our assumption that $\mathcal{T}(1)$ is bounded this shows that

$$\|\mathcal{T}(1)(x_N)\| \ge \frac{1}{4}\int_0^1 \left|\sum_{m=1}^{N} r_m(\omega)\right| d\omega \ge \frac{C^{-1}}{4}\sqrt{N}$$

for some constant $C > 0$ by the Khintchine inequality (Theorem A.3.1). Hence, $(T(t))_{t\ge 0}$ is not $\mathcal{R}$-bounded on $[0,1]$. □

The Space ℓ_1 Let $(e_m)_{m\in\mathbb{N}}$ denote the standard unit vector basis of ℓ_1. Then $(f_m)_{m\in\mathbb{N}}$ given by $f_1 = e_1$ and $f_m = e_m - e_{m-1}$ for $m \ge 2$ is a conditional basis for ℓ_1 [Sin70, Example 14.2]. Notice that this can also been seen as follows: ℓ_1 can be identified with the space BV of all sequences with bounded variation. Under the natural identification $(f_m)_{m\in\mathbb{N}}$ corresponds to the standard basis on BV which is conditional by Remark 2.1.4. Again this conditional basis can be used to construct a counterexample.

Proposition 2.1.21. *Let $(f_m)_{m\in\mathbb{N}}$ be the basis of ℓ_1 defined above. Then $-A$ given by*

$$D(A) = \left\{ x = \sum_{m=1}^{\infty} a_m f_m : \sum_{m=1}^{\infty} 2^m a_m f_m \quad \text{exists} \right\}$$
$$A\left(\sum_{m=1}^{\infty} a_m f_m\right) = \sum_{m=1}^{\infty} 2^m a_m f_m$$

generates an analytic C_0-semigroup $(T(z))_{z\in\Sigma_{\pi/2}}$ that is not $\mathcal{R}$-bounded on $[0,1]$.

Proof. One proceeds as in the proof of Proposition 2.1.20. This time one looks at the vector $x_N = \sum_{n,m=1}^{N} r_n \otimes f_m = \sum_{n=1}^{N} r_n \otimes e_N$ in $\mathrm{Rad}(\ell_1)$ for $N \in \mathbb{N}$. By the Khintchine inequality (Theorem A.3.1), its norm in $\mathrm{Rad}(\ell^1)$ is

$$\left\|\sum_{n,m=1}^{N} r_n f_m\right\| = \int_0^1 \left\|\sum_{n=1}^{N} r_n(\omega) e_N\right\|_{\ell^1} d\omega = \int_0^1 \left|\sum_{n=1}^{N} r_n(\omega)\right| d\omega \le C\sqrt{N}$$

for some constant $C > 0$. A short calculation using (2.1) shows that $\mathcal{T}(1)x_N$ is given by

$$\sum_{n,m=1}^{N} e^{-2^m q_n} r_n \otimes f_m = \sum_{n=1}^{N} e^{-2^N q_n} r_n \otimes e_N + \sum_{m=1}^{N-1}\sum_{n=1}^{N} (e^{-2^m q_n} - e^{-2^{m+1} q_n}) r_n \otimes e_m$$

Now, a second application of the Khintchine inequality shows that one has

$$\begin{aligned}\|\mathcal{T}(1)x_N\| &\geq \sum_{m=1}^{N-1} \int_0^1 \left| \sum_{n=1}^{N} (e^{-2^m q_n} - e^{-2^{m+1} q_n}) r_n(\omega) \right| d\omega \\ &\geq C^{-1} \sum_{m=1}^{N-1} \left(\sum_{n=1}^{N} \left| e^{-2^m q_n} - e^{-2^{m+1} q_n} \right|^2 \right)^{1/2}.\end{aligned}$$

We again choose $q_m = \frac{\log 2}{2^m}$. By estimating the right hand side, we obtain

$$\|\mathcal{T}(1)x_N\| \geq C^{-1} \sum_{m=1}^{N-1} \frac{1}{4} = \frac{N-1}{4C}.$$

As in the proof of Proposition 2.1.20 this implies that the semigroup $(T(t))_{t\geq 0}$ is not $\mathcal{R}$-bounded on $[0,1]$. □

2.1.4.3 The L_p-case

Before we give our proof of the Kalton–Lancien Theorem in full generality, we give the first self-contained counterexample for the maximal regularity problem for individual L_p-spaces following [Fac13a]. This special case already contains all the main ideas used in the proof of the general case. Moreover, it is easier to understand for non-experts in the geometric theory of Banach spaces because the used tools are of a rather elementary nature, whereas the general case uses deep sophisticated tools. Additionally, from the point of view of applications, the case of L_p-spaces is surely the most important case and it is therefore desirable to have a presentation as elementary as possible.

A key role in what follows is played by L_p-functions which stay away from zero in a sufficiently large set. More precisely, for $p \in [1,\infty)$ and $\varepsilon > 0$ we consider

$$M_\varepsilon^p := \left\{ f \in L_p([0,1]) : \lambda\left(\left\{ x \in [0,1] : |f(x)| \geq \varepsilon \|f\|_p \right\} \right) \geq \varepsilon \right\}.$$

Functions belonging to these sets have a very important summability property which is comparable to the L_2-case. For the proofs of the next two lemmata we follow closely the main ideas in [Sin70, §21].

Lemma 2.1.22. *For $p \in [2,\infty)$ and $\varepsilon > 0$ let $(f_m)_{m\in\mathbb{N}} \subset L_p([0,1])$ be a sequence in M_ε^p such that $\sum_{m=1}^\infty f_m$ converges unconditionally in $L_p([0,1])$. Then one has $\sum_{m=1}^\infty \|f_m\|_p^2 < \infty$.*

Proof. Since $p \in [2,\infty)$, it follows from Hölder's inequality that for all $f \in L_p([0,1])$ one has $\|f\|_2 \leq \|f\|_p$. This shows that the series $\sum_{m=1}^\infty f_m$ converges unconditionally in $L_2([0,1])$ as well. By the unconditionality of the series there exists a $K \geq 0$ such that $\left\|\sum_{m=1}^{\infty} \varepsilon_m f_m\right\|_2 \leq K$ for all $(\varepsilon_m)_{m\in\mathbb{N}} \in \{-1,1\}^{\mathbb{N}}$. Now, for all $N \in \mathbb{N}$ one has

$$\sum_{m=1}^{N} \|f_m\|_2^2 = \int_0^1 \left\|\sum_{m=1}^{N} r_m(t) f_m\right\|_2^2 dt \leq K^2.$$

Hence, $\sum_{m=1}^{\infty} \|f_m\|_2^2 < \infty$. Notice that the assumption $f_m \in M_\varepsilon^p$ implies that for all $m \in \mathbb{N}$

$$\|f_m\|_2^2 \geq \int_{|f_m| \geq \varepsilon \|f_m\|_p} |f_m(x)|^2 \, dx \geq \varepsilon^3 \|f_m\|_p^2.$$

Together with the summability shown above this yields $\sum_{m=1}^{\infty} \|f_m\|_p^2 < \infty$. □

The next lemma shows that unconditional basic sequences formed out of elements in M_ε^p behave like Hilbert space bases.

Lemma 2.1.23. *For $p \in [2,\infty)$ let $(e_m)_{m\in\mathbb{N}}$ be an unconditional normalized basic sequence in $L_p([0,1])$ for which there exists an $\varepsilon > 0$ such that $e_m \in M_\varepsilon^p$ for all $m \in \mathbb{N}$. Then the expansion*

$$\sum_{m=1}^{\infty} a_m e_m$$

converges if and only if $(a_m)_{m\in\mathbb{N}} \in \ell_2$.

Proof. Assume that the expansion $\sum_{m=1}^{\infty} a_m e_m$ converges. Since $(e_m)_{m\in\mathbb{N}}$ is an unconditional basic sequence, the series $\sum_{m=1}^{\infty} a_m e_m$ converges unconditionally in $L_p([0,1])$. By Lemma 2.1.22, one has

$$\sum_{m=1}^{\infty} |a_m|^2 = \sum_{m=1}^{\infty} \|a_m e_m\|_p^2 < \infty.$$

Conversely, we have to show that the expansion converges for all $(a_m)_{m\in\mathbb{N}} \in \ell_2$. This can be done by using similar ideas as in the proof of Lemma 2.1.22. Indeed, one has $\left\|\sum_{m=1}^{N} a_m e_m\right\| \leq K \left\|\sum_{m=1}^{N} \varepsilon_m a_m e_m\right\|$ for all $(\varepsilon_m)_{m\in\mathbb{N}} \in \{-1,1\}^N$ and all $N \in \mathbb{N}$, where $K \geq 0$ denotes the unconditional basis constant of $(e_m)_{m\in\mathbb{N}}$. Now, since for $p \geq 2$ the space $L_p([0,1])$ has type 2 (see Appendix A.3.1), we have for all $N, M \in \mathbb{N}$

$$\left\|\sum_{m=M}^{N} a_m e_m\right\|_p \leq K \int_0^1 \left\|\sum_{m=M}^{N} r_m(t) a_m e_m\right\|_p dt \leq KC \left(\sum_{m=M}^{N} |a_m|^2\right)^{1/2}$$

for some constant $C > 0$. From this it is immediate that the sequence of partial sums $(\sum_{m=1}^{N} a_m e_m)_{N\in\mathbb{N}}$ is Cauchy in $L_p([0,1])$. □

Remark 2.1.24. Of course, the lemma remains true for a semi-normalized unconditional basic sequence $(e_m)_{m\in\mathbb{N}}$ as in this case the basic sequence $(e_m)_{m\in\mathbb{N}}$ is equivalent to the normalized basic sequence $\left(\frac{e_m}{\|e_m\|_p}\right)$.

For the following counterexample on L_p-spaces our starting point is a particular basis given by the Haar system.

Definition 2.1.25. The *Haar system* is the sequence $(h_n)_{n\in\mathbb{N}}$ of functions defined by $h_1 = 1$ and for $n = 2^k + s$ (where $k = 0,1,2,\ldots$ and $s = 1,2,\ldots,2^k$) by

$$h_n(t) = \mathbb{1}_{[\frac{2s-2}{2^{k+1}},\frac{2s-1}{2^{k+1}})}(t) - \mathbb{1}_{[\frac{2s-1}{2^{k+1}},\frac{2s}{2^{k+1}})}(t) = \begin{cases} 1 & \text{if } t \in [\frac{2s-2}{2^{k+1}},\frac{2s-1}{2^{k+1}}) \\ -1 & \text{if } t \in [\frac{2s-1}{2^{k+1}},\frac{2s}{2^{k+1}}) \\ 0 & \text{otherwise} \end{cases}.$$

We now present some elementary properties of the Haar system.

Proposition 2.1.26. *The Haar system has the following properties.*

(a) *The Haar system is a basis for $L_p([0,1])$ for all $1 \le p < \infty$.*

(b) *The Haar system is an unconditional basis for $1 < p < \infty$.*

We therefore also speak of the Haar basis *instead of the Haar system.*

Remark 2.1.27. Note that the Haar system is not normalized in $L_p([0,1])$ for $p \in [1,\infty)$. Of course, we can always work with $(h_n/\|h_n\|_p)$ instead which is a normalized basis. It is however important to note that the normalization constant $\|h_n\|_p = 2^{-k/p}$ depends on p and we can therefore not simultaneously normalize $(h_n)_{n\in\mathbb{N}}$ on the L_p-scale. In the construction of the following counterexample it is crucial for the argument (more precisely for the application of the block perturbation result Proposition A.1.11) that the used basis is normalized. Therefore the construction below yields counterexamples for the maximal regularity problem on L_p for individual p, it is however not possible to obtain consistent semigroups on the L_p-scale along the line of the arguments below. This normalization issue was overlooked in the beginning of the proof in [Fac13a], nevertheless [Fac13a] still gives a rather easy counterexample for individual L_p-spaces which we present now. Later in Section 3.2.3 we will use a more involved approach to construct consistent counterexamples to the maximal regularity extrapolation problem on L_p.

Now, we are ready to give the first explicit counterexample to the maximal regularity problem on the spaces $L_p([0,1])$ for $p > 2$ or even on UMD-spaces as published in [Fac13a]. It is an interesting point that we start with the Haar basis as an unconditional basis out of which we construct a conditional one.

Theorem 2.1.28. *For $p \in (2,\infty)$ there exists a bounded analytic C_0-semigroup $(T(z))_{z\in\Sigma_{\frac{\pi}{2}}}$ on $L_p([0,1])$ which is not $\mathcal{R}$-analytic.*

Proof. Until the rest of the proof let $(h_n)_{n\in\mathbb{N}}$ denote the normalized Haar system. Choose a subsequence $(n_k)_{k\in\mathbb{N}} \subset 2\mathbb{N}$ such that the functions h_{n_k} have pairwise disjoint supports. Then $(h_{n_k})_{k\in\mathbb{N}}$ is an unconditional basic sequence equivalent to the standard basis in ℓ_p. Indeed, for any finite sequence $a_1,\dots,a_N$ we have by the disjointness of the supports

$$\left\|\sum_{k=1}^{N} a_k h_{n_k}\right\|_p^p = \sum_{k=1}^{N} \left\|a_k h_{n_k}\right\|_p^p = \sum_{k=1}^{N} |a_k|^p.$$

Choose a permutation $\pi\colon \mathbb{N} \to \mathbb{N}$ of the even numbers such that $\pi(4k) = n_k$ for all $k \in \mathbb{N}$. We now define a new system $(f_n)_{n\in\mathbb{N}}$ as

$$f_n := \begin{cases} h_{\pi(n)} \\ h_{\pi(n)} + h_{\pi(n-1)} \end{cases} = \begin{cases} h_n & n \text{ odd} \\ h_{\pi(n)} + h_{n-1} & n \text{ even.} \end{cases}$$

Notice that by the unconditionality of the Haar basis, $(h_{\pi(n)})_{n\in\mathbb{N}}$ is a Schauder basis of $L_p([0,1])$ as well. As a block perturbation of the normalized basis $(h_{\pi(n)})_{n\in\mathbb{N}}$, by Proposition A.1.11, $(f_n)_{n\in\mathbb{N}}$ is a basis for $L_p([0,1])$ as well. Further, let A be the closed linear operator on $L_p([0,1])$ given by

$$D(A) = \left\{ x = \sum_{n=1}^{\infty} a_n f_n : \sum_{n=1}^{\infty} 2^n a_n f_n \text{ exists} \right\}$$
$$A\left(\sum_{n=1}^{\infty} a_n f_n\right) = \sum_{n=1}^{\infty} 2^n a_n f_n.$$

Since $(2^n)_{n\in\mathbb{N}}$ is positive and increasing, Proposition 2.1.5 shows that $-A$ generates an analytic C_0-semigroup $(T(z))_{z\in\Sigma_{\pi/2}}$ on $L_p([0,1])$ which is given by the Schauder multipliers associated to the sequences $(e^{-2^n z})_{n\in\mathbb{N}}$. We now show that this semigroup is not $\mathcal{R}$-analytic.

Since $p \in (2,\infty)$ the basic sequences $(h_{\pi(4n)})_{n\in\mathbb{N}}$ and $(h_{4n+1})_{n\in\mathbb{N}}$ are not equivalent. Indeed, assume that this would be the case. Then on the one hand for $(h_{4n+1})_{n\in\mathbb{N}}$ the block basic sequence

$$b_k = \sum_{\substack{m:4m+1 \\ \in[2^k+1,2^{k+1}]}} h_{4m+1}$$

satisfies by the disjointness of the summands for $k \ge 2$

$$\|b_k\|_p^p = \sum_{\substack{m:4m+1 \\ \in[2^k+1,2^{k+1}]}} \|h_{4m+1}\|_p^p = \sum_{\substack{m:4m+1 \\ \in[2^k+1,2^{k+1}]}} 1 = \frac{1}{4}\cdot 2^k = 2^{k-2}.$$

Moreover, for $k \geq 2$ the function b_k satisfies $|b_k(t)| = 2^{k/p}$ on its non-vanishing part . Hence, for the normalized block basic sequence $(\tilde{b}_k)_{k\geq 2} = (\frac{b_k}{\|b_k\|_p})_{k\geq 2}$ one has $|\tilde{b}_k(t)| = 2^{2/p}$. Therefore we have

$$\lambda\Big(\Big\{t \in [0,1] : |\tilde{b}_k(t)| \geq \varepsilon \|\tilde{b}_k\|_p\Big\}\Big) = \lambda\Big(\Big\{t \in [0,1] : |\tilde{b}_k(t)| \geq \varepsilon\Big\}\Big) = \frac{1}{4}$$

for $\varepsilon \leq 2^{2/p}$. In particular, for $\varepsilon \leq \frac{1}{4}$ we have $\tilde{b}_k \in M_\varepsilon^p$. By Lemma 2.1.23 this implies that $(\tilde{b}_k)_{k\geq 2}$ is equivalent to the standard basis of ℓ_2.

Since we have assumed that the basic sequence $(h_{\pi(4n)})_{n\in\mathbb{N}}$ is equivalent to $(h_{4k+1})_{k\in\mathbb{N}}$, the block basic sequence $(c_k)_{k\geq 2}$ defined by

$$c_k = \|b_k\|_p^{-1} \sum_{\substack{m:4m+1 \\ \in[2^k+1,2^{k+1}]}} h_{\pi(4m)}$$

is semi-normalized. Recall that $(h_{\pi(4n)})_{n\in\mathbb{N}}$ is equivalent to the standard basis of ℓ_p. Therefore by Theorem A.1.6 the semi-normalized block basic sequence $(c_k)_{k\geq 2}$ is equivalent to the standard basis of ℓ_p as well. Altogether we have shown that the standard basic sequences of ℓ_p and ℓ_2 are equivalent, which is obviously wrong.

In particular, the above arguments show that there is a sequence $(a_n)_{n\in\mathbb{N}}$ which converges with respect to $(h_{\pi(2n)})_{n\in\mathbb{N}}$ but not with respect to $(h_{2n+1})_{n\in\mathbb{N}}$. Now, assume that $(T(t))_{t\geq 0}$ is $\mathcal{R}$-bounded on $[0,1]$. Then for every sequence $(q_m)_{m\in\mathbb{N}} \subset (0,1)$ we consider the associated semigroup $(\mathcal{T}(t))_{t\geq 0}$ defined on $\mathrm{Rad}(L_p([0,1]))$. In particular,

$$\mathcal{T}(1)\colon \sum_{k=1}^{N} r_k x_k \mapsto \sum_{k=1}^{N} r_k T(q_k) x_k$$

extends to a bounded operator on $\mathrm{Rad}(L_p([0,1]))$ by Theorem 2.1.18(a). We now show that

$$\sum_{m=1}^{\infty} a_m h_{\pi(2m)} r_m \tag{2.2}$$

converges in $\mathrm{Rad}(L_p([0,1]))$. Indeed, for fixed $\omega \in [0,1]$ the infinite series $\sum_{m=1}^{\infty} a_m r_m(\omega) h_{\pi(2m)}$ converges by the unconditionality of the basic sequence $(h_{\pi(2m)})_{m\in\mathbb{N}}$ as $r_m(\omega) \in \{-1,1\}$ for every $m \in \mathbb{N}$. Hence, the series (2.2) defines a measurable function as the pointwise limit of measurable functions. Moreover, if K denotes the unconditional constant of $(h_{\pi(2m)})_{m\in\mathbb{N}}$, one has for each $\omega \in [0,1]$

$$\left\| \sum_{m=1}^{\infty} r_m(\omega) a_m h_{\pi(2m)} \right\| \leq K \left\| \sum_{m=1}^{\infty} a_m h_{\pi(2m)} \right\|. \tag{2.3}$$

This shows that the series (2.2) lies in $L_1([0,1];L_p([0,1]))$. Using an analogous estimate as (2.3), one sees that the sequence of partial sums $\sum_{m=1}^{N} a_m h_{\pi(2m)} \otimes r_m$ converges to $\sum_{m=1}^{\infty} a_m h_{\pi(2m)} \otimes r_m$ in $\operatorname{Rad}(L_p([0,1]))$. Notice that $h_{\pi(2m)} = f_{2m} - f_{2m-1}$. By the continuity of $\mathcal{T}(1)$ we obtain from (2.1) that

$$\begin{aligned} g &:= \mathcal{T}(1)\left(\sum_{m=1}^{\infty} a_m (f_{2m} - f_{2m-1}) r_m\right) \\ &= \lim_{N\to\infty} \sum_{m=1}^{N} a_m \left(e^{-2^{2m} q_m} f_{2m} - e^{-2^{2m-1} q_m} f_{2m-1}\right) r_m \\ &= \lim_{N\to\infty} \sum_{m=1}^{N} e^{-2^{2m} q_m} a_m h_{\pi(2m)} r_m + a_m (e^{-2^{2m} q_m} - e^{-2^{2m-1} q_m}) h_{2m-1} r_m \end{aligned}$$

exists in $\operatorname{Rad}(L_p([0,1]))$. Now choose $q_m = \frac{\log 2}{2^{2m-1}}$ as discussed and motivated in Lemma 2.1.19. Then after choosing a subsequence (N_k) there exists a set $N \subset [0,1]$ of measure zero such that

$$\frac{1}{4} \sum_{m=1}^{N_k} \left(a_m r_m(\omega) h_{\pi(2m)} - a_m r_m(\omega) h_{2m-1}\right) \xrightarrow[k\to\infty]{} g(\omega) \quad \text{for all } \omega \in N^c. \tag{2.4}$$

Applying the coordinate functionals for $(h_m)_{m\in\mathbb{N}}$ to (2.4) we see that for $\omega \in N^c$ the unique coefficients $(h_m^*(h(\omega)))_{m\in\mathbb{N}}$ of the expansion of $g(\omega)$ with respect to $(h_m)_{m\in\mathbb{N}}$ satisfy $h_{2m-1}^*(g(\omega)) = -\frac{a_m}{4} r_m(\omega)$. Since $(h_m)_{m\in\mathbb{N}}$ is unconditional,

$$\sum_{m=1}^{\infty} a_m r_m(\omega) h_{2m-1} \quad \text{and therefore} \quad \sum_{m=1}^{\infty} a_m h_{2m-1} \quad \text{converge.}$$

This contradicts our assumptions on $(a_n)_{n\in\mathbb{N}}$ and shows that the semigroup $(T(t))_{t\geq 0}$ cannot be $\mathcal{R}$-bounded on $[0,1]$. □

Remark 2.1.29. There is nothing special about the restriction $p > 2$ above. Indeed, notice that for $1 < p < 2$ the dual semigroup $(T^*(z))_{z\in\Sigma_{\frac{\pi}{2}}}$ of the above counterexample is an analytic C_0-semigroup without maximal regularity by Proposition 1.2.3(b).

2.1.4.4 The Case of a Banach Space with an Unconditional Basis: The Kalton–Lancien Theorem

Recall that the key argument in the L_p-case is the existence of an unconditional basis for which a certain permutation of the basis has different convergence properties than the original basis. Such bases are known in the geometric theory of Banach spaces as *symmetric bases* and have been studied thoroughly.

Replacing the Haar basis of L_p with a general non-symmetric unconditional Schauder basis allows us to prove the full Kalton–Lancien Theorem with our completely new approach as done in [Fac14]. We first present the necessary results on symmetric Schauder bases. We assume that the reader knows the basic concepts from the theory of Schauder bases as presented in Appendix A.

Recall that if $(e_m)_{m\in\mathbb{N}}$ is an unconditional basis for a Banach space X, then $(e_{\pi(m)})_{m\in\mathbb{N}}$ is an unconditional basis for X for all permutations $\pi\colon \mathbb{N}\to\mathbb{N}$.

Definition 2.1.30 (Symmetric Basis). An unconditional basis $(e_m)_{m\in\mathbb{N}}$ for a Banach space X is called *symmetric* if $(e_m)_{m\in\mathbb{N}}$ is equivalent to $(e_{\pi(m)})_{m\in\mathbb{N}}$ for all permutations π of $\mathbb{N}$.

Easy examples of symmetric Schauder bases are the standard bases of the classical Banach spaces c_0 and ℓ_p for $p\in[1,\infty)$. Symmetric bases have the following well-known fundamental property.

Proposition 2.1.31. *Let $(e_m)_{m\in\mathbb{N}}$ be a symmetric basis for a Banach space X. Then there exists a constant $C\geq 0$ such that for all permutations $\pi\colon\mathbb{N}\to\mathbb{N}$, arbitrary scalars $a_1,\dots,a_N$ and all $N\in\mathbb{N}$ one has*

$$C^{-1}\left\|\sum_{m=1}^{N} a_m e_{\pi(m)}\right\| \leq \left\|\sum_{m=1}^{N} a_m e_m\right\| \leq C\left\|\sum_{m=1}^{N} a_m e_{\pi(m)}\right\|. \tag{2.5}$$

Proof. For symmetry reasons it suffices to show the first inequality. Moreover, by the uniform boundedness principle it is sufficient to show that the inequality holds for every $x=\sum_{m=1}^{\infty} a_m e_m$ with an x-dependent constant.

Therefore from now on let $x=\sum_{m=1}^{\infty} a_m e_m\in X$ be fixed. We first obverse that a symmetric basis is norm-bounded from below and from above. Indeed, assume that $(e_m)_{m\in\mathbb{N}}$ is not bounded from above. Then for $m\in\mathbb{N}$ there exist $a_{2m}\in\mathbb{R}$ and a strictly increasing sequence $(k_m)_{m\in\mathbb{N}}$ such that $\|a_{2m}e_{2m}\|=2^{-m}$ and $\|a_{2m}e_{k_m}\|\geq 1$. Then choose a permutation $\pi\colon\mathbb{N}\to\mathbb{N}$ such that $\pi(2m)=k_m$. Then $\sum_{m=1}^{\infty} a_{2m}e_{2m}$ converges, whereas $\sum_{m=1}^{\infty} a_{2m}e_{\pi(2m)}$ clearly does not. This contradicts the symmetry of the basis. Analogously, one can show that $(e_m)_{m\in\mathbb{N}}$ is bounded from below as well.

Let $M_0\in\mathbb{N}$ be fixed. For all permutations $\pi\colon\mathbb{N}\to\mathbb{N}$ we have the two inequalities

$$\left\|\sum_{m=1}^{M_0} a_m e_{\pi(m)}\right\| \leq \sup_{m\in\mathbb{N}}\|e_m\|\sum_{m=1}^{M_0}|a_m| \leq K_1 M_0\,\|x\|\sup_{m\in\mathbb{N}}\|e_m\|, \tag{2.6}$$

$$\left\|\sum_{\substack{m>M_0:\\ \pi(m)\leq M_0}} a_m e_{\pi(m)}\right\| \leq K_1 M_0\,\|x\|\sup_{m\in\mathbb{N}}\|e_m\|, \tag{2.7}$$

where K_1 is given by $\sup_{m\in\mathbb{N}} \|e_m^*\|$. As a prior step we now show that inequality (2.5) holds (x-dependent) if and only if it holds for the truncated basis $(e_m)_{m>M_0}$.

Assume that (2.5) holds for the truncated sequence $(e_m)_{m>M_0}$ and let $\pi\colon \mathbb{N} \to \mathbb{N}$ be an arbitrary permutation. Notice that the two sets $S_1 = \{m > M_0 : \pi(m) \le M_0\}$ and $S_2 = \{m \le M_0 : \pi(m) > M_0\}$ have the same cardinality. This allows us to choose a bijection τ from S_1 onto S_2. Now let $\tilde{\pi} : \mathbb{N} \to \mathbb{N}$ be given by $\tilde{\pi}(m) = \pi(m)$ for $m \in \mathbb{N} \setminus S_1$ and $\tilde{\pi}(m) = \pi(\tau(m))$ for $m \in S_1$. Then $\tilde{\pi}$ is a permutation of the set $\{m \in \mathbb{N} : m > M_0\}$. Moreover, we have

$$\sum_{\substack{m>M_0:\\ \pi(m)>M_0}} a_m e_\pi(m) = \sum_{\substack{m>M_0:\\ \tilde{\pi}(m)>M_0}} a_m e_{\tilde{\pi}(m)}.$$

Now, by the unconditionality of the basis $(e_m)_{m\in\mathbb{N}}$ one has for some $K_2 \ge 0$

$$\begin{aligned}\Big\| \sum_{\substack{m>M_0:\\ \pi(m)>M_0}} a_m e_\pi(m) \Big\| &= \Big\| \sum_{\substack{m>M_0:\\ \tilde{\pi}(m)>M_0}} a_m e_{\tilde{\pi}(m)} \Big\| \\ &\le K_2 \Big\| \sum_{m>M_0} a_m e_{\tilde{\pi}(m)} \Big\| \le C K_2 \Big\| \sum_{m>M_0}^{\infty} a_m e_m \Big\|.\end{aligned} \tag{2.8}$$

Altogether inequalities (2.6), (2.7) and (2.8) show inequality (2.5) with an x-dependent constant C.

Now assume that the inequality does not hold. Then by the prior step just proved there exists a strictly increasing sequence $(p_k)_{k\in\mathbb{N}}$ of positive integers such that for every $k \in \mathbb{N}$ there exists a permutation π_k of $[p_k + 1, p_{k+1}] \cap \mathbb{N}$ such that

$$\Big\| \sum_{m=p_k+1}^{p_{k+1}} a_m e_{\pi_k(m)} \Big\| \le 2^{-k} \qquad \text{and} \qquad \Big\| \sum_{m=p_k+1}^{p_{k+1}} a_m e_m \Big\| \ge 1.$$

Choose a permutation $\pi\colon \mathbb{N} \to \mathbb{N}$ such that $\pi_{|[p_k+1,p_{k+1}]} = \pi_k$ for all $k \in \mathbb{N}$. Then $\sum_{m=1}^{\infty} a_m e_{\pi(m)}$ clearly converges, whereas $\sum_{m=1}^{\infty} a_m e_m$ does not. This is a contradiction to the symmetry of the basis $(e_m)_{m\in\mathbb{N}}$. □

There is also a slightly weaker notion of symmetry for bases, namely that of a subsymmetric basis.

Definition 2.1.32 (Subsymmetric Basis). An unconditional basis $(e_m)_{m\in\mathbb{N}}$ for a Banach space X is called *subsymmetric* if for every strictly increasing sequence $(m_k)_{k\in\mathbb{N}}$ of positive integers the subbasis $(e_{m_k})_{k\in\mathbb{N}}$ is equivalent to $(e_m)_{m\in\mathbb{N}}$.

We now observe that every symmetric basis is subsymmetric.

Proposition 2.1.33. *Every symmetric basis is subsymmetric.*

Proof. Let $(e_m)_{m\in\mathbb{N}}$ be a symmetric basis for a Banach space and $(m_k)_{k\in\mathbb{N}}$ a strictly increasing sequence of positive integers. Let $N \in \mathbb{N}$ and $a_1, \ldots, a_N \in \mathbb{C}$. Choose a permutation π such that $\pi(k) = m_k$ for all $k \leq N$. Then by the symmetry of the basis $(e_m)_{m\in\mathbb{N}}$ and Proposition 2.1.31 there exists a constant $C \geq 0$ independent of π such that

$$\begin{aligned} C^{-1}\left\|\sum_{k=1}^{N} a_k e_{m_k}\right\| = C^{-1}\left\|\sum_{k=1}^{N} a_k e_{\pi(k)}\right\| &\leq \left\|\sum_{k=1}^{N} a_k e_k\right\| \\ &\leq C\left\|\sum_{k=1}^{N} a_k e_{\pi(k)}\right\| = C\left\|\sum_{k=1}^{N} a_k e_{m_k}\right\|. \end{aligned}$$

Hence, the two basic sequences are equivalent, which shows that $(e_m)_{m\in\mathbb{N}}$ is subsymmetric. □

The following proposition takes the role of the explicit calculations done for the Haar basis in the proof of Theorem 2.1.28. We follow the proof found in [Sin70, Proposition 23.2].

Proposition 2.1.34. *Let X be a Banach space which admits an unconditional, non-symmetric semi-normalized Schauder basis $(e_m)_{m\in\mathbb{N}}$. Then there exists a permutation of the basic sequence $(e_{2m})_{m\in\mathbb{N}}$ which is not equivalent to the basic sequence $(e_{2m-1})_{m\in\mathbb{N}}$.*

Proof. Assume that every permutation of $(e_{2m})_{m\in\mathbb{N}}$ is equivalent to the basic sequence $(e_{2m-1})_{m\in\mathbb{N}}$. Then $(e_{2m})_{m\in\mathbb{N}}$ is a symmetric basic sequence. A fortiori by Proposition 2.1.33, $(e_{2m})_{m\in\mathbb{N}}$ is subsymmetric, that is all subsequences of $(e_{2m})_{m\in\mathbb{N}}$ are equivalent to $(e_{2m})_{m\in\mathbb{N}}$. In particular, one has

$$(e_{4m}) \sim (e_{4m-2}) \sim (e_{2m}) \sim (e_{2m-1}).$$

Now, the mapping defined by

$$\begin{aligned} e_{2m-1} &\mapsto e_{4m-2} \\ e_{2m} &\mapsto e_{4m} \end{aligned}$$

is an equivalence between the two basic sequences $(e_m)_{m\in\mathbb{N}}$ and $(e_{2m})_{m\in\mathbb{N}}$. Indeed, by the unconditionality of the basis one has for a sequence $(a_m)_{m\in\mathbb{N}}$ that

$$\sum_{m=1}^{\infty} a_m e_m \text{ converges} \quad \Leftrightarrow \quad \sum_{m=1}^{\infty} a_{2m-1} e_{2m-1} \text{ and } \sum_{m=1}^{\infty} a_{2m} e_{2m} \text{ converge.}$$

Using the equivalences $(e_{2m-1}) \sim (e_{4m-2})$ and $(e_{2m}) \sim (e_{4m})$ this is equivalent to

$$\sum_{m=1}^{\infty} a_{2m-1} e_{4m-2} \text{ and } \sum_{m=1}^{\infty} a_{2m} e_{4m} \text{ converge} \quad \Leftrightarrow \quad \sum_{m=1}^{\infty} a_m e_{2m} \text{ converge.}$$

Altogether we have shown that $(e_m)_{m\in\mathbb{N}}$ is equivalent to $(e_{2m})_{m\in\mathbb{N}}$ and therefore symmetric. However, this contradicts our assumptions made on $(e_m)_{m\in\mathbb{N}}$. Therefore there must exist a permutation π of the even numbers restricting to the identity on the odd numbers such that $(e_{\pi(2m)})_{m\in\mathbb{N}}$ is not equivalent to $(e_{2m-1})_{m\in\mathbb{N}}$. $\square$

We are now ready to construct explicit counterexamples for general Banach spaces that admit a non-symmetric normalized unconditional Schauder basis.

Theorem 2.1.35. *Let X be a Banach space which admits an unconditional, non-symmetric normalized Schauder basis $(e_m)_{m\in\mathbb{N}}$. Then there exists a generator $-A$ of an analytic C_0-semigroup $(T(z))_{z\in\Sigma_{\pi/2}}$ on X that is not $\mathcal{R}$-bounded on $[0,1]$. More precisely, there exists a Schauder basis $(f_m)_{m\in\mathbb{N}}$ of X such that A is given by*

$$D(A) = \left\{ x = \sum_{m=1}^{\infty} a_m f_m : \sum_{m=1}^{\infty} 2^m a_m f_m \quad \text{exists} \right\}$$

$$A\left(\sum_{m=1}^{\infty} a_m f_m\right) = \sum_{m=1}^{\infty} 2^m a_m f_m.$$

Proof. Let $\pi\colon \mathbb{N} \to \mathbb{N}$ be a permutation as given by Proposition 2.1.34. We now let

$$f_m' = \begin{cases} e_{\pi(m)} & m \text{ odd} \\ e_{\pi(m-1)} + e_{\pi(m)} & m \text{ even} \end{cases} = \begin{cases} e_m & m \text{ odd} \\ e_{m-1} + e_{\pi(m)} & m \text{ even} \end{cases}$$

and

$$f_m'' = \begin{cases} e_{\pi(m)} + e_{\pi(m+1)} & m \text{ odd} \\ e_{\pi(m)} & m \text{ even} \end{cases} = \begin{cases} e_m + e_{\pi(m+1)} & m \text{ odd} \\ e_{\pi(m)} & m \text{ even} \end{cases}.$$

Then both f_m' and f_m'' are block perturbations of the basis $(e_{\pi(m)})_{m\in\mathbb{N}}$. Hence, by Proposition A.1.11 both are Schauder bases for X. Since $(e_{2m-1})_{m\in\mathbb{N}}$ and $(e_{\pi(2m)})_{m\in\mathbb{N}}$ are not equivalent, there exists a sequence $(a_m)_{m\in\mathbb{N}}$ such that the expansion for the coefficients $(a_m)_{m\in\mathbb{N}}$ converges with respect to $(e_{2m-1})_{m\in\mathbb{N}}$ or $(e_{\pi(2m)})_{m\in\mathbb{N}}$ but not for both. For the rest of the proof we will assume without loss of generality that the expansion converges for $(e_{\pi(2m)})_{m\in\mathbb{N}}$ (in the other case simply replace f_m' by f_m'' in the next steps). Let $f_m \coloneqq f_m'$. We now define A as in the statement of the proposition. By Proposition 2.1.5, $-A$ generates an analytic semigroup $(T(z))_{z\in\Sigma_{\pi/2}}$. Assume that $(T(t))_{t\geq 0}$ is

$\mathcal{R}$-bounded on $[0,1]$. Then for each choice of $(q_n)_{n\in\mathbb{N}} \subset (0,1)$ the associated semigroup $(\mathcal{T}(t))_{t\geq 0}$ – as defined in Definition 2.1.16 – is a C_0-semigroup on $\mathrm{Rad}(X)$ by Theorem 2.1.18(a). We now show that

$$\sum_{m=1}^{\infty} a_m e_{\pi(2m)} \otimes r_m \tag{2.9}$$

converges in $\mathrm{Rad}(X)$. Indeed, for fixed $\omega \in [0,1]$ the series $\sum_{m=1}^{\infty} a_m r_m(\omega) e_{\pi(2m)}$ converges by the unconditionality of $(e_{\pi(2m)})_{m\in\mathbb{N}}$ as $r_m(\omega) \in \{-1,1\}$ for every $m \in \mathbb{N}$. Hence, the above series defines a measurable function as the pointwise limit of measurable functions. Moreover, if K denotes the unconditional basis constant of $(e_{\pi(2m)})_{m\in\mathbb{N}}$, one has for each $\omega \in [0,1]$

$$\left\| \sum_{m=1}^{\infty} r_m(\omega) a_m e_{\pi(2m)} \right\| \leq K \left\| \sum_{m=1}^{\infty} a_m e_{\pi(2m)} \right\|. \tag{2.10}$$

This shows that the series (2.9) lies in $L_1([0,1];X)$. Using an analogous estimate as (2.10), one sees that the sequence of partial sums $\sum_{m=1}^{N} a_m e_{\pi(2m)} \otimes r_m$ converges to $\sum_{m=1}^{\infty} a_m e_{\pi(2m)} \otimes r_m$ in $\mathrm{Rad}(X)$. Notice that $e_{\pi(2m)} = f_{2m} - f_{2m-1}$. By the continuity of $\mathcal{T}(1)$, we obtain from (2.1) that

$$\begin{aligned} h &:= \mathcal{T}(1)\left(\sum_{m=1}^{\infty} a_m (f_{2m} - f_{2m-1}) \otimes r_m \right) \\ &= \lim_{N\to\infty} \sum_{m=1}^{N} a_m \left(e^{-2^{2m} q_m} f_{2m} - e^{-2^{2m-1} q_m} f_{2m-1} \right) \otimes r_m \\ &= \lim_{N\to\infty} \sum_{m=1}^{N} e^{-2^{2m} q_m} a_m e_{\pi(2m)} \otimes r_m + a_m (e^{-2^{2m} q_m} - e^{-2^{2m-1} q_m}) e_{2m-1} \otimes r_m \end{aligned}$$

exists in $\mathrm{Rad}(X)$. Now choose $q_m = \frac{\log 2}{2^{2m-1}}$ as discussed in Lemma 2.1.19. Then after choosing a subsequence (N_k) there exists a set $N \subset [0,1]$ of measure zero such that

$$\frac{1}{4} \sum_{m=1}^{N_k} \left(a_m r_m(\omega) e_{\pi(2m)} - a_m r_m(\omega) e_{2m-1} \right) \xrightarrow[k\to\infty]{} h(\omega) \quad \text{for all } \omega \in N^c. \tag{2.11}$$

Applying the coordinate functionals for $(e_m)_{m\in\mathbb{N}}$ to (2.11) shows that for $\omega \in N^c$ the unique coefficients $e_m^*(h(\omega))$ of the expansion of $h(\omega)$ with respect to $(e_m)_{m\in\mathbb{N}}$ satisfy $e_{2m-1}^*(h(\omega)) = -\frac{a_m}{4} r_m(\omega)$. Since $(e_m)_{m\in\mathbb{N}}$ is unconditional,

$$\sum_{m=1}^{\infty} a_m r_m(\omega) e_{2m-1} \quad \text{and therefore} \quad \sum_{m=1}^{\infty} a_m e_{2m-1} \quad \text{converge.}$$

This contradicts our choice of $(a_m)_{m\in\mathbb{N}}$ and therefore the semigroup $(T(t))_{t\geq 0}$ cannot be $\mathcal{R}$-bounded on $[0,1]$. $\square$

Remark 2.1.36. Notice that in the spaces $L_p([0,1])$ for $p \in (1,\infty) \setminus \{2\}$ every unconditional basis is non-symmetric [Sin70, Theorem 21.1]. So the above construction in particular works for the normalized Haar basis. This was exactly done a little more explicitly in the proof of Theorem 2.1.28.

Remark 2.1.37. Let X be a Banach space which admits a non-symmetric unconditional normalized Schauder basis. Then we can choose two Schauder bases $(f'_m)_{m\in\mathbb{N}}$ and $(f''_m)_{m\in\mathbb{N}}$ as in the proof of Theorem 2.1.35. It is a direct consequence of Corollary 2.1.7 that if X is a Banach space with property (Δ), then at least one of these two bases must be conditional. For if both bases were unconditional, Proposition 2.1.6 would imply that both the Schauder multipliers associated to the sequence $(2^m)_{m\in\mathbb{N}}$ with respect to the bases $(f'_m)_{m\in\mathbb{N}}$ and $(f''_m)_{m\in\mathbb{N}}$ would have a bounded H^∞-calculus of angle 0. Moreover, if X has property (Δ), Corollary 2.1.7 shows that this implies that both have maximal regularity. This, however, contradicts the assertion of Theorem 2.1.35.

One can also obtain this result directly without any geometric restrictions on the Banach space. Indeed, assume that both $(f'_m)_{m\in\mathbb{N}}$ and $(f''_m)_{m\in\mathbb{N}}$ are unconditional. By the unconditionality of the two bases there exists a constant $M \ge 0$ such that the projections P_A for the two bases are uniformly bounded by M for all $A \in \mathcal{P}(\mathbb{N})$. In particular, one has

$$\begin{aligned}\left\|\sum_{m=1}^N a_m e_{\pi(2m)}\right\| &= \left\|\sum_{m=1}^N a_m e_{\pi(2m-1)} - \left(\sum_{m=1}^N a_m e_{\pi(2m)} + a_m e_{\pi(2m-1)}\right)\right\| \\ &= \left\|\sum_{m=1}^N a_m f'_{2m-1} - \sum_{m=1}^N a_m f'_{2m}\right\| \ge M^{-1}\left\|P_{2\mathbb{N}-1}\left(\sum_{m=1}^N a_m f'_{2m-1} - \sum_{m=1}^N a_m f'_{2m}\right)\right\| \\ &= M^{-1}\left\|\sum_{m=1}^N a_m f'_{2m-1}\right\| = M^{-1}\left\|\sum_{m=1}^N a_m e_{2m-1}\right\|.\end{aligned}$$

Completely analogously, one has

$$\begin{aligned}\left\|\sum_{m=1}^N a_m e_{2m-1}\right\| &= \left\|\sum_{m=1}^N a_m e_{\pi(2m)} - \left(\sum_{m=1}^N a_m e_{\pi(2m-1)} + a_m e_{\pi(2m)}\right)\right\| \\ &= \left\|\sum_{m=1}^N a_m f''_{2m} - \sum_{m=1}^N a_m f''_{2m-1}\right\| \ge M^{-1}\left\|P_{2\mathbb{N}}\left(\sum_{m=1}^N a_m f''_{2m} - \sum_{m=1}^N a_m f''_{2m-1}\right)\right\| \\ &= M^{-1}\left\|\sum_{m=1}^N a_m f''_{2m}\right\| = M^{-1}\left\|\sum_{m=1}^N a_m e_{\pi(2m)}\right\|.\end{aligned}$$

Both inequalities together show that the two basic sequences $(e_{2m-1})_{m\in\mathbb{N}}$ and $(e_{\pi(2m)})_{m\in\mathbb{N}}$ are equivalent. This, however, contradicts our choice of the permutation π. Therefore we have shown that a Banach space admitting an unconditional non-symmetric normalized basis has a conditional basis as well.

We have almost proved the Kalton–Lancien Theorem. However, at the moment we do not yet know which Banach spaces admit an unconditional normalized non-symmetric basis. We will soon give a very satisfying answer to this question. For this we need the following technical lemma.

Lemma 2.1.38. *Let X be a Banach space with a normalized symmetric basis $(e_m)_{m\in\mathbb{N}}$ and $(u_m)_{m\in\mathbb{N}}$ a normalized constant coefficient block basic sequence. Then the closed subspace spanned by $(u_m)_{m\in\mathbb{N}}$ is complemented.*

Proof. For a strictly increasing sequence $(p_m)_{m\in\mathbb{N}}$ let the constant block basic sequence be given by

$$u_m = c_m \sum_{k\in A_m} e_k$$

for some subset A_m of $(p_{m-1}, p_m] \cap \mathbb{N}$. For each $m \in \mathbb{N}$ let Π_m denote the set of all permutations $\pi\colon \mathbb{N} \to \mathbb{N}$ such that for each $k \in \{1,\dots,m\}$ the mapping π restricted to A_k is a cyclic permutation of the elements of A_k and $\pi(j) = j$ for all $j \notin \cup_{k=1}^m A_k$. For every $\pi \in \Pi_m$ for some $m \in \mathbb{N}$ we let

$$T_{m,\pi}\left(\sum_{j=1}^{\infty} a_j e_j\right) = \sum_{j=1}^{\infty} a_j e_{\pi(j)}.$$

It follows from Proposition 2.1.31 that there exists a constant $C \geq 0$ such that $\|T_{m,\pi}\| \leq C$ for all $m \in \mathbb{N}$ and $\pi \in \Pi_m$. We now define the following operator which averages over all $\pi \in \Pi_m$: for $m \in \mathbb{N}$ let

$$T_m = \frac{1}{|\Pi_m|} \sum_{\pi\in\Pi_m} T_{m,\pi}.$$

For $k \in \mathbb{N}$ let Π_{A_k} be the set of all cyclic permutations of A_k. Notice that for $X \ni x = \sum_{j=1}^{\infty} a_j e_j$ one has

$$\begin{aligned}
T_m(x) &= \frac{1}{|\Pi_m|} \sum_{\pi\in\Pi_m} \sum_{k=1}^{m} \sum_{j\in A_k} a_j e_{\pi(j)} + \sum_{j\notin \cup_{k=1}^m A_k} a_j e_j \\
&= \sum_{k=1}^{m} \frac{1}{|\Pi_m|} \frac{|\Pi_m|}{|A_k|} \sum_{\pi\in\Pi_{A_k}} \sum_{j\in A_k} a_j e_{\pi(j)} + \sum_{j\notin \cup_{k=1}^m A_k} a_j e_j \\
&= \sum_{k=1}^{m} \left(\frac{1}{|A_k|} \sum_{j\in A_k} a_j\right) \sum_{j\in A_k} e_j + \sum_{j\notin \cup_{k=1}^m A_k} a_j e_j.
\end{aligned}$$

Moreover, we have the estimate $\|T_m\| \leq C$. Now, for $m \in \mathbb{N}$ we let

$$P_m(x) = \sum_{k=1}^{m} \left(\frac{1}{|A_k|} \sum_{j \in A_k} a_j \right) \sum_{j \in A_k} e_j.$$

It is easy to see that P_m is a projection onto $\overline{\operatorname{span}}\{u_k : k \in \{1, \ldots, m\}\}$. By the unconditionality of the basis, we have for all $m \in \mathbb{N}$

$$\|P_m\| = \|P_{\cup_{k=1}^m A_k} T_m\| \leq \|P_{\cup_{k=1}^m A_k}\| \|T_m\| \leq C'$$

for some $C' \geq 0$. From this it follows directly that

$$P = \sum_{k=1}^{\infty} \left(\frac{1}{|A_k|} \sum_{j \in A_k} a_j \right) \sum_{j \in A_k} e_j = \sum_{k=1}^{\infty} \left(\frac{1}{c_k |A_k|} \sum_{j \in A_k} a_j \right) u_k$$

is a projection onto $\overline{\operatorname{span}}\{u_m : m \in \mathbb{N}\}$ with $\|P\| \leq C'$. □

The following theorem due to J. Lindenstrauss and M. Zippin [LZ69, Note (1) at the end] shows that our methods developed so far are applicable to all Banach spaces with an unconditional basis with the exception of c_0, ℓ_1 and ℓ_2. It is interesting to note that the theorem is not formulated in the main body of the article and was later added in proof. This may be the reason for this result to be not mentioned in the standard references on Banach space geometry and to be not widely known. Nevertheless, the result can be proved using known arguments. We adapt the proof on the uniqueness of unconditional bases found in [AK06, Theorem 9.3.1].

Theorem 2.1.39 (Existence of Non-Symmetric Bases). *Let X be a Banach space with an unconditional basis. If X is not isomorphic to c_0, ℓ_1 or ℓ_2, then X has a normalized unconditional, non-symmetric basis.*

Proof. Assume that every normalized unconditional Schauder basis for X is symmetric. Let $(e_m)_{m \in \mathbb{N}}$ be a normalized unconditional, hence symmetric, basis for X. We next show that $(e_m)_{m \in \mathbb{N}}$ is equivalent to all of its normalized constant coefficient block basic sequences, i.e. $(e_m)_{m \in \mathbb{N}}$ is perfectly homogeneous (see also Definition A.1.8).

This can be seen as follows. Fix a normalized constant coefficient block basic sequence $(u_m)_{m \in \mathbb{N}}$ with respect to $(e_m)_{m \in \mathbb{N}}$ that has infinitely many blocks of size k for all $k \in \mathbb{N}$. That is

$$|S_k| := |\{n \in \mathbb{N} : |\operatorname{supp} u_n| = k\}| = \infty$$

for every $k \in \mathbb{N}$. The main idea of the proof is that $(u_m)_{m \in \mathbb{N}}$ can be seen as a universal constant coefficient block basic sequence that contains copies of all constant coefficient block basic sequences. The general case can then be reduced to the case of this fixed block basic sequence. We now give the details.

Let Y be the closed linear span of the sequence $(u_m)_{m\in\mathbb{N}}$. Then Y is complemented in X by Lemma 2.1.38. Moreover, the subsequence of $(u_m)_{m\in\mathbb{N}}$ which consists of all blocks whose supports have size 1 is, since $(u_m)_{m\in\mathbb{N}}$ is normalized, exactly a subsequence of the form $(e_{m_k})_{k\in\mathbb{N}}$ of the basis $(e_m)_{m\in\mathbb{N}}$. Since $(e_m)_{m\in\mathbb{N}}$ is symmetric and therefore a fortiori subsymmetric by Proposition 2.1.33, this subsequence of $(u_m)_{m\in\mathbb{N}}$ is equivalent to $(e_m)_{m\in\mathbb{N}}$. This shows that the closed subspace spanned by this sequence is isometrically isomorphic to X. Moreover, by the unconditionality of $(u_m)_{m\in\mathbb{N}}$ this subspace is complemented in Y. By the subsymmetry of the basis $(e_m)_{m\in\mathbb{N}}$ one has

$$X \oplus X = [e_m] \oplus [e_m] \simeq [e_{2m}] \oplus [e_{2m-1}] \simeq [e_m] = X.$$

In a similar spirit, we can split the natural numbers into two subsets S_1 and S_2 such that both contain for every $k \in \mathbb{N}$ infinitely many blocks of size k. That is

$$|S_{1,k}| := |\{n \in S_1 : |\operatorname{supp} u_n| = k\}| = \left|S_{2,k}\right| := |\{n \in S_1 : |\operatorname{supp} u_n| = k\}| = \infty$$

for all $k \in \mathbb{N}$. Choosing bijections between $S_{1,k}$ and S_k and respectively between $S_{2,k}$ and S_k, we see by the symmetry of the basis $(e_m)_{m\in\mathbb{N}}$ that $[u_m]_{m\in\mathbb{N}}$ is isomorphic to $[u_m]_{m\in S_1}$ and $[u_m]_{m\in S_2}$. Then we have

$$Y \oplus Y = [u_m]_{m\in\mathbb{N}} \oplus [u_m]_{m\in\mathbb{N}} \simeq [u_m]_{m\in S_1} \oplus [u_m]_{m\in S_2} \simeq [u_m]_{m\in\mathbb{N}} = Y.$$

Now by Pełczyński's decomposition technique (Theorem A.2.2) we get $X \simeq Y$. This shows that $(u_m)_{m\in\mathbb{N}}$ can be identified with an unconditional basis of X. Hence, by assumption, $(u_m)_{m\in\mathbb{N}}$ is symmetric. In particular, $(u_m)_{m\in\mathbb{N}}$ is subsymmetric by Proposition 2.1.33 and therefore equivalent to all of its subsequences. We have already seen above that $(u_m)_{m\in\mathbb{N}}$ contains a subsequence which is equivalent to $(e_m)_{m\in\mathbb{N}}$. This shows that $(u_m)_{m\in\mathbb{N}}$ is equivalent to $(e_m)_{m\in\mathbb{N}}$. On the other hand the fact that $|S_k| = \infty$ for all $k \in \mathbb{N}$ guarantees by the symmetry of the basis $(e_m)_{m\in\mathbb{N}}$ that every normalized constant block basic sequence is equivalent to a subsequence of $(u_m)_{m\in\mathbb{N}}$, which in turn is equivalent to $(u_m)_{m\in\mathbb{N}}$. Altogether we have shown that every normalized constant block basic sequence is equivalent to $(e_m)_{m\in\mathbb{N}}$. This is exactly the perfect homogeneity of $(e_m)_{m\in\mathbb{N}}$.

By Zippin's characterization of Banach spaces with a perfectly homogeneous basis (Theorem A.1.9), X is isomorphic to c_0 or ℓ_p for some $1 \le p < \infty$. However, we will see later in the self-contained Proposition 3.2.11 the well-known result that for $p \in (1,\infty) \setminus \{2\}$ the spaces ℓ_p have a non-symmetric, unconditional Schauder basis. □

Remark 2.1.40. Conversely, all normalized unconditional bases in the spaces c_0, ℓ_1 and ℓ_2 are symmetric for the simple reason that all normalized unconditional bases in these spaces are unique up to equivalence [LT77, Theorem 2.b.10].

Remark 2.1.41. With the so far developed results one can also show that each Banach space X which admits a Schauder basis even has a conditional basis. Indeed, if X admits an unconditional basis and is not isomorphic to c_0, ℓ_1 or ℓ_2, then by Theorem 2.1.39 X has a normalized, non-symmetric unconditional basis. In this case Remark 2.1.37 shows that X has a conditional basis. Moreover, we have already essentially used in Proposition 2.1.20 and Proposition 2.1.21 that c_0 and ℓ_1 have conditional bases. It therefore remains to show that ℓ_2 has a conditional basis. Since all separable Hilbert spaces are isomorphic, it suffices to find a conditional basis for an arbitrary separable Hilbert space. It follows from Example 2.1.15 that for suitable A_2-weights $w \in \mathcal{A}_2(\mathbb{T})$ the trigonometric basis with respect to the enumeration $\{0,-1,1,-2,2,\ldots\}$ is a conditional basis for the Hilbert space $L_2([0,2\pi],w)$.

Now, the main result of this section (published in [Fac14]) is an easy consequence of our developed methods. The first part of the following theorem is the famous result by Kalton and Lancien. The advantage of our method is that we obtain a concrete representation of the counterexamples for the maximal regularity problem.

Theorem 2.1.42 (Kalton–Lancien Theorem). *Let X be a Banach space with an unconditional basis. Assume that X has (MRP). Then $X \simeq \ell_2$. More precisely, for $X \not\simeq \ell_2$ there exists a Schauder basis $(f_m)_{m\in\mathbb{N}}$ for X such that $-A$ given by*

$$D(A) = \left\{ x = \sum_{m=1}^{\infty} a_m f_m : \sum_{m=1}^{\infty} 2^m a_m f_m \quad \text{exists} \right\}$$
$$A\left(\sum_{m=1}^{\infty} a_m f_m\right) = \sum_{m=1}^{\infty} 2^m a_m f_m$$

generates an analytic C_0-semigroup $(T(z))_{z\in\Sigma_{\pi/2}}$ on X that is not $\mathcal{R}$-bounded on $[0,1]$ and in particular does not have maximal regularity.

Proof. If X is not isomorphic to c_0 or ℓ_1, Theorem 2.1.39 shows that we can apply Theorem 2.1.35 which yields the desired counterexample. In the cases of $X \simeq c_0$ or $X \simeq \ell_1$ we showed the theorem by hand in Proposition 2.1.20 and Proposition 2.1.21. □

2.1.4.5 Non-$\mathcal{R}$-Sectoriality of the Counterexamples

Let X be a Banach space with an unconditional basis which is not isomorphic to a Hilbert space. Notice that Theorem 2.1.42 gives us a rather explicit method to construct a sectorial operator A with $\omega(A) = 0$ such that $-A$ generates a semigroup which is not $\mathcal{R}$-bounded on $[0,1]$. This shows by Theorem 1.2.9 and the last part of Proposition 1.2.6 that $\omega_R(A) \geq \frac{\pi}{2}$. Note that this does not yet show that A is not $\mathcal{R}$-sectorial. In this subsection we want to clarify that this is always the case.

For this note that if A is a Schauder multiplier associated to the sequence $(2^m)_{m\in\mathbb{N}}$, then for $\alpha > 0$ the fractional powers A^α are Schauder multipliers associated to the sequences $(2^{m\alpha})_{m\in\mathbb{N}}$. One can now directly verify that all the above results also work with $(2^m)_{m\in\mathbb{N}}$ replaced by $(2^{m\alpha})_{m\in\mathbb{N}}$ and the above assertion is proved. As an alternative approach one can also directly mimic the construction of the associated semigroup as introduced in Theorem 2.1.18. We now present the details of this approach. We replace Theorem 2.1.18 by the following analogue.

Proposition 2.1.43. *Let A be an $\mathcal{R}$-sectorial operator. Then there exists a constant $C \geq 0$ such that for all $(q_n)_{n\in\mathbb{N}} \subset \mathbb{R}_-$ the associated operator*

$$\mathcal{R}\colon \sum_{n=1}^{N} r_n x_n \mapsto \sum_{n=1}^{N} r_n q_n R(q_n, A) x_n$$

defined on the finite Rademacher sums extends to a bounded operator on $\operatorname{Rad}(X)$ *with operator norm at most C.*

Proof. Since A is $\mathcal{R}$-sectorial, one has $C := \mathcal{R}\{\lambda R(\lambda, A) : \lambda \in \mathbb{R}_-\} < \infty$. Hence, for all finite Rademacher sums we have by the definition of $\mathcal{R}$-boundedness

$$\left\| \sum_{n=1}^{N} r_n q_n R(q_n, A) x_n \right\| \leq C \left\| \sum_{n=1}^{N} r_n x_n \right\|. \qquad \square$$

One now again uses the freedom in the choice of the sequence $(q_n)_{n\in\mathbb{N}}$. We maximize the difference between two entries of the multiplier sequences for the resolvents. The following result then replaces Lemma 2.1.19. However, for later use we work this time with a general sequence $(\gamma_m)_{m\in\mathbb{N}}$ instead of the particular sequence $(2^m)_{m\in\mathbb{N}}$. We think that this also helps to understand the special role played by lacunary sequences as $(2^m)_{m\in\mathbb{N}}$.

Lemma 2.1.44. *For $\gamma_m > \gamma_{m-1} > 0$ consider the function $d(t) := t[(t+\gamma_{m-1})^{-1} - (t+\gamma_m)^{-1}]$ on $\mathbb{R}_+$. Then d has a maximum which is bigger than $\frac{1}{2}\frac{\gamma_m - \gamma_{m-1}}{\gamma_m + \gamma_{m-1}}$.*

Proof. By the mean value theorem we have for some $\xi \in (\gamma_{m-1}, \gamma_m)$ and all $t > 0$ that

$$\frac{1}{t+\gamma_{m-1}} - \frac{1}{t+\gamma_m} = (\gamma_m - \gamma_{m-1})\frac{1}{(t+\xi)^2} \geq (\gamma_m - \gamma_{m-1})\frac{1}{(t+\gamma_m)^2}.$$

One now easily verifies that the function $t \mapsto (\gamma_m - \gamma_{m-1})\frac{t}{(t+\gamma_m)^2}$ has a unique maximum for $t = \gamma_m$. In particular, one has

$$\max_{t>0} d(t) \geq d(\gamma_m) = \frac{1}{2}\frac{\gamma_m - \gamma_{m-1}}{\gamma_m + \gamma_{m-1}}. \qquad \square$$

We can now prove the following analogue of Theorem 2.1.35 dealing with $\mathcal{R}$-sectoriality.

Theorem 2.1.45. *Let X be a Banach space which admits a normalized non-symmetric unconditional basis. Then there exists a sectorial operator A with $\omega(A) = 0$ which is not $\mathcal{R}$-sectorial. More precisely, there exists a Schauder basis $(f_m)_{m\in\mathbb{N}}$ of X such that A is given by*

$$D(A) = \left\{ x = \sum_{m=1}^{\infty} a_m f_m : \sum_{m=1}^{\infty} 2^m a_m f_m \quad \text{exists} \right\}$$
$$A\left(\sum_{m=1}^{\infty} a_m f_m\right) = \sum_{m=1}^{\infty} 2^m a_m f_m.$$

Proof. Choose the Schauder basis $(f_m)_{m\in\mathbb{N}}$ and the sequence $(a_m)_{m\in\mathbb{N}}$ as in the proof of Theorem 2.1.35. We again only consider the case where the expansion for $(a_m)_{m\in\mathbb{N}}$ converges with respect to $(e_{\pi(2m)})_{m\in\mathbb{N}}$, but not with respect to $(e_{2m-1})_{m\in\mathbb{N}}$. Let $(q_m)_{m\in\mathbb{N}} \subset \mathbb{R}_-$ be a sequence to be chosen later. Then it follows from Proposition 2.1.43 that the operator $\mathcal{R}\colon \operatorname{Rad}(X) \to \operatorname{Rad}(X)$ associated to the sequence $(q_m)_{m\in\mathbb{N}}$ is bounded. We now apply $\mathcal{R}$ to the element $x = \sum_{m=1}^{\infty} a_m e_{\pi(2m)} \otimes r_m$ of $\operatorname{Rad}(X)$. Because of $e_{\pi(2m)} = f_{2m} - f_{2m-1}$ we obtain

$$\begin{aligned}
\mathcal{R}(x) = \mathcal{R}\left(\sum_{m=1}^{\infty} a_m(f_{2m} - f_{2m-1})r_m\right) &= \sum_{m=1}^{\infty} r_m \frac{a_m q_m}{q_m - \gamma_{2m}} f_{2m} - r_m \frac{a_m q_m}{q_m - \gamma_{2m-1}} f_{2m-1} \\
&= \sum_{m=1}^{\infty} r_m \frac{a_m q_m}{q_m - \gamma_{2m}} (e_{\pi(2m)} + e_{2m-1}) - r_m \frac{a_m q_m}{q_m - \gamma_{2m-1}} e_{2m-1} \\
&= \sum_{m=1}^{\infty} r_m \frac{a_m q_m}{q_m - \gamma_{2m}} e_{\pi(2m)} + r_m a_m q_m \left(\frac{1}{q_m - \gamma_{2m}} - \frac{1}{q_m - \gamma_{2m-1}}\right) e_{2m-1}.
\end{aligned}$$

We now want to choose $(q_m)_{m\in\mathbb{N}}$ in such a way that the last term in the bracket is large. Notice that if we set $\gamma_m = 2^m$, then by Lemma 2.1.44 for $t = \gamma_{2m}$ one has $t[(t+\gamma_{2m-1})^{-1} - (t+\gamma_{2m})^{-1}] = \frac{1}{6}$. Hence, for the choice $q_m = -\gamma_{2m}$ we obtain

$$\mathcal{R}(x) = \sum_{m=1}^{\infty} r_m \frac{1}{2} a_m e_{\pi(2m)} - r_m \frac{1}{6} a_m e_{2m-1}.$$

As in the proof of Theorem 2.1.35 one deduces from the above equality that $\sum_{m=1}^{\infty} a_m e_{2m-1}$ converges. This contradicts the choice of $(a_m)_{m\in\mathbb{N}}$ and therefore A cannot be $\mathcal{R}$-sectorial. □

2.1.4.6 Positive Analytic Semigroups without Maximal Regularity

Let X be a Banach space that admits a non-symmetric normalized unconditional Schauder basis $(e_m)_{m\in\mathbb{N}}$ and $\pi\colon \mathbb{N} \to \mathbb{N}$ a permutation of the even numbers such that $(e_{\pi(2m)})_{m\in\mathbb{N}}$ and $(e_{2m-1})_{m\in\mathbb{N}}$ are not equivalent. Recall that the existence of such a permutation is guaranteed by Proposition 2.1.34. We further can assume after equivalent renorming that the basis $(e_m)_{m\in\mathbb{N}}$ is 1-unconditional, i.e. the unconditional basis constant of $(e_m)_{m\in\mathbb{N}}$ is equal to one. Then one can easily verify that $(e_m)_{m\in\mathbb{N}}$ via

$$x = \sum_{m=1}^{\infty} a_m e_m \geq 0 \quad :\Leftrightarrow \quad a_m \geq 0 \quad \text{for all } m \in \mathbb{N}$$

the space X becomes a Banach lattice. For the basis $(f_m)_{m\in\mathbb{N}}$ of X given by

$$f_m = \begin{cases} e_m & m \text{ odd} \\ e_{m-1} + e_{\pi(m)} & m \text{ even} \end{cases}$$

we again consider the Schauder multiplier associated to some real sequence $(\gamma_m)_{m\in\mathbb{N}}$, that is $A(\sum_{m=1}^{\infty} a_m f_m) = \sum_{m=1}^{\infty} \gamma_m a_m f_m$ with its natural domain. We now want to study the positivity of the formal semigroup $(e^{-tA})_{t\geq 0}$ generated by $-A$ with respect to the lattice structure induced by $(e_m)_{m\in\mathbb{N}}$. For the positivity it is necessary and sufficient that $e^{-tA} e_m \geq 0$ for all $m \in \mathbb{N}$ and all $t \geq 0$. For odd m this is satisfied because of $e^{-tA} e_m = e^{-tA} f_m = e^{-\gamma_m t} e_m$. However, if m is even, one has

$$\begin{aligned} e^{-tA} e_m &= e^{-tA}(f_{\pi^{-1}(m)} - e_{\pi^{-1}(m)-1}) = e^{-tA}(f_{\pi^{-1}(m)} - f_{\pi^{-1}(m)-1}) \\ &= e^{-t\gamma_{\pi^{-1}(m)}} f_{\pi^{-1}(m)} - e^{-t\gamma_{\pi^{-1}(m)-1}} f_{\pi^{-1}(m)-1} \\ &= e^{-t\gamma_{\pi^{-1}(m)}} (e_{\pi^{-1}(m)-1} + e_m) - e^{-t\gamma_{\pi^{-1}(m)-1}} e_{\pi^{-1}(m)-1} \\ &= (e^{-t\gamma_{\pi^{-1}(m)}} - e^{-t\gamma_{\pi^{-1}(m)-1}}) e_{\pi^{-1}(m)-1} + e^{-t\gamma_{\pi^{-1}(m)}} e_m. \end{aligned} \tag{2.12}$$

Therefore $(e^{-tA})_{t\geq 0}$ is positive if and only if $\gamma_m \leq \gamma_{m-1}$ for all even $m \in \mathbb{N}$. Note that the usual lacunary sequences such as $(2^m)_{m\in\mathbb{N}}$ used until now do not have this property. However, in Lemma 2.1.44 and in the proof of

Theorem 2.1.45 only the difference between γ_m and γ_{m-1} plays a role. This allows us to obtain positive semigroups by rather easy modifications of the original argument. The following result is new and has not been published yet. Note that also this time the exact structure of the counterexamples to the maximal regularity problem obtained with our new methods allows us to give such precise information on the nature of the counterexamples.

Theorem 2.1.46. *Let X be a Banach space which admits a normalized non-symmetric 1-unconditional Schauder basis $(e_m)_{m\in\mathbb{N}}$. We consider X as a Banach lattice with the order induced by $(e_m)_{m\in\mathbb{N}}$. Then there exists a non-$\mathcal{R}$-sectorial operator A with $\omega(A) = 0$ such that $-A$ generates a positive analytic C_0-semigroup on X.*

Proof. One proceeds as in the proof of Theorem 2.1.35. We again only consider the case where the expansion for $(a_m)_{m\in\mathbb{N}}$ converges with respect to $(e_{\pi(2m)})_{m\in\mathbb{N}}$, but not with respect to $(e_{2m-1})_{m\in\mathbb{N}}$. The other case can be proved analogously. We use the following twisted version of the lacunary sequence $(2^m)_{m\in\mathbb{N}}$:

$$\gamma_m = \begin{cases} 2^{m+1} & m \text{ odd} \\ 2^{m-1} & m \text{ even.} \end{cases}$$

The first elements of this sequence are given by $\gamma_1 = 4$, $\gamma_2 = 2$, $\gamma_3 = 16$ and $\gamma_4 = 8$. By definition, one has $\gamma_m < \gamma_{m-1}$ for all even $m \in \mathbb{N}$. By the above observation this implies that the formal semigroup generated by the Schauder multiplier associated to $(-\gamma_m)_{m\in\mathbb{N}}$ with respect to $(f_m)_{m\in\mathbb{N}}$ is positive. It therefore remains to show that the multiplier A associated to $(\gamma_m)_{m\in\mathbb{N}}$ is a sectorial operator which is not $\mathcal{R}$-sectorial. For this let us consider the sequence $(e^{-t\gamma_m})_{m\in\mathbb{N}}$ for $t > 0$. For its total variation one obtains

$$\begin{aligned} &\sum_{m=1}^{\infty} e^{-t2^{2m-1}} - e^{-t2^{2m}} + e^{-t2^{2m-1}} - e^{-t2^{2(m+1)}} \\ &\quad \le t \sum_{m=1}^{\infty} (2^{2m} - 2^{2m-1}) e^{-t2^{2m-1}} + (2^{2m+2} - 2^{2m-1}) e^{-t2^{2m-1}} \\ &\quad = 8t \sum_{m=1}^{\infty} 2^{2m-1} e^{-t2^{2m-1}} = 8t \sum_{m=1}^{\infty} \int_{2^{2m-1}}^{2^{2m}} e^{-t2^{2m-1}} \, ds \le 8t \sum_{m=1}^{\infty} \int_{2^{2m-1}}^{2^{2m}} e^{-ts/2} \, ds \\ &\quad = 8t \int_2^{\infty} e^{-ts/2} \, ds = 16 e^{-t}. \end{aligned}$$

Of course, an analogous estimate can be made if one replaces $(\gamma_m)_{m\in\mathbb{N}}$ by $(\gamma_m^{\alpha})_{m\in\mathbb{N}}$ for some $\alpha > 0$. From these observations it follows as in the proof of Proposition 2.1.5 that A is sectorial with $\omega(A) = 0$. Moreover, with our choice of $(\gamma_m)_{m\in\mathbb{N}}$ we can repeat the proof of Proposition 2.1.45 for the non-$\mathcal{R}$-analyticity of A involving only minor changes. Indeed using the same notation, one sees that

$$\sum_{m=1}^{\infty} r_m \frac{a_m q_m}{q_m - \gamma_{2m}} e_{\pi(2m)} + r_m a_m q_m \left(\frac{1}{q_m - \gamma_{2m}} - \frac{1}{q_m - \gamma_{2m-1}} \right) e_{2m-1}$$

exists in $\operatorname{Rad}(X)$ for all $(q_m)_{m\in\mathbb{N}} \subset \mathbb{R}_-$. Now choose $q_m = -2^{2m-1}$. Then we obtain that

$$\sum_{m=1}^{\infty} \frac{1}{2} r_m a_m e_{\pi(2m)} + \frac{1}{6} r_m a_m e_{2m-1}.$$

exists in $\operatorname{Rad}(X)$. Now, one obtains a contradiction as in the proof of Proposition 2.1.5. This shows that A is not $\mathcal{R}$-sectorial. □

We now give some concrete examples of spaces for which the above theorem can be applied directly.

Example 2.1.47. For $p, q \in (1, \infty)$ consider the UMD-spaces $\ell_p(\ell_q)$ with their natural lattice structure. Its ordering is induced by the standard unit vector basis $(e_m)_{m\in\mathbb{N}}$ of $\ell_p(\ell_q)$ for some enumeration of $\mathbb{N} \times \mathbb{N}$. Clearly, $\ell_p(\ell_q)$ contains both copies of ℓ_p and ℓ_q and therefore for $p \neq q$ the basis $(e_m)_{m\in\mathbb{N}}$ is 1-unconditional and non-symmetric. Hence for $p \neq q$, Theorem 2.1.46 yields a sectorial operator A on $\ell_p(\ell_q)$ with $\omega(A) = 0$ and $\omega_R(A) = \infty$ such that $-A$ generates a positive analytic C_0-semigroup.

Example 2.1.48. We will later see in Proposition 3.2.11 that for $p \in (1, \infty)$ the space ℓ_p admits after equivalent renorming a non-symmetric 1-unconditional basis. If one uses the ordering induced by this basis, one sees with the help of Theorem 2.1.46 that one can give ℓ_p after equivalent renorming a non-standard lattice structure for which there exists a generator $-A$ of a positive analytic C_0-semigroup satisfying $\omega(A) = 0$ and $\omega_R(A) = \infty$. Further, these arguments apply to every normalized unconditional basis of $L_p([0,1])$ for $p \in (1, \infty)$ as such bases are automatically non-symmetric [Sin70, Ch. II, Theorem 21.1], a result whose proof partially relies on the methods used for the counterexample on $L_p([0,1])$ presented before the proof of Theorem 2.1.28. Furthermore note that the proof of this theorem implicitly contains a proof of the special case that for $p \in (1, \infty)$ the normalized Haar basis is non-symmetric.

2.1.4.7 Consequences of the Kalton–Lancien Theorem

The Kalton–Lancien Theorem (Theorem 2.1.42) for which we have given a new proof implies that among further classes of Banach spaces, namely separable Banach lattices and vector-valued L_p-spaces for $p \in [1, \infty)$, the Hilbert spaces are the only spaces having the maximal regularity property (MRP). This observation goes back to the original work [KL00] by N.J. Kalton and G. Lancien. We can also deduce their results rather directly from our new techniques developed so far.

As a first step we prove a partial extension of the Kalton–Lancien Theorem for spaces with an unconditional Schauder decomposition. Before, however, we need some preparatory results. The following observation will be useful at several places.

Proposition 2.1.49. *Let X be a Banach space with (MRP). Then every complemented subspace of X has (MRP).*

Proof. Let Y be a complemented subspace of X and let $-A$ be the infinitesimal generator of an analytic C_0-semigroup on Y. Then it is easy to see that $-B = -A \oplus 0$ is the infinitesimal generator of an analytic C_0-semigroup on X. Let $f \in L_p([0,T];Y) \subset L_p([0,T];X)$. Since X has (MRP), there exists a unique $u \in W_p^1([0;T];X)$ such that

$$\dot{u}(t) + Bu(t) = f(t).$$

It follows from the equation that u only takes values in Y. Hence, we obtain $u \in W_p^1([0;T];Y)$ which implies that Y has (MRP). □

Remark 2.1.50. It follows from the Kalton–Lancien Theorem (see Theorem 2.1.42) and Proposition 2.1.49 that no Banach space with an unconditional basis which is not isomorphic to a Hilbert space can be complemented in a space with (MRP). In the particular case of ℓ_∞, this is a special case of the fact that ℓ_∞ is *prime*, i.e. every infinite-dimensional complemented subspace of ℓ_∞ is isomorphic to ℓ_∞.

A variant of the following lemma can be found in [KL00, Theorem 3.1]. We give a proof based on the concept of associated semigroups, for a comparison with the original statement see Section 2.4. In this step our approach is conceptually different from the one of Kalton and Lancien.

Lemma 2.1.51. *Let X be a Banach space with (MRP). Further let $(\Delta_m)_{m\in\mathbb{N}}$ be a Schauder decomposition for X. Then for all sequences $(x_n)_{n\in\mathbb{N}}$ with $\sum_{n=1}^{\infty} r_n x_n \in \operatorname{Rad}(X)$ and $x_n \in \operatorname{Rg}\Delta_{2n-1} \oplus \operatorname{Rg}\Delta_{2n}$ one has*

$$\Big\| \sum_{n=1}^{\infty} r_n \Delta_{2n} x_n \Big\|_{\operatorname{Rad}(X)} \le C \Big\| \sum_{n=1}^{\infty} r_n x_n \Big\|_{\operatorname{Rad}(X)}.$$

Proof. For a positive non-decreasing sequence $(\gamma_m)_{m\in\mathbb{N}}$ let A be the Schauder multiplier

$$A = \sum_{m=1}^{\infty} \gamma_m \Delta_m$$

with its natural domain. By Proposition 2.1.5, A generates a bounded analytic C_0-semigroup $(T(z))_{z\in\Sigma_{\pi/2}}$ on X. Since X has the maximal regularity property (MRP), it follows from Theorem 1.2.9 and Proposition 1.2.6 that $(T(z))_{z\in\Sigma_{\pi/2}}$ is $\mathcal{R}$-analytic. It then follows from Proposition 2.1.18 that for all $(q_n)_{n\in\mathbb{N}} \subset (0,1)$ the associated semigroup $(\mathcal{T}(t))_{t\geq 0}$ on $\operatorname{Rad}(X)$

$$\mathcal{T}(t)\biggl(\sum_{n=1}^{\infty} r_n x_n\biggr) := \sum_{n=1}^{\infty} r_n T(q_n t) x_n$$

is well-defined and strongly continuous. In particular, for $(x_n)_{n\in\mathbb{N}}$ as in the assumptions we obtain

$$\begin{aligned} \mathcal{T}(1)\biggl(\sum_{n=1}^{\infty} r_n x_n\biggr) &= \sum_{n=1}^{\infty} r_n \biggl(\sum_{m=1}^{\infty} e^{-\gamma_m q_n} \Delta_m\biggr) x_n \\ &= \sum_{n=1}^{\infty} r_n (e^{-\gamma_{2n-1} q_n} \Delta_{2n-1} + e^{-\gamma_{2n} q_n} \Delta_{2n}) x_n. \end{aligned} \tag{2.13}$$

In particular, on the one hand, choosing $\gamma_m = 2^m$ and $q_n = \frac{\log 2}{2^{2n-1}}$, the right hand side of (2.13) becomes

$$\sum_{n=1}^{\infty} r_n \Bigl(\frac{1}{2}\Delta_{2n-1} + \frac{1}{4}\Delta_{2n}\Bigr) x_n. \tag{2.14}$$

On the other hand, choosing $\gamma_{2m} = \gamma_{2m-1} = 2^{2m}$ and $q_n = \frac{\log 2}{2^{2n}}$, the right hand side of (2.13) becomes

$$\sum_{n=1}^{\infty} r_n \Bigl(\frac{1}{2}\Delta_{2n-1} + \frac{1}{2}\Delta_{2n}\Bigr) x_n. \tag{2.15}$$

Subtracting (2.14) from (2.15) and using the boundedness of both associated semigroups, we see that for some $C \geq 0$ one obtains the desired estimate

$$\biggl\|\sum_{n=1}^{\infty} r_n \Delta_{2n} x_n\biggr\| \leq C \biggl\|\sum_{n=1}^{\infty} r_n x_n\biggr\|. \qquad \square$$

We can now prove the announced variant of the Kalton–Lancien Theorem for unconditional Schauder decompositions [KL00, Theorem 3.5]. Here our new approach makes the proof very natural as it only involves Rademacher averages.

Proposition 2.1.52. *Let X be a Banach space with an unconditional Schauder decomposition $(\Delta_m)_{m\in\mathbb{N}}$. Further assume that X has (MRP). Then X is isomorphic to $\oplus_{\ell_2}^m \operatorname{Rg} \Delta_m$.*

Proof. Let $(e_m)_{m\in\mathbb{N}}$ be a normalized sequence such that $e_m \in \operatorname{Rg}\Delta_m$ for all $m \in \mathbb{N}$. Let Y be the closed linear span of $\{e_m : m \in \mathbb{N}\}$. Then $(e_m)_{m\in\mathbb{N}}$ is a normalized unconditional Schauder basis for the Banach space Y. By the Hahn–Banach theorem, for each $m \in \mathbb{N}$, there exists a norm-one projection e_m^* onto the one-dimensional subspace spanned by e_m. We now show that that the series $\sum_{m=1}^{\infty} e_m^*$ converges in the strong operator topology towards a projection onto Y. For this let $D_{2m-1} = (1-e_m^*)\Delta_m$ and $D_{2m} = e_m^*\Delta_m$ for $m \in \mathbb{N}$. Then $(D_m)_{m\in\mathbb{N}}$ is a family of projections satisfying $D_n D_m = 0$ for all $m \neq n$ and $x = \sum_{m=1}^{\infty} D_m x$ for all $x \in X$. Hence, $(D_m)_{m\in\mathbb{N}}$ is a Schauder decomposition for X. We now apply Lemma 2.1.51 to the decomposition $(D_m)_{m\in\mathbb{N}}$ and obtain a constant $C \geq 0$ such that for all $(x_n)_{n\in\mathbb{N}}$ with $\sum_{n=1}^{\infty} r_n x_n \in \operatorname{Rad}(X)$ and $x_n \in \operatorname{Rg} D_{2n-1} \oplus D_{2n} = \operatorname{Rg}\Delta_n$ one has

$$\left\| \sum_{n=1}^{\infty} r_n e_n^* \Delta_n x_n \right\| = \left\| \sum_{n=1}^{\infty} r_n D_{2n} x_n \right\| \leq C \left\| \sum_{n=1}^{\infty} r_n x_n \right\|. \tag{2.16}$$

Now, as in the proof of Theorem 2.1.35, one sees that for all $x \in X$ the series $\sum_{n=1}^{\infty} r_n \Delta_n x$ converges in $\operatorname{Rad}(X)$ by the unconditionality of the Schauder decomposition $(\Delta_m)_{m\in\mathbb{N}}$. Hence, it follows from inequality (2.16) and the unconditionality of the Schauder decomposition $(\Delta_m)_{m\in\mathbb{N}}$ that

$$\begin{aligned} \frac{1}{K} \left\| \sum_{n=1}^{\infty} e_n^* x \right\|_X &= \frac{1}{K} \left\| \sum_{n=1}^{\infty} e_n^* \Delta_n x \right\|_X \leq \left\| \sum_{n=1}^{\infty} r_n e_n^* \Delta_n x \right\|_{\operatorname{Rad}(X)} \\ &\leq C \left\| \sum_{n=1}^{\infty} r_n \Delta_n x \right\|_{\operatorname{Rad}(X)} \leq CK \left\| \sum_{n=1}^{\infty} \Delta_n x \right\|_X = CK \|x\|_X, \end{aligned}$$

where K denotes the unconditional basis constant of the decomposition $(\Delta_m)_{m\in\mathbb{N}}$. This shows that $\sum_{n=1}^{\infty} e_n^* = \sum_{n=1}^{\infty} e_n^* \Delta_n$ defines a bounded projection onto Y. Hence, Y is a complemented subspace of X and therefore has (MRP) by Proposition 2.1.49. Since Y has an unconditional basis, it follows from the Kalton–Lancien Theorem (Theorem 2.1.42) that $Y \simeq \ell_2$. In particular, $(e_m)_{m\in\mathbb{N}}$ is equivalent to the standard Hilbert space basis of ℓ_2. Since this argument holds for arbitrary normalized sequences $(e_m)_{m\in\mathbb{N}}$ with $e_m \in \operatorname{Rg}\Delta_m$, we obtain a bijection between X and $\oplus_{\ell_2}^m \operatorname{Rg}\Delta_m$ via $x \mapsto (\Delta_m x)_{m\in\mathbb{N}}$ which is an isomorphism of Banach spaces by the open mapping theorem. □

Now the following characterizations of spaces with (MRP) follow exactly as in [KL00]. We need the following characterization of UMD-spaces due to T. Coulhon & D. Lamberton [CL86].

Proposition 2.1.53. *Let X be a Banach space and $(T(t))_{t\geq 0}$ the vector-valued extension of the Poisson semigroup on $L_2(\mathbb{R};X)$, i.e.*

$$(T(t)f)(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{t}{t^2 + (x-y)^2} f(y)\, dy.$$

Then the negative generator of $(T(t))_{t\geq 0}$ has maximal regularity if and only if X is a UMD-space.

This allows us to prove the following characterization of spaces with (MRP) in the class of vector-valued Lebesgue spaces [KL00, Theorem 3.6]. It is remarkable that for this result no assumptions on the Banach space X are needed.

Theorem 2.1.54. *Let X be a Banach space and $p \in [1, \infty)$. Then $L_p([0,1];X)$ has (MRP) if and only if $p = 2$ and X is isomorphic to a Hilbert space.*

Proof. It follows from the result of D. Simon (Theorem 1.2.12) that $L_2([0,1];H)$ has (MRP) for every Hilbert space H. For the converse note that $L_p([0,1])$ is complemented in $L_p([0,1];X)$. Hence, by Proposition 2.1.49 the space $L_p([0,1])$ has (MRP) as well, which can only hold for $p = 2$ (for $p \neq 1$ this follows from the Kalton–Lancien Theorem (Theorem 2.1.42) or the explicit counterexamples constructed in Section 2.1.4.3, for $p = 1$ one can use the fact that ℓ_1 which does not have (MRP) by the counterexample given in Section 2.1.4.2 is complemented in $L_1([0,1])$). Further, it follows from Proposition 2.1.53 that X is a UMD-space. A fortiori, by Theorem A.3.10, the space X is K-convex which means that $\operatorname{Rad}(X)$ is a complemented subspace of $L_p([0,1];X)$. Hence, $\operatorname{Rad}(X)$ has (MRP) by Proposition 2.1.49. For $m \in \mathbb{N}$ let $\Delta_m\colon \operatorname{Rad}(X) \to \operatorname{Rad}(X)$ be given by

$$\Delta_m(x) := r_m \int_0^1 r_m(t)x(t)\,dt,$$

the projection onto the m-th component. Then $(\Delta_m)_{m\in\mathbb{N}}$ clearly is an unconditional Schauder decomposition of $\operatorname{Rad}(X)$. Hence, by Proposition 2.1.52, one has $\operatorname{Rad}(X) \simeq \ell_2(X)$. This means that X has both type and cotype equal to 2. By Kwapień's characterization A.3.5, X is isomorphic to a Hilbert space. □

Further, (MRP) characterizes Hilbert spaces in a broad class of Banach lattices [KL00, Theorem 3.7]. For a very short overview of the theory of Banach lattices we refer to Appendix A.4.

Theorem 2.1.55. *An order continuous Banach lattice has (MRP) if and only if it is isomorphic to a Hilbert space.*

Proof. Let X be an order continuous Banach lattice. By [LT79, Lemma 1.b.13] it suffices to show that every normalized sequence of disjoint elements in X is equivalent to the standard unit vector basis of ℓ_2. Let $(x_n)_{n\in\mathbb{N}} \subset X$ be such a sequence. By taking the disjoint ideals generated by the elements $(x_n)_{n\in\mathbb{N}}$, one obtains an unconditional Schauder decomposition $(\Delta_m)_{m\in\mathbb{N}}$ of X such that $x_n \in \operatorname{Rg}\Delta_n$ for all $n \in \mathbb{N}$ (for an analogous argument see [LT79, Proposition 1.a.9]). Now, Proposition 2.1.52 shows that $X \simeq \oplus_{\ell_2}^{m} \operatorname{Rg} \Delta_m$. Hence, $(x_n)_{n\in\mathbb{N}}$ is equivalent to the standard unit vector basis of ℓ_2. $\square$

In particular, for separable Banach lattices one obtains a complete characterization [KL00, Corollary 3.8].

Corollary 2.1.56. *A separable Banach lattice has (MRP) if and only if it is isomorphic to a Hilbert space.*

Proof. Let X be a separable Banach lattice. If X is order continuous, the assertion follows from Theorem 2.1.55. If X is not order continuous, it follows that X is not σ-order continuous, as otherwise the separability of X would imply the order continuity of X. Moreover, X cannot be σ-complete either, as otherwise X must contain a copy of ℓ_∞ [LT79, Proposition 1.a.7], which is impossible. By a result of P. Meyer-Nieberg [LT79, Theorem 1.a.5], the space c_0 is contained in X. Since X is separable, it follows from Sobczyk's Theorem [LT77, Theorem 2.f.5] that c_0 is complemented in X. Now, Proposition 2.1.49 implies that c_0 has (MRP), which contradicts the counterexample given in Section 2.1.4.2. $\square$

2.1.5 A First Application: Existence of Schauder Bases which are not $\mathcal{R}$-Bases

In this and the following section we give further applications of the techniques developed to give explicit counterexamples to the maximal regularity problem. The first application deals with the existence of Schauder bases (on classical function spaces) that are not $\mathcal{R}$-bases. We will see that our methods directly apply to this open problem.

We start by introducing the necessary terminology. Consider a Schauder basis $(e_m)_{m\in\mathbb{N}}$ for some Banach space X. Then for $N \in \mathbb{N}$ one has the partial sum projections $P_N : X \to X$ with respect to the expansion given by

$$P_N\left(\sum_{m=1}^{\infty} a_m e_m\right) = \sum_{m=1}^{N} a_m e_m.$$

It follows from the uniform boundedness principle that the family $(P_N)_{N\in\mathbb{N}}$ is uniformly bounded in operator norm [AK06, Proposition 1.1.4]. One can now consider Schauder bases for which the associated family $(P_N)_{N\in\mathbb{N}}$ is even $\mathcal{R}$-bounded.

Definition 2.1.57. A Schauder basis $(e_m)_{m\in\mathbb{N}}$ of a Banach space X is called an $\mathcal{R}$-*basis* if the set $\{P_N : N \in \mathbb{N}\} \subset \mathcal{B}(X)$ of projections is $\mathcal{R}$-bounded.

On sufficiently regular Banach spaces unconditional Schauder bases are always $\mathcal{R}$-bases. Indeed, the following result holds which is a special case of [KW01, Theorem 3.3 (4)].

Theorem 2.1.58. *Let $(e_m)_{m\in\mathbb{N}}$ be an unconditional Schauder basis for a Banach space with property (Δ). Then $(e_m)_{m\in\mathbb{N}}$ is an $\mathcal{R}$-basis.*

In particular, the above theorem holds for all L_p-spaces with $p \in (1,\infty)$ or more generally for all UMD-spaces. We now use our techniques to give explicit examples of Schauder bases that are not $\mathcal{R}$-bases. This goes back to [Fac14]. In particular, the theorem below applies for the spaces $L_p([0,1])$ for $p \in (1,\infty)\setminus\{2\}$ and answers an open problem stated at the end of [KLM10].

It is remarkable that most of the examples of Schauder bases (as the Fourier basis) which appear in analysis are $\mathcal{R}$-bases. However, the result shows that one can always construct a basis which is not an $\mathcal{R}$-basis.

Theorem 2.1.59. *Let X be a Banach space that admits an unconditional, non-symmetric normalized Schauder basis $(e_m)_{m\in\mathbb{N}}$. Then X has a Schauder basis that is not an $\mathcal{R}$-basis.*

Proof. As in the proof of Theorem 2.1.35 we choose the Schauder bases f'_m respectively f''_m. Again, we only consider the case of f'_m. Suppose that $(f'_m)_{m\in\mathbb{N}}$ is an $\mathcal{R}$-basis and let $(P_N)_{N\in\mathbb{N}}$ be the associated projections. Then $\sum_{m=1}^N r_m x_m \mapsto \sum_{m=1}^N r_m P_{2m-1} x_m$ extends to a bounded operator $\tilde{P} \in \mathcal{B}(\operatorname{Rad}(X))$. In particular, we have

$$
\begin{aligned}
-\sum_{m=1}^N r_m a_m e_{2m-1} = -\sum_{m=1}^N r_m a_m f'_{2m-1} &= \sum_{m=1}^N r_m a_m P_{2m-1}(f'_{2m} - f'_{2m-1}) \\
&= \sum_{m=1}^N r_m a_m P_{2m-1} e_{\pi(2m)} = \tilde{P}\biggl(\sum_{m=1}^N r_m a_m e_{\pi(2m)}\biggr).
\end{aligned}
$$

The boundedness of $\tilde{P}$ implies that the left hand side converges in $\operatorname{Rad}(X)$ whenever $\sum_{m=1}^N r_m a_m e_{\pi(2m)}$ converges for $N \to \infty$, which exactly as in the proof of Theorem 2.1.35 yields a contradiction. □

2.1.6 A Second Application: Sectorial Operators whose Sum is not Closed

The aim of this section is to present a connection between the maximal regularity problem and the problem of the closedness of the sum of two sectorial operators. We will show that the explicit counterexamples to the maximal regularity problem constructed so far directly yield explicit counterexamples for a variant of the closedness of the sum problem for which until now no explicit counterexamples have been known on UMD-spaces. Before discussing the counterexample, we start with some fundamental results.

Given two (densely defined) closed operators A and B on a Banach space X, one can define their sum $A+B$ as the linear operator with domain $D(A+B) = D(A) \cap D(B)$. In general, the sum $A+B$ may not even be closable. However, for sectorial operators one has the following partial positive result [DPG75, Théorème 3.7] (see [Prü93, Theorem 8.5] for the spectral inclusion). In the following we say that two sectorial operators A and B *commute* if $R(\lambda, A)R(\lambda, B) = R(\lambda, B)R(\lambda, A)$ for one (equivalently all) $\lambda \in \rho(A) \cap \rho(B)$.

Theorem 2.1.60. *Let A and B be two commuting sectorial operators on a Banach space with*

$$\omega(A) + \omega(B) < \pi.$$

Then the sum $A+B$ is closable and for its closure $\overline{A+B}$ one has the spectral inclusion

$$\sigma(\overline{A+B}) \subset \sigma(A) + \sigma(B).$$

It is, however, far less clear whether the above sum is closed. There are two celebrated sufficient conditions for the closedness of the sum of two commuting sectorial operators.

Theorem 2.1.61. *Let A and B be two commuting sectorial operators on a UMD-space. Suppose further that one of the two following conditions holds.*

(a) $\omega_{\mathrm{BIP}}(A) + \omega_{\mathrm{BIP}}(B) < \pi$.

(b) $\omega_{H^\infty}(A) + \omega_R(B) < \pi$.

Then the sum $A+B$ is closed.

The closedness under the first assumption is the statement of the Dore–Veni Theorem [DV87, Theorem 2.1], whereas the second assumption is sufficient by a theorem of L. Weis and N.J. Kalton [KW01, Theorem 6.3]. It is now a natural question whether one of the above conditions on the sectorial operators can be weakened. There are several possibilities, among them are:

(a) $\omega_R(A) + \omega_R(B) < \pi$,

(b) $\omega_{\mathrm{BIP}}(A) + \omega_R(B) < \pi$,

(c) $\omega_{H^\infty}(A) + \omega(B) < \pi$.

We will see that in general one cannot obtain the closedness of the sum under one of the three above assumptions. This gives a complete description of all possible cases that can be obtained by combining the angles $\omega(\cdot)$, $\omega_R(\cdot)$, $\omega_{\mathrm{BIP}}(\cdot)$

and $\omega_{H^\infty}(\cdot)$ for the operators A and B as each combination is a special case either of the positive results or the counterexamples.

We start with (a). In the Hilbert space case this assumption reduces to $\omega(A) + \omega(B) < \pi$. A classical counterexample by J.-B. Baillon and P. Clément [BC91, Example B] shows, however, that this assumption is not sufficient for the sum to be closed.

For (b) observe that in the Hilbert space case the assumption is equivalent to $\omega_{H^\infty}(A) + \omega(B) < \infty$ or $\omega_{H^\infty}(A) + \omega_R(B) < \infty$, which implies the closedness by Theorem 2.1.61(b). This Hilbert space result is also a part of the Dore–Veni Theorem [DV87, Remark 2.11]. On L_p-spaces, for $p \in (1,\infty) \setminus \{2\}$, condition (b) is, however, not sufficient as shown by a counterexample of G. Lancien [Lan98, Theorem 2.1 1)]. Let $(e_m)_{m\in\mathbb{N}}$ be the trigonometric basis of $L_p([0,2\pi])$ and let A be the Schauder multiplier associated to the sequence $(2^m)_{m\in\mathbb{Z}}$. We have seen in Example 2.1.11 that A is a sectorial operator with $\omega_{\mathrm{BIP}}(A) = 0$ that does not have a bounded H^∞-calculus. Since the (partial) trigonometric basis $(e_m)_{m\in\mathbb{N}}$ is conditional, there exists a sequence $(\varepsilon_m)_{m\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}$ such that the Schauder multiplier with respect to $(\varepsilon_m)_{m\in\mathbb{N}}$ is not bounded. Lancien then considers the Schauder multiplier associated to the sequence $(b_m)_{m\in\mathbb{N}}$, where $b_m = 1$ for $m \le 0$, and $b_m = 2^m$ if $\varepsilon(m) = 1$ and $b_m = 2^{m-1}$ if $\varepsilon(m) = -1$. By Proposition 2.1.5 the operator B is sectorial with $\omega(B) = 0$. He then shows that the sum of the two operators A and B is not closed. To obtain a counterexample to the closedness of the sum under assertion (b), it therefore suffices to additionally verify that B is $\mathcal{R}$-sectorial with $\omega_R(B) = 0$. By the boundedness of the Hilbert transform and an analogous calculation as in Example 2.1.15, this reduces to the fact that the sequence $(\frac{ae^{i\theta}}{ae^{i\theta}-b_m})_{m\in\mathbb{N}}$ satisfies the assumptions of the Marcinkiewicz multiplier theorem (Theorem 2.1.14) uniformly in $\theta \in [\theta_0, 2\pi)$ and in $|a| \in [1,2]$, which can be checked directly.

It remains to find a counterexample to the closedness of the sum under the assumption (c). We can directly obtain an explicit counterexample as a consequence of the constructed explicit counterexamples to the maximal regularity problem. For this we now explain the connection between maximal regularity and the problem of the closedness of the sum of two sectorial operators as already indicated in the introduction of this thesis. Let X be a Banach space, $T > 0$ and $-B = \frac{d}{dt}$ the infinitesimal generator of the shift semigroup $(S(t))_{t\ge 0}$ on $L_p([0,T])$ with domain $D(B) = \{u \in W_p^1([0,T]) : u(0) = 0\}$. Then for all $p \in (1,\infty)$ the sectorial operator B has a bounded $H^\infty(\Sigma_\theta)$-calculus for all $\theta \in (\frac{\pi}{2}, \pi)$ because $(S(t))_{t\ge 0}$ clearly has a strict dilation to a shift group on $L_p(\mathbb{R})$. One can now consider the vector-valued extension of the shift semigroup $(S(t) \otimes \mathrm{Id}_X)_{t\ge 0}$ on $L_p([0,T];X)$. Its negative infinitesimal generator is given by $\mathcal{B} = B \otimes \mathrm{Id}_X$ and if X is a UMD-space, it has a bounded $H^\infty(\Sigma_\theta)$-calculus for all $\theta \in (\frac{\pi}{2}, \pi)$ by the same dilation argument. Now, let further A be a sectorial operator with domain $D(A)$ which is a Banach space when endowed with its graph norm. We can now lift A to a multiplication operator $\mathcal{A}$ by letting

$$\begin{aligned} D(\mathcal{A}) &= L_p([0,T];D(A)) \\ (\mathcal{A}u)(s) &:= A(u(s)). \end{aligned}$$

Then $-\mathcal{A}$ generates the C_0-semigroup $(e^{-t\mathcal{A}}u)(s) = e^{-tA}(u(s))$ for $s \in [0,T]$ and one has $\omega(\mathcal{A}) = \omega(A)$. One directly sees that the C_0-semigroups $(e^{-t\mathcal{A}})_{t\ge 0}$ and $(e^{-t\mathcal{B}})_{t\ge 0}$ commute, which implies that the sectorial operators $\mathcal{A}$ and $\mathcal{B}$ commute. Since the intersection of the domains $D(\mathcal{A}) \cap D(\mathcal{B})$ is dense in $L_p([0,T];X)$ and is left invariant by the product semigroup, it follows that $D(\mathcal{A}) \cap D(\mathcal{B})$ is a core of the generator of the product semigroup. Hence, the negative generator of the product semigroup is given by the closure of $\mathcal{A} + \mathcal{B}$. We are now interested in the following equivalent characterization of maximal regularity.

Theorem 2.1.62. *Let X be a Banach space and $-A$ the infinitesimal generator of a C_0-semigroup on X. Then the following are equivalent.*

(i) *$-A$ has maximal regularity, i.e. for all $f \in L_p([0,T];X)$ there exists a unique $u \in L_p([0,T];D(A)) \cap W_p^1([0,T];X)$ with*

$$\begin{cases} \dot{u}(t) + A(u(t)) = f(t) \\ \qquad\qquad u(0) = 0. \end{cases}$$

(ii) *The operator $\mathcal{A} + \mathcal{B}$ is closed.*

Proof. Note that the intersection of the domains $D(\mathcal{A}) \cap D(\mathcal{B})$ consists of those $u \in W_p^1([0,T];X) \cap L_p([0;T];D(A))$ that satisfy $u(0) = 0$. Since one has $0 \in \rho(\mathcal{B})$, it follows from the spectral inclusion of Theorem 2.1.60 that $0 \notin \sigma(\overline{\mathcal{A} + \mathcal{B}})$, i.e. the closure of $\mathcal{A} + \mathcal{B}$ is invertible. Now, if A has maximal regularity, it follows that $D(\mathcal{A}) \cap D(\mathcal{B})$ is mapped bijectively onto $L_p([0,T];X)$. This shows that $D(\overline{\mathcal{A} + \mathcal{B}}) = D(\mathcal{A} + \mathcal{B})$, i.e. the sum $\mathcal{A} + \mathcal{B}$ is closed. Conversely, if $\mathcal{A} + \mathcal{B}$ is closed, it is invertible and maximal regularity follows immediately. □

Now, the constructed explicit counterexamples to the maximal regularity problem yield directly the first explicit counterexamples to the problem of the closedness of the sum of two sectorial operators on UMD-spaces, for example on L_p-spaces.

Corollary 2.1.63. *On $L_p([0,1];L_p([0,1]) \simeq L_p([0,1]^2)$ let $\mathcal{A}$ be the multiplication operator obtained from the counterexample to the maximal regularity problem given in Theorem 2.1.28 and let $\mathcal{B} = \frac{d}{dt} \otimes \mathrm{Id}$ be as before. Then one has*

$$\omega_{H^\infty}(\mathcal{A}) + \omega(\mathcal{B}) = \frac{\pi}{2}$$

and the sum $\mathcal{A} + \mathcal{B}$ is not closed.

2.2 Using Pisier's Counterexample to the Halmos Problem

We now present a counterexample to the last implication from the introduction of this chapter which has been left open, namely that there exists a C_0-semigroup with generator $-A$ and $\omega_{H^\infty}(A) = \frac{\pi}{2}$ which does not have a loose dilation. The key ingredient here is G. Pisier's counterexample to the Halmos problem [Pis97] (for a more elementary approach towards the counterexample see [DP97]). We now shortly explain the content of this counterexample. Pisier showed that there exists a Hilbert space H and an operator $T \in \mathcal{B}(H)$ that is *polynomially bounded*, that is for some $K \geq 0$ one has $\|p(T)\| \leq K \sup_{|z| \leq 1} |p(z)|$ for all polynomials $p \in \mathcal{P}$, but that is not similar to a contraction, i.e. there does not exist any invertible $S \in \mathcal{B}(H)$ such that $S^{-1}TS$ is a contraction.

Using an observation of C. Le Merdy made in [LM98] and the theory of operator spaces (for a short overview see Appendix B), Pisier's counterexample gives the following example concerning dilations. This seems to be new, but I have learned from a personal communication that this was also known to C. Arhancet. The following proof can also be found in [Facb].

Theorem 2.2.1. *There exists a generator $-A$ of a C_0-semigroup $(T(t))_{t\geq 0}$ on some Hilbert space with $\omega_{H^\infty}(A) = \frac{\pi}{2}$ such that $(T(t))_{t\geq 0}$ does not have a loose dilation in the class of all Hilbert spaces.*

Proof. Let T and H be as above from Pisier's counterexample to the Halmos problem. It is explained in [LM98, Proposition 4.8] that the concrete structure of T allows one to define $A = (I+T)(I-T)^{-1}$ which turns out to be a sectorial operator with $\omega(A) = \frac{\pi}{2}$. Moreover, it is shown that $-A$ generates a bounded C_0-semigroup $(T(t))_{t\geq 0}$ on H. Further, it follows from the polynomial boundedness of T with a conformal mapping argument that A has a bounded H^∞-calculus with $\omega_{H^\infty}(A) = \frac{\pi}{2}$ [LM98, Remark 4.4]. Now assume that $(T(t))_{t\geq 0}$ has a loose dilation in the class of all Hilbert spaces. Then it follows from Dixmier's unitarization theorem [Pau02, Theorem 9.3] that

$(T(t))_{t\geq 0}$ has a loose dilation to a unitary C_0-group $(U(t))_{t\in\mathbb{R}}$ on some Hilbert space K, i.e. there exist bounded operators $J\colon H\to K$ and $Q\colon K\to H$ such that

$$T(t) = QU(t)J \qquad \text{for all } t\geq 0.$$

Now let $\mathcal{A}$ be the unital subalgebra of $L_\infty([0,\infty))$ generated by the functions $x\mapsto e^{-itx}$ for $t\geq 0$, where we identify elements in $L_\infty([0,\infty))$ with multiplication operators on the Hilbert space $L_2([0,\infty))$. This gives $\mathcal{A}$ the structure of an operator space. We now show that the algebra homomorphism

$$u\colon \mathcal{A}\to\mathcal{B}(H), \qquad e^{-it\cdot}\mapsto T(t)$$

is completely bounded with respect to this operator space structure for $\mathcal{A}$. Indeed, observe that by Stone's theorem on unitary groups and the spectral theorem for self-adjoint operators there exists a measure space Ω and a measurable function $m\colon\Omega\to\mathbb{R}$ such that after unitary equivalence $U(t)$ is the multiplication operator with respect to the function e^{-itm} for all $t\in\mathbb{R}$. Now for $n\in\mathbb{N}$ let $[f_{ij}]\in M_n(\mathcal{A})$ with $f_{ij}=\sum_{k=1}^N a_k^{(ij)}e^{-it_k\cdot}$. Then one has

$$\begin{aligned}\left\|u_n([f_{ij}])\right\|_{M_n(\mathcal{B}(X))} &= \left\|\Big[\sum_{k=1}^N a_k^{(ij)}T(t_k)\Big]\right\|_{M_n(\mathcal{B}(X))} = \left\|\Big[Q\sum_{k=1}^N a_k^{(ij)}U(t_k)J\Big]\right\|_{M_n(\mathcal{B}(X))} \\ &\leq \|Q\|\|J\|\left\|\Big[\sum_{k=1}^N a_k^{(ij)}e^{-it_k m}\Big]\right\|_{M_n(\mathcal{B}(L_2(\Omega)))} \leq \|J\|\|Q\|\sup_{x\in\mathbb{R}}\left\|\Big[\sum_{k=1}^N a_k^{(ij)}e^{-it_k x}\Big]\right\|_{M_n} \\ &= \|J\|\|Q\|\left\|[f_{ij}]\right\|_{M_n(L_\infty[0,\infty))}.\end{aligned}$$

Here we have used the (isometric) identification of the two C^*-algebras $M_n(L_\infty(\Omega))\simeq L^\infty(\Omega;M_n)$ for all $n\in\mathbb{N}$. We deduce from Theorem B.0.11 that $(T(t))_{t\geq 0}$ is similar to a semigroup of contractions. However, since by construction T is the cogenerator of $(T(t))_{t\geq 0}$, this holds if and only if T is similar to a contraction [SNFBK10, III,8]. This is a contradiction to our choice of T. □

2.3 Using Monniaux's Theorem

In this section we present a new alternative method to construct counterexamples to certain questions concerning the introduced regularity properties of sectorial operators. The method is based on a theorem of S. Monniaux. Recall that if A is a sectorial operator with dense range and bounded imaginary powers, then $t\mapsto A^{it}$ is a strongly continuous group (Proposition 1.5.2). Conversely, one may ask which C_0-groups can be written in this form. The next theorem of S. Monniaux [Mon99, Theorem 4.3] (for an alternative proof see [Haa07, Section 4]) in its generalized version proved by M. Haase [Haa03, Theorem 5.2] gives a very satisfying answer to this question.

For its formulation we need to consider the following straightforward analogue of sectorial operators on strips. For details see [Haa06, Chapter 4].

Definition 2.3.1. For $\omega > 0$ let $H_\omega := \{z \in \mathbb{C} : |\operatorname{Im} z| < \omega\}$ be the *horizontal strip* of height 2ω. A closed densely defined operator B on some Banach space X is called a *strip type operator* of height $\omega > 0$ if $\sigma(B) \subset \overline{H_\omega}$ and

$$\sup\{\|R(\lambda, B)\| : |\operatorname{Im}\lambda| \geq \omega + \varepsilon\} < \infty \qquad \text{for all } \varepsilon > 0. \tag{H_ω}$$

Further, we define the *spectral height* of B as $\omega_{st}(B) := \inf\{\omega > 0 : (H_\omega) \text{ holds}\}$.

On UMD-spaces one then obtains the following correspondence. Recall that the group type of a C_0-group $(U(t))_{t\in\mathbb{R}}$ on some Banach space is the infimum of those $\omega > 0$ for which $t \mapsto e^{-\omega|t|}\|U(t)\|$ is bounded on the real line.

Theorem 2.3.2. *Let X be a UMD-space. Then there is an one-to-one correspondence*

$$\left\{ \begin{array}{c} \textit{A sectorial operator with dense} \\ \textit{range, BIP and } \omega_{\mathrm{BIP}}(A) < \pi \end{array} \right\} \underset{e^B}{\overset{\log A}{\longleftrightarrow}} \left\{ \begin{array}{c} \textit{B strip type operator with} \\ iB \sim C_0\textit{-group of type} < \pi \end{array} \right\}.$$

Proof. For the surjectivity let B be a strip type operator such that iB generates a C_0-group $(U(t))_{t\in\mathbb{R}}$ of type $< \pi$. Then by Monniaux's theorem [Mon99, Theorem 4.3] there exists a sectorial operator A with dense range and bounded imaginary powers such that $A^{it} = U(t)$ for all $t \in \mathbb{R}$. Moreover, $(U(t))_{t\in\mathbb{R}}$ is generated by $i \log A$. It then follows from the uniqueness of the generator that $B = \log A$.

For the injectivity assume that $\log A = \log B$ for two sectorial operators from the left-hand side. Then by [Haa06, Corollary 4.2.5] one has $A = e^{\log A} = e^{\log B} = B$. $\square$

Remark 2.3.3. In [Haa03] M. Haase shows that for every strip type operator B with $\omega_{st}(B) < \pi$ such that iB generates a C_0-group $(U(t))_{t\in\mathbb{R}}$ there exists a sectorial operator A with $A^{it} = U(t)$ for all $t \in \mathbb{R}$. If one chooses B as above such that $(U(t))_{t\in\mathbb{R}}$ has group type bigger than π (which is possible on some UMD-spaces) one sees that there exists a sectorial operator A with $\omega_{\mathrm{BIP}}(A) > \pi$. By taking suitable fractional powers of A one then obtains a sectorial operator $\tilde{A}$ with $\omega(\tilde{A}) < \omega_{\mathrm{BIP}}(\tilde{A}) < \pi$.

Because of the above results, for a moment, we restrict our attention to a UMD-space X. A particular class of sectorial operators with dense range which have bounded imaginary powers are those with a bounded H^∞-calculus. Recall that a sectorial operator A on X with dense range and a bounded H^∞-calculus satisfies $\omega_R(A) = \omega_{\mathrm{BIP}}(A) = \omega_{H^\infty}(A)$ by Theorem 1.3.4. In particular one has $\omega_{\mathrm{BIP}}(A) < \pi$. For sectorial operators with a bounded H^∞-calculus one can formulate an analogous correspondence which essentially follows from Monniaux's theorem and an unpublished result of N.J. Kalton & L. Weis.

In the following we call for a C_0-group $(U(t))_{t\in\mathbb{R}}$ on some Banach space the infimum of those $\omega > 0$ for which $\mathcal{R}\left\{e^{-\omega|t|}U(t) : t \in \mathbb{R}\right\} < \infty$ the $\mathcal{R}$-group type of $(U(t))_{t\in\mathbb{R}}$.

Theorem 2.3.4. *Let X be a Banach space with Pisier's property (α). Then there is an one-to-one correspondence*

$$\left\{ \begin{array}{c} \text{A sectorial operator with dense} \\ \text{range and bounded } H^\infty\text{-calculus} \end{array} \right\} \underset{e^B}{\overset{\log A}{\longleftrightarrow}} \left\{ \begin{array}{c} B \text{ strip type operator with } iB \sim \\ C_0\text{-group of } \mathcal{R}\text{-type} < \pi \end{array} \right\}.$$

Proof. Let A be a sectorial operator with dense range and a bounded H^∞-calculus. Then it follows from Theorem 1.3.3 and the fact that the norm of $\lambda \mapsto \lambda^{it}$ in $H^\infty(\Sigma_\theta)$ is bounded by $\exp(|t|\theta)$ for $t \in \mathbb{R}$ that $\{e^{-|t|\theta}A^{it} : t \in \mathbb{R}\}$ is $\mathcal{R}$-bounded for all $\theta \in (\omega_{H^\infty}(A), \pi)$. In particular $(A^{it})_{t\in\mathbb{R}}$ is of $\mathcal{R}$-type $< \pi$.

Conversely, let B be from the right hand side. It then follows from an unpublished result in [KWb] (see [Haa11, Theorem 6.5] for a proof, here one has to additionally use the equivalence of $\mathcal{R}$- and γ-boundedness for Banach spaces with finite cotype) that the $\mathcal{R}$-type assumption implies that B has a bounded H^∞-calculus on some strip of height smaller than π. By [Haa06, Proposition 5.3.3], the operator e^B is sectorial and has a bounded H^∞-calculus.

The one-to-one correspondence then follows as in the proof of Theorem 2.3.2. □

From the above theorems it follows immediately that on L_p for $p \in (1,\infty) \setminus \{2\}$ there exist sectorial operators with bounded imaginary powers which do not have a bounded H^∞-calculus.

Corollary 2.3.5. *Let $p \in (1,\infty) \setminus \{2\}$. Then there exists a sectorial operator A on $L_p(\mathbb{R})$ with dense range and $\omega(A) = \omega_{\mathrm{BIP}}(A) = 0$ that does not have a bounded H^∞-calculus.*

Proof. Let $(U(t))_{t\in\mathbb{R}}$ be the shift group on $L_p(\mathbb{R})$. One sees with the same argument as in Example 1.2.7 that the set $\{U(t) : t \in [0,1]\}$ is not $\mathcal{R}$-bounded. By Theorem 2.3.2 there exists a sectorial operator A with dense range and bounded imaginary powers such that $A^{it} = U(t)$ for all $t \in \mathbb{R}$. Then one has $\omega(A) \le \omega_{\mathrm{BIP}}(A) = 0$. However, by construction, A^{it} is not $\mathcal{R}$-bounded on $[0,1]$ and therefore Theorem 2.3.4 implies that A cannot have a bounded H^∞-calculus. □

Note that the constructed counterexample is exactly the same as Example 2.1.11 which was obtained by different methods except for the fact that we worked in Example 2.1.11 with the periodic shift. Of course, we could have started with the same periodic shift in Corollary 2.3.5.

2.3.1 Some Results on Exotic Banach Spaces

Although from the point of view of applications, the main interest lies in the properties of sectorial operators defined on spaces built from L_p-spaces, there are good reasons to study these operators in more general Banach spaces. On the one hand such studies can help to extract the essential ingredients in the proofs of results which are important in the L_p-case and therefore can be a guide for streamlined proofs and presentations of the core theory. On the other hand, even counterexamples on rather exotic Banach spaces may lead to the insight that certain approaches are doomed for failure. In this spirit we want to study shortly sectorial operators in exotic Banach spaces. In the past twenty years Banach spaces have been constructed whose algebra of operators have an extremely different structure compared to those of the well-known classical Banach spaces. The most prominent examples are probably the indecomposable Banach spaces.

Definition 2.3.6 (Hereditarily Indecomposable Banach Space (H.I.)). A Banach space X is called *indecomposable* if it cannot be written as the sum of two closed infinite-dimensional subspaces. Further, X is is called *hereditarily indecomposable (H.I.)* if every infinite dimensional closed subspace of X is indecomposable.

It is a deep result of B. Maurey and T. Gowers that such (separable) spaces do actually exist [GM93]. We are now interested in the properties of C_0-semigroups on such spaces. We will use the following theorem proved in [RR96, Theorem 2.3].

Theorem 2.3.7. *Let X be a H.I. Banach space. Then every C_0-group on X has a bounded generator.*

The above result can be directly used to show the following result on sectorial operators with bounded imaginary powers.

Corollary 2.3.8. *Let A be a sectorial operator with dense range and bounded imaginary powers on an H.I. Banach space. Then A is bounded.*

Proof. Let A be as in the assertion. Note that $(A^{it})_{t\in\mathbb{R}}$ is a C_0-group with generator $i\log A$. By Theorem 2.3.7 $\log A$ is a bounded operator. This implies that $e^{\log A} = A$ is bounded. □

In particular, on H.I. Banach spaces the structure of sectorial operators with a bounded H^∞-calculus is rather trivial. The following observation seems to be new. In a certain sense it contrasts the meta-mathematical observation that on most spaces all common examples of sectorial operators have a bounded H^∞-calculus or at least bounded imaginary powers. Notice that there exist H.I. Banach spaces with a conditional basis. In this case there exist unbounded generators of analytic semigroups.

Corollary 2.3.9. *Let A be an invertible operator on a H.I. Banach space. Then the following assertions are equivalent.*

(i) *A is a bounded operator.*

(ii) *A has bounded imaginary powers.*

(iii) *A has a bounded H^∞-calculus.*

Proof. The implication (i) $\Rightarrow$ (iii) follows from Example 1.3.5, (iii) $\Rightarrow$ (ii) always holds as discussed in Section 1.5 and the implication (ii) $\Rightarrow$ (i) follows from Corollary 2.3.8. $\square$

2.4 Notes & Open Problems

We start with some comments on Kalton and Lancien's original approach to the maximal regularity problem. The key result obtained by the authors from which all the other results are deduced is the following [KL00, Theorem 3.1].

Proposition 2.4.1. *Let X be a Banach space with (MRP). Further let $(\Delta_m)_{m\in\mathbb{N}}$ be a Schauder decomposition for X. Then there exists a constant $C > 0$ such that for all $N \in \mathbb{N}$ and all $x_1, \ldots, x_N \in \operatorname{Rg}\Delta_{2n-1} \oplus \operatorname{Rg}\Delta_{2n}$ one has*

$$\left(\frac{1}{2\pi}\int_0^{2\pi}\left\|\sum_{n=1}^N \Delta_{2n}x_n e^{i2^n t}\right\| dt\right)^{1/2} \le C\left(\frac{1}{2\pi}\int_0^{2\pi}\left\|\sum_{n=1}^N x_n e^{i2^n t}\right\| dt\right)^{1/2}.$$

Notice that it follows from an inequality going back to G. Pisier [Pis78a, Théorème 2.1] that this statement is actually equivalent to the statement of Lemma 2.1.51. Our approach is however totally different in the way one deduces the existence of counterexamples to the maximal regularity problem. In fact, Kalton and Lancien use Proposition 2.4.1 to show that the closed subspace spanned by an arbitrary block basic sequence of an arbitrary permutation of an arbitrary unconditional basis of a Banach space with (MRP) is complemented. It then follows from a result of J. Lindenstrauss and L. Tzafriri that such an unconditional basis is equivalent to the standard unit vector basis of c_0 or ℓ_p for some $p \in [1,\infty)$. The fact that there exist two non-equivalent unconditional bases on ℓ_p for $p \in (1,\infty)$ (Proposition 3.2.11) and explicit computations excluding the spaces c_0 and ℓ_1 then finish the original proof of Theorem 2.1.42.

The use of the abstract result of Lindenstrauss and Tzafriri makes it almost impossible to construct explicit counterexamples to the maximal regularity problem. This was our main motivation to find a new approach to the maximal regularity problem which at least at the level of the constructed semigroup uses explicit calculations. Furthermore, notice that if a Banach space X admits an unconditional (in a certain sense) non-symmetric Schauder decomposition, then one can also construct explicit counterexamples to the maximal regularity problem if one uses the perturbation result Proposition A.1.13 as a replacement in the proof of Theorem 2.1.35.

Further Results In the follow-up article [KL02] Kalton and Lancien proved the following refinement of Proposition 2.1.52. For the formulation of the result note that a Schauder decomposition $(\Delta_m)_{m\in\mathbb{N}}$ of a Banach space is called *finite dimensional* if $\operatorname{Rg}\Delta_m < \infty$ for all $m \in \mathbb{N}$.

Theorem 2.4.2. *Let X be a UMD-space with (MRP). If X has a finite dimensional Schauder decomposition, then there exist finite-dimensional spaces $(X_n)_{n\in\mathbb{N}}$ with $X \simeq \oplus_{\ell_2}^n X_n$.*

The proof of this result is also based on Proposition 2.4.1, however it makes further use of sophisticated results from the geometric theory of Banach spaces. If there exists a uniform bound on the dimensions of the spaces X_n for $n \in \mathbb{N}$, the above theorem of course implies that X is isomorphic to a Hilbert space. Nevertheless the following problems are still open.

Problem 2.4.3. Let X be a separable Banach space with (MRP). Is then X isomorphic to a Hilbert space?

If this is not true, one may ask whether the following weaker version does hold.

Problem 2.4.4. Let X be a (separable) UMD-space with (MRP). Is then X isomorphic to a Hilbert space?

An interesting class of Banach spaces to study for this question (compare with the results obtained in Section 2.3.1 for the H^∞-calculus and bounded imaginary powers) are the hereditarily indecomposable Banach spaces. This has not been done yet by the author of the thesis so far.

There are many possible scenarios. It may be possible that every generator of an analytic C_0-semigroup already has maximal regularity. On the other hand, it could also be possible that on some of these spaces every generator of an $\mathcal{R}$-sectorial semigroup is already bounded as it was the case for bounded imaginary powers.

Here the following interesting question arises: do there exist hereditarily indecomposable Banach spaces for which every generator of a C_0-semigroup is already bounded? Notice that this is not true for every H.I. Banach space as there exist H.I. Banach spaces with a conditional Schauder basis. Answers to these problems may provide counterexamples to the two problems above.

The Maximal Regularity Problem and the Complemented Subspace Problem At the very end of this chapter we want to mention to the reader the close connection between the maximal regularity problem and the *complemented subspace problem*, one of the most famous problems in the theory of Banach spaces. It asks whether a Banach space in which every closed subspace is complemented is isomorphic to a Hilbert space. This problem indeed has a positive solution given in the celebrated article [LT71] by J. Lindenstrauss and L. Tzafriri. Now, notice that the short description of Kalton and Lancien's original proof we have just given is an (almost) complete proof of the fact that every Banach space that admits an unconditional basis and satisfies the complemented subspace property is isomorphic to a Hilbert space. In fact, in F. Albiac and N.J. Kalton's student text on Banach space theory exactly this proof is given [AK06, Theorem 9.4.4].

However, one must be careful with this connection as the complemented subspace problem has a positive solution on ℓ_∞, whereas ℓ_∞ is a Banach space with (MRP) that is not isomorphic to a Hilbert space. The crucial point that makes it more difficult to obtain positive results for the maximal regularity problem is that for the maximal regularity problem one can only rely on the information obtained from the C_0-semigroups living on a given space.

3 Extrapolation of Regularity Properties

We now change the focus of our studies. After we have studied the connections between the different regularity properties of sectorial operators on a single space in the previous chapter, we now ask how such properties extrapolate if considered on scales of Banach spaces. The main new result of this chapter gives a complete negative answer to the extrapolation problem for maximal regularity. In simplified terms it says that maximal regularity does not extrapolate from L_2 to L_p. We now give a detailed motivation for the extrapolation problems studied in the following.

The most prominent and important example for applications is the following. Suppose one has given a consistent family of C_0-semigroups $(T_p(t))_{t\geq 0}$ on $L_p(\Omega)$ for some measure space (Ω, Σ, μ) and for $p \in I$, where $I \subset [1, \infty]$ is an interval. This means that one has $T_{p_1}(t)f = T_{p_2}(t)f$ for all $f \in L_{p_1}(\Omega) \cap L_{p_2}(\Omega)$, all $t \geq 0$ and all pairs $p_1, p_2 \in I$. Further suppose that for some $p_0 \in I$ the semigroup $(T_{p_0}(t))_{t\geq 0}$ satisfies some additional regularity property, for example is analytic or $\mathcal{R}$-analytic. Then one can ask under which conditions this regularity property extrapolates to all (or some) $p \in I$, which means that $(T_p(t))_{t\geq 0}$ automatically has the same regularity property for all $p \in I$. This is of particular relevance for applications in the case $p = 2$. As $L_2(\Omega)$ is a Hilbert space, one can rely on the well-established Hilbert space theory which is both powerful and often relatively easily to apply. Extrapolation results then give us for free the same regularity properties on L_p-spaces which are more difficult to study individually. After this abstract treatment we now discuss two concrete problems which are studied in the following sections.

First of all we study the extrapolation properties of analytic semigroups. In concrete examples one often obtains the analyticity of a semigroup on Hilbert spaces, for example by using form methods or Fourier analysis. It is then a classical well-known result that analyticity always extrapolates from $L_2(\Omega)$ or some different $L_{p_0}(\Omega)$ to $L_p(\Omega)$ for all $p \in (1, \infty)$. This is usually proved by invoking the Stein interpolation theorem. Using generalizations of the classical Stein interpolation theorem, we present this method for complex interpolation spaces. Because of the use of classical results from complex analysis this approach is naturally limited to the complex interpolation functor. Therefore, afterwards, we develop a new approach to the extrapolation problem which is based on a characterization of analytic semigroups by T. Kato and A. Beurling. We give a generalization of Kato's sufficient criterion for the analyticity of a C_0-semigroup which then gives a complete characterization of analyticity in terms of the semigroup. We then use this characterization to prove a generalization of the classical extrapolation result for a rather general class of interpolation spaces. In particular, this includes all interpolation spaces obtained by the real interpolation method except for some endpoint cases.

Afterwards, we study the extrapolation properties of $\mathcal{R}$-analyticity or maximal regularity. Recall that in the Hilbert space case maximal regularity holds automatically if the semigroup is analytic. In this case one can ask whether maximal regularity automatically extrapolates to the L_p-scale for consistent semigroups as it is the case for analyticity. We refer to this problem as the *maximal regularity extrapolation problem*. Note that although the Kalton–Lancien Theorem shows the existence of analytic semigroups that are not $\mathcal{R}$-analytic on a fixed L_p-space, it is not clear whether a consistent family of analytic semigroups on the L_p-scale can lose maximal regularity for some $p \neq 2$. This problem is open since the early beginnings of the theory [Are04, 7.2.2]. In the following we will answer this question negatively by giving a counterexample to the maximal regularity problem. In fact, we show in Theorem 3.2.22 that for every interval $I \subset (1, \infty)$ with $2 \in I$ there exists a family of consistent (analytic) C_0-semigroups $(T_p(z))_{z \in \Sigma_{\frac{\pi}{2}}}$ for $p \in (1, \infty)$ such that $(T_p(z))$ has maximal regularity if and only if $p \in I$. In particular, the interval may be $I = \{2\}$.

As these examples show that one does not get maximal regularity for free from the Hilbert space case as it is the case for analyticity, it is important to give sufficient general criteria for the extrapolation of maximal regularity. Here we profit from the study of the extrapolation properties of analyticity from the first part. Indeed, both the approaches using the generalized Stein interpolation theorem and Kato–Beurling type characterizations have natural analogues in the $\mathcal{R}$-analytic case. Whereas the first uses a reduction to the analytic case via the use of associated semigroups as introduced in Definition 2.1.16, the second develops a new characterization of maximal regularity from scratch.

Throughout the presentation we make use of interpolation theory. The used results and references to the literature can be found in Appendix A.5. This chapter contains material from the published articles [Fac13b] and [Fac14] and new unpublished results.

3.1 Extrapolation of Analyticity

We begin with the study of the extrapolation properties of analytic semigroups. The central result in this section is that one essentially gets the extrapolation of analyticity for free. This result is well-known at least in the case of L_p-

spaces and is usually proved with the help of Stein's interpolation theorem. We start by giving a proof of this result in the general context of complex interpolation spaces. Since this approach makes heavy use of methods from complex analysis, it is naturally limited to spaces obtained from the complex interpolation method. We therefore give a new approach to the extrapolation result which is completely free from such limitations. This will finally allow us to generalize the extrapolation result to a rather general class of interpolation spaces. In particular, we can deal with spaces obtained from the real interpolation method. Moreover, from a conceptual point of view this shows that the extrapolation of analyticity is a general phenomenon which is almost independent of the concrete technique used for interpolation.

Our new approach is based on a complete characterization of the analyticity of a C_0-semigroup $(T(t))_{t\geq 0}$ in terms of the behaviour of the mapping $t \mapsto T(t)$ in zero developed in [Fac13b]. In particular, the characterization does not involve the generator of $(T(t))_{t\geq 0}$. Our characterization essentially is a reformulation of a theorem obtained by A. Beurling. However, for the main part we do not follow Beurling's original argument. Instead, we follow ideas of T. Kato which allow us to give an extremely simplified proof of Beurling's result for strongly continuous semigroups. Along the way we also prove a zero-two law for cosine families with our developed techniques.

3.1.1 Via the Abstract Stein Interpolation Theorem

In this preliminary subsection we give the short classical proof of the fact that the analyticity of semigroups extrapolates. The most important application of this theorem arises in the following setting: Suppose that one has given two consistent C_0-semigroups $(T_p(t))_{t\geq 0}$ on some L_p-space $L_p(\Omega)$ for $p \neq 2$ and $(T_2(t))_{t\geq 0}$ on $L_2(\Omega)$ over the same measure space. Moreover, we assume that $(T_2(t))_{t\geq 0}$ is analytic. In applications this can often be obtained rather easily by using Hilbert space methods, e.g. by applying form methods or Fourier transforms. The result below then guarantees that $(T_2(t))_{t\geq 0}$ can be continued to a semigroup on $L_q(\Omega)$ for all q between 2 and p which is automatically analytic as well.

This result is usually proved as a direct consequence of the Stein interpolation theorem. We prove it in a generalized version for complex interpolation spaces. For the proof we rely on a generalized version of the classical Stein interpolation theorem for complex interpolation couples. We have decided to present the result in this generality as this allows us to formulate it more closely in the spirit of the extrapolation result for general interpolation spaces presented in the next subsections. For the special case of L_p-spaces the proof below is well-known in the literature, for example see [Lun09, 6.2].

Theorem 3.1.1. *Let (X_1, X_2) be an interpolation couple of Banach spaces. Assume that $(T(t))_{t\geq 0}$ is a semigroup on $X_1 + X_2$ which leaves both X_1 and X_2 invariant and is bounded analytic of angle δ on X_2 and bounded and immediately norm-continuous on X_1. If $(T(t))_{t\geq 0}$ is strongly continuous on X_1 or X_2, then for all $\alpha \in (0,1)$ the semigroup $(T(t))_{t\geq 0}$ induces an analytic C_0-semigroup on the complex interpolation space $X_\alpha = (X_1, X_2)_\alpha$ of angle at least $\alpha\delta$.*

Proof. Let S denote the strip $S := \{z \in \mathbb{C} : 0 \leq \operatorname{Re} z \leq 1\}$ in the complex plane. Further let $\theta \in (0, \delta)$ and $t > 0$. By assumption, one has the estimate $\|T(se^{i\theta})\|_{\mathcal{B}(X_2)} \leq M$ for all $s \geq 0$ and some constant $M > 0$. Moreover, since $(T(s))_{s\geq 0}$ defines a bounded semigroup on X_1, by possibly enlarging M one has $\|T(s)\|_{\mathcal{B}(X_1)} \leq M$ for all $s \geq 0$. Now let

$$\begin{aligned} N\colon S &\to X_1 + X_2 \\ \lambda &\mapsto T(te^{i\theta\lambda}). \end{aligned}$$

Clearly, for all x in the dense set $X_1 \cap X_2$ the function $N(\cdot)x\colon S \to X_1 + X_2$ is continuous, bounded and analytic on the interior of S. Moreover, for $j = 1, 2$ one has

$$\sup\{\|N(j-1+is)x\|_{X_j} : s \in \mathbb{R}, \|x\|_{X_j} \leq 1\} \leq M.$$

Hence, the abstract Stein interpolation theorem (Theorem A.5.8) shows that for all $\alpha \in (0,1)$ one has $N(\alpha)(X_1 \cap X_2) \subset X_\alpha$ and that $N(\alpha)$ extends to a bounded operator on X_α with

$$\|T(te^{i\theta\alpha})\|_{X_\alpha} = \|N(\alpha)x\|_{X_\alpha} \leq M \|x\|_{X_\alpha}$$

for all $t > 0$. Since an analogous estimate holds for θ replaced by $-\theta$, it follows from a well-known characterization of analytic semigroups [EN00, Theorem 4.6(b)] that for all $\alpha \in (0,1)$ the semigroup $(T(t))_{t\geq 0}$ extrapolates to an analytic semigroup on X_α of angle at least $\alpha\delta$. □

3.1.2 Via a Kato–Beurling Type Characterization

We now present a new approach for the extrapolation of analyticity of C_0-semigroups that does not make use of a variant of Stein's interpolation theorem. Instead, we use a complete characterization of analyticity of semigroups proved in [Fac13b] and originally going back to T. Kato and A. Beurling in terms of the behaviour of the operator-norm of polynomials of the semigroup at zero. The key step done in [Fac13b] is here to obtain a new proof of Beurling's results for C_0-semigroups that is based on the simple ideas of Kato. In particular, a key feature of the used arguments is the fact that they can be easily adapted to the case of $\mathcal{R}$-analytic semigroups. Moreover, the extrapolation results can then be deduced for arbitrary interpolation spaces of order

$\theta \in (0,1)$ by the mere definition of those interpolation spaces. In particular, we see that the extrapolation property does not only hold for the complex interpolation method, a restriction previously imposed by the usage of Stein's interpolation theorem.

3.1.2.1 Kato–Beurling Type Characterizations of Analytic Semigroups

In this subsection we prove a characterization of the analyticity of a strongly continuous semigroup $(T(t))_{t\geq 0}$ in terms of a zero-two law which is purely based on the behaviour of the semigroup at zero. It is notable that the characterization is valid without any restrictions on the underlying Banach space or on the structure of the semigroup. Zero-two laws for semigroups have a very rich history, which we present shortly in Section 3.3. In particular, notable similar characterizations were also found by A. Beurling [Beu70] and T. Kato [Kat70]. We therefore speak of Kato–Beurling type theorems. In the following we will prove a variant of the result in [Beu70] whose proof relies on the far more easy techniques used in [Kat70].

In particular, the following result is due to T. Kato [Kat70, p. 495].

Lemma 3.1.2. *Let X be a Banach space and $(T(t))_{t\geq 0}$ a C_0-semigroup on X. Then $(T(t))_{t\geq 0}$ extends to an analytic C_0-semigroup if and only if there exist constants $|\zeta| = 1$, $t_0 > 0$ and $K > 0$ such that*

$$\zeta \in \rho(T(t)) \quad \textit{and} \quad \|(\zeta - T(t))^{-1}\| \leq K \qquad \textit{for all } 0 < t \leq t_0. \tag{3.1}$$

In this case, (3.1) *holds for all $|\zeta| \geq 1$ with $\zeta \neq 1$.*

Proof. Let A be the infinitesimal generator of $(T(t))_{t\geq 0}$. Assume that (3.1) holds for some $|\zeta| = 1$. Choose $\theta > 0$ such that $e^{i\theta} = \zeta$ and for $t > 0$ the unique α such that $t\alpha = \theta$. Then for $\alpha \geq \frac{\theta}{t_0}$ we have $t \leq t_0$. Further

$$e^{-it\alpha}T(t)x - x = (A - i\alpha)\int_0^t e^{-is\alpha}T(s)x\,ds = \int_0^t e^{-is\alpha}T(s)(A - i\alpha)x\,ds,$$

where the first equality holds for all $x \in X$ and the second for all $x \in D(A)$. Hence, $A - i\alpha$ is invertible and

$$(A - i\alpha)^{-1} = -e^{it\alpha}(\zeta - T(t))^{-1}\int_0^t e^{-is\alpha}T(s)\,ds.$$

Choose $M > 0$ such that $t \mapsto \|T(t)\|$ is bounded by M on $[0, t_0]$. Then $\|(A - i\alpha)^{-1}\| \leq KMt = KM\theta\alpha^{-1}$. The same argument works for negative values of α if one replaces θ by $\tilde{\theta} < 0$ with $e^{i\tilde{\theta}} = \zeta$. Hence, there exist $\alpha_0 > 0$ and $C > 0$ such that $\{i\alpha : |\alpha| > \alpha_0\} \subseteq \rho(A)$ and $\|\alpha(A - i\alpha)^{-1}\| \leq C$ for all $|\alpha| > \alpha_0$. By [ABHN11, Corollary 3.7.18], this implies the analyticity of the semigroup.

Conversely, let $(T(t))_{t\geq 0}$ be analytic. Then for some $\alpha > 0$ the negative infinitesimal generator $-A+\alpha$ of $(e^{-\alpha t}T(t))_{t\geq 0}$ is sectorial with $\omega(-A+\alpha) < \frac{\pi}{2}$. Using the holomorphic functional calculus and a deformation argument, one has the representation

$$T(t) = \frac{1}{2\pi i}\int_{\gamma} e^{-(\lambda-\alpha)t} R(\lambda, -A+\alpha)\, d\lambda$$

for a curve γ that surrounds $\Sigma_{\omega(-A+\alpha)}$ and agrees with $\partial\Sigma_\theta$ for some $\theta \in (\omega(-A+\alpha), \frac{\pi}{2})$ except in a small neighbourhood of the origin of the complex plane, where $\partial\Sigma_\theta$ is deformed in such a way that it surrounds the singularity at the origin. Moreover, for a given $\zeta \neq 0$ the equality $e^{-zt} = \zeta$ holds if and only if $z = \frac{-\log|\zeta| + i(\arg\zeta + 2k\pi)}{t}$ for some $k \in \mathbb{Z}$. We deduce that for $|\zeta| \geq 1$ with $\zeta \neq 1$ there exists a $\delta > 0$ such that for $t \in (0,\delta)$ the singularity of the function $\lambda \to (e^{-(\lambda-\alpha)t} - \zeta)^{-1}$ lies outside the region to the right of the curve γ. Hence, for $t \in (0,\delta)$ we can define via the holomorphic functional calculus

$$B(t) = \frac{1}{2\pi i}\int_{\gamma} e^{-(\lambda-\alpha)t}(e^{-(\lambda-\alpha)t} - \zeta)^{-1} R(\lambda, -A+\alpha)\, d\lambda.$$

By the change of variables $z = (\lambda-\alpha)t$, we obtain

$$B(t) = \frac{1}{2\pi i}\int_{\gamma_t} e^{-z}(e^{-z} - \zeta)^{-1} t^{-1} R(z/t + \alpha, -A+\alpha)\, dz$$

for a t-dependent curve γ_t. Cauchy's integral formula now allows us to replace the family γ_t by a fixed curve γ_{t_0}. Let $d := \inf_{z\in\gamma_{t_0}} |e^z - \zeta| > 0$. Estimating the operator norm of the above integral, we see that for sufficiently small $t \in (0,\delta)$ and some constants $M, M' \geq 0$ one has

$$\begin{aligned}
\|B(t)\| &\leq \frac{M}{2\pi}\int_{\gamma_{t_0}} e^{-\operatorname{Re} z} |e^{-z} - \zeta|^{-1}\, t^{-1} \left|\frac{z}{t} + \alpha\right|^{-1} d|z| \\
&\leq \frac{M}{2\pi d}\int_{\gamma_{t_0}} e^{-\operatorname{Re} z} |z + t\alpha|^{-1}\, d|z| \leq \frac{M'}{d}\int_{\gamma_{t_0}} e^{-\operatorname{Re} z} |z|^{-1}\, d|z| < \infty.
\end{aligned}$$

Notice that one has $e^{-z} \cdot e^{-z}(e^{-z} - \zeta)^{-1} = e^{-z}(1 + \zeta(e^{-z} - \zeta)^{-1})$. Applying the holomorphic functional calculus, we therefore obtain for $t \in (0,\delta)$ that

$$T(t)B(t) = B(t)T(t) = T(t) + \zeta B(t).$$

Hence, for $t \in (0,\delta)$ we have

$$(\operatorname{Id} - B(t))(\zeta - T(t)) = (\zeta - T(t))(\operatorname{Id} - B(t)) = \zeta.$$

Altogether, we have $R(\zeta, T(t)) = \zeta^{-1}(\operatorname{Id} - B(t))$, which is uniformly bounded in operator norm for $t \in (0,\delta)$. $\square$

Remark 3.1.3. Consider the unitary multiplication group $U(t)f(x) := e^{itx}f(x)$ on $L_2(\mathbb{R})$. Then for every $|\zeta| \neq 1$ the norm of the inverses $\|(\zeta - T(t))^{-1}\|$ is uniformly bounded in t. Since $(U(t))_{t\geq 0}$ clearly not extend to an analytic semigroup, this example shows that one cannot weaken the assumption $|\zeta| = 1$ in the above theorem.

We can directly apply Lemma 3.1.2 to prove one implication of a Kato–Beurling type theorem for C_0-semigroups. The proof of the converse implication goes back to A. Beurling [Beu70, p. 398] and uses a variant of Bernstein's inequality for trigonometric polynomials.

Theorem 3.1.4 (Bernstein's Inequality). *Let $f(x) = \sum_{k=0}^{n} c_k e^{ikx}$ be a trigonometric polynomial of degree n. Then*

$$\sup_{x\in\mathbb{R}}|f'(x)| \leq n \sup_{x\in\mathbb{R}}|f(x)|.$$

Proof. Calculating the coefficients of the Fourier series for f' one obtains

$$\widehat{f'}(m) = im\hat{f}(m).$$

For $k \geq 0$ let $F_k(x) = \frac{1}{k+1}\sum_{p=0}^{k}\sum_{q=-p}^{p} e^{iqx}$ be the k-th Féjer-kernel. One sees that $\widehat{F_k}$ is given by $\widehat{F_k}(m) = \frac{k+1-|m|}{k+1}$ for $|m| \leq k+1$. In particular, one has $n\widehat{F_{n-1}}(m-n) = m$ for $m = 0,\dots,n$. Since multiplying with the phase e^{ix} shifts the Fourier coefficients one to the right, we have

$$n\widehat{e^{in\cdot}F_{n-1}}(m) = m \qquad \text{for } m = 0,\dots,n.$$

Since f is a trigonometric polynomial of degree n, $\hat{f}(m) \neq 0$ only holds for $m = 0,\dots,n$. Hence, we obtain for all $m \in \mathbb{Z}$ that

$$\widehat{f'}(m) = in \cdot \widehat{e^{in\cdot}F_{n-1}}(m) \cdot \hat{f}(m).$$

Thus we can write f' as the convolution $f' = in(e^{in\cdot}F_{n-1}) * f$ for which we have the estimate

$$\sup_{x\in\mathbb{R}}|f'(x)| \leq n \sup_{x\in\mathbb{R}}|f(x)| \|e^{in\cdot}F_{n-1}\|_1$$

Using the well-known fact that the Féjer-kernel is normalized, i.e.

$$\|F_k\|_1 = \frac{1}{2\pi}\int_0^{2\pi}|F_k(x)|\,dx = 1,$$

we obtain the desired inequality. □

Inductively, we now obtain a variant of Bernstein's inequality for higher derivatives.

Lemma 3.1.5. *Let $f(x) = \sum_{k=0}^{n} c_k e^{ikx}$ be a trigonometric polynomial of degree n with $|f(x)| \leq 1$ for all $x \in \mathbb{R}$. Then for $N \in \mathbb{N}$ and $x \in \mathbb{R}$ one has for all $l \in \mathbb{N}_0$ the following estimate for the l-th derivative of the N-th power of f:*

$$\left| \left(\frac{d}{dx}\right)^l f^N(x) \right| \leq \begin{cases} (Nn)^l \, |f(x)|^{N-l} & \text{if } l \leq N, \\ (Nn)^l & \text{if } l > N. \end{cases}$$

Proof. Since f^N and all of its derivatives are trigonometric polynomials of degree Nn, the case $l > N$ follows directly from Bernstein's inequality (Theorem 3.1.4). For the case $l \leq N$ we write $\left(\frac{d}{dx}\right)^l f^N = f^{N-l} r_l$. Here r_l is a trigonometric polynomial of degree $Nn - (N-l)n = ln$. Bernstein's inequality (Theorem 3.1.4) yields

$$\|r_l'\|_\infty \leq ln \|r_l\|_\infty,$$

where $\|\cdot\|_\infty$ denotes the supremum norm on $C_b(\mathbb{R})$. Further, we have for $l \leq N-1$

$$\left(\frac{d}{dx}\right)^{l+1} f^N = (N-l) f^{N-(l+1)} f' r_l + f^{N-l} r_l'.$$

This shows that $r_{l+1} = (N-l) f' r_l + f r_l'$. Hence, we have

$$\begin{aligned} \|r_{l+1}\|_\infty &\leq (N-l)\|f'\|_\infty \|r_l\|_\infty + \|f\|_\infty \|r_l'\|_\infty \\ &\leq ((N-l)\|f'\|_\infty + ln\|f\|_\infty) \|r_l\|_\infty \leq ((N-l)n + ln) \|f\|_\infty \|r_l\|_\infty \\ &= Nn \|f\|_\infty \|r_l\|_\infty, \end{aligned}$$

where we have again used Bernstein's inequality in the last inequality. Inductively, we obtain $\|r_l\|_\infty \leq (Nn)^l$. This finishes the proof of the second case. ☐

In the following $\mathcal{P}_1$ denotes the space of all polynomials $p \in \mathcal{P}$ with $|p(1)| < \|p\|_{\mathbb{D}}$, where $\|p\|_{\mathbb{D}} := \sup_{|z| \leq 1} |p(z)|$ is the norm of p in the disc algebra $A(\mathbb{D})$. We can now give a new and easy proof of the following Kato–Beurling type characterization presented in [Fac13b]. We are grateful to S. Król who informed us that part (a) of the following theorem is also contained in [vC85, Theorem 5.1] using the same line of proof.

Theorem 3.1.6 (Kato–Beurling). *Let $(T(t))_{t \geq 0}$ be a C_0-semigroup on a Banach space X. Then*

(a) *$(T(t))_{t \geq 0}$ extends to an analytic C_0-semigroup if there is $p \in \mathcal{P}$ such that*

$$\limsup_{t \downarrow 0} \|p(T(t))\| < \|p\|_{\mathbb{D}}.$$

(b) *Conversely, if* $(T(z))_{z\in\Sigma}$ *is analytic, then for each* $p \in \mathcal{P}_1$ *there exists* $N_0 \in \mathbb{N}$ *such that for every* $N \geq N_0$ *there exists* $K_0 \in \mathbb{R}_+$ *with*

$$\limsup_{t\downarrow 0} \left\| p^N(T(t))T(Kt) \right\| < \left\| p^N \right\|_{\mathbb{D}} \quad \textit{for all } K \geq K_0.$$

Proof. We start with (a). Notice that by the assumptions p cannot be constant. Further, as a consequence of the maximum principle we may assume that $\|p\|_{\mathbb{D}} = p(\zeta) = 1$ for some $\zeta \in \partial\mathbb{D}$ (replace p by $\tilde{p}(z) = e^{-i\arg p(\zeta)}\|p\|_{\mathbb{D}}^{-1}p(z)$ if necessary). By assumption, there exist constants $t_0 > 0$ and $0 < \rho < 1$ such that $\|p(T(t))\| \leq \rho$ for all $0 < t \leq t_0$. By the expansion of the Neumann series, $\mathrm{Id} - p(T(t))$ is invertible with

$$\|[\mathrm{Id} - p(T(t))]^{-1}\| \leq \frac{1}{1-\rho} \qquad \text{for all } 0 < t \leq t_0. \tag{3.2}$$

Factorization of $1 - p$ yields $\mathrm{Id} - p(T(t)) = (\zeta - T(t))q(T(t))$ for some $q \in \mathcal{P}$. Observe that $\zeta - T(t)$ is invertible for $0 < t < t_0$ and that its inverse is given by $q(T(t))\,[\mathrm{Id} - p(T(t))]^{-1}$. The boundedness of $t \mapsto \|T(t)\|$ on $[0,1]$ together with (3.2) shows that there exists a $K > 0$ such that

$$\|(\zeta - T(t))^{-1}\| \leq K \qquad \text{for all } 0 < t \leq t_0.$$

Now, Kato's result (Lemma 3.1.2) shows that $(T(t))_{t\geq 0}$ extends to an analytic semigroup. This proves (a).

Conversely for part (b), let $(T(z))_{z\in\Sigma_{\tilde{\delta}}}$ be an analytic C_0-semigroup and let $0 < \delta < \tilde{\delta}$. Note that this implies that for all $t > 0$ the ball around t with radius $t\sin\delta$ is contained in Σ_δ. Let $M := \sup\{\|T(z)\| : z \in \Sigma_\delta, |z| \leq 2\}$. For a polynomial $p(z) = \sum_{k=0}^n a_k z^k$ in $\mathcal{P}$ we obtain for $s > 0$ that $p(T(t))T(s) = \sum_{k=0}^n a_k T(s + kt)$. Since $z \mapsto T(z)$ is holomorphic in Σ_δ, we can write for $0 < nt \leq s\sin\delta$

$$\sum_{k=0}^n a_k T(s+kt) = \frac{1}{2\pi i}\sum_{l=0}^\infty \int_{|z-s|=r} \frac{T(z)}{(z-s)^{l+1}}\,dz\,t^l \sum_{k=0}^n a_k k^l \qquad (r \leq s\sin\delta).$$

For $N \in \mathbb{N}$ we now replace p by $p^N = \sum_{k=0}^{Nn} a_{k,N} z^k$, where we assume $p \in \mathcal{P}_1$. After scaling we may assume that $\|p\|_{\mathbb{D}} = 1$. Further, we choose $s = Kt$ for $K \in \mathbb{R}_+$. Then, for $K \geq Nn(\sin\delta)^{-1}$ and $r_t = Kt\sin\delta$ we obtain for $t \leq K^{-1}$

$$\|p^N(T(t))T(Kt)\| \leq M \sum_{l=0}^\infty \left| \sum_{k=0}^{Nn} a_{k,N} k^l \right| t^l (tK\sin\delta)^{-l}. \tag{3.3}$$

Now, using equation (3.3) together with Lemma 3.1.5 (for the trigonometric polynomial $f(x) = p(e^{ix})$ and $x = 0$) in the second inequality below, we obtain for $t \le K^{-1}$ the estimate

$$
\begin{aligned}
\left\|p^N(T(t))T(Kt)\right\| &\le M \sum_{l=0}^{\infty} \left| \sum_{k=0}^{Nn} a_{k,N} k^l \right| (K \sin\delta)^{-l} \\
&\le M \left(|p(1)|^N \sum_{l=0}^{N} \left(\frac{Nn}{|p(1)| K \sin\delta} \right)^l + \sum_{l=N+1}^{\infty} \left(\frac{Nn}{K\sin\delta} \right)^l \right) \\
&= M \left(|p(1)|^N \frac{1 - C_{1,K}^{N+1}}{1 - C_{1,K}} + \frac{C_{2,K}^{N+1}}{1 - C_{2,K}} \right),
\end{aligned}
$$

where $C_{1,K} = \frac{Nn}{|p(1)|K\sin\delta}$ and $C_{2,K} = \frac{Nn}{K\sin\delta}$ (of course for $p(1) = 0$ the first term vanishes). Now, since $|p(1)| < 1$, the right hand side is arbitrarily small provided first N and then K are chosen large enough. $\square$

Notice that by the maximum principle $\|p\|_{\mathbb{D}}$ equals $\|z \mapsto p(z)z^m\|_{\mathbb{D}}$ for every $m \in \mathbb{N}$. This gives us the following characterization of analyticity.

Corollary 3.1.7 (A Characterization of Analyticity on the Real Line). *A C_0-semigroup $(T(t))_{t\ge0}$ on a Banach space X is analytic if and only if there is $p \in \mathcal{P}$ such that*

$$\limsup_{t\downarrow 0} \|p(T(t))\| < \|p\|_{\mathbb{D}}.$$

Moreover, taking $p(z) = (z-1)^N$ for some $N \in \mathbb{N}$ we obtain a variant of Kato's original result which is invariant under equivalent renorming of the underlying Banach space.

Corollary 3.1.8. *Let $(T(t))_{t\ge0}$ be a C_0-semigroup on a Banach space X.*

(a) *$(T(t))_{t\ge0}$ extends to an analytic semigroup if for some $N \in \mathbb{N}$*

$$\limsup_{t\downarrow 0} \left\|(T(t) - \mathrm{Id})^N\right\|^{1/N} < 2. \tag{3.4}$$

(b) *Conversely, if $(T(t))_{t\ge0}$ is analytic, there exists $N_0 \in \mathbb{N}$ such that for every $N \ge N_0$ there exists $K_0 \in \mathbb{R}_+$ such that*

$$\limsup_{t\downarrow 0} \left\|(T(t) - \mathrm{Id})^N T(t)^K\right\|^{1/N} < 2 \quad \textit{for all } \mathbb{N} \ni K \ge K_0.$$

We remark that rescaling the semigroup does not affect both the analyticity of a semigroup and the validity of inequality (3.4). Therefore we can assume in the following remark that all semigroups are bounded.

Remark 3.1.9. It is a natural question whether already the condition (3.4)

$$\limsup_{t\downarrow 0}\left\|(T(t)-\mathrm{Id})^N\right\|^{1/N}<2$$

for some $N\in\mathbb{N}$ is equivalent for a C_0-semigroup $(T(t))_{t\geq 0}$ to be analytic. In fact, by an observation of A. Pazy [Paz83, Corollary 5.8] an analytic C_0-semigroup $(T(t))_{t\geq 0}$ on a uniformly convex space X which is contractive on the real line satisfies $\limsup_{t\downarrow 0}\|T(t)-\mathrm{Id}\|<2$. This can be seen as follows: Assume that $\limsup_{t\downarrow 0}\|T(t)-\mathrm{Id}\|=2$. Then there exist sequences $(x_n)_{n\in\mathbb{N}}\subset X$ and $(t_n)_{n\in\mathbb{N}}\subset\mathbb{R}_+$ such that $t_n\to 0$ for $n\to\infty$ and

$$\|T(t_n)x_n-x_n\|\geq 2-\frac{1}{n}.$$

Since $\|T(t_n)x_n\|\leq 1$ by the contractivity of the semigroup, it follows from the uniform convexity of X that

$$\|T(t_n)x_n+x_n\|\to 0\qquad\text{for } n\to\infty.$$

By Kato's criterion (Lemma 3.1.2) this contradicts the fact that $(T(t))_{t\geq 0}$ is analytic. Further notice that the Gaussian semigroup on $L_1(\mathbb{R})$ shows that Pazy's converse does not hold on general Banach spaces (for details see [Fac11]). Even more, it does not even hold on reflexive spaces. For this let $(G_p(z))_{z\in\Sigma_{\frac{\pi}{2}}}$ denote the Gaussian semigroup on $L_p(\mathbb{R})$ for $p\in[1,\infty)$ (for details see [ABHN11, Example 3.7.6]). Choose a sequence $(p_n)_{n\in\mathbb{N}}\subset(1,\infty)$ with $p_n\to 1$ for $n\to\infty$. Then on the reflexive space given by the direct ℓ_2-sum $\oplus_{\ell_2}^n L_{p_n}(\mathbb{R})$ (see Definition A.2.1) we define the diagonal semigroup

$$G(z)((f_n)_{n\in\mathbb{N}}):=(G_{p_n}(z)(f_n))_{n\in\mathbb{N}}.$$

It follows from interpolation that $\|G_{p_n}(z)\|$ is uniformly bounded in $n\in\mathbb{N}$ and in $z\in\Sigma_\delta$ for all $\delta\in(0,\frac{\pi}{2})$. Hence, $(G(z))_{z\in\frac{\pi}{2}}$ defines a bounded analytic C_0-semigroup on $\oplus_{\ell_2}^n L_{p_n}(\mathbb{R})$ which is contractive on the real line. We now show that one has $\limsup_{t\downarrow 0}\|G(t)-I\|=2$. Since $\limsup_{t\downarrow 0}\|G_1(t)-\mathrm{Id}\|_{\mathcal{B}(L_1(\mathbb{R}))}=2$, there exists a sequence $(t_n)_{n\in\mathbb{N}}$ with $t_n\downarrow 0$ and a sequence of simple functions $(f_n)_{n\in\mathbb{N}}$ with $\|f_n\|_1=1$ and $\|G_1(t_n)f_n-f_n\|_1\to 2$. Because of the continuity of $p\mapsto\|f\|_p$ for $f\in L_1(\mathbb{R})\cap L_\infty(\mathbb{R})$, for every $\varepsilon>0$ there exists an $N_0\in\mathbb{N}$ such that for every $n\geq N_0$ there exists an $m_n\in\mathbb{N}$ such that $\|f_n\|_{p_{m_n}}\leq 1+\varepsilon$ and $\|G_{p_{n_m}}(t_n)f_n-f_n\|_{p_{m_n}}\geq 2-\varepsilon$. This shows that for all $n\geq N_0$ one has

$$\|G(t_n)-\mathrm{Id}\|\geq(1+\varepsilon)^{-1}\|G_{p_{n_m}}(t_n)f_n-f_n\|_{p_{m_n}}\geq\frac{2-\varepsilon}{1+\varepsilon}.$$

Since $\varepsilon>0$ is arbitrary, this shows $\limsup_{t\downarrow 0}\|G(t)-\mathrm{Id}\|=2$.

In some situations it is possible to reduce the general problem to the case of a contractive C_0-semigroup on a uniformly convex space after an equivalent renorming. Assume that $(T(z))$ is an analytic C_0-semigroup (not necessarily contractive on the real line) on a Banach space X for which there exists an equivalent uniformly convex norm $\|\cdot\|_2$ such that $\|T(t)\|_2 \le 1$ for all $t \ge 0$. We will see in Theorem 5.5.14 and in the remarks given in Section 5.6 that this is possible if X is super-reflexive and the negative generator A of $(T(t))_{t\ge 0}$ has a bounded H^∞-calculus with $\omega_{H^\infty}(A) < \frac{\pi}{2}$ 5.5.14. Now the above considerations show $\limsup_{t\downarrow 0} \|T(t) - \mathrm{Id}\|_2 < 2$. Hence, for the original norm we obtain

$$\limsup_{t\downarrow 0} \|(T(t) - \mathrm{Id})^N\|^{1/N} < 2$$

if $N \in \mathbb{N}$ is chosen sufficiently large. However, in the above case of a bounded H^∞-calculus a direct proof is possible. Let e_t be the analytic function $z \mapsto e^{-tz}$. Then one has for all $N \in \mathbb{N}$

$$\|(T(t) - \mathrm{Id})^N\| = \|(e_t - 1)^N(A)\| \le M \, \|e_t - 1\|^N_{H^\infty(\Sigma_\theta)}$$

for some $\theta \in (0, \frac{\pi}{2})$ and some $M \ge 0$ by the boundedness of the H^∞-functional calculus. Further, one has

$$e^{-tz} = e^{-t \operatorname{Re} z} e^{-it \operatorname{Im} z}.$$

Note that the image of $\overline{\Sigma_\theta}$ under e_t, which is independent of $t \ge 0$, has positive distance $\delta > 0$ from the point -1 in the complex plane because of the restriction $|\operatorname{Im} z| \le \tan\theta \operatorname{Re} z$. Since $|e^{-tz}| \le 1$ for all $t \ge 0$ and $z \in \Sigma_\theta$, one has

$$2 \ge \|e_t + 1\|_{H^\infty(\Sigma_\theta)} = \|e_t - (-1)\|_{H^\infty(\Sigma_\theta)} \ge \delta.$$

Altogether, for all $N \in \mathbb{N}$ we obtain the estimate

$$\|(T(t) - \mathrm{Id})^N\|^{1/N} \le M^{1/N}(2 - \delta)$$

for all $t \ge 0$, whose right hand side is strictly smaller than 2 for sufficiently large $N \in \mathbb{N}$.

To summarize, we have seen in rather general cases that we can omit the additional factor of z^K in Corollary 3.1.8(b). However, we do not know whether this is true for all analytic semigroups.

3.1.2.2 Extrapolation of Analytic Semigroups

We now apply the Kato–Beurling Theorem (Theorem 3.1.6) to show extrapolation results for analytic semigroups. The advantage of this new approach over the classical approach via Stein's interpolation theorem is that the argument is valid for arbitrary interpolation spaces of exponent $\theta \in (0,1)$. In this section we make again use of the theory of interpolation spaces whose basic properties are summarized in Appendix A.5. The general extrapolation theorem proved in [Fac13b] then reads as follows.

Theorem 3.1.10. *Let $(T_1(t))_{t\geq 0}$ and $(T_2(t))_{t\geq 0}$ be consistent semigroups on an interpolation couple (X_1, X_2) and let X be a regular interpolation space of exponent $\theta \in (0,1)$ with respect to (X_1, X_2). Assume further that one of the semigroups is strongly continuous and that the other is locally bounded. If one of the two semigroups is analytic, then there exists a unique analytic C_0-semigroup $(T(z))$ on X which is consistent with $(T_1(t))_{t\geq 0}$ and $(T_2(t))_{t\geq 0}$.*

Proof. Of course, the semigroup $(T(t))_{t\geq 0}$ is obtained by interpolation. Moreover, since X is a regular interpolation space, the semigroup $(T(t))_{t\geq 0}$ agrees with $(T_1(t))_{t\geq 0}$ and $(T_2(t))_{t\geq 0}$ on a dense subset and is therefore uniquely determined. Moreover, by the exactness of the interpolation space X one has for some constant $C \geq 0$

$$\|T(t)x - x\|_X \leq C \|T_1(t)x - x\|_{X_1}^{1-\theta} \|T_2(t)x - x\|_{X_2}^{\theta}$$

for all $t \geq 0$ and all x in the dense subset $X_1 \cap X_2$ of X. Since, by assumption one of the factors tends to zero as $t \downarrow 0$ and the other is locally bounded, the semigroup $(T(t))_{t\geq 0}$ is strongly continuous. It remains to show that $(T(t))_{t\geq 0}$ extends to an analytic semigroup on X. For this we assume without loss of generality that $(T_1(t))_{t\geq 0}$ is analytic. By Corollary 3.1.7 there exists a polynomial $p \in \mathcal{P}$ with $\|p\|_{\mathbb{D}} = 1$ of degree $n = \deg p$, $\rho \in (0,1)$ and $t_0 > 0$ such that for all $t \in (0, t_0)$ one has

$$\|p(T_1(t))\| \leq \rho.$$

Notice that for this implication no strong continuity at zero is needed in the proof of Theorem 3.1.6(b). Since $(T_2(t))_{t\geq 0}$ is locally bounded, there exists a constant $M \geq 0$ such that $\|T_2(t)\|_{\mathcal{B}(X_2)} \leq M$ for all $t \in (0, t_0)$. Let $N \in \mathbb{N}$. We write $p^N(z) = \sum_{k=0}^{Nn} a_{k,N} z^k$. Then we have for sufficiently small $t > 0$

$$\begin{aligned}
\|p^N(T(t))\|_{\mathcal{B}(X)} &\leq C \|p^N(T_1(t))\|_{\mathcal{B}(X_1)}^{1-\theta} \|p^N(T_2(t))\|_{\mathcal{B}(X_2)}^{\theta} \\
&\leq C\rho^{(1-\theta)N} \left(\sum_{k=0}^{Nn} |a_{k,N}| \left\| T_{p_2}(kt) \right\| \right)^{\theta} \leq CM^{\theta} \rho^{(1-\theta)N} \left(\sum_{k=0}^{Nn} |a_{k,N}| \right)^{\theta} \\
&= CM^{\theta} \rho^{(1-\theta)N} \left(\sum_{k=0}^{Nn} \frac{1}{k!} \left| \left(\frac{d}{dx} \right)^k p^N(0) \right| \right)^{\theta} \leq CM^{\theta} (Nn+1)^{\theta} \rho^{(1-\theta)N},
\end{aligned}$$

where we have used the standard estimate obtained from Cauchy's integral formula in the last inequality. In particular, we see that for sufficiently large N one has

$$\limsup_{t\downarrow 0} \left\|p^N(T(t))\right\|_{\mathcal{B}(X)} < 1.$$

By Corollary 3.1.7, the semigroup $(T(t))_{t\geq 0}$ is analytic. □

Remark 3.1.11. The idea to use the Kato–Beurling Theorem to show extrapolation results for analytic semigroups goes back to W. Arendt (see [FGG⁺10, Remark 2.7] or [Are04, 7.2.3]).

3.1.3 An Application: A Zero-Two Law for Cosine Families

In this subsection we present as an application of the Kato–Beurling Theorem (Theorem 3.1.6) a zero-two law for strongly continuous cosine families on UMD-spaces. In the following we need some results from the theory of cosine families. As this theory does not touch the main themes of this thesis, we depart here from our aim to be self-contained and simply refer to the literature in the proofs whenever necessary.

However, before we turn our attention to cosine families we start with the easier case of strongly continuous groups.

Proposition 3.1.12 (Zero-Two Law For Groups). *Let $(U(t))_{t\in\mathbb{R}}$ be a C_0-group on a Banach space X. Then*

$$\limsup_{t\downarrow 0} \|U(t) - \mathrm{Id}\| < 2 \tag{3.5}$$

implies that $(U(t))_{t\in\mathbb{R}}$ has a bounded generator. In particular, the left hand side of (3.5) *equals* 0.

Proof. It follows from the Kato–Beurling Theorem (Theorem 3.1.6) applied to the polynomial $p(z) = z - 1$ that the C_0-semigroup $(U(t))_{t\geq 0}$ is analytic. In particular, $(U(t))_{t\geq 0}$ is immediately norm-continuous. This implies that

$$U(t) - \mathrm{Id} = U(-1)(U(t+1) - U(1)) \to 0 \qquad \text{for } t \downarrow 0$$

in operator norm. This shows that $(U(t))_{t\geq 0}$ is norm-continuous which implies that its generator is a bounded operator. □

Notice that the shift group on various function spaces shows that the above assumption cannot be improved. The zero-two law for groups does in fact hold for arbitrary groups $(U(t))_{t\in\mathbb{R}}$ on Banach spaces without any assumptions on the mapping $t \mapsto U(t)$. This result goes to back to J. Esterle [Est03] (see also the comments in [Dub06]) and its proof is of course far more less elementary.

For semigroups similar zero-one or zero-two laws have been investigated in the literature: for an arbitrary semigroup $(T(t))_{t\geq 0}$ of bounded linear operators $\limsup_{t\downarrow 0}\|T(t)-I\| < 1$ implies the uniform continuity of $(T(t))_{t\geq 0}$ and the left hand side to be equal to zero. The proof of this theorem only involves elementary algebraic manipulations and therefore holds without any regularity assumptions on the semigroup whatsoever. Similar laws were also investigated in the more general context of Banach algebras. For this as well as the stated results see [Est04].

We now want to prove an analogous statement for cosine families. Cosine families play an important role in the study of abstract second order Cauchy problems (see for example [ABHN11, Sec. 3.14] and [Gol85, Ch. 2, Sec. 8]). We now recall their definition.

Definition 3.1.13 (Cosine Family). A strongly continuous map $\mathrm{Cos}\colon \mathbb{R} \to \mathcal{B}(X)$ for a Banach space X is called a strongly continuous *cosine family* if it satisfies the following properties:

(i) $\mathrm{Cos}(0) = \mathrm{Id}$,

(ii) $2\,\mathrm{Cos}(t)\mathrm{Cos}(s) = \mathrm{Cos}(t+s) + \mathrm{Cos}(t-s) \qquad$ for all $t, s \geq 0$.

We now shortly present some basic properties of cosine families that will be useful. Given a strongly continuous cosine family $\mathrm{Cos}\colon \mathbb{R} \to \mathcal{B}(X)$, there exists a uniquely determined closed operator A called the *generator* of Cos satisfying $(\omega^2, \infty) \subset \rho(A)$ for some $\omega > 0$ such that

$$\lambda R(\lambda^2, A) = \int_0^\infty e^{-\lambda t}\,\mathrm{Cos}(t)\,dt \qquad \text{for } \lambda > \omega. \tag{3.6}$$

As in the case of C_0-semigroups it is known that a cosine family Cos has a bounded generator if and only if $\lim_{t\downarrow 0}\|\mathrm{Cos}(t) - \mathrm{Id}\| = 0$. In that case, $\mathrm{Cos}\colon \mathbb{R} \to \mathcal{B}(X)$ is continuous in operator norm [ABHN11, Corollary 3.14.9]. There is a systematic way to build cosine families out of groups: suppose $(U(t))_{t\in\mathbb{R}}$ is a C_0-group on a Banach space. Then $C(t) = \frac{1}{2}(U(t) + U(-t))$ defines a cosine family whose generator is given by the square of the group generator [ABHN11, Example 3.14.15].

We are now ready to prove a preliminary version of the zero-two law for cosine families.

Lemma 3.1.14. *Let $C(t) = \frac{1}{2}(U(t) + U(-t))$ be the cosine family induced by a strongly continuous group $(U(t))_{t\in\mathbb{R}}$. Suppose*

$$\limsup_{t\downarrow 0}\|C(t) - \mathrm{Id}\| < 2. \tag{3.7}$$

Then $(C(t))_{t\in\mathbb{R}}$ is uniformly continuous and the left hand side of (3.7) *equals* 0.

Proof. Assumption (3.7) means that $\|C(t)-\mathrm{Id}\| < \rho$ for some $0<\rho<2$ and all sufficiently small t, say $0<t<t_0$. Let $p(z)=\frac{1}{2}(z-1)^2$. Then $\|p^N\|_{\mathbb{D}} = p^N(-1) = 2^N$ for all $N \in \mathbb{N}$. Further, we observe that if $M \geq 0$ and $\omega \in \mathbb{R}$ are chosen such that $\|U(t)\| \leq Me^{\omega t}$ for $t \geq 0$, then

$$\begin{aligned}\left\|p^N(U(t))\right\| &= \left\|\left(U(t)\left[\frac{U(t)+U(-t)}{2}-\mathrm{Id}\right]\right)^N\right\| \\ &\leq \|U(Nt)\| \left\|\left[\frac{U(t)+U(-t)}{2}-\mathrm{Id}\right]^N\right\| \leq Me^{\omega N t}\rho^N \\ &\leq (M^{1/N}e^{\omega t}\rho)^N \leq \tilde{\rho}^N < \|p^N\|_{\mathbb{D}}\end{aligned}$$

for every $0<\rho<\tilde{\rho}<2$ provided N is big and t is small enough. Hence, Theorem 3.1.6 applied to p^N yields the analyticity of $(U(t))_{t\in\mathbb{R}}$. It then follows as in the proof of the zero-two law for groups (Proposition 3.1.12) that $(U(t))_{t\in\mathbb{R}}$ and therefore $(C(t))_{t\in\mathbb{R}}$ are norm-continuous. □

In particular, for cosine families on UMD-spaces we obtain the following result proved in [Fac13b]. It solves partially the problem raised by W. Arendt whether a zero-two law holds for general cosine families on Banach spaces.

Theorem 3.1.15 (Zero-Two Law for Cosine Families). *Let* $\mathrm{Cos} = (C(t))_{t\in\mathbb{R}}$ *be a strongly continuous cosine family on a UMD-space X such that* (3.7) *holds. Then* Cos *is uniformly continuous and the left hand side of* (3.7) *equals* 0.

Proof. Let A denote the generator of the cosine family Cos. Since A is defined on a UMD-space, by Fattorini's Theorem there exists a generator B of a C_0-group $(U(t))_{t\in\mathbb{R}}$ on X and $\omega \geq 0$ such that $A = B^2+\omega$ [ABHN11, Corollary 3.16.8]. Let $D(t)=\frac{1}{2}(U(t)+U(-t))$ be the cosine family generated by B^2. It is shown in [Fat69, Lemma 6.1] that from the iteration given by

$$C_0(t) = C(t), \quad C_n(t) = \int_0^t S(t-s)C_{n-1}(s)\,ds$$

in the strong operator topology, where $S(t) \coloneqq \int_0^t C(s)\,ds$ is the associated so-called sine function, one obtains the cosine family $(D(t))_{t\in\mathbb{R}}$ in the strong sense as the series $D(t)=\sum_{n=0}^{\infty}(-\omega)^n C_n(t)$. Moreover, it is shown in the proof of [Fat69, Lemma 6.1] that for all $n \in \mathbb{N}$ one has $\|C_n(t)\| \leq Me^{\alpha t}\frac{t^{2n}}{(2n)!}$ for some constants $M \geq 1$ and $\alpha \geq 0$. Consequently, we obtain

$$\limsup_{t\downarrow 0}\|D(t)-\mathrm{Id}\| \leq \limsup_{t\downarrow 0}\left(\|C(t)-\mathrm{Id}\| + Me^{\omega t}\sum_{n=1}^{\infty}\frac{(\alpha t^2)^n}{(2n)!}\right) < 2.$$

Hence, by Lemma 3.1.14, B^2 and therefore A are bounded operators which in turn is equivalent to the claim [ABHN11, Corollary 3.14.9]. □

It is an interesting and natural question whether the zero-two law for cosine families can be strengthened. Using elementary algebraic operations as in the case of the zero-one law for semigroups, W. Arendt [Are12] observed that a 0-3/2 law holds for arbitrary cosine families without any regularity assumptions on the mapping $\operatorname{Cos}\colon \mathbb{R} \to \mathcal{B}(X)$. Very recently, it has been shown in the preprint [SZ14] by means of spectral theory that the zero-two law for strongly continuous cosine functions as in Theorem 3.1.15 holds for arbitrary Banach spaces.

3.2 Extrapolation of $\mathcal{R}$-Analyticity

We now turn our attention to the extrapolation of $\mathcal{R}$-analyticity of semigroups, which in general is a strictly stronger property than analyticity. Here we will benefit from the methods developed so far in the previous section for the extrapolation of analyticity. Indeed, we will see that we can give analogous proofs in the $\mathcal{R}$-analytic case. This holds for both approaches presented in the analytic case, namely via variants of the Stein interpolation theorem and the approach via Kato–Beurling type characterizations. The first approach is again restricted to the complex interpolation method, whereas the second works for a general class of interpolation functors of order $\theta \in (0,1)$. However, it will only be possible to show positive extrapolation results if one additionally requires the semigroup at the non-regular part of the interpolation couple to be locally $\mathcal{R}$-bounded on the real line. In view of this result one may ask whether maximal regularity does still extrapolate – at least on the L_p-scale – if the non-regular part is only locally bounded in operator-norm. In particular, for applications this result would be very useful as maximal regularity does hold for every analytic semigroup on a Hilbert space, a condition which can often be easily checked in applications, e.g. by form methods or Fourier analytic methods. Moreover, non-linear partial differential equations often need to be treated in higher L_p-spaces. This extrapolation problem is sometimes called the *maximal regularity extrapolation problem* and has been open since the beginnings of the study of maximal regularity. In the last part of this chapter we will give a negative answer to this problem based on the article [Fac14]. Here we will strongly benefit from the methods developed in Section 2.1.4 which allow us to construct rather concrete explicit counterexamples to the maximal regularity problem.

3.2.1 Via Abstract Stein Interpolation

In this subsection we show how one can apply the Stein interpolation theorem to obtain extrapolation results for $\mathcal{R}$-analyticity. The approach presented here goes back to S. Bu and W. Arendt [AB03]. The key idea is to use Theorem 2.1.18

which gives an equivalent description of the $\mathcal{R}$-analyticity of a semigroup on a Banach space X in terms of the analyticity of an associated semigroup on $\mathrm{Rad}(X)$. This allows one to apply the known results for the analytic case.

Theorem 3.2.1. *Let (X_1, X_2) be an interpolation couple of Banach spaces with non-trivial type. Assume that $(T(t))_{t\geq 0}$ is a semigroup on $X_1 + X_2$ which leaves both X_1 and X_2 invariant and is bounded $\mathcal{R}$-analytic of angle δ on X_2 and $\mathcal{R}$-bounded on X_1. If $(T(t))_{t\geq 0}$ is strongly continuous on X_1 or X_2, then for all $\alpha \in (0,1)$ the semigroup $(T(t))_{t\geq 0}$ induces a bounded $\mathcal{R}$-analytic C_0-semigroup on the complex interpolation space $X_\alpha = (X_1, X_2)_\alpha$ of angle at least $\alpha\delta$.*

Proof. We first notice that $(\mathrm{Rad}(X_1), \mathrm{Rad}(X_2))$ is an interpolation couple. By Theorem 2.1.18 one obtains an associated semigroup $(\mathcal{T}(t))_{t\geq 0}$ on the sum $\mathrm{Rad}(X_1) + \mathrm{Rad}(X_2)$. More precisely, $(\mathcal{T}(t))_{t\geq 0}$ is a bounded analytic semigroup of angle δ on $\mathrm{Rad}(X_2)$ and a compatible bounded semigroup on $\mathrm{Rad}(X_1)$. It then follows from Theorem 3.1.1 that $(\mathcal{T}(t))_{t\geq 0}$ induces an analytic C_0-semigroup on the complex interpolation space $(\mathrm{Rad}(X_1), \mathrm{Rad}(X_2))_\alpha$ of angle at least $\alpha\delta$. Since both X_1 and X_2 have non-trivial type, it follows from Corollary A.5.7 that one has the identification $(\mathrm{Rad}(X_1), \mathrm{Rad}(X_2))_\alpha = \mathrm{Rad}(X_\alpha)$. Applying Theorem 2.1.18 again, we see that $(T(t))_{t\geq 0}$ is bounded $\mathcal{R}$-analytic of angle at least $\alpha\delta$ on X_α. □

3.2.2 Via a Kato–Beurling Type Characterization

In this section we use our second approach based on a Kato–Beurling type characterization of $\mathcal{R}$-analytic semigroups in order to obtain extrapolation results for $\mathcal{R}$-analytic semigroups. One could again work with the induced semigroup on $\mathrm{Rad}(X)$ as done in Section 3.2.1 to deduce the extrapolation results directly from the known results for analytic semigroups. However, we think that a direct approach proving analogues of the Kato–Beurling characterizations as done in [Fac13b] is nevertheless desirable because the so-obtained characterizations of $\mathcal{R}$-analytic semigroups are of independent interest.

We start with generalizing the main results, namely Lemma 3.1.2 and Theorem 3.1.6, to the $\mathcal{R}$-analytic case.

Lemma 3.2.2. *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on some Banach space X. Then $(T(t))_{t\geq 0}$ is an $\mathcal{R}$-analytic semigroup if and only if there exist constants $|\zeta| = 1$, $t_0 > 0$ and $K > 0$ such that $\mathcal{R}\{T(t) : 0 < t < t_0\} < \infty$ and*

$$\zeta \in \rho(T(t)) \text{ for all } 0 < t < t_0 \quad \text{and} \quad \mathcal{R}\left\{(\zeta - T(t))^{-1} : 0 < t < t_0\right\} \leq K. \tag{3.8}$$

In this case the above condition holds for all $|\zeta| \geq 1$, $\zeta \neq 1$.

Proof. Let A be the infinitesimal generator of $(T(t))_{t\geq 0}$. We first assume that $(T(t))_{t\geq 0}$ satisfies (3.8). Using the same notation as in the proof of Lemma 3.1.2 (which we use freely without further notice), we obtain with the stronger assumptions of this theorem that for $t\alpha = \theta$ and $\alpha > \frac{\theta}{t_0}$

$$(A - i\alpha)^{-1} = -e^{it\alpha}(\zeta - T(t))^{-1} \int_0^t e^{-is\alpha} T(s)\, ds.$$

Again, making essentially the same estimate as before, we obtain that for $\alpha_1, \dots, \alpha_n$ with $t_k\alpha_k = \theta$ and $\alpha_k > \frac{\theta}{t_0}$ $(1 \leq k \leq n)$

$$\begin{aligned}\mathcal{R}\Big\{\alpha_k(A - i\alpha_k)^{-1} : 1 \leq k \leq n\Big\} &\leq K\mathcal{R}\left\{\int_0^{t_k} \alpha_k e^{-is\alpha_k} T(s)\, ds\right\} \\ &= K\mathcal{R}\left\{\int_0^{\theta} e^{-is} T\left(\frac{s}{\alpha_k}\right) ds\right\}.\end{aligned}$$

Now let $R := \mathcal{R}\{T(t) : 0 < t < t_0\}$. For $x_1, \dots, x_n \in X$ we see that

$$\begin{aligned}\left\|\sum_{k=1}^n r_k \int_0^{\theta} e^{-is} T\left(\frac{s}{\alpha_k}\right) ds\, x_k\right\| &\leq \int_0^{\theta} \left\|\sum_{k=1}^n r_k T\left(\frac{s}{\alpha_k}\right) x_k\right\| ds \\ &\leq R\theta \left\|\sum_{k=1}^n r_k x_k\right\|.\end{aligned}$$

Hence, $\mathcal{R}\Big\{\alpha(A - i\alpha)^{-1} : \alpha > \frac{\theta}{t_0}\Big\} \leq KR\theta$. As before, the same argument works for negative α. Thus there exists an $\alpha_0 \geq 0$ such that $\mathcal{R}\Big\{\alpha(A - i\alpha)^{-1} : |\alpha| > \alpha_0\Big\} < \infty$. Now, Proposition 1.2.6 implies that $(T(t))_{t\geq 0}$ is $\mathcal{R}$-analytic.

Conversely, let $(T(z))_{z\in\Sigma}$ be $\mathcal{R}$-analytic. Then it follows from Proposition 1.2.6 that $\{\lambda R(\lambda, -A + \alpha) : \lambda \notin \overline{\Sigma_\theta}\}$ is $\mathcal{R}$-bounded for some $\alpha > 0$ and $\theta \in (0, \frac{\pi}{2})$. Let $|\zeta| \geq 1$, $\zeta \neq 1$. As shown in the proof of Lemma 3.1.2 one can choose a path γ such that

$$B(t) = \frac{1}{2\pi i} \int_\gamma e^{-(\lambda-\alpha)t}(e^{-(\lambda-\alpha)t} - \zeta)^{-1} R(\lambda, -A + \alpha)\, d\lambda.$$

is a bounded operator for sufficiently small $t \in (0, t_0]$. Again, by the change of variables $z = (\lambda - \alpha)t$, we obtain

$$B(t) = \frac{1}{2\pi i} \int_{\gamma_t} e^{-z}(e^{-z} - \zeta)^{-1} t^{-1} R(z/t + \alpha, -A + \alpha)\, dz$$

for a t-dependent curve γ_t. By Cauchy's integral formula, we can again replace the family γ_t by a fixed curve γ_{t_0}. Let $d := \inf_{z\in\gamma_{t_0}} |e^z - \zeta| > 0$. For $x_1, \dots, x_n \in X$ and $t_1, \dots, t_n \in [0, t_0)$ we have (provided t_0 is chosen small enough)

$$\left\|\sum_{k=1}^n r_k B(t_k) x_k\right\| \leq \frac{1}{2\pi} \int_{\gamma_{t_0}} \left|\frac{e^{-z}}{z(e^{-z} - \zeta)}\right| \left\|\sum_{k=1}^n r_k \frac{z}{t_k} R\left(\frac{z}{t_k} + \alpha, -A + \alpha\right) x_k\right\| |dz|$$

$$
\begin{aligned}
&\leq \frac{1}{\pi d} \sup_{(t,z)\in[0,t_0]\times\gamma_{t_0}} \left|\frac{z}{z+t\alpha}\right| \int_{\gamma_{t_0}} \left|\frac{e^z}{z}\right| \left\| \sum_{k=1}^n r_k \left(\frac{z}{t_k}+\alpha\right) R\left(\frac{z}{t_k}+\alpha, -A+\alpha\right) x_k \right\| |dz| \\
&\leq C_1 \int_{\gamma_{t_0}} \left|\frac{e^z}{z}\right| |dz| \left\| \sum_{k=1}^n r_k x_k \right\| \leq C_2 \left\| \sum_{k=1}^n r_k x_k \right\|
\end{aligned}
$$

for some constants $C_1, C_2 \geq 0$, where we have used Kahane's contraction principle (Proposition 1.2.2) in the second inequality. This shows that one has $\mathcal{R}\{B(t) : 0 < t < t_0\} < \infty$. Further, by the argument in the proof of Lemma 3.1.2, using the functional calculus one obtains $R(\zeta, T(t)) = \zeta^{-1}(\mathrm{Id} - B(t))$, which yields (3.8). $\square$

We can now give the desired characterization of $\mathcal{R}$-analyticity in terms of the behaviour of certain polynomials of the semigroup at zero obtained in [Fac13b].

Theorem 3.2.3. *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on a Banach space X.*

(a) *If $\mathcal{R}\{T(t) : 0 < t < 1\} < \infty$ and there is a polynomial $p \in \mathcal{P}$ such that*

$$\lim_{\varepsilon \downarrow 0} \mathcal{R}\{p(T(t)) : 0 < t < \varepsilon\} < \|p\|_{\mathbb{D}},$$

then $(T(t))_{t\geq 0}$ extends to an $\mathcal{R}$-analytic semigroup.

(b) *Conversely, if $(T(z))_{z\in\Sigma}$ is $\mathcal{R}$-analytic, for each $p \in \mathcal{P}_1$ there exists $N_0 \in \mathbb{N}$ such that for every $N \geq N_0$ there exists $K_0 \in \mathbb{R}_+$ such that*

$$\lim_{\varepsilon\downarrow 0} \mathcal{R}\left\{p^N(T(t))T(Kt) : 0 < t < \varepsilon\right\} < \|p^N\|_{\mathbb{D}} \quad \textit{for all } K \geq K_0.$$

Proof. For (a) we may suppose without loss of generality that $p(\zeta) = 1 = \|p\|_{\mathbb{D}}$ for some $\zeta \in \partial\mathbb{D}$. Then the theorem can be proved exactly along the lines of the proof of Theorem 3.1.6 from which we borrow the notation: We obtain that

$$\mathcal{R}\left\{[\mathrm{Id} - p(T(t))]^{-1} : 0 < t < t_0\right\} \leq \frac{1}{1-\rho}$$

for some $t_0 < 1$ and $0 < \rho < 1$ such that $\mathcal{R}\{p(T(t)) : 0 < t < t_0\} < \rho$. From this we see by factorization that one has $\mathcal{R}\left\{(\zeta - T(t))^{-1} : 0 < t < t_0\right\} < \infty$. Finally, Lemma 3.2.2 shows the claim.

Conversely, let $z \mapsto T(z)$ be $\mathcal{R}$-analytic in the sector $\Sigma_{\tilde{\delta}}$ and let $0 < \delta < \tilde{\delta}$. Note that this implies that for $t \in \mathbb{R}_+$ the ball around t with radius $t \sin\delta$ is contained in Σ_δ. Let $R := \mathcal{R}\{T(z) : z \in \Sigma_\delta, |z| \leq 2\}$. For a polynomial $p(z) = \sum_{k=0}^n a_k z^k$ we obtain for $s > 0$ that $p(T(t))T(s) = \sum_{k=0}^n a_k T(s+kt)$. Since $z \mapsto T(z)$ is an analytic mapping, we can write for $0 < nt \leq s\sin\delta$

$$\sum_{k=0}^n a_k T(s+kt) = \frac{1}{2\pi i} \sum_{l=0}^\infty \int_{|z-s|=r} \frac{T(z)}{(z-s)^{l+1}}\, dz\, t^l \sum_{k=0}^n a_k k^l, \qquad (r \leq s\sin\delta).$$

We now replace p by $p^N = \sum_{k=0}^{Nn} a_{k,N} z^k$, where $p \in \mathcal{P}_1$. After scaling we may again assume that $\|p\|_{\mathbb{D}} = 1$. Further, we choose $s = Kt$ for $K \in \mathbb{R}_+$. Then, for $K \geq Nn(\sin\delta)^{-1}$ and $r_t = Kt \sin\delta$ we obtain

$$\begin{aligned} &\mathcal{R}\left\{p^N(T(t))T(Kt) : t \leq K^{-1}\right\} \\ &\quad \leq \sum_{l=0}^{\infty} \left|\sum_{k=0}^{Nn} a_{k,N} k^l\right| (2\pi)^{-1} \mathcal{R}\left\{t^l \int_{|z-Kt|=r_t} \frac{T(z)}{(z-Kt)^{l+1}}\, dz : t \leq K^{-1}\right\} \end{aligned} \tag{3.9}$$

Moreover,

$$\begin{aligned} \int_{|z-Kt|=r_t} \frac{T(z)}{(z-Kt)^{l+1}}\, dz &= i \int_0^{2\pi} \frac{T(Kt + r_t e^{i\theta})}{(r_t e^{i\theta})^{l+1}} r_t e^{i\theta}\, d\theta \\ &= i(tK\sin\delta)^{-l} \int_0^{2\pi} T(Kt + r_t e^{i\theta}) e^{-il\theta}\, d\theta. \end{aligned} \tag{3.10}$$

Now, using equations (3.9) and (3.10) together with the estimate obtained from Lemma 3.1.5 (for $f(x) = p(e^{ix})$ and $x = 0$) in the second inequality below, we obtain the estimate

$$\begin{aligned} \mathcal{R}\left\{p^N(T(t))T(Kt) : t \leq K^{-1}\right\} &\leq R \sum_{l=0}^{\infty} \left|\sum_{k=0}^{Nn} a_{k,N} k^l\right| (K\sin\delta)^{-l} \\ &\leq R\left(|p(1)|^N \sum_{l=0}^{N} \left(\frac{Nn}{|p(1)|K\sin\delta}\right)^l + \sum_{l=N+1}^{\infty} \left(\frac{Nn}{K\sin\delta}\right)^l\right) \\ &= R\left(|p(1)|^N \frac{1 - C_{1,K}^{N+1}}{1 - C_{1,K}} + \frac{C_{2,K}^{N+1}}{1 - C_{2,K}}\right), \end{aligned}$$

where $C_{1,K} = \frac{Nn}{|p(1)|K\sin\delta}$ and $C_{2,K} = \frac{Nn}{K\sin\delta}$ (of course for $p(1) = 0$ the first term vanishes). Now, since $|p(1)| < 1$, the right hand side is arbitrarily small provided first N and then K are chosen large enough. □

Again, we obtain the following characterization of $\mathcal{R}$-analyticity via polynomials proved in [Fac13b].

Corollary 3.2.4. *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on a Banach space X satisfying $\mathcal{R}\{T(t) : 0 < t < 1\} < \infty$. Then $(T(t))_{t\geq 0}$ is $\mathcal{R}$-analytic if and only if there is $p \in \mathcal{P}$ such that*

$$\lim_{\varepsilon \downarrow 0} \mathcal{R}\{p(T(t)) : 0 < t < \varepsilon\} < \|p\|_{\mathbb{D}}.$$

Now, we use the characterization via polynomials just obtained in Theorem 3.2.3 to show the announced extrapolation theorem for $\mathcal{R}$-analyticity – the analogue of Theorem 3.1.10 – for general classes of interpolation functors which are exact of order $\theta \in (0,1)$. In [Fac13b, Theorem 6.1] the following theorem is stated without the additional assumption of L_p-compability using [HHK06, Remark 6.8] which the author can not verify without assuming L_p-compability.

Theorem 3.2.5. *Let $(T_1(t))_{t\geq 0}$ and $(T_2(t))_{t\geq 0}$ be two consistent semigroups on an interpolation couple (X_1, X_2) of Banach spaces X_1 and X_2 with non-trivial type and let X be an interpolation space with respect to (X_1, X_2) obtained by a regular L_p-compatible interpolation functor $\mathcal{F}$ of exponent $\theta \in (0,1)$. Assume that $(T_1(t))_{t\geq 0}$ is $\mathcal{R}$-analytic and $\mathcal{R}\{T_2(t) : 0 < t < 1\} < \infty$. If at least one of the two semigroups is strongly continuous, then there exists a unique $\mathcal{R}$-analytic C_0-semigroup $(T(z))$ on X that is consistent with $(T_1(t))_{t\geq 0}$ and $(T_2(t))_{t\geq 0}$.*

Proof. We immediately obtain from Theorem 3.1.10 that $(T(t))_{t\geq 0}$ is uniquely determined and strongly continuous. It therefore remains to show the $\mathcal{R}$-analyticity of $(T(t))_{t\geq 0}$. Since $(T_1(z))$ is $\mathcal{R}$-analytic, there is $p \in \mathcal{P}$ with $\|p\|_{\mathbb{D}} = 1$ and $\deg p = n$ and a constant $\varepsilon > 0$ such that

$$\mathcal{R}\{p(T_1(t)) : 0 < t < \varepsilon\} < \rho < 1.$$

Moreover, since X_1 and X_2 have non-trivial type and $\mathcal{F}$ is L_p-compatible, one has

$$\mathcal{F}((\mathrm{Rad}(X_1), \mathrm{Rad}(X_2))) \simeq \mathrm{Rad}(\mathcal{F}((X_1, X_2)))$$

by Corollary A.5.7. Now, let $R := \mathcal{R}\{T_2(t) : 0 < t < 1\}$ and $p^N(z) = \sum_{k=0}^{Nn} a_{k,N} z^k$. Then for sufficiently small $\varepsilon_0(N)$ one has for some constant $C \geq 0$

$$\begin{aligned}
\mathcal{R}\left\{p^N(T(t)) : 0 < t < \varepsilon_0(N)\right\} &\leq C\mathcal{R}\left\{p^N(T_1(t))\right\}^{1-\theta} \mathcal{R}\left\{p^N(T_2(t))\right\}^{\theta} \\
&\leq C\rho^{N(1-\theta)} \left(\sum_{k=0}^{Nn} \left|a_{k,N}\right| \mathcal{R}\{T_2(kt)\} \right)^{\theta} \leq CR^{\theta} \rho^{N(1-\theta)} \left(\sum_{k=0}^{Nn} \left|a_{k,N}\right| \right)^{\theta} \\
&\leq CR^{\theta} (Nn+1)^{\theta} \rho^{N(1-\theta)}.
\end{aligned}$$

The right hand side tends to zero as N tends to infinity. Hence, Theorem 3.2.3 applied to p^N for sufficiently large N shows the $\mathcal{R}$-analyticity of $(T(t))_{t\geq 0}$. □

The by far most important application of Theorem 3.2.5 is the following special case for the scale of L_p-spaces which was first proved by W. Arendt & S. Bu [AB03, Theorem 4.3].

Corollary 3.2.6. *Let $(T_2(z))_{z\in\Sigma}$ be an analytic C_0-semigroup on $L_2(\Omega)$ for some σ-finite measure space $(\Omega, \mathcal{F}, \mu)$ and $(T_p(t))_{t\geq 0}$ a consistent semigroup on $L_p(\Omega)$ for $p \neq 2$. If $\mathcal{R}\left\{T_p(t) : 0 < t < 1\right\} < \infty$, then the semigroups $(T_q(t))_{t\geq 0}$ obtained by (complex) interpolation can be extended to $\mathcal{R}$-analytic C_0-semigroups on $L_q(\Omega)$ for all q strictly between 2 and p.*

Exactly as in [AB03, Corollary 4.5], the above corollary can directly be applied to semigroups with Gaussian estimates. For the sake of completeness we repeat the main notions and arguments. Let $\Omega \subseteq \mathbb{R}^N$ be an open set. We say that a semigroup $(T_p(t))_{t\geq 0}$ on $L_p(\Omega)$ has *Gaussian estimates* if there exist constants $C > 0$ and $a > 0$ such that for all $f \in L_p(\Omega)$

$$|T_p(t)f|(x) \leq CG_p(at)|f|(x) \quad \text{for almost all } x \in \Omega \text{ and all } 0 < t \leq 1,$$

where G_p denotes the Gaussian semigroup

$$G_p(t)f = k_t * f \quad (f \in L_p(\mathbb{R}^N)) \quad \text{with} \quad k_t(x) = \frac{1}{(4\pi t)^{N/2}} e^{-|x|^2/4t}.$$

on $L_p(\mathbb{R}^N)$. Moreover, given a measure space $(\Omega, \mathcal{F}, \mu)$, the following equivalent characterization of the $\mathcal{R}$-boundedness of a family $(T_n)_{n\in\mathbb{N}}$ of operators on $L_p(\Omega)$ (for $1 \leq p < \infty$) is often useful: The *square function estimate*

$$\Big\| \Big(\sum_{k=1}^{n} |T_k f_k|^2 \Big)^{1/2} \Big\|_{L_p} \leq C \Big\| \Big(\sum_{k=1}^{n} |f_k|^2 \Big)^{1/2} \Big\|_{L_p}$$

holds for some $C \geq 0$, all $n \in \mathbb{N}$ and all sequences $(f_k)_{k=1}^n \subset L_p(\Omega)$. The equivalence with the general definition of $\mathcal{R}$-boundedness follows from the Khintchine inequality (Theorem A.3.1) and Fubini's theorem.

Corollary 3.2.7. *Let $\Omega \subset \mathbb{R}^N$ be an open set and let $(T_p(t))_{t\geq 0}$ be consistent C_0-semigroups on $L_p(\Omega)$ for $1 < p < \infty$. Assume that $(T_2(t))_{t\geq 0}$ is analytic and the semigroups $(T_p(t))_{t\geq 0}$ have Gaussian estimates. Then $(T_p(t))_{t\geq 0}$ is $\mathcal{R}$-analytic for all $1 < p < \infty$.*

Proof. By Corollary 3.2.6 and the above remarks, it is sufficient to show square function estimates for the Gaussian semigroup. For this notice that by [Ste93, p. 24, Proposition] the maximal function associated to the convolution with the Gaussian kernel can be estimated by

$$\sup_{0<t<1} |k_t * f|(x) \leq \sup_{t>0} t^{-N/2} |k_t * f|(x) \leq cMf(x) \qquad \text{for almost all } x \in \mathbb{R}^N$$

for some constant $c \geq 0$ only depending on k_1. Here M denotes the maximal operator given by

$$(Mf)(x) := \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy.$$

Hence, the boundedness of the vector-valued maximal operator [Ste93, p. 51, Theorem 1] yields for all $n \in \mathbb{N}$, $t_1, \dots, t_n \in (0,1)$ and $f_1, \dots, f_n \in L_p(\mathbb{R}^N)$

$$\Big\| \Big(\sum_{k=1}^{n} \big|G_p(t_k) f_k\big|^2 \Big)^{1/2} \Big\|_p \leq c \Big\| \Big(\sum_{k=1}^{n} |Mf_k|^2 \Big)^{1/2} \Big\|_p \leq C \Big\| \Big(\sum_{k=1}^{n} |f_k|^2 \Big)^{1/2} \Big\|_p. \qquad \square$$

Remark 3.2.8. The above result for Gaussian estimates is a particular instance of a general domination principle. Suppose that one has given a family $(S(t))_{t\in(0,1)}$ of operators on $L_p(\Omega)$ for $p \in (1,\infty)$ and some σ-finite measure space Ω. If there exists a second $\mathcal{R}$-bounded family of positive operators $(T(t))_{t\in(0,1)}$ such that $|S(t)| \leq T(t)$ for all $t \in (0,1)$, then the family $(S(t))_{t\in(0,1)}$ is $\mathcal{R}$-bounded as well. In particular, this holds if $(S(t))_{t\in(0,1)}$ is order bounded, i.e. dominated by a single bounded linear operator. In the above setting of Gaussian estimates, the given semigroup is dominated by the Gaussian semigroup which is itself domainted by the (non-linear) maximal operator M. Notice that the fact that M is non-linear makes the validity of the square function estimate for the single operator M non-trivial. As a second example one obtains a similar result for Poisson estimates.

3.2.3 A Counterexample to the Maximal Regularity Extrapolation Problem

The maximal regularity extrapolation problem asks whether in the obtained extrapolation results for $\mathcal{R}$-analyticity (see Theorem 3.2.5) one can weaken – at least in special cases – the assumption of local $\mathcal{R}$-boundedness on the non-regular end of the couple to the local boundedness of the semigroup. In particular, one is interested in the following L_p-space situation.

Problem 3.2.9 (Maximal Regularity Extrapolation Problem). Suppose that $(T_2(t))_{t\geq 0}$ and $(T_{p_0}(t))_{t\geq 0}$ are compatible C_0-semigroups on L_2 and L_{p_0} for $p_0 \in (1,\infty)\setminus\{2\}$ respectively. Further assume that $(T_2(t))_{t\geq 0}$ is analytic. Do then the induced C_0-semigroups $(T_p(t))_{t\geq 0}$ for p between 2 and p_0 have maximal regularity?

Note that $(T_2(t))_{t\geq 0}$ is a semigroup on a Hilbert space and therefore has maximal regularity because of the analyticity of the semigroup. Moreover, we have seen in Theorem 3.1.10 that in the setting of Problem 3.2.9 the semigroups $(T_p(t))_{t\geq 0}$ are analytic for all $p \in (1,\infty)$. The maximal regularity problem asks therefore whether this extrapolation property can be improved to $\mathcal{R}$-analytic.

The maximal regularity extrapolation problem has been open since the emergence of the theory of maximal regularity. Indeed, a positive answer to the problem would have striking applications to the study on non-linear partial differential equations as maximal regularity on Hilbert spaces or equivalently analyticity is often easily established, for example by using form methods, whereas the verification of maximal regularity usually requires far more sophisticated tools and can be very difficult to check on general domains or for general boundary conditions.

The aim of this subsection is to give a negative answer to this problem based on the presentation in [Fac14]. We construct consistent semigroups $(T_p(t))_{t\geq 0}$ on L_p for $p \in (1,\infty)$ with the following properties: $(T_2(t))_{t\geq 0}$ and therefore all $(T_p(t))_{t\geq 0}$ are analytic but $(T_p(t))_{t\geq 0}$ has maximal regularity if and only if $p = 2$. This is done by using a well-known concrete non-symmetric basis for ℓ_p for which the general results of Section 2.1.4.4 on individual spaces apply. The basis is obtained from a non-standard representation of the space ℓ_p which we now present.

We consider the spaces $X_p \coloneqq \oplus_{\ell^p}^n \ell_n^2$ given by the ℓ_p-sum of finite dimensional Hilbert spaces of increasing dimension (see Appendix A.2.1). We always identify X_p with a sequence space in the canonical way. Observe that the standard unit vector basis of X_p is normalized and unconditional. For $p \in (1,\infty) \setminus \{2\}$ it is not equivalent to the standard basis of ℓ_p. Indeed, for $1 < p < 2$ consider the sequence given by $x_k = 2^{-n/p}$ for $k = \frac{(2^n-1)2^n}{2} + 1, \ldots, \frac{2^n(2^n+1)}{2}$ and for $n \in \mathbb{N}$ and by $x_k = 0$ in any other case. Then one has $\sum_{k=1}^{\infty} |x_k|^p = \sum_{n=1}^{\infty} 2^n \frac{1}{2^n} = \infty$, but

$$\|(x_k)\|_{X^p}^p = \sum_{n=1}^{\infty} (2^{n(1-\frac{2}{p})})^{p/2} = \sum_{n=1}^{\infty} 2^{n(\frac{p}{2}-1)} < \infty \quad \text{as } p < 2.$$

In the case $2 < p < \infty$ let $(x_k)_{k\in\mathbb{N}}$ be the sequence obtained by inserting the sequence $(\frac{1}{\sqrt{k}})_{k\in\mathbb{N}}$ into the set $\cup_{n=1}^{\infty} [\frac{(2^n-1)2^n}{2} + 1, \frac{2^n(2^n+1)}{2}]$. Then $\sum_{k=1}^{\infty} |x_k|^p = \sum_{k=1}^{\infty} k^{-p/2} < \infty$ as $p > 2$. However, one has

$$\sum_{k=\frac{(2^n-1)2^n}{2}+1}^{\frac{2^n(2^n+1)}{2}} |x_k|^2 \geq \frac{1}{2}$$

by the well-known argument for the divergence of the harmonic series. This yields $\|(x_k)\|_{X_p} = \infty$.

Moreover, using Pełczyński's decomposition technique, one has the following identifications.

Proposition 3.2.10. *For $p \in (1,\infty)$ one has the isomorphism $X_p \simeq \ell_p$.*

Proof. By the Khintchine inequality (Theorem A.3.1), there exists a constant $C_p > 0$ such that

$$C_p^{-1}\left(\sum_{k=1}^{n} |a_k|^2\right)^{1/2} \leq \left(\int_0^1 \left|\sum_{k=1}^{n} r_k(\omega) a_k\right|^p d\omega\right)^{1/p} \leq C_p\left(\sum_{k=1}^{n} |a_k|^2\right)^{1/2}$$

for all $n \in \mathbb{N}$ and $a_1, \ldots, a_n \in \mathbb{C}$. Hence, the spaces ℓ_n^2 are uniformly isomorphic to the subspaces Rad_n of $L_p([0,1])$ spanned by the first n Rademacher functions. Furthermore, Rad_n is a subspace of the space $G_n \subset L_p([0,1])$ spanned by the indicator functions $\mathbb{1}_{[\frac{k}{2^n},\frac{k+1}{2^n}]}$ for $k = 0,\ldots,2^n - 1$. Clearly, one has a (p-dependent) canonical isometric isomorphism $G_n \simeq \ell_p^{2^n}$. Hence, by the uniform boundedness of the isomorphisms $\ell_2^n \simeq R_n$ one obtains

$$X_p = \oplus_{\ell_p}^n \ell_2^n \simeq \oplus_{\ell_p}^n \operatorname{Rad}_n \subset \oplus_{\ell_p}^n G_{2^n} \simeq \oplus_{\ell_p}^n \ell_p^{2^n} \simeq \ell_p.$$

Since the family of Rademacher projections $L_p([0,1]) \supset G_{2^n} \to \operatorname{Rad}_n$ are uniformly bounded, $X_p = \oplus_{\ell^p}^n \ell_n^2$ is a complemented subspace of ℓ_p. We now use the Pełczyński decomposition technique (Theorem A.2.2) to show that X_p is even isomorphic to ℓ_p. For this notice that $F_p := [e_{\frac{(k-1)k}{2}+1}]_{k\in\mathbb{N}}$ is a complemented subspace of X_p isomorphic to ℓ_p. Since one clearly has $\ell_p(\ell_p) \simeq \ell_p$, the assertion indeed follows from the Pełczyński decomposition technique (Theorem A.2.2). □

It follows that the standard unit vector basis of X_p can be identified with an unconditional basis of ℓ_p. Since the standard basis $(e_m)_{m\in\mathbb{N}}$ of X_p is not equivalent to the standard basis of ℓ_p (for $p \in (1,\infty)\setminus\{2\}$), the general theory shows that $(e_m)_{m\in\mathbb{N}}$ cannot be symmetric [Sin70, Proposition 21.5]. More easily, choose $\pi(2m) = \frac{(m-1)m}{2} + 1$ and use successively $\pi(2m+1)$ to fill up the rest. Then $[e_{\pi(2m)}]_{m\in\mathbb{N}}$ is isometrically isomorphic to ℓ_p and versions of the counterexamples above show that $(e_{\pi(2m)})_{m\in\mathbb{N}}$ is not equivalent to $(e_{2m})_{m\in\mathbb{N}}$. Note that one sees directly that X_2 is isometrically isomorphic to ℓ_2 and that the standard unit vector basis of X_2 is equivalent to the standard Hilbert space basis of ℓ_2. We have therefore shown the following

Proposition 3.2.11. *Under the identification $\ell_p \simeq X_p$ for $p \in (1,\infty)\setminus\{2\}$ the standard unit vector basis of X_p is a semi-normalized non-symmetric unconditional basis of ℓ_p.*

One can now use Theorem 2.1.35 and the calculations just made to obtain an explicit counterexample on $X_p \simeq \ell_p$ for $p \in (1,\infty)\setminus\{2\}$. However, we want to do more: we want to define a consistent family of counterexamples on X_p on the scale $p \in (1,\infty)\setminus\{2\}$. For this it is necessary to find explicitly p-independent choices of both the permutations and the bases f_m used in the proof of Theorem 2.1.35. This is the goal of the next proposition.

Proposition 3.2.12. *Let $(e_m)_{m\in\mathbb{N}}$ be the standard unit vector basis of X_p ($p \in (1,\infty)$). Then there exists a p-independent permutation π of the even numbers such that the choice f'_m in the proof of Theorem 2.1.35 yields semigroups without maximal regularity for all $p \in (2,\infty)$.*

Proof. The permutation π of the even numbers is defined as follows. Let $b_0, b_1, b_2, \ldots$ be the first even numbers in the blocks $B_k := [\frac{(k-1)k}{2} + 1, \frac{k(k+1)}{2}]$

($k \in \mathbb{N}$), so $b_0 = 2$, $b_1 = 4$, $b_2 = 8$, $b_3 = 12$, $b_4 = 16$, $b_5 = 22$, $b_6 = 30$ and so on. Now, we define

$$\pi(m) = \begin{cases} m & m \text{ odd} \\ b_k & m = 4k+2 \\ \min 2\mathbb{N} \setminus (\{b_n : n \in \mathbb{N}\} \cup \pi([1, m-1])) & m = 4k. \end{cases}$$

The permutation π jumps to the first even number of some block B_k in every second permutation step of the even numbers and collects all other even numbers in the other steps. Notice that a sequence of the form $(a_1, 0, a_2, 0, a_3, 0, \ldots)$ converges with respect to $(e_{\pi(2m)})_{m\in\mathbb{N}}$ if and only if $(a_m)_{m\in\mathbb{N}} \in \ell_p$. This observation together with a slight modification of the above counterexamples shows that $(e_{\pi(2m)})_{m\in\mathbb{N}}$ is not equivalent to $(e_{2m+1})_{m\in\mathbb{N}}$.

Moreover, the above arguments show that for $p \in (2, \infty)$ there exists a sequence $(a_m)_{m\in\mathbb{N}}$ which converges with respect to $(e_{\pi(2m)})_{m\in\mathbb{N}}$ but not with respect to $(e_{2m+1})_{m\in\mathbb{N}}$. Thus in the case $p \in (2, \infty)$ one can use $(f'_m)_{m\in\mathbb{N}}$ to construct a counterexample. □

Proposition 3.2.12 leaves open what happens in the case $p \in (1, 2)$. Since we can construct a counterexample to the extrapolation problem without addressing this issue and we want to stay as elementary as possible first, we will postpone the discussion of this question to the next subsection where we will give refined counterexamples.

Notice that we have not yet found a counterexample to the extrapolation problem although we have found consistent semigroups on X_p with the desired properties. For this $X_p \simeq \ell_p$ is not sufficient because we also need the consistency of the isomorphisms for different p. Sadly, the argument given in Proposition 3.2.10 using Pełczyński's decomposition technique does not seem to yield such consistent isomorphisms. In a different direction one could try to apply Theorem 2.1.35 to the normalized Haar basis of $L_p([0,1])$. This works perfectly for a fixed $p \in (1, \infty)$, but the Haar basis cannot be simultaneously normalized for all or two different choices of p. Please be aware that the assumption on the basis to be (semi-)normalized is crucial in the proof of Theorem 2.1.35 in order to show that the constructed sequences $(f')_{m\in\mathbb{N}}$ and $(f'')_{m\in\mathbb{N}}$ are Schauder bases. This issue was overlooked in the presentation given by the author in [Fac13a] and clarified in [Fac14].

Nevertheless, there is a way to embed the above family of counterexamples on X_p consistently into a scale of L_p-spaces.

Theorem 3.2.13. *There exist consistent analytic C_0-semigroups $(T_p(z))_{z\in\Sigma_{\pi/2}}$ on $L_p([0,\infty))$ for $p \in (1, \infty)$ such that $(T_p(z))_{z\in\Sigma_{\pi/2}}$ does not have maximal regularity for $p \in (2, \infty)$.*

Proof. Notice that for each $n \in \mathbb{N}$ one has an isomorphism

$$\ell_2^n \simeq \operatorname{span}\{r_1, \dots, r_n\} = \mathrm{Rad}_n \qquad (a_1, \dots, a_n) \mapsto \sum_{k=1}^n a_k r_k.$$

It follows from this explicit representation that the above isomorphisms are consistent. Moreover, by the Khintchine inequality (Theorem A.3.1) there exists $C_p > 0$ such that for all $n \in \mathbb{N}$, $a_1, \dots, a_n \in \mathbb{C}$

$$C_p^{-1} \Big(\sum_{k=1}^n |a_k|^2 \Big)^{1/2} \le \Big\| \sum_{k=1}^n a_k r_k \Big\|_{L_p([0,1])} \le C_p \Big(\sum_{k=1}^n |a_k|^2 \Big)^{1/2}.$$

Therefore the isomorphisms are uniformly bounded in n. Hence, one has consistent isomorphisms $i_p \colon X_p = \oplus_{\ell_p}^n \ell_2^n \xrightarrow{\sim} \oplus_{\ell_p}^n \mathrm{Rad}_n$. The right hand side is a subspace of $\oplus_{\ell_p}^n L_p([0,1]) \simeq L_p([0,\infty))$. We now show that there exist consistent projections Q_p from $L_p([0,\infty))$ onto this subspace. Indeed, for a fixed $p \in (1,\infty)$ the subspace Rad_n of $L_p([0,1])$ is uniformly complementable in $n \in \mathbb{N}$ (since $\mathbb{C}$ is clearly K-convex by Theorem A.3.7 and the Khintchine inequality A.3.1), where the projections explicitly given by

$$P_n \colon f \mapsto \sum_{k=1}^n r_k \int_0^1 f(\omega) r_k(\omega) \, d\omega$$

are consistent for all $p \in (1,\infty)$. By the uniform boundedness of these projections one obtains consistent projections Q_p as desired. From these we obtain consistent decompositions $L_p([0,\infty)) \simeq (\oplus_{\ell_p}^n \mathrm{Rad}_n) \oplus Z_p$. Let $(T_p(z))_{z \in \Sigma_{\pi/2}}$ be the family of semigroups obtained from Proposition 3.2.12. Using the above decomposition we can define consistent analytic C_0-semigroups $(S_p(z))_{z \in \Sigma_{\pi/2}}$ ($p \in (1,\infty)$) on $L_p([0,\infty))$ as

$$S_p(z) \coloneqq i_p \circ T_p(z) \circ i_p^{-1} \oplus \mathrm{Id}.$$

Clearly, $(S_p(z))_{z \in \Sigma_{\pi/2}}$ has maximal regularity if and only if $(T_p(z))_{z \in \Sigma_{\pi/2}}$ has maximal regularity. Hence, by Proposition 3.2.12 the analytic semigroup $(S_p(z))_{z \in \Sigma_{\pi/2}}$ does not have maximal regularity for $p \in (2,\infty)$. □

We can now easily modify the above counterexample to obtain the main result of this subsection published in [Fac14].

Corollary 3.2.14 (A Counterexample to the Maximal Regularity Extrapolation Problem). *There exist consistent analytic C_0-semigroups $(R_p(z))_{z \in \Sigma_{\pi/2}}$ on $L_p(\mathbb{R})$ for $p \in (1,\infty)$ such that $(R_p(z))_{z \in \Sigma_{\pi/2}}$ has maximal regularity if and only if $p = 2$.*

Proof. Let $(S_p^*(z))_{z\in\Sigma_{\pi/2}}$ be the adjoint semigroups (which are again strongly continuous and analytic) of the semigroups $(S_p(z))_{z\in\Sigma_{\pi/2}}$ constructed in Theorem 3.2.13. By Proposition 1.2.3(b), $(S_p^*(z))_{z\in\Sigma_{\pi/2}}$ does not have maximal regularity for $p \in (1,2)$. Now, the direct sum $R_p(z) = S_p(z) \oplus S_p^*(z)$ of both semigroups has the desired properties. □

3.2.4 Exact Control of the Extrapolation Scale

In this section we continue with a more detailed study of the extrapolation problem for maximal regularity. We have already seen that maximal regularity does not extrapolate from L_2 to the L_p-scale. However, it may theoretically be possible, for example, that it extrapolates to all $p > 2$ as soon as maximal regularity holds for one $p_0 > 2$. For the moment suppose that one has given a family $(T_p(z))$ of consistent analytic C_0-semigroups on L_p for $p \in (1,\infty)$ and let $M \subset (1,\infty)$ be the set of all $p \in (1,\infty)$ for which the semigroup $(T_p(z))$ has maximal regularity. We now collect some basic facts on the set M. First of all an analytic C_0-semigroup on a Hilbert space has maximal regularity by Theorem 1.2.12, so $2 \in M$ holds. Moreover, it follows from interpolation (apply Corollary A.5.7 with the complex interpolation method) that M is a convex set. In other words, M is a subinterval of $(1,\infty)$ that contains 2. The goal of this section is to show that apart from this obvious structural restrictions one cannot obtain any further positive results for the maximal regularity extrapolation problem. In fact, we show that for every interval $I \subset (1,\infty)$ with $2 \in I$ there exists a family of consistent C_0-semigroups $(T_p(z))_{z\in\Sigma_{\frac{\pi}{2}}}$ on $L_p(\mathbb{R})$ such that $(T_p(z))_{z\in\Sigma_{\frac{\pi}{2}}}$ has maximal regularity if and only if $p \in I$. The results of this section are new and have not yet been published.

As a first step we now return to the basis $(f_m')_{m\in\mathbb{N}}$ obtained in Proposition 3.2.12. Recall that we have seen so far that for $p \in (2,\infty)$ the basis $(f_m')_{m\in\mathbb{N}}$ is conditional and yields counterexamples to the maximal regularity problem. In the case $p \in (1,2)$ left open we have the following technical result.

Proposition 3.2.15. *The basis $(f_m')_{m\in\mathbb{N}}$ constructed in Proposition 3.2.12 is unconditional for $p \in (1,2]$.*

Proof. Assume that $\sum_{m=1}^{\infty} a_m f_m'$ converges. We have to show that $\sum_{m=1}^{\infty} \varepsilon_m a_m f_m'$ converges for any choice of signs $(\varepsilon_m)_{m\in\mathbb{N}} \in \{-1,1\}^{\mathbb{N}}$. Obviously, the even part $\sum_{m=1}^{\infty} \varepsilon_{2m} a_{2m} e_{\pi(2m)}$ converges because there is no interference between two different components of $(e_m)_{m\in\mathbb{N}}$. For the odd part we have to check the convergence of the series

$$\Bigg(\sum_{k=1}^{\infty}\Bigg(\sum_{\substack{l\in B_k\\ l \text{ odd}}} |\varepsilon_l a_l + \varepsilon_{l+1} a_{l+1}|^2\Bigg)^{p/2}\Bigg)^{1/p} \le \Bigg(\sum_{k=1}^{\infty}\Bigg(\sum_{\substack{l\in B_k\\ l \text{ odd}}} |a_l + a_{l+1}|^2\Bigg)^{p/2}\Bigg)^{1/p}$$

$$+\left(\sum_{k=1}^{\infty}\left(\sum_{\substack{l\in B_k\\ l\text{ odd}}}|(\varepsilon_{l+1}-\varepsilon_l)a_{l+1}|^2\right)^{p/2}\right)^{1/p}$$

The first term converges by assumption. Again, we split the second term. First notice that $(a_{4m+2})_{m\in\mathbb{N}}\in\ell_p$. Observe that for $p\le 2$ we have the inclusion $\ell_p\hookrightarrow\ell_2$ which yields $\ell_p\hookrightarrow\oplus_{\ell_p}^n\ell_2^n$. From this inclusion we deduce that the part of the second series where l runs over the numbers l with $l+1\equiv 2\mod 4$ converges. Finally, we have to show that the part of the second series where l runs over the numbers with $l+1\equiv 0\mod 4$ converges. This part is essentially built from the convergent series

$$\sum_{m=1}^{\infty}a_{4m}e_{\pi(4m)}$$

by eventually inserting zeros. The following lemma shows that this procedure does not destroy the convergence and finishes the proof. $\square$

Let $(a_m)_{m\in\mathbb{N}}$ be a sequence and $(b_m)=(0,\dots,0,a_1,0,\dots,0,a_2,\dots)$ be a sequence built from $(a_m)_{m\in\mathbb{N}}$ by inserting zeros. We can then introduce a mapping $\varphi\colon\mathbb{N}\to\mathbb{N}$ which maps k to the position of a_k in the new sequence $(b_m)_{m\in\mathbb{N}}$.

Lemma 3.2.16. *Let $p\in[1,\infty)$, $(a_m)_{m\in\mathbb{N}}$ be a sequence, $(b_m)_{m\in\mathbb{N}}$ and $\varphi\colon\mathbb{N}\to\mathbb{N}$ be as above and suppose that*

$$M:=\sup_{k\in\mathbb{N}}\varphi(k+1)-\varphi(k)<\infty.$$

If $(a_m)_{m\in\mathbb{N}}\in X_p$, then $(b_m)_{m\in\mathbb{N}}\in X_p$ as well. Conversely, if $(b_m)_{m\in\mathbb{N}}\in X_p$, then $(a_m)_{m\in\mathbb{N}}\in X_p$.

Proof. We only prove the first implication as the proof of the second is analogous. Furthermore it suffices to consider the case $M=2$ as the general case then follows inductively. Let $\mathcal{B}:=\{B_n : n\in\mathbb{N}\}$ be the set of all blocks. By considering the worst cases, one sees that for each $A\in\mathcal{B}$ there exist at most three different $B\in\mathcal{B}$ such that $\varphi(A)\cap B\neq\emptyset$ and likewise for each $B\in\mathcal{B}$ there exist at most three different $A\in\mathcal{B}$ such that $\varphi(A)\cap B\neq\emptyset$. Choose $C\ge 1$ such that for every triple $\alpha,\beta,\gamma\in\mathbb{R}$ one has $(\alpha^2+\beta^2+\gamma^2)^{1/2}\le C(\alpha^p+\beta^p+\gamma^p)^{1/p}$. Then for each $B\in\mathcal{B}$ one has

$$\left(\sum_{m\in B}|b_m|^2\right)^{1/2}\le\left(\sum_{\substack{A\in\mathcal{B}\\ \varphi(A)\cap B\neq\emptyset}}\sum_{m\in A}|a_m|^2\right)^{1/2}\le C\left(\sum_{\substack{A\in\mathcal{B}\\ \varphi(A)\cap B\neq\emptyset}}\left(\sum_{m\in A}|a_m|^2\right)^{p/2}\right)^{1/p}.$$

Therefore one obtains

$$\begin{aligned}\|(b_m)\|_{X^p}^p &= \sum_{B\in\mathcal{B}}\Big(\sum_{m\in B}|b_m|^2\Big)^{p/2} \le C^p \sum_{B\in\mathcal{B}} \sum_{\substack{A\in\mathcal{B}\\ \varphi(A)\cap B\neq\emptyset}} \Big(\sum_{m\in A}|a_m|^2\Big)^{p/2} \\ &= C^p \sum_{A\in\mathcal{B}} \sum_{\substack{B\in\mathcal{B}\\ \varphi(A)\cap B\neq\emptyset}} \Big(\sum_{m\in A}|a_m|^2\Big)^{p/2} \le 3C^p \sum_{A\in\mathcal{B}}\Big(\sum_{m\in A}|a_m|^2\Big)^{p/2}. \qquad \square\end{aligned}$$

From this we obtain the following preliminary result which shows that maximal regularity can only extrapolate to one side of the L_p-scale.

Corollary 3.2.17. *There exist consistent analytic C_0-semigroups $(T_p(z))_{z\in\Sigma_{\pi/2}}$ on $L_p([0,\infty))$ for $p\in(1,\infty)$ such that $(T_p(z))_{z\in\Sigma_{\pi/2}}$ has maximal regularity if and only if $p\in(1,2]$ (respectively if and only if $p\in[2,\infty)$).*

So far we have always chosen lacunary sequences $(\gamma_m)_{m\in\mathbb{N}}$ for which the associated multipliers with respect to the perturbed bases yield counterexamples to the maximal regularity problem or respectively to $\mathcal{R}$-sectoriality. As seen in Lemma 2.1.44 the key property is that for such sequences the ratios

$$\frac{\gamma_m-\gamma_{m-1}}{\gamma_m+\gamma_{m-1}} \tag{3.11}$$

are bounded from below, which can easily be verified for the examples $\gamma_m=b^m$ for $b>1$ used until now for the case $b=2$. We now want to study more precisely the Schauder multipliers associated to various sequences $(\gamma_m)_{m\in\mathbb{N}}$ for the concrete basis $(f'_m)_{m\in\mathbb{N}}$ obtained from the standard unit vector basis of X_p as described in the proof of Proposition 3.2.12. In particular, we are interested in multipliers associated to sequences $(\gamma_m)_{m\in\mathbb{N}}$ which grow slower than the lacunary sequences considered in the general situation.

As a starting point we make the very elementary observation that one can find sequences $(\gamma_m)_{m\in\mathbb{N}}$ for which the ratio (3.11) has a prescribed growth.

Lemma 3.2.18. *Let $(c_m)_{m\geq 2}$ be a sequence of real numbers with $c_m\in(0,\frac{1}{2})$ for all $m\in\mathbb{N}$. Then there exists a unique strictly increasing sequence $(\gamma_m)_{m\in\mathbb{N}}$ of real numbers with $\gamma_1=1$ and*

$$\frac{1}{2}\frac{\gamma_m-\gamma_{m-1}}{\gamma_m+\gamma_{m-1}}=c_m \qquad \textit{for all } m\geq 2. \tag{3.12}$$

Proof. Rewriting the defining relation, one obtains

$$\gamma_m=\gamma_{m-1}\left(\frac{2}{1-2c_m}-1\right) \qquad \text{for all } m\geq 2,$$

which recursively uniquely determines $(\gamma_m)_{m\in\mathbb{N}}$ together with the initial condition $\gamma_m=1$. It is then clear that $(\gamma_m)_{m\in\mathbb{N}}$ is strictly increasing. $\square$

We now formulate a necessary condition for the sequence $(c_m)_{m\in\mathbb{N}}$ that implies that the Schauder multiplier associated to the sequence $(\gamma_m)_{m\in\mathbb{N}}$ with respect to the basis $(f'_m)_{m\in\mathbb{N}}$ given by (3.12) is $\mathcal{R}$-sectorial.

Proposition 3.2.19. *Let $(c_m)_{m\geq 2}$ be a sequence with $c_m \in (0, \frac{1}{2})$ for all $m \geq 2$ and $(\gamma_m)_{m\in\mathbb{N}}$ the sequence given by Lemma 3.2.18. Suppose that for $p > 2$ the sectorial operator A on X_p given as the Schauder multiplier*

$$D(A) = \left\{ x = \sum_{m=1}^{\infty} a_m f_m : \sum_{m=1}^{\infty} \gamma_m a_m f_m \quad \text{exists} \right\}$$

$$A\left(\sum_{m=1}^{\infty} a_m f_m \right) = \sum_{m=1}^{\infty} \gamma_m a_m f_m$$

is $\mathcal{R}$-sectorial, where $(f_m)_{m\in\mathbb{N}}$ is the conditional basis for X_p constructed in Proposition 3.2.12. Then $(a_m c_{4m+2})_{m\in\mathbb{N}} \in X_p$ for all $(a_m)_{m\in\mathbb{N}} \in \ell_p$.

Proof. Recall that the basic sequence $(e_{\pi(4m+2)})_{m\in\mathbb{N}}$ is isometrically equivalent to the standard unit vector basis of ℓ_p. Let $(a_m)_{m\in\mathbb{N}} \in \ell_p$. Then the Rademacher series $x = \sum_{m=1}^{\infty} r_m a_m e_{\pi(4m+2)}$ lies in $\mathrm{Rad}(X_p)$. One can now argue as in the proof of Theorem 2.1.45:

Let $(q_m)_{m\in\mathbb{N}} \subset \mathbb{R}_-$ be a sequence to be chosen later. Since A is $\mathcal{R}$-sectorial by assumption, it follows from Proposition 2.1.43 that the operator $\mathcal{R}\colon \mathrm{Rad}(X) \to \mathrm{Rad}(X)$ associated to the sequence $(q_n)_{n\in\mathbb{N}}$ is bounded. We now apply $\mathcal{R}$ to x. Because of $e_{\pi(4m+2)} = f_{4m+2} - f_{4m+1}$ we obtain

$$\begin{aligned}
\mathcal{R}(x) &= \mathcal{R}\left(\sum_{m=1}^{\infty} r_m a_m (f_{4m+2} - f_{4m+1}) \right) \\
&= \sum_{m=1}^{\infty} r_m \frac{a_m q_m}{q_m - \gamma_{4m+2}} f_{4m+2} - r_m \frac{a_m q_m}{q_m - \gamma_{4m+1}} f_{4m+1} \\
&= \sum_{m=1}^{\infty} r_m \frac{a_m q_m}{q_m - \gamma_{4m+2}} (e_{\pi(4m+2)} + e_{4m+1}) - r_m \frac{a_m q_m}{q_m - \gamma_{4m+1}} e_{4m+1} \\
&= \sum_{m=1}^{\infty} r_m \frac{a_m q_m}{q_m - \gamma_{4m+2}} e_{\pi(4m+2)} + r_m a_m q_m \left(\frac{1}{q_m - \gamma_{4m+2}} - \frac{1}{q_m - \gamma_{4m+1}} \right) e_{4m+1}.
\end{aligned}$$

Again, we now want to choose $(q_m)_{m\in\mathbb{N}}$ in such a way that the last term in the bracket is big. By Lemma 2.1.44 one has for $t = \gamma_{4m+2}$

$$t[(t + \gamma_{4m+2})^{-1} - (t + \gamma_{4m+1})^{-1}] = \frac{1}{2} \frac{\gamma_{4m+2} - \gamma_{4m+1}}{\gamma_{4m+2} + \gamma_{4m+1}} = c_{4m+2}.$$

Hence, for the choice $q_m = -\gamma_{4m+2}$ we obtain

$$\mathcal{R}(x) = \sum_{m=1}^{\infty} \frac{1}{2} r_m a_m e_{\pi(4m+2)} - c_{4m+2} r_m a_m e_{4m+1}.$$

As in the proof of Theorem 2.1.35 one deduces from the above equality that $\sum_{m=1}^{\infty} c_{4m+1} a_m e_{2m-1}$ converges in X_p. By Lemma 3.2.16 this implies that $(a_m c_{4m+2})_{m\in\mathbb{N}} \in X_p$. □

In the next step we prove a sufficient criterion for maximal regularity. As it seems difficult to verify maximal regularity directly, we will establish the boundedness of the imaginary powers which seems easier. Let $(e_m)_{m\in\mathbb{N}}$ denote the standard unit vector basis of X_p and let A be the Schauder multiplier associated to some sequence $(\gamma_m)_{m\in\mathbb{N}}$ as above. It then follows from formula (2.12) that the imaginary powers A^{it} for $t \in \mathbb{R}$ act formally as

$$\sum_{m=1}^{\infty} a_m e_m \mapsto \sum_{m=1}^{\infty} \tilde{\gamma}_m^{it} a_m e_m + \sum_{m=1}^{\infty} a_{2m}(\gamma_{\pi^{-1}(2m)}^{it} - \gamma_{\pi^{-1}(2m)-1}^{it}) e_{\pi^{-1}(2m)-1}, \tag{3.13}$$

where

$$\tilde{\gamma}_m = \begin{cases} \gamma_m, & m \text{ odd} \\ \gamma_{\pi^{-1}(m)}, & m \text{ even} \end{cases}.$$

It is clear that the first series of the right hand side of (3.13) converges for all $(a_m)_{m\in\mathbb{N}} \in X_p$. The crucial point is therefore the question whether the second series of (3.13), which by the unconditionality of the basis $(e_m)_{m\in\mathbb{N}}$ can be rewritten as

$$\sum_{m=1}^{\infty} a_{2m}(\gamma_{\pi^{-1}(2m)}^{it} - \gamma_{\pi^{-1}(2m)-1}^{it}) e_{\pi^{-1}(2m)-1} = \sum_{m=1}^{\infty} a_{\pi(2m)}(\gamma_{2m}^{it} - \gamma_{2m-1}^{it}) e_{2m-1},$$

converges in X_p for all $(a_m)_{m\in\mathbb{N}} \in X_p$. Equivalently by Lemma 3.2.16, the sequence $(a_{\pi(2m)}(\gamma_{2m}^{it} - \gamma_{2m-1}^{it}))_{m\in\mathbb{N}}$ must lie in X_p for all $(a_m)_{m\in\mathbb{N}} \in X_p$. We now give a sufficient condition.

Proposition 3.2.20. *Let $(c_m)_{m\in\mathbb{N}}$ be a sequence with $c_m \in (0, \frac{1}{8})$ for all $m \geq 2$ and let $(\gamma_m)_{m\in\mathbb{N}}$ be the sequence given by Lemma 3.2.18. Consider for $p > 2$ the sectorial operator A on X_p defined as*

$$D(A) = \left\{ x = \sum_{m=1}^{\infty} a_m f_m : \sum_{m=1}^{\infty} \gamma_m a_m f_m \quad \text{exists} \right\}$$
$$A\left(\sum_{m=1}^{\infty} a_m f_m\right) = \sum_{m=1}^{\infty} \gamma_m a_m f_m,$$

where $(f_m)_{m\in\mathbb{N}}$ is the conditional basis of X_p constructed in Proposition 3.2.12. If $(b_m c_{2m})_{m\in\mathbb{N}}$ lies in X_p for all $(b_m)_{m\in\mathbb{N}} \in \ell_p$, then A has bounded imaginary powers with $\omega_{\mathrm{BIP}}(A) = 0$. In particular, A is $\mathcal{R}$-sectorial with $\omega_R(A) = 0$.

Proof. A short calculation shows that one has for all $m \in \mathbb{N}$

$$\begin{aligned} |\gamma_{2m}^{it} - \gamma_{2m-1}^{it}|^2 &= \left|\exp(it \log \gamma_{2m}) - \exp(it \log \gamma_{2m-1})\right|^2 \\ &= |\exp(it \log \gamma_{2m})|^2 + |\exp(it \log \gamma_{2m-1})|^2 \\ &\qquad - 2 \operatorname{Re} \exp(it(\log(\gamma_{2m-1} - \log \gamma_{2m}))) \\ &= 2(1 - \cos(t(\log \gamma_{2m-1} - \log \gamma_{2m}))). \end{aligned}$$

Here we have used the identity

$$|z-w|^2 = (z-w)(\overline{z} - \overline{w}) = |z|^2 + |w|^2 - (z\overline{w} + \overline{z}w) = |z|^2 + |w|^2 - 2\operatorname{Re} z\overline{w}.$$

Further, one has

$$\begin{aligned} \left|\log \gamma_{2m-1} - \log \gamma_{2m}\right| &= \left|\log\left(\frac{\gamma_{2m-1}}{\gamma_{2m}}\right)\right| = \left|\log\left(1 - \frac{\gamma_{2m} - \gamma_{2m-1}}{\gamma_{2m}}\right)\right| \\ &\le \left|\log\left(1 - 2\frac{\gamma_{2m} - \gamma_{2m-1}}{\gamma_{2m} + \gamma_{2m-1}}\right)\right| = |\log(1 - 4c_{2m})|. \end{aligned}$$

It follows from elementary calculus that $1 - \cos x \le \frac{x^2}{2}$ for all $x \in \mathbb{R}$. In particular, we obtain the estimate

$$2(1 - \cos(t(\log \gamma_{2m-1} - \log \gamma_{2m})) \le t^2 \log^2(1 - 4c_m).$$

A further elementary estimate from calculus is that $|\log(1-4x)| \le 8x$ holds for all $x \in [0, \frac{1}{8}]$. Therefore we see that for all $m \in \mathbb{N}$ one has

$$|\gamma_{2m}^{it} - \gamma_{2m-1}^{it}| \le 8\,|t|\,c_{2m}. \tag{3.14}$$

Now, let $(a_m)_{m\in\mathbb{N}} \in X_p$. Since $p > 2$, we have the inclusion $X_p \hookrightarrow \ell_p$. Hence, $(a_{\pi(2m)})_{m\in\mathbb{N}} \in \ell_p$. By assumption, the mapping $(b_m)_{m\in\mathbb{N}} \mapsto (b_m c_{2m})_{m\in\mathbb{N}}$ from ℓ_p into X_p is well-defined and closed. Hence, by the closed graph theorem there exists a constant $C \ge 0$ such that $\|(c_{2m}b_m)\|_{X_p} \le C\,\|(b_m)\|_{\ell_p}$ for all $(b_m)_{m\in\mathbb{N}}$ in ℓ_p. Hence, we obtain that $(a_{\pi(2m)}c_{2m})_{m\in\mathbb{N}} \in X_p$ with

$$\|(a_{\pi(2m)}c_{2m})\|_{X_p} \le C\|(a_{\pi(2m)})\|_{\ell_p} \le C\,\|(a_m)\|_{X_p}.$$

It is a now a direct consequence of equation (3.14) that $((\gamma_{2m}^{it} - \gamma_{2m-1}^{it})a_{\pi(2m)}) \in X_p$ with

$$\|((\gamma_{2m}^{it} - \gamma_{2m-1}^{it})a_{\pi(2m)})\|_{X_p} \le 8C\,|t|\,\|(a_m)\|_{X_p}.$$

Altogether this shows that A has bounded imaginary powers with $\|A^{it}\| \le K(1+|t|)$ for some constant $K > 0$. Hence, $\omega_{\mathrm{BIP}}(A) = 0$. ☐

For a special type of sequences $(c_m)_{m\in\mathbb{N}}$ one can use the above results to even obtain a complete characterization of maximal regularity.

Corollary 3.2.21. *Let $(c_m)_{m\in\mathbb{N}}$ be an eventually decreasing sequence with $c_m \in (0, \frac{1}{4})$ for all $m \geq 2$ and $(\gamma_m)_{m\in\mathbb{N}}$ the sequence given by Lemma 3.2.18. Consider for $p > 2$ the sectorial operator A on X_p defined by*

$$D(A) = \left\{ x = \sum_{m=1}^{\infty} a_m f_m : \sum_{m=1}^{\infty} \gamma_m a_m f_m \quad \text{exists} \right\}$$

$$A\left(\sum_{m=1}^{\infty} a_m f_m\right) = \sum_{m=1}^{\infty} \gamma_m a_m f_m,$$

where $(f_m)_{m\in\mathbb{N}}$ is the conditional basis for X_p constructed in Proposition 3.2.12. Then A is $\mathcal{R}$-sectorial if and only if $(c_m)_{m\in\mathbb{N}} \in \oplus^n_{\ell_\infty} \ell^n_q$, where $\frac{1}{2} = \frac{1}{p} + \frac{1}{q}$. Moreover, in this case one has $\omega_R(A) = 0$.

Proof. Clearly, it suffices to show the corollary for decreasing sequences. As a first observation we show that both the conditions of Proposition 3.2.19 and Proposition 3.2.20 are equivalent to: $(a_m c_m)_{m\in\mathbb{N}} \in X_p$ for all $(a_m)_{m\in\mathbb{N}} \in \ell_p$. We show only the non-trivial implication for the condition of Proposition 3.2.20. Of course, for the condition of Proposition 3.2.19 the proof is completely analogous. So assume that $(a_m c_{2m})_{m\in\mathbb{N}} \in X_p$ for all $(a_m)_{m\in\mathbb{N}} \in \ell_p$. Now let $(a_m)_{m\in\mathbb{N}} \in \ell_p$. In order to show that $(a_m c_m)_{m\in\mathbb{N}} \in X_p$ it suffices to show by Lemma 3.2.16 that $(a_{2m} c_{2m})_{m\in\mathbb{N}}$ and $(a_{2m+1} c_{2m+1})_{m\in\mathbb{N}}$ lie in X_p. For the first sequence this follows directly from the assumption and for the second this follows from the monotonicity of $(c_m)_{m\in\mathbb{N}}$ and the elementary estimate

$$\|(a_{2m+1} c_{2m+1})\|_{X_p} \leq \|(a_{2m+1} c_{2m})\|_{X_p}.$$

Hence, we have shown that A is $\mathcal{R}$-sectorial if and only if $(a_m c_m)_{m\in\mathbb{N}} \in X_p$ for all $(a_m)_{m\in\mathbb{N}} \in \ell_p$. In this case, by the closed graph theorem, there exists a constant $M \geq 0$ such that

$$\|(a_m c_m)\|_{X_p} \leq M \|(a_m)\|_{\ell_p} \tag{3.15}$$

for all $(a_m)_{m\in\mathbb{N}} \in \ell_p$. Now, we show that this condition is equivalent to $(c_m)_{m\in\mathbb{N}} \in \oplus_{\ell_\infty} \ell^n_q$. On the one hand it follows from Hölder's inequality that for $(c_m)_{m\in\mathbb{N}} \in \oplus_{\ell_\infty} \ell^n_q$ one has

$$\begin{aligned}
\|(a_m c_m)\|_{X_p} &= \left(\sum_{m=1}^{\infty} \left(\sum_{k\in B_m} |a_k c_k|^2 \right)^{p/2} \right)^{1/p} \\
&\leq \left(\sum_{m=1}^{\infty} \left(\sum_{k\in B_m} |a_k|^p \right) \left(\sum_{k\in B_m} |a_k|^q \right)^{p/q} \right)^{1/p} \\
&\leq \sup_{m\in\mathbb{N}} \left(\sum_{k\in B_m} |a_k|^q \right)^{1/q} \left(\sum_{m=1}^{\infty} |a_m|^p \right)^{1/p},
\end{aligned}$$

which is (3.15). On the other hand it follows from (3.15) that for all $n \in \mathbb{N}$

$$\sup_{\|(a_m)\|_{\ell_p} \le 1} \left(\sum_{k \in B_n} |a_k c_k|^2 \right)^{1/2} \le M.$$

This implies that for all $n \in \mathbb{N}$ one has

$$\left(\sum_{k \in B_n} |c_k|^q \right)^{1/q} \le M.$$

In other words one has $(c_m)_{m \in \mathbb{N}} \in \oplus_{\ell_\infty} \ell_q^n$. This finishes the proof. □

We now check for special sequences $(c_m)_{m \in \mathbb{N}}$ whether they lie in $\oplus_{\ell_\infty}^n \ell_q^n$. We start with the choice $c_m = m^{-\alpha}$ for $\alpha > 0$. Then one has for all $n \in \mathbb{N}$

$$\begin{aligned} \sum_{k=\frac{n(n-1)}{2}+1}^{\frac{n(n+1)}{2}} k^{-\alpha q} &\simeq [n(n+1)]^{-\alpha q+1} - [n(n-1)]^{-\alpha q+1} \\ &= n^{-\alpha q+1}((n+1)^{-\alpha q+1} - (n-1)^{-\alpha q+1}) \simeq n^{-\alpha q+1} n^{-\alpha q} \simeq n^{-2\alpha q+1}. \end{aligned}$$

Hence, $(m^{-\alpha})_{m \in \mathbb{N}} \in \oplus_{\ell_\infty}^n \ell_q^n$ if and only if $-2\alpha q + 1 \le 0$. This holds if and only if

$$\alpha \ge \frac{1}{2q} = \frac{p-2}{4p} \qquad \Leftrightarrow \qquad p \le \frac{2}{1-4\alpha},$$

where the last inequality holds for $\alpha \in (0, \frac{1}{4})$.

A second interesting choice is the variant $c_m = m^{-\alpha} \log^{q_0}(m)$ for $m \ge 2$ and some $q_0 \in (0, \infty)$. It is then clear from the above calculations that $(c_m)_{m \in \mathbb{N}} \in \oplus_{\ell_\infty}^n \ell_q^n$ for all $\alpha > \frac{1}{2q}$ and all $q_0 \in (0, \infty)$. However, for the choice $\alpha = \frac{1}{2q}$ and $q_0 = \frac{1}{q}$ one has

$$\begin{aligned} \sum_{k=\frac{n(n-1)}{2}+1}^{\frac{n(n+1)}{2}} c_k^q = \sum_{k=\frac{n(n-1)}{2}+1}^{\frac{n(n+1)}{2}} k^{-1/2} \log k \\ &\simeq \sqrt{n(n+1)}(\log(n(n+1)) - 2) - \sqrt{n(n-1)} \log(n(n-1) - 2) \\ &= \sqrt{n}(\sqrt{n+1}(\log(n(n+1)) - 2) - \sqrt{n-1} \log(n(n-1) - 2)) \\ &\simeq \sqrt{n} \frac{1}{\sqrt{n}} \log n = \log n. \end{aligned}$$

Hence, $(m^{-\alpha} \log^{2\alpha}(m))_{m \in \mathbb{N}} \in \oplus_{\ell_\infty}^n \ell_q^n$ if and only if $-2\alpha q + 1 < 0$. This holds if and only if

$$\alpha > \frac{1}{2q} = \frac{p-2}{4p} \qquad \Leftrightarrow \qquad p < \frac{2}{1-4\alpha},$$

These two families of sequences can now be used to obtain the following complete answer to the maximal regularity extrapolation problem.

Theorem 3.2.22. *Let $I \subset (1,\infty)$ be an arbitrary interval with $2 \in I$. Then there exists a family of consistent analytic C_0-semigroups $(T_p(z))_{z\in\Sigma_{\frac{\pi}{2}}}$ on $L_p(\mathbb{R})$ for $p \in (1,\infty)$ such that $(T_p(z))_{z\in\Sigma_{\frac{\pi}{2}}}$ has maximal regularity if and only if $p \in I$.*

Proof. Let I be such an interval and let p_0 be the right end of I. We first construct a family $(T_p(z))_{z\in\Sigma_{\frac{\pi}{2}}}$ that has maximal regularity if and only if $p \in (1,2) \cup I$. For $p_0 = 2$ this has already been done in Corollary 3.2.17. So we may assume $p_0 > 2$. Choose $c_m = m^{-\alpha}$ for $\alpha = \frac{p_0-2}{4p_0}$ if $p_0 \in I$ or $c_m = m^{-\alpha}\log^{2\alpha} m$ for $\alpha = \frac{p_0-2}{4p_0}$ if $p_0 \notin I$ multiplied by appropriate scaling constants such that $c_m \in (0, \frac{1}{4})$ for all $m \geq 2$. Then it follows from Corollary 3.2.21 and the above calculations that the analytic semigroups on X_p for $p \in (1,\infty)$ whose negative generators are the Schauder multipliers associated the sequence $(\gamma_m)_{m\in\mathbb{N}}$ given by Lemma 3.2.18 with respect to the basis $(f_m)_{m\in\mathbb{N}}$ have maximal regularity for $p \in (2,\infty)$ if and only if $p \in I \cap (2,\infty)$. Moreover, it follows from Proposition 3.2.15 and Corollary 2.1.7 that these semigroups have maximal regularity for $p \in (1,2]$. One can now use the consistent mappings from X_p to $L_p([0,\infty))$ from the proof of Theorem 3.2.13 to obtain the desired consistent semigroups $(T_p(z))_{z\in\Sigma_{\frac{\pi}{2}}}$ on $L_p([0,\infty))$ which have maximal regularity if and only if $p \in (1,2) \cup I$. Taking the dual semigroups, it follows from Proposition 1.2.3(b) and the first part of the proof that there exist consistent analytic C_0-semigroups $(S_p(z))_{z\in\Sigma_{\frac{\pi}{2}}}$ on $L_p([0,\infty))$ for $p \in (1,\infty)$ such that $(S_p(z))_{z\in\Sigma_{\frac{\pi}{2}}}$ that has maximal regularity if and only if $p \in (2,\infty) \cup I$. Taking the direct sum of $(T_p(z))_{z\in\Sigma_{\frac{\pi}{2}}}$ and $(S_p(z))_{z\in\Sigma_{\frac{\pi}{2}}}$ one obtains the desired family of semigroups. □

3.3 Notes & Open Problems

Kato–Beurling type theorems as Theorem 3.1.6 have an interesting and rich literature which we now want to outline shortly. For an overview mainly presenting Beurling's impact we refer to [Neu93].

In [Neu70] J. W. Neuberger proved the following theorem.

Theorem 3.3.1. *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on a real Banach space with generator A satisfying*

$$\limsup_{t\downarrow 0} \|T(t) - \mathrm{Id}\| < 2.$$

Then $AT(t)$ is a bounded operator for all $t > 0$.

Further, the article contains a second result based on the work of A. Beurling which we only formulate in a special case. Suppose that the semigroup just considered satisfies the weaker condition

$$\limsup_{n\to\infty} \|T(3^{-n}) - \mathrm{Id}\| < 2. \tag{3.16}$$

Then the collection of functions $\mathcal{T} := \{t \mapsto \langle x^*, T(t)x\rangle : x \in X, x^* \in X^*\}$ forms a *quasi-analytic* collection on $(0,\infty)$ which means that two members $f, g \in \mathcal{T}$ agree on a non-empty open subset of $(0,\infty)$ if and only if $g = h$. Note that this property is clearly also satisfied if the set $\mathcal{T}$ would consist of analytic functions. Moreover, Neuberger gives an example of a C_0-semigroup that satisfies (3.16), but for which the set $\mathcal{T}$ contains elements that are not analytic functions. In particular, this shows that

$$\liminf_{t\downarrow 0} \|T(t) - \mathrm{Id}\| < 2$$

does not imply the analyticity of the semigroup $(T(t))_{t\ge 0}$. Remarkably, the same issue of the journal Proc. Amer. Math. Soc. contains right before the article of Neuberger the article of T. Kato [Kat70] on which our approach is based. Here the following strengthening of Theorem 3.3.1 is shown.

Theorem 3.3.2. *Let $(T(t))_{t\ge 0}$ be a C_0-semigroup on a complex Banach space with*

$$\limsup_{t\downarrow 0} \|T(t) - \mathrm{Id}\| < 2.$$

Then the semigroup $(T(t))_{t\ge 0}$ is analytic.

Apparently, when A. Beurling heard of the work of Neuberger he quickly proved the following more general result which was also published in the same year [Beu70].

Theorem 3.3.3. *Let $(T(t))_{t\ge 0}$ be a weakly measurable and exponentially bounded semigroup on some complex Banach space X. Then $(T(t))_{t\ge 0}$ is analytic if and only if for one (and then for all) $p \in \mathcal{P}_1$ one has*

$$\limsup_{\xi+\frac{\alpha}{\xi}+\frac{1}{n}\to 0} \left\| p^n\left(T\left(\frac{\alpha}{n}\right)\right) T(\xi)\right\|^{1/n} < \|p\|_{\mathbb{D}}.$$

One can show that the above condition in Beurling's theorem is essentially equivalent to the condition in Corollary 3.1.7. Beurling's proof uses deep hard analysis and is completely independent of Kato's and therefore our approach. Indeed, it is essentially based on ideas originating from *Beurling's analyticity theorem*, a deep theorem giving sufficient conditions for a scalar-valued continuous function defined on an interval to be analytic in some rhombus containing the interval. This allows Beurling to prove a variant of the above result without requiring strong continuity for the semigroup. For details on Beurling's result and a statement of the analyticity theorem together with many historical information and references we refer to J. W. Neuberger's review article [Neu93].

Although so much work has been done, the following problem discussed in Remark 3.1.9 is still open.

Problem 3.3.4. Let $(T(t))_{t\geq 0}$ be an analytic C_0-semigroup on some complex Banach space X. Does there exist an $N \in \mathbb{N}$ such that

$$\limsup_{t\downarrow 0} \left\|(T(t)-\mathrm{Id})^N\right\|^{1/N} < 2?$$

Recall, however, that we have seen that the question has a positive answer if the negative generator of $(T(t))_{t\geq 0}$ has a bounded H^∞-calculus for some angle smaller than $\frac{\pi}{2}$.

Part II

Structural Characterizations of Sectorial Operators with a Bounded H^∞-calculus

4 Classes of Semigroups with a Bounded H^∞-Calculus

In this chapter we present general classes of negative generators of strongly continuous semigroups that have a bounded H^∞-calculus. We will see that the H^∞-calculus is closely related to the existence of dilations. In fact, apart from explicit calculations and perturbation arguments for concrete differential operators, the approach via dilations seems to be the only one available to obtain the boundedness of the H^∞-calculus for classes of sectorial operators. To give a first flavour of the methods and expected results, we shortly recall the Hilbert space case which is essentially completely understood. After that we turn our focus to the case of L_p-spaces where matters get way more complicated. Here we give a complete proof of Fendler's dilation theorem in the strongest form possible, namely for so-called r-contractive semigroups. In fact we will generalize Fendler's theorem to r-contractive semigroups on closed subspaces of L_p-spaces and prove as an application a pointwise ergodic theorem on those subspaces. This allows us to obtain a complete characterization of those semigroups having a strict dilation. Further, we can deduce the boundedness of the H^∞-calculus for those semigroups.

4.1 Contractive Semigroups on Hilbert Spaces

In this introductory section we want to present shortly the well-understood Hilbert space case in order to motivate the results in the following sections. For semigroups on Hilbert spaces one has the following classical dilation result [SNFBK10, Theorem 8.1].

Theorem 4.1.1. *Let $(T(t))_{t\geq 0}$ be a contractive C_0-semigroup on a Hilbert space H. Then there exists a second Hilbert space K, an isometric embedding $J\colon H \to K$, an orthogonal projection $P\colon K \to H$ and a unitary group $(U(t))_{t\in\mathbb{R}}$ on K such that*

$$T(t) = PU(t)J \qquad \text{for all } t \geq 0.$$

In particular, we obtain the following characterization for which we will later prove an L_p-analogue.

Corollary 4.1.2. *A C_0-semigroup on a Hilbert space has a strict dilation in the class of all Hilbert spaces if and only if the semigroup is contractive.*

This gives the following powerful result on the boundedness of the H^∞-calculus.

Corollary 4.1.3. *Let $-A$ be the infinitesimal generator of a bounded analytic C_0-semigroup $(T(z))_{z\in\Sigma}$ on some Hilbert space that is contractive on the real line. Then A has a bounded H^∞-calculus with $\omega_{H^\infty}(A) < \frac{\pi}{2}$.*

Proof. It follows from Theorem 4.1.1 together with Corollary 1.3.10 that A has a bounded H^∞-calculus with $\omega_{H^\infty}(A) \le \frac{\pi}{2}$. Now, on Hilbert spaces by Theorem 1.3.4 this automatically improves to $\omega_{H^\infty}(A) = \omega(A) < \frac{\pi}{2}$. □

In the next chapter we will present a converse result due to C. Le Merdy.

4.2 Positive Contractive Semigroups on L_p-Spaces

In this section we show that if $-A$ generates a bounded analytic C_0-semigroup on L_p for some $p \in (1,\infty)$ which is r-contractive (see Definition 4.2.6) on the real line, then A has a bounded H^∞-calculus with $\omega_{H^\infty}(A) < \frac{\pi}{2}$. In the case of positive contractive semigroups this is due to L. Weis [Wei01b, Remark 4.9c)]. Still Weis' approach seems to be the only one known. It makes heavy use of Fendler's dilation theorem [Fen97]. We have decided to give a detailed proof of this result on the boundedness of the H^∞-calculus including a proof of Fendler's dilation theorem for two reasons. First of all we do not know of any reference in the literature where a complete presentation of the proof can be found. This is even more the case for the general version proved here, although the validity of the result is probably well-known among experts. The main result is that Fendler's dilation theorem can be naturally generalized to semigroups acting on a closed subspace of some L_p-space. As we will see, this new result has applications to ergodic theory which are interesting in their own right. Furthermore, we will see that the r-contractive analytic semigroups even characterize those semigroups on L_p which have a strict dilation.

4.2.1 Some Operator Theoretic Results for L_p-Spaces

Before we can give the proofs of the main results of this chapter, we need some operator theoretic background on the structure of bounded operators, in particular isometries, on L_p-spaces. Background on general Banach lattices can be found in Appendix A.4.

Definition 4.2.1. A bounded linear operator $T\colon L_p(\Omega_1) \to L_p(\Omega_2)$ between two L_p-spaces is called *disjointness preserving* if $f \cdot g = 0$ implies $Tf \cdot Tg = 0$ for all $f, g \in L_p(\Omega_1)$.

Isometries are an important class of separation preserving operators on L_p. The next lemma can be found in J. Lamperti's work [Lam58] on the structure of isometries on L_p-spaces.

Lemma 4.2.2. *For $p \in [1,\infty) \setminus \{2\}$ let $T\colon L_p(\Omega_1) \to L_p(\Omega_2)$ be a linear isometry. Then T is disjointness preserving.*

Proof. Let $f,g \in L_p(\Omega_1)$. Note that $f \cdot g = 0$ implies

$$\|f+g\|^p + \|f-g\|^p = 2(\|f\|^p + \|g\|^p). \tag{4.1}$$

Conversely, the above equality (4.1) implies $f \cdot g = 0$. Indeed, for $p \geq 2$ one has by the convexity of the mapping $y \mapsto y^{p/2}$ for $y \geq 0$

$$\begin{aligned}
\frac{1}{2}(\|f+g\|^p + \|f-g\|^p) &= \int_{\Omega_1} \frac{|f(x)+g(x)|^p + |f(x)-g(x)|^p}{2}\, dx \\
&\geq \int_{\Omega_1} \left(\frac{|f(x)+g(x)|^2 + |f(x)-g(x)|^2}{2}\right)^{p/2} dx = \int_{\Omega_1} (|f(x)|^2 + |g(x)|^2)^{p/2}\, dx \\
&\geq \int_{\Omega_1} |f(x)|^p + |g(x)|^p\, dx = \|f\|^p + \|g\|^p.
\end{aligned}$$

Moreover, note that for $p > 2$ the inequality $(x+y)^{p/2} \geq x^{p/2} + y^{p/2}$ for non-negative x and y is strict if $xy \neq 0$. This shows that equality (4.1) can only hold if $f(x)g(x) = 0$ almost everywhere, i.e. $f \cdot g = 0$. Further observe that in the case $p < 2$ the above inequalities reverse and one can then use the same argument.

Now, since T is an isometry equality (4.1) holds for f and g if and only if it holds for Tf and Tg. This implies the assertion. □

As a consequence we obtain that a positive isometry T on L_p is a lattice homomorphism, i.e. T respects the lattice structure of L_p.

Lemma 4.2.3. *For $p \in [1,\infty) \setminus \{2\}$ let $T\colon L_p(\Omega_1) \to L_p(\Omega_2)$ be a positive isometry. Then T is a lattice homomorphism.*

Proof. Let $f,g \in L_p(\Omega_1)$. Since T is positive, one has $T(f \vee g) \geq Tf \vee Tg$. Now consider the disjoint sets

$$\begin{aligned}
F &:= \{x \in \Omega_1 : f(x) \geq g(x)\} \\
G &:= \{x \in \Omega_1 : g(x) > f(x)\}.
\end{aligned}$$

By decomposing $f \vee g$ along the above sets we obtain

$$T(f \vee g) = T(f\mathbb{1}_F + g\mathbb{1}_G) = T(f\mathbb{1}_F) + T(g\mathbb{1}_G). \tag{4.2}$$

Since T is disjointness preserving by Lemma 4.2.2, on the support of $T(f\mathbb{1}_F)$ one has

$$T(f\mathbb{1}_F) = T(f\mathbb{1}_F + f\mathbb{1}_{F^c}) = Tf \leq Tf \vee Tg.$$

Of course, an analogous estimate holds for the other summand of the right hand side of equation (4.2). Since the two summands have disjoint supports, we have shown the converse inequality $T(f \vee g) \leq Tf \vee Tg$. □

The definition of a direct sum of multiplication operators is intuitively clear, nevertheless it needs some care in the case of non-σ-finite measure spaces because of measurability problems.

Definition 4.2.4 (Direct Sum of Multiplication Operators). Let (Ω, Σ, μ) be a measure space and $\mathcal{M}$ a family of measurable functions $g\colon \Omega \to \mathbb{C}$ satisfying $\mu(\operatorname{supp}(g_1) \cap \operatorname{supp}(g_2)) = 0$ for all $g_1, g_2 \in \mathcal{M}$. Then on the space M_σ of all measurable functions $f\colon \Omega \to \mathbb{C}$ with σ-finite support one can define a linear operator $T_{\mathcal{M}}$ as follows. Let $f \in M_\sigma$. Then f has σ-finite support and therefore the family of functions $g \in \mathcal{M}$ for which $\mu(\operatorname{supp}(f) \cap \operatorname{supp}(g)) > 0$ is at most countable. Let $(g_n)_{n\in\mathbb{N}}$ be an enumeration of these elements and let $N = \bigcup_{n \neq m} \operatorname{supp}(g_n) \cap \operatorname{supp}(g_m)$, a measurable set of measure zero that satisfies: for all $x \in \operatorname{supp}(f) \setminus N$ there exists a unique $n \in \mathbb{N}$ with $x \in \operatorname{supp}(g_n)$. We set

$$(T_{\mathcal{M}} f)(x) := \begin{cases} \sum_{n=1}^\infty g_n(x) f(x) & \text{if } x \in N \\ 0 & \text{if } x \notin N. \end{cases}$$

Then $T_{\mathcal{M}} f \in M_\sigma$. We call $T_{\mathcal{M}}$ the *direct sum of multiplication operators* induced by the family $\mathcal{M}$. Notice that in particular $T_{\mathcal{M}} f$ is well-defined for all $f \in L_p(\Omega)$ for all $p \in [1, \infty)$. If $\operatorname{Im}|g| \subset \{0, 1\}$ for all $g \in \mathcal{M}$, we say that $T_{\mathcal{M}}$ is a *direct sum of unitary multiplication operators*.

The following technical lemma has important consequences for the structure of operators on L_p-spaces. We follow the presentation in [Lac74, Chapter 6, §17].

Lemma 4.2.5. *For $p \in [1, \infty) \setminus \{2\}$ let $T\colon L_p(\Omega_1, \Sigma_1, \mu_1) \to L_p(\Omega_2)$ be a linear isometry. Then there exists a linear surjective isometric operator $U\colon L_p(\Omega_2) \to L_p(\Omega_2)$ such that UT is a positive mapping. Moreover, U can be chosen as a direct sum of unitary multiplication operators.*

Proof. We assume that $L_p(\Omega_1, \Sigma_1, \mu) \neq 0$. Otherwise there is nothing to show. Let $\mathfrak{A}$ be the set of all families of sets $\mathcal{F} \subset \mathcal{P}(\Sigma_1)$ that have the following properties:

1. For every $A \in \mathcal{F}$ one has $\mu_1(A) \in (0, \infty)$.
2. For every pair $A \neq B \in \mathcal{F}$ one has $\mu_1(A \cap B) = 0$.

Then $\mathfrak{A}$ is partially ordered by inclusion. Let $\mathcal{M}$ be a maximal element in $\mathfrak{A}$. Note that if $(\Omega_1, \Sigma_1, \mu_1)$ is σ-finite, such an element can be constructed directly. The case of a general measure space follows from Zorn's lemma.

Let $\operatorname{sgn}\colon \mathbb{C} \to \mathbb{C}$ be the complex signum function defined by

$$\operatorname{sgn} z := \begin{cases} \frac{z}{|z|} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}.$$

Now let $A \in \mathcal{M}$ and $M \subset A$ measurable. Then $\mathbb{1}_A - \mathbb{1}_M$ and $\mathbb{1}_M$ are disjoint which implies by Lemma 4.2.2 that $T(\mathbb{1}_A - \mathbb{1}_M)$ and $T\mathbb{1}_M$ are disjoint as well. This implies

$$\begin{aligned} T\mathbb{1}_M &= |T\mathbb{1}_M|\operatorname{sgn}(T\mathbb{1}_M) = |T\mathbb{1}_M|(\operatorname{sgn}(T\mathbb{1}_A) - \operatorname{sgn}(T\mathbb{1}_{A\setminus M})) \\ &= |T\mathbb{1}_M|\operatorname{sgn}(T\mathbb{1}_A). \end{aligned} \tag{4.3}$$

Since T is separation preserving, the above identity extends to all positive step functions $f = \sum_{k=1}^{n} a_k \mathbb{1}_{M_k}$ with $a_k \geq 0$ and $M_k \subset A$ pairwise disjoint and measurable. Further, by an approximation argument the above identity extends to all positive f in $L_p(A)$. On the band $(T\mathbb{1}_A)^{\perp\perp}$ generated by $T\mathbb{1}_A$ we define the isometric unitary multiplication operator

$$\begin{aligned} U_A\colon (T\mathbb{1}_A)^{\perp\perp} &\to L_p(\Omega_2) \\ g &\mapsto \overline{\operatorname{sgn} T\mathbb{1}_A} \cdot g \end{aligned}$$

Since T is separation preserving by Lemma 4.2.2, the bands $(T\mathbb{1}_A)^{\perp\perp}$ and $(T\mathbb{1}_B)^{\perp\perp}$ for $A \neq B \in \mathcal{M}$ have trivial intersection. For $A \in \mathcal{M}$ let P_A be the band projection onto $(T\mathbb{1}_A)^{\perp\perp}$. Then P_A is positive and contractive and by the observation just made one has $P_A P_B = P_B P_A = 0$ for all $A \neq B \in \mathcal{M}$.

Further notice that the net of finite sums of the form $\sum_A P_A$ converges to a band projection P in the strong operator topology. Now, it follows that the net of finite sums $\sum_A U_A P_A$ converges strongly to a unitary multiplication operator $\hat{U}\colon L_p(\Omega_2) \to L_p(\Omega_2)$. Observe that U is neither necessarily surjective nor isometric. However, we can replace $\hat{U}$ by $U = \hat{U}P + (\mathrm{Id} - P)$. Then U is the direct sum of unitary multiplication operators induced by the family $\{\overline{\operatorname{sgn} T\mathbb{1}_A} : A \in \mathcal{M}\}$ whose boundedness we have just verified directly. Further notice that the discussion after equation (4.3) implies $UTf \geq 0$ for all positive $f \in L_p(\Omega_1)$ as desired. □

We now introduce the class of regular operators on a subspace of a Banach lattice. Note that there exist several equivalent definitions of regular operators. We follow the approach by G. Pisier [Pis94] which allows for a definition which includes the subspace setting.

Definition 4.2.6 (Regular Operator). A linear operator $T\colon S \to F$ between a subspace S of a Banach lattice E and a Banach lattice F is called *regular* if there is a constant $C \geq 0$ such that for all finite sequences $x_1, \ldots, x_n$ in S one has

$$\| \sup_{i=1,\ldots,n} |T(x_i)| \|_F \leq C \| \sup_{i=1,\ldots,n} |x_i| \|_E. \tag{4.4}$$

The smallest constant $C \geq 0$ such that (4.4) holds is the *regular norm* of T and is denoted by $\|T\|_r$. We call T *r-contractive* if $\|T\|_r \leq 1$ holds.

In the case of an operator $T\colon L_p(\Omega_1) \to L_p(\Omega_2)$ between two L_p-spaces one can show that the above definition is equivalent to the more common definition of a regular operator as the difference of two positive operators [Pis10, Chapter 1]. In this case there exist positive operators that dominate T. The infimum of those operators with respect to the natural order structure on $\mathcal{B}(L_p)$ exists and is the *modulus* $|T|$. One can then show that $\|T\|_r = \||T|\|$.

In the following we are only interested in regular operators on subspaces of L_p-spaces for $p \in (1,\infty)$. Nevertheless we decided to give the definition for general Banach lattices to clarify the core of the definition.

We now obtain the classical result that in the non-Hilbert space case every isometry on L_p is regular and even r-contractive.

Corollary 4.2.7. *For $p \in [1,\infty) \setminus \{2\}$ let $T\colon L_p(\Omega_1) \to L_p(\Omega_2)$ be a linear isometry. Then T is r-contractive.*

Proof. By Lemma 4.2.5 there exists a surjective isometric unitary multiplication operator $U\colon L_p(\Omega_2) \to L_p(\Omega_2)$ such that $UT\colon L_p(\Omega_1) \to L_p(\Omega_2)$ is a positive operator. Clearly, one has $|U^{-1}f| = |f|$ for all $f \in L_p(\Omega_2)$. Hence, one has
$$|Tf| = |U^{-1}UTf| = |UTf| \le UT|f|$$
for all $f \in L_p(\Omega_1)$. This shows that T is regular with $\|T\|_r = \||T|\| \le \|UT\| = \|T\| = 1$. □

The key property of regular operators on subspaces of L_p is the following extension property shown by G. Pisier [Pis94, Theorem 3] which even holds for regular maps between subspaces of more general Banach lattices.

Proposition 4.2.8. *For $p \in [1,\infty]$ let $L_p(\Omega_1)$ and $L_p(\Omega_2)$ be two arbitrary L_p-spaces and $S \subset L_p(\Omega_1)$ a closed subspace. Then every regular operator $T\colon S \to L_p(\Omega_2)$ admits a regular extension $\tilde{T}\colon L_p(\Omega_1) \to L_p(\Omega_2)$ with $\|\tilde{T}\|_r = \|T\|_r$.*

4.2.2 Fendler's Dilation Theorem for Subspaces of L_p

We now turn our attention to the proof of Fendler's dilation theorem for r-contractive semigroups [Fen97]. Here we call a semigroup $(T(t))_{t\ge 0}$ *r-contractive* if $T(t)$ is an r-contractive operator for all $t \ge 0$.

For discrete r-contractive semigroups on L_p the following celebrated dilation theorem independently proved by V.V. Peller [Pel81, Theorem 3 & Remark 1] and by R. R. Coifman, R. Rochberg and G. Weiss [CRW78, p. 58-59]) holds.

Theorem 4.2.9. *For $p \in (1,\infty) \setminus \{2\}$ let $T\colon L_p(\Omega) \to L_p(\Omega)$ be an r-contractive operator on some L_p-space $L_p(\Omega)$. Then there exists an L_p-space $L_p(\Omega')$, an isometric isomorphism $U\colon L_p(\Omega') \to L_p(\Omega')$, a positive isometric embedding*

$D\colon L_p(\Omega) \to L_p(\Omega')$ and a positive contractive projection $P\colon L_p(\Omega') \to L_p(\Omega')$ such that

$$DT^n = PU^nD \qquad \text{for all } n \geq 0.$$

Note that in the first reference it is assumed that the underlying measure space is σ-finite. However, one can check that the same argument works for general measure spaces.

We need the following decomposition result for group representations on Banach spaces. The result goes in far more generality back to I. Glicksberg and K. de Leeuw [dLG65]. A self-contained proof for the special case below can be found in [Fen97, Proposition 1].

Theorem 4.2.10 (Glicksberg–de Leeuw). *Let X be a reflexive Banach space, G a commutative topological group and $\pi\colon G \to \mathcal{B}(X)$ a uniformly bounded representation of G. Then there exists a projection $Q \in \mathcal{B}(X)$ onto the closed subspace Y of all elements $x \in X$ for which $g \mapsto \pi(g)x$ is continuous from G to X. Moreover, Q satisfies*

(a) *$\pi(g)Q = Q\pi(g)$ for all $g \in G$,*

(b) $\|Q\| \leq \sup_{g\in G} \|\pi(g)\|$.

We are now ready to prove a new generalization to subspaces of L_p-spaces of Fendler's celebrated dilation theorem for semigroups on L_p-spaces [Fen97]. For this we need some basics on ultraproducts of Banach spaces which can be found in Appendix A.2.1. The dilation theorem was originally proved by Fendler for r-contractive semigroups on L_p for $p \in (1, \infty)$. Using the same methods and Pisier's extension result for regular operators (Proposition 4.2.8), we can extend the result to the subspace case.

Theorem 4.2.11. *Let $(T(t))_{t\geq 0}$ be an r-contractive C_0-semigroup on a closed subspace S of some L_p-space $L_p(\Omega)$ for $p \in (1, \infty) \setminus \{2\}$. Then there exist an L_p-space $L_p(\Omega')$, a C_0-group $(U(t))_{t\in\mathbb{R}}$ of isometries on $L_p(\Omega')$ together with a positive isometry $D\colon L_p(\Omega) \to L_p(\Omega')$ and a contractive projection $P\colon L_p(\Omega') \to L_p(\Omega')$ such that*

$$DT(t)f = PU(t)Df \qquad \text{for all } t \geq 0 \text{ and all } f \in S.$$

Proof. By Pisier's extension result for regular operators (Proposition 4.2.8) one can extend each operator $T(t)$ of the semigroup individually to an r-contractive operator $\tilde{T}(t)$ on $L_p(\Omega)$. Of course, we cannot expect $(\tilde{T}(t))_{t\geq 0}$ neither to be strongly continuous nor a semigroup on $L_p(\Omega)$. Nevertheless, by the dilation theorem for r-contractive operators (Theorem 4.2.9), for every $n \in \mathbb{N}$ there exists a dilation of the operator $\tilde{T}(\frac{1}{n})$. More precisely, there exist a measure space $\Omega_{1/n}$, an invertible isometry $U_{1/n}\colon L_p(\Omega_{1/n}) \to L_p(\Omega_{1/n})$

together with a positive isometry $D_{1/n}\colon L_p(\Omega) \to L_p(\Omega_{1/n})$ and a positive contractive projection $P\colon L_p(\Omega_{1/n}) \to L_p(\Omega_{1/n})$ such that

$$D_{1/n}\tilde{T}(1/n)^k = PU_{1/n}^k D_{1/n} \qquad \text{for all } k \geq 0.$$

For finite sets $B \subset \mathbb{Q}$ we define $F_B := \{n \in \mathbb{N} : ns \in \mathbb{Z} \text{ for all } s \in B\}$. Then the collection $\{F_B : B \subset \mathbb{Q}\}$ is a basis of some filter $\mathcal{F}$ on $\mathbb{N}$. Now choose an ultrafilter $\mathcal{U}$ on $\mathbb{N}$ refining $\mathcal{F}$. Further, we set

$$U_{n,s} = \begin{cases} U_{1/n}^{ns} & \text{if } ns \in \mathbb{Z} \\ \mathrm{Id} & \text{if } ns \notin \mathbb{Z}. \end{cases}$$

Let $\hat{D}$ be the composition of the canonical inclusion of $L_p(\Omega)$ into $\prod_{\mathcal{U}} L_p(\Omega)$ with $\prod_{\mathcal{U}} D_{1/n}$, $\hat{P} = \prod_{\mathcal{U}} P_{1/n}$ and $\hat{U}(s) = \prod_{\mathcal{U}} U_{n,s}$ for all $s \in \mathbb{Q}$. Notice that the operators $\hat{U}(s)$ live on $\prod_{\mathcal{U}} L_p(\Omega)$ which is order isometric to some L_p-space $L_p(\hat{\Omega})$ by Kakutani's theorem (Theorem A.4.7). Further, $\hat{D}$ is a positive isometric embedding, $\hat{P}$ is a positive projection and the operators $\hat{U}(s)$ are isometric isomorphisms. We now show that the mapping $\hat{U}\colon \mathbb{Q} \to \mathcal{B}(L_p(\hat{\Omega}))$ is a homomorphism. For this choose $s, t \in \mathbb{Q}$ and let $(f_n)_{n\in\mathbb{N}}$ be a representant of an element in $\prod_{\mathcal{U}} L_p(\Omega_{1/n})$. Then for $n \in F_{\{s,t\}}$ one has

$$U_{n,s+t}(f_n) = U_{1/n}^{n(s+t)}(f_n) = U_{1/n}^{ns}(U_{1/n}^{nt}(f_n)) = U_{n,s}(U_{n,t}f_n).$$

Since $\mathcal{U}$ is finer than $\mathcal{F}$, we see that both $(U_{n,s+t}(f_n))_{n\in\mathbb{N}}$ and $(U_{n,s}(U_{n,t}f_n))_{n\in\mathbb{N}}$ represent the same element in $\prod_{\mathcal{U}} L_p(\Omega_{1/n})$. Thus $\hat{U}(t+s) = \hat{U}(s)\hat{U}(t)$ is shown. Analogously, one has for $f \in S \subset L_p(\Omega)$, $t \in \mathbb{Q}_{\geq 0}$ and $n \in F_{\{t\}}$

$$\begin{aligned} P_{1/n}U_{n,t}D_{1/n}f &= P_{1/n}U_{1/n}^{nt}D_{1/n}f = D_{1/n}\tilde{T}(1/n)^{nt}f \\ &= D_{1/n}T(1/n)^{nt}f = D_{1/n}T(t)f. \end{aligned}$$

This shows that for all $f \in S$ one has $\hat{P}\hat{U}(t)\hat{D}f = \hat{D}T(t)f$.

In the next step we show that for all $f \in S$ the map $t \mapsto \hat{U}(t)\hat{D}f$ is continuous from $\mathbb{Q}$ endowed with the Euclidean topology to $L_p(\hat{\Omega})$. First observe that since $(\hat{U}(t))_{t\in\mathbb{Q}}$ is a group of isometries, it suffices to show strong continuity from the right at $t = 0$. Without loss of generality, we assume $\|f\| = 1$. Then for $\varepsilon > 0$ there exists a $\delta > 0$ such that $\|T(t)f - f\| \leq \varepsilon$ for all $t \in [0, \delta)$. Hence, for $t \in \mathbb{Q} \cap [0, \delta)$ one has

$$\begin{aligned} \|\hat{U}(t)\hat{D}f + \hat{D}f\| &\geq \|\hat{P}\hat{U}(t)\hat{D}f + \hat{P}\hat{D}f\| = \|\hat{D}T(t)f + \hat{D}f\| = \|T(t)f + f\| \\ &= \|2f - (f - T(t)f\| \geq \|2f\| - \|T(t)f - f\| \geq 2 - \varepsilon. \end{aligned}$$

Since $\|\hat{U}(t)\hat{D}f\| = \|\hat{D}f\| = 1$ for all $t \in \mathbb{Q}$ and $L_p(\Omega)$ is uniformly convex, the above estimate implies that $\|\hat{U}(t)\hat{D}f - \hat{D}f\| \to 0$ for $t \downarrow 0$. This shows the desired continuity.

Furthermore, it follows from Theorem 4.2.10 that there exists a contractive projection Q from $L_p(\hat{\Omega})$ to the space Y of elements $f \in L_p(\hat{\Omega})$ for which $t \mapsto \hat{U}(t)f$ is continuous. Notice that by the above observation the range of $\hat{D}$ is contained in Y. Since Y is invariant under the action of $\hat{U}$, we can restrict $\hat{U}$ to a strongly continuous representation of $\mathbb{Q}$ on Y. Hence, we can extend $\hat{U}$ to a continuous representation of $\mathbb{R}$ on Y which we will still denote by $\hat{U}$.

We are finished except for the fact that $\hat{U}$ acts on $Y = QL_p(\hat{\Omega})$ instead of an L_p-space. As Y is the image of a contractive projection on L_p, there exists an L_p-space $L_p(\Omega_1)$ and an isometric isomorphism $L\colon L_p(\Omega_1) \to Y$ [Lac74, Chapter 6, §17, Theorem 3(ii)]. Consider the isometric mapping $L^{-1} \circ \hat{D}\colon L_p(\Omega) \to L_p(\Omega_1)$, which is well-defined because of $\hat{D}L_p(\Omega) \subset Y$. By Lemma 4.2.5 there is an isometric isomorphism $V\colon L_p(\Omega_1) \to L_p(\Omega_1)$ such that $V \circ L^{-1} \circ \hat{D}\colon L_p(\Omega) \to L_p(\Omega_1)$ is a positive mapping. Applying Lemma 4.2.5 again to the isometric mapping $L \circ V^{-1}\colon L_p(\Omega_1) \to L_p(\hat{\Omega})$, there exists a surjective isometric direct sum of unitary multiplication operators $W\colon L_p(\hat{\Omega}) \to L_p(\hat{\Omega})$ such that $W \circ L \circ V^{-1} \geq 0$. Observe that for positive $f \in L_p(\Omega)$ one has

$$W\hat{D}f = (W \circ L \circ V^{-1})(V \circ L^{-1} \circ \hat{D})f \geq 0.$$

Since $\hat{D}$ is positive, one also has $\hat{D}f \geq 0$. This implies that W acts as the identity on the image of $\hat{D}$. Moreover, $(W \circ L \circ V^{-1})(L_p(\Omega_1)) = WY$ is a closed sublattice of $L_p(\hat{\Omega})$ because a positive isometric mapping on L_p for $p \in (1,\infty) \setminus \{2\}$ is automatically a lattice homomorphism by Lemma 4.2.3. Therefore WY is order isometric to some L_p-space $L_p(\Omega')$ by a mapping $\Phi\colon WY \to L_p(\Omega')$. Now set

$$\begin{aligned} D &= \Phi \circ \hat{D} \\ U(t) &= \Phi \circ W \circ Q \circ \hat{U}(t) \circ W^{-1} \circ \Phi^{-1} \\ P &= \Phi \circ W \circ \hat{P} \circ W^{-1} \circ \Phi^{-1}. \end{aligned}$$

Then D, P and $U(t)$ for $t \geq 0$ have the desired properties. We only verify the dilation property explicitly. Since W acts as the identity on the image of $\hat{D}$, one has for all $f \in S$

$$\begin{aligned} PU(t)Df &= (P \circ \Phi \circ W \circ Q \circ \hat{U}(t) \circ W^{-1} \circ \hat{D})f = (P \circ \Phi \circ W \circ Q \circ \hat{U}(t) \circ \hat{D})f \\ &= (P \circ \Phi \circ W \circ \hat{W}(t) \circ \hat{D})f = (\Phi \circ W \circ \hat{P} \circ \hat{U}(t) \circ \hat{D})f \\ &= (\Phi \circ W \circ \hat{D} \circ T(t))f = (\Phi \circ \hat{D} \circ T(t))x = DT(t)f. \qquad \square \end{aligned}$$

Sometimes it is more useful to use a different representation of the dilation just obtained.

Corollary 4.2.12. *Let $(T(t))_{t \geq 0}$ be an r-contractive C_0-semigroup on a closed subspace S of some L_p-space $L_p(\Omega)$ for $p \in (1,\infty) \setminus \{2\}$. Then there exist an*

L_p-space $L_p(\Omega')$, a C_0-group $(U(t))_{t\in\mathbb{R}}$ of isometries on $L_p(\Omega')$ together with a positive isometric embedding $J\colon L_p(\Omega) \to L_p(\Omega')$ and a contractive mapping $Q\colon L_p(\Omega') \to L_p(\Omega)$ such that

$$T(t)f = QU(t)Jf \qquad \text{for all } t \geq 0 \text{ and all } f \in S.$$

Proof. Consider the dilation $DT(t) = PU(t)D$ obtained in Theorem 4.2.11. Since D is an isometry, D is an isometric isomorphism onto the closed subspace $\operatorname{Rg} D$ of $L_p(\Omega')$. The dilation property shows that $(P \circ U(t) \circ D)(L_p(\Omega)) \subset \operatorname{Rg} D$ for all $t \geq 0$. Hence, we can apply $D^{-1}\colon \operatorname{Rg} D \to L_p(\Omega)$ to both sides of the dilation equation and obtain

$$T(t) = D^{-1}PU(t)D \qquad \text{for all } t \geq 0.$$

Now, we set $J = D$ and $Q = D^{-1}P$ and we obtain the desired form of the dilation. □

Hence, we see that every r-contractive C_0-semigroup on a closed subspace of some L_p-space for $p \in (1,\infty) \setminus \{2\}$ has a strict dilation in the class of all L_p-spaces. We now show that the r-contractive semigroups are exactly those having a strict dilation in the class of all L_p-spaces. This seems to be a new semigroup variant of the known characterization for discrete semigroups due to V.V. Peller [Pel81, §3] which is probably known to experts.

Theorem 4.2.13. *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on an L_p-space $L_p(\Omega)$ for $p \in (1,\infty) \setminus \{2\}$. Then $(T(t))_{t\geq 0}$ has a strict dilation in the class of all L_p-spaces if and only if $(T(t))_{t\geq 0}$ is an r-contractive semigroup.*

Proof. On the one hand we have just seen in Corollary 4.2.12 that every r-contractive C_0-semigroup on L_p has a strict dilation in the class of all L_p-spaces. On the other hand suppose that $(T(t))_{t\geq 0}$ has a strict dilation in the class of all L_p-spaces, i.e. there exists a C_0-group $(U(t))_{t\in\mathbb{R}}$ of contractions on some L_p-space $L_p(\Omega')$ and contractive mappings $J\colon L_p(\Omega) \to L_p(\Omega')$ and $Q\colon L_p(\Omega') \to L_p(\Omega)$ such that

$$T(t) = QU(t)J \qquad \text{for all } t \geq 0.$$

Note that one directly sees that $U(t)$ is an isometric isomorphism for all $t \in \mathbb{R}$. In particular, it follows from Lemma 4.2.7 that $U(t)$ is r-contractive for all $t \geq 0$. Moreover, for $t = 0$ one obtains the identity $\operatorname{Id} = QJ$. Since Q and J are contractions, this implies that J is an isometry. Moreover, taking adjoints we obtain $\operatorname{Id} = J^*Q^*$. By the same argument we see that $Q^*\colon L_q(\Omega') \to L_q(\Omega)$ is an isometry, where q is the adjoint index given by $1 = \frac{1}{p} + \frac{1}{q}$. Applying Lemma 4.2.7 again to J and Q^*, we see that both are r-contractive. We now show that the r-contractivity of Q^* implies that Q is r-contractive as well.

Let $P^*\colon L_q(\Omega) \to L_q(\Omega)$ be a positive contraction such that $|Q^*g| \le P^*|g|$ for all $g \in L_q(\Omega)$. We show that $|Qf| \le P^{**}|f|$ for all $f \in L_p(\Omega)$ as desired. Notice that P^{**} is a positive contraction and that it suffices to show that $\int_{\Omega'} |Qf| \mathbb{1}_A \le \int_{\Omega'} P^{**}|f| \mathbb{1}_A$ for all $f \in L_p(\Omega)$ and all $A \subset \Omega'$ of finite measure. For $A \subset \Omega'$ of finite measure one has

$$\begin{aligned}\int_{\Omega'} |Qf| \mathbb{1}_A &= \int_{\Omega'} Qf \operatorname{sgn}(Qf) \mathbb{1}_A = \int_{\Omega} f Q^*(\operatorname{sgn}(Qf) \mathbb{1}_A) \\ &\le \int_{\Omega} |f||Q^*(\operatorname{sgn}(Qf) \mathbb{1}_A)| \le \int_{\Omega} |f| P^* |\operatorname{sgn}(Qf) \mathbb{1}_A| \\ &= \int_{\Omega} |f| P^* \mathbb{1}_A = \int_{\Omega'} P^{**} |f| \mathbb{1}_A.\end{aligned}$$

It follows that for all $t \ge 0$ the operator $T(t)$ is r-contractive as the composition of r-contractive operators. This finishes the proof. □

Remark 4.2.14. Notice that the same characterization is valid in the subspace case if one additionally requires that J extends to a mapping $J\colon L_p(\Omega) \to L_p(\Omega')$ and that one has $\mathrm{Id}_{L_p} = QJ$. This property is clearly satisfied by the dilation constructed in Corollary 4.2.12. For the proof of the characterization in the subspace case just notice that with the same proof one obtains that Q, J and $U(t)$ for all $t \ge 0$ are r-contractive. Hence, $T(t) = QU(t)J_{|S}$ is r-contractive as the restriction of an r-contractive operator to the closed subspace S.

On L_p-spaces (and on more general Banach lattices) one can generalize the definition of $\mathcal{R}$-boundedness seen as square function estimates to the notion of $\mathcal{R}_q$-boundedness. This notion seems to go back to L. Weis [Wei01a] and opens the door for interpolation arguments which are crucial in the proof of the main result of this section.

Definition 4.2.15 ($\mathcal{R}_q$-Boundedness). Let p, q in $[1, \infty]$ and $S \subset L_p(\Omega)$ be a closed subspace of an L_p-space. A subset $\mathcal{T}$ of $\mathcal{B}(S)$ is called *$\mathcal{R}_q$-bounded* if there is a constant $C \ge 0$ such that for all $n \in \mathbb{N}$, all $T_1, \ldots, T_n \in \mathcal{T}$ and $f_1, \ldots, f_n \in L_p(\Omega)$ one has

$$\begin{aligned}\Big\| \Big(\sum_{i=1}^n |T_i f_i|^q \Big)^{1/q} \Big\|_p &\le C \Big\| \Big(\sum_{i=1}^n |f_i|^q \Big)^{1/q} \Big\|_p \qquad \text{if } q \in [1, \infty), \\ \| \sup_{i=1,\ldots,n} |T_i f_i| \|_p &\le C \| \sup_{i=1,\ldots,n} |f_i| \|_p \qquad \text{if } q = \infty.\end{aligned}$$

The smallest constant $C \ge 0$ such that the above inequality holds is denoted by $\mathcal{R}_q(\mathcal{T})$. We give some remarks on the basic properties of $\mathcal{R}_q$-bounded sets and their connection with $\mathcal{R}$-boundedness.

Remark 4.2.16. A subset $\mathcal{T} \subset \mathcal{B}(S)$ for a closed subspace S of $L_p(\Omega)$ for $p \in [1,\infty)$ is $\mathcal{R}_p$-bounded if and only if it is norm-bounded. For this observe that one has

$$\begin{aligned}\left\| \left(\sum_{i=1}^n |T_i f_i|^p \right)^{1/p} \right\|_p^p &= \int_\Omega \sum_{i=1}^n |(T_i f_i)(x)|^p \, dx = \sum_{i=1}^n \int_\Omega |(T_i f_i)(x)|^p \, dx = \sum_{i=1}^n \|T_i f_i\|^p \\ &\le \sup_{i=1,\dots,n} \|T_i\| \sum_{i=1}^n \|f_i\|^p = \sup_{i=1,\dots,n} \|T_i\| \left\| \left(\sum_{i=1}^n |f_i|^p \right)^{1/p} \right\|_p^p.\end{aligned}$$

Moreover, a subset $\mathcal{T} \subset \mathcal{B}(L_p(\Omega))$ is $\mathcal{R}$-bounded if and only if $\mathcal{T}$ is $\mathcal{R}_2$-bounded. Indeed, by the Khintchine inequality (Theorem A.3.1) one has the equivalence

$$\begin{aligned}\left\| \left(\sum_{i=1}^n |T_i f_i|^2 \right)^{1/2} \right\|_{L_p(\Omega)}^p &= \int_\Omega \left(\sum_{i=1}^n |T_i f_i(x)|^2 \right)^{p/2} dx \simeq \int_\Omega \left\| \sum_{i=1}^n r_i (T_i f_i)(x) \right\|_{L_p([0,1])}^p dx \\ &= \int_0^1 \left\| \sum_{i=1}^n r_i(\omega) T_i f_i \right\|_{L_p(\Omega)}^p d\omega.\end{aligned}$$

Let $(V(s)f)(t) := f(t-s)$ be the right bilateral shift group on $L_p(\mathbb{R})$. For $b \in L_1(\mathbb{R})$ one defines the convolution operator

$$\begin{aligned} T_b \colon L_p(\mathbb{R}) &\to L_p(\mathbb{R}) \\ f &\mapsto b * f = \int_{-\infty}^{\infty} b(s) f(\cdot - s) \, ds = \int_{-\infty}^{\infty} b(s) V(s) f \, ds.\end{aligned}$$

In the following we denote by $L_{1,c}(\mathbb{R})$ the space of all functions $f \in L_1(\mathbb{R})$ with compact support.

We now prove an order theoretic version of the transference principle for C_0-groups (Theorem 1.3.9). Here and for the rest of this subsection our point of view is very similar to the presentations in [Blu01] (and [Fen12]). There it is assumed that the semigroup $(T(t))_{t \ge 0}$ is dominated by a contractive positive semigroup. In our treatment we only use that the semigroup is r-contractive (or r-bounded) throughout the presentation. It is known, but not at all obvious, that both assumptions are equivalent [BG86, Proposition 2.3].

Proposition 4.2.17 (Transference of $\mathcal{R}_\infty$-boundedness). *Let $(U(t))_{t \in \mathbb{R}}$ be a C_0-group of uniformly r-bounded operators on some L_p-space $L_p(\Omega)$ for $p \in [1,\infty)$ and let $M := \sup\{\|U(t)\|_r : t \in \mathbb{R}\}$. Then for arbitrary subsets $B \subset L_{1,c}(\mathbb{R})$ one has*

$$\mathcal{R}_\infty \left\{ \int_0^\infty b(s) U(s) \, ds : b \in B \right\} \le M^2 \mathcal{R}_\infty \{ T_b : b \in B \},$$

where the integral on the left hand side is understood in the strong sense.

Proof. We first assume that the underlying measure space Ω is σ-finite. Choose a compact set K such that $b_1, \ldots, b_n \in B$ have support in K. Further for $t > 0$ let $K_t := K + [0,t]$. Since $|f| \le |U(-t)||U(t)f|$ holds for all $f \in L_p(\Omega)$ and all $t > 0$, we have for all $f_1, \ldots, f_n \in L_p(\Omega)$ that

$$\begin{aligned}
&\left\| \max_{i=1,\ldots,n} \left| \int_0^\infty b_i(s) U(s) f_i \, ds \right| \right\|^p \\
&\quad \le \left\| \max_{i=1,\ldots,n} \frac{1}{t} \int_0^t |U(-r)| \left| \int_0^\infty U(r) b_i(s) U(s) f_i \, ds \right| dr \right\|^p \\
&\quad \le \left\| \frac{1}{t} \int_0^t |U(-r)| \max_{i=1,\ldots,n} \left| \int_0^\infty b_i(s) U(r+s) f_i \, ds \right| dr \right\|^p \\
&\quad \le \frac{1}{t} \int_0^t |||U(-r)|||^p \left\| \max_{i=1,\ldots,n} \left| \int_0^\infty b_i(s) U(r+s) f_i \, ds \right| \right\|^p dr \\
&\quad \le \frac{M^p}{t} \int_{-\infty}^\infty \left\| \max_{i=1,\ldots,n} \left| \int_0^\infty \mathbb{1}_{K_t}(r+s) b_i(s) U(r+s) f_i \, ds \right| \right\|^p dr,
\end{aligned}$$

where we have used Jensen's inequality in the second to last inequality. Further, one has by Fubini's theorem

$$\begin{aligned}
&\int_{-\infty}^\infty \left\| \max_{i=1,\ldots,n} \left| \int_0^\infty b_i(s) \mathbb{1}_{K_t}(r+s) U(r+s) f_i \, ds \right| \right\|^p dr \\
&\quad = \int_\Omega \int_{-\infty}^\infty \max_{i=1,\ldots,n} \left| \int_0^\infty b_i(s) V(-s) (\mathbb{1}_{K_t}(\cdot)(U(\cdot) f_i)(x))(r) \, ds \right|^p dr \, dx \\
&\quad = \int_\Omega \left\| \max_{i=1,\ldots,n} \left| \int_0^\infty b_i(s) V(-s) (\mathbb{1}_{K_t}(\cdot)(U(\cdot) f_i)(x)) \, ds \right| \right\|_{L_p(\mathbb{R})}^p dx \\
&\quad \le \mathcal{R}_\infty \left\{ \int_0^\infty b(s) V(-s) \, ds : b \in B \right\}^p \int_\Omega \left\| \max_{i=1,\ldots,n} \left| (\mathbb{1}_{K_t}(\cdot) U(\cdot) f_i)(x) \right| \right\|_{L_p(\mathbb{R})}^p dx \\
&\quad = \mathcal{R}_\infty \left\{ \int_0^\infty b(s) V(-s) \, ds : b \in B \right\}^p \int_\mathbb{R} \left\| \max_{i=1,\ldots,n} \left| \mathbb{1}_{K_t}(r) U(r) f_i \right| \right\|^p dr \\
&\quad \le \mathcal{R}_\infty \left\{ \int_0^\infty b(s) V(-s) \, ds : b \in B \right\}^p \int_\mathbb{R} \mathbb{1}_{K_t}(r) \left\| \max_{i=1,\ldots,n} |U(r) f_i| \right\|^p dr \\
&\quad \le \mathcal{R}_\infty \left\{ \int_0^\infty b(s) V(-s) \, ds : b \in B \right\}^p \int_\mathbb{R} \mathbb{1}_{K_t}(r) \left\| |U(r)| \max_{i=1,\ldots,n} |f_i| \right\|^p dr \\
&\quad \le |K_t| M^p \mathcal{R}_\infty \left\{ \int_0^\infty b(s) V(-s) \, ds : b \in B \right\}^p \left\| \max_{i=1,\ldots,n} |f_i| \right\|^p.
\end{aligned}$$

Note that by change of variables one has for all $b_1, \ldots, b_n \in B$ and all $f_1, \ldots, f_n \in L_p(\mathbb{R})$

$$\left\| \max_{i=1,\ldots,n} \left| \int_0^\infty b_i(s) V(-s) f_i \, ds \right| \right\|^p = \int_\mathbb{R} \max_{i=1,\ldots,n} \left| \int_0^\infty b_i(s) f(t+s) \, ds \right|^p dt$$

$$= \int_{\mathbb{R}} \max_{i=1,\dots,n} \left| \int_0^\infty b_i(s) f_i(-(t-s))\, ds \right|^p dt = \left\| \max_{i=1,\dots,n} \left| \int_0^\infty b_i(s) V(s) f_i(-\cdot)\, ds \right| \right\|^p.$$

From the above identity one directly deduces

$$\begin{aligned} \mathcal{R}_\infty \left\{ \int_0^\infty b(s) V(-s)\, ds : b \in B \right\} &= \mathcal{R}_\infty \left\{ \int_0^\infty b(s) V(s)\, ds : b \in B \right\} \\ &= \mathcal{R}_\infty \{T_b : b \in B\}. \end{aligned}$$

Altogether, we have shown that for all $t > 0$

$$\left\| \max_{i=1,\dots,n} \left| \int_0^\infty b_i(s) U(s) f_i\, ds \right| \right\| \le \left(\frac{|K_t|}{t} \right)^{1/p} M^2 \mathcal{R}_\infty \{T_b : b \in B\} \| \max_{i=1,\dots,n} |f_i| \|.$$

For $t \to \infty$ we obtain the desired inequality in the σ-finite case. If Ω is an arbitrary measure space we consider for given $f_1, \dots, f_n \in L_p(\Omega)$ the separable sublattice L of $L_p(\Omega)$ generated by $f_1, \dots, f_n$ and their images under $U(t)$ for $t \in \mathbb{R}$ which is order isometric to a σ-finite L_p-space [LT79, Theorem 1.b.2]. Hence, the above estimate holds and is independent of L. This shows the general case. □

The transference principle for $\mathcal{R}_\infty$-boundedness together with Fendler's dilation theorem now yields the following.

Theorem 4.2.18. *Let $(T(t))_{t \ge 0}$ be an r-contractive C_0-semigroup on a closed subspace S of some L_p-space $L_p(\Omega)$ for $p \in (1, \infty) \setminus \{2\}$. Further let $M \colon (0, \infty) \to \mathcal{B}(X)$ be given by*

$$M(t) = \frac{1}{t} \int_0^t T(s)\, ds$$

in the strong sense. Then the set $\{M(t) : t \in (0, \infty)\}$ is $\mathcal{R}_\infty$-bounded.

Proof. We first start with the special case where $(T(t))_{t \ge 0}$ is an r-contractive C_0-group on $L_p(\Omega)$, i.e. $S = L_p(\Omega)$. Notice that for $b_t = \frac{1}{t} \mathbb{1}_{[0,t]}$ one has

$$(T_{b_t} f)(x) = \frac{1}{t} \int_0^t f(x-y)\, dy = \frac{1}{t} \int_0^t (V(y) f)(x)\, dy = \int_0^\infty b_t(y) (V(y) f)(x)\, dy.$$

By the transference principle for $\mathcal{R}_\infty$-boundedness (Theorem 4.2.17) one has

$$\mathcal{R}_\infty \{M(t) : t \in (0, \infty)\} \le \mathcal{R}_\infty \left\{ T_{b_t} : t \in (0, \infty) \right\}.$$

It therefore remains to show that for all $n \in \mathbb{N}$, $t_1, \dots, t_n > 0$ and $f_1, \dots, f_n \in L_p(\mathbb{R})$ one has

$$\left\| \sup_{i=1,\dots,n} \left| \frac{1}{t_i} \int_0^{t_i} f_i(\cdot - y)\, dy \right| \right\| \le C \| \sup_{i=1,\dots,n} |f_i| \|.$$

for some constant $C \geq 0$. For this one has

$$\left\| \sup_{i=1,\ldots,n} \left| \frac{1}{t_i} \int_0^{t_i} f_i(\cdot - y)\, dy \right| \right\| \leq \left\| \sup_{t>0} \frac{1}{t} \int_0^t \sup_{i=1,\ldots,n} \left| f_i(\cdot - y) \right| dy \right\|,$$

where the right hand side is the left-sided Hardy-Littlewood maximal operator. It is a classical result in harmonic analysis that this operator is bounded on L_p for $p \in (1, \infty)$ [HS75, (21.76)].

Now, we consider the general case, where $(T(t))_{t \geq 0}$ is an r-contractive C_0-semigroup on a closed subspace of $L_p(\Omega)$. Then it follows from the dilation theorem for such semigroups (Theorem 4.2.11) that there exist an r-contractive C_0-group $(U(t))_{t \in \mathbb{R}}$ on some L_p-space $L_p(\hat{\Omega})$, a positive isometric embedding $D \colon L_p(\Omega) \to L_p(\hat{\Omega})$ and an r-contractive mapping $P \colon L_p(\hat{\Omega}) \to L_p(\hat{\Omega})$ such that

$$PU(t)Df = DT(t)f \qquad \text{for all } f \in S.$$

Notice that Lemma 4.2.3 shows that D is an isometric lattice homomorphism. Therefore one has for all $n \in \mathbb{N}$, $t_1, \ldots, t_n > 0$ and $f_1, \ldots, f_n \in S$ by the first part of the proof

$$\begin{aligned} \left\| \sup_{i=1,\ldots,n} |M(t_i) f_i| \right\| &= \left\| D \sup_{i=1,\ldots,n} \left| \frac{1}{t_i} \int_0^{t_i} T(s) f_i \, ds \right| \right\| = \left\| \sup_{i=1,\ldots,n} \left| \frac{1}{t_i} \int_0^{t_i} DT(s) f_i \, ds \right| \right\| \\ &= \left\| \sup_{i=1,\ldots,n} \left| \frac{1}{t_i} \int_0^{t_i} PU(s) Df_i \, ds \right| \right\| \leq |||P||| \left\| \sup_{i=1,\ldots,n} \left| \frac{1}{t_i} \int_0^{t_i} U(s) Df_i \right| ds \right\| \\ &\leq C \left\| \sup_{i=1,\ldots,n} |Df_i| \right\| = C \left\| D \left(\sup_{i=1,\ldots,n} |f_i| \right) \right\| = C \left\| \sup_{i=1,\ldots,n} |f_i| \right\|. \end{aligned}$$

□

4.2.3 An Application to Ergodic Theory

Before proving the main result of this chapter, we present some direct consequences of the results proved so far. As a corollary of the $\mathcal{R}_\infty$-boundedness just shown one obtains a maximal ergodic inequality for r-contractive semigroups. The validity of this inequality is known to experts in the L_p-case (see for example [LMX12, Remark 4.3]) and can be obtained along the lines of the arguments in [Fen12, Theorem 5.4.3]. However, the result below seems to be new in the subspace case.

Corollary 4.2.19 (Maximal Ergodic Inequality). *Suppose $(T(t))_{t \geq 0}$ is an r-contractive C_0-semigroup on a closed subspace S of some L_p-space $L_p(\Omega)$ ($p \in (1, \infty) \setminus \{2\}$). Then there exists a $C \geq 0$ such that for all $f \in S$ the semigroup $(T(t))_{t \geq 0}$ satisfies the maximal ergodic inequality*

$$\left\| \sup_{t>0} \left| \frac{1}{t} \int_0^t T(s) f \, ds \right| \right\| \leq C \, \|f\|,$$

where the supremum is formed in the pointwise sense.

Proof. It follows as a special case from Theorem 4.2.18 that for all $n \in \mathbb{N}$, $t_1, \ldots, t_n > 0$ and all $f \in S$ one has

$$\left\| \sup_{i=1,\ldots,n} \left| \frac{1}{t_i} \int_0^{t_i} T(s)f \, ds \right| \right\| \le C \|f\|$$

for some constant $C \ge 0$. It then follows from the monotone convergence theorem that

$$\left\| \sup_{t \in \mathbb{Q}_+} \left| \frac{1}{t} \int_0^{t} T(s)f \, ds \right| \right\| \le C \|f\|$$

for all $f \in S$. Taking suitable representatives, one sees that the expression $x \mapsto \frac{1}{t} \int_0^t (T(s)f)(x) \, ds$ exists and is continuous in x except for a set of measure zero (see the detailed discussion in [DS58, p. 686f.] and the proof of Corollary 4.2.20 for a similar argument). This shows that

$$\left\| \sup_{t>0} \left| \frac{1}{t} \int_0^{t} T(s)f \, ds \right| \right\| = \left\| \sup_{t \in \mathbb{Q}_+} \left| \frac{1}{t} \int_0^{t} T(s)f \, ds \right| \right\| \le C \|f\|. \qquad \square$$

The maximal ergodic inequality is the key ingredient to prove pointwise almost everywhere convergence of the Cesàro means of the semigroup. We use a continuous variant of the argument in [Kre85, Section 5.2, Lemma 2.1] for which we do not know any explicit reference in the literature. The maximal ergodic inequality is known to experts in the L_p-case although we could not find any explicit reference. It seems to be new in the subspace case.

Corollary 4.2.20. *Let $(T(t))_{t \ge 0}$ be an r-contractive C_0-semigroup on a closed subspace S of some L_p-space $L_p(\Omega, \Sigma, \mu)$ ($p \in (1, \infty)$). Then for all $f \in S$ the ergodic mean*

$$\frac{1}{t} \int_0^t T(s)f \, ds$$

converges almost everywhere for $t \to \infty$.

Proof. Since S is reflexive, one has $S = \operatorname{Ker}(A) \oplus \overline{\operatorname{Rg}(A)}$, where A denotes the generator of the semigroup $(T(t))_{t \ge 0}$. Now, if $f \in \operatorname{Ker}(A)$, one clearly has $T(t)f = f$ for all $t \ge 0$ and the ergodic mean obviously converges pointwise. Further, if $f \in \operatorname{Rg}(A)$, there is a $g \in S$ with $f = Ag$ and one obtains

$$M(t)f = \frac{1}{t} \int_0^t T(s)f \, ds = \frac{1}{t} \int_0^t T(s)Ag \, ds = \frac{1}{t}(T(t)g - g).$$

For the pointwise convergence almost everywhere it clearly suffices to show that $\frac{1}{t} T(t)g$ converges pointwise almost everywhere. For this notice that we may assume $L_p(\Omega)$ to be σ-finite. Indeed, since the space generated by the functions $T(t)g$ for $t \ge 0$ is separable by the strong continuity of $(T(t))_{t \ge 0}$, [DS58, Lemma III.8.5] shows that this space can be seen as a subspace of an L_p-space over a σ-finite measure space. Now, notice that because of $g \in D(A)$ the map $t \mapsto T(t)g$ lies in $W_p^1(L_p(\Omega))$. By measure theoretic considerations [DS58, Theorem III.11.17] there exists a measurable function $h\colon [1,\infty) \times \Omega \to \mathbb{C}$ uniquely determined up to a set of product measure zero such that $h(t,\cdot) = T(t)g$ for t almost everywhere. Moreover, except for a set $N \subset \Omega$ of measure zero one has $h(\cdot,x) \in W_p^1([1,\infty))$. In particular, by the Sobolev embedding theorem one has the inclusion $W_p^1([1,\infty)) \hookrightarrow C_0([1,\infty))$ [Bre11, Theorem 8.12]. Hence for all $x \in \Omega \setminus N$, one has $h(t,x) \to 0$ as $t \to \infty$.

Altogether we have shown that the ergodic mean converges pointwise almost everywhere in the dense set $\operatorname{Ker}(A) \oplus \operatorname{Rg}(A)$. We now finish the proof by showing that the subspace of all functions in S for which pointwise convergence almost everywhere holds is closed. This follows from Banach's principle which we now present. Let $f, g \in S$ and $t, s > 0$. Then

$$\begin{aligned} |M(t)f - M(s)f| &\le |M(t)f - M(t)g| + |M(t)g - M(s)g| + |M(s)g - M(s)f| \\ &\le 2\sup_{t\ge 0} |M(t)(f-g)| + |M(t)g - M(s)g|. \end{aligned}$$

Now, if $M(t)g$ converges pointwise almost everywhere for $g \in S$, one has by taking the limes superior in the measurable functions

$$\Delta := \limsup_{t,s\to\infty} |M(t)f - M(s)f| \le 2\sup_{t\ge 0} |M(t)(f-g)|.$$

Now, employing the maximal ergodic inequality shown in Corollary 4.2.19 one has for every $\varepsilon > 0$

$$\begin{aligned} \mu\{\omega : |\Delta|(\omega) > 2\varepsilon\} &\le \mu\Big\{\omega : \sup_{t\ge 0} |M(t)(f-g)| > \varepsilon\Big\} \le \varepsilon^{-p} \|\sup_{t\ge 0} |M(t)(f-g)|\|^p \\ &\le C^p \varepsilon^{-p} \|f-g\|^p. \end{aligned}$$

Now, taking $f_n \in S$ for which $M(t)f_n$ converges almost everywhere and which converge in norm towards f (such a sequence exists because of the density of $\operatorname{Ker}(A) \oplus \operatorname{Rg}(A)$ in S), it follows from the above inequality that $\Delta = 0$ almost everywhere. Hence, except for a set of measure zero, $(M(t)f)(x)$ is a Cauchy sequence and therefore converges. This shows that for all $f \in S$ the ergodic means $M(t)f$ converge for $t \to \infty$ almost everywhere. □

4.2.4 Bounded H^∞-Calculus for r-Contractive Semigroups

We now return to the main goal of this section, namely the study of the H^∞-calculus. In the following we prove on subspaces of L_p-spaces a generalization of Weis' celebrated theorem (see [Wei01b, Remark 4.9c)] and [Wei01a, Section 4d)]) on the boundedness of the H^∞-calculus for r-contractive analytic

C_0-semigroups on L_p-spaces. This theorem was originally proved by Weis for positive contractive semigroups, it is however known among experts that the methods extend to r-contractive semigroups on L_p (see for example [LMX12, Proposition 2.2]). However, it is new that the same approach also works for the subspace case (without the improvement of the angle) as until now the validity of Fendler's dilation result has not been known for the subspace case. Notice that the improvement of the angle to $\omega_{H^\infty}(A) < \frac{\pi}{2}$ in the theorem is crucial as this implies the maximal regularity of A.

Theorem 4.2.21. *Let $-A$ be the generator of an r-contractive C_0-semigroup $(T(t))_{t\geq 0}$ on a closed subspace S of $L_p(\Omega)$ for $p \in (1,\infty)$. Then the negative generator A of $(T(t))_{t\geq 0}$ has a bounded H^∞-calculus with $\omega_{H^\infty}(A) \leq \frac{\pi}{2}$. Moreover, if $S = L_p(\Omega)$ and $(T(t))_{t\geq 0}$ extends to a bounded analytic semigroup, then one has*

$$\omega_{H^\infty}(A) \leq \begin{cases} \frac{\pi}{2}(1-\frac{p}{2}) + \frac{p}{2}\omega(A) & \text{for } p \in (1,2] \\ \frac{\pi}{2}\frac{p-2}{2(p-1)} + \frac{p}{2(p-1)}\omega(A) & \text{for } p \in [2,\infty) \end{cases}.$$

In particular, one has $\omega_{H^\infty}(A) < \frac{\pi}{2}$.

Proof. Since the semigroup $(T(t))_{t\geq 0}$ dilates to a group on some L_p-space by Theorem 4.2.11, it follows from Proposition 1.4.2 that A has a bounded H^∞-calculus with $\omega_{H^\infty}(A) \leq \frac{\pi}{2}$.

Moreover, we have shown in Theorem 4.2.18 that the family $\{M(t) : t \in (0,\infty)\}$, where $M(t) = \frac{1}{t}\int_0^t T(s)\,ds$, satisfies $\mathcal{R}_\infty\{M(t) : t \in (0,\infty)\} < \infty$. Now, assume that $(T(t))_{t\geq 0}$ can be continued to a bounded analytic semigroup on some sector $\Sigma_{\tilde{\delta}}$ for some $\tilde{\delta} \in (0,\frac{\pi}{2})$. Then it follows from Remark 4.2.16 that $\mathcal{R}_p\{M(z) : z \in \Sigma_\delta\} < \infty$ for all $\delta \in (0,\tilde{\delta})$. Choose such a $\delta \in (0,\tilde{\delta})$.

Let us now assume that $S = L_p(\Omega)$ for $p \in (1,2]$ and an underlying σ-finite measure space. Further for $\lambda \in \mathbb{C}$ such that $e^{i\lambda} \in \Sigma_\delta$, $q \in [1,\infty]$ and $n \in \mathbb{N}$, $t_1,\ldots,t_n > 0$ consider the mappings

$$\begin{aligned} N(\lambda)\colon L_p(\Omega;\ell_q^n) &\to L_p(\Omega;\ell_q^n) \\ (f_1,\ldots,f_n) &\mapsto (M(t_1e^{i\lambda})f_1,\ldots,M(t_ne^{i\lambda})f_n). \end{aligned}$$

Then the above arguments show that

1. $\{N(is) : s \in \mathbb{R}\}$ is uniformly bounded in $\mathcal{B}(L_p(\Omega;\ell_\infty^n))$ and that
2. for every $\delta \in (0,\tilde{\delta})$ the set $\{N(\delta + is) : s \in \mathbb{R}\}$ is uniformly bounded in $\mathcal{B}(L_p(\Omega;\ell_p^n))$

Let $\alpha \in (0,1)$ be such that $\frac{1}{2} = \alpha\frac{1}{p} + (1-\alpha)\frac{1}{\infty}$. Then it follows from the abstract Stein interpolation theorem (Theorem A.5.8) that for every $\delta \in (0,\tilde{\delta})$ the family $\{N(\alpha\delta + is) : s \in \mathbb{R}\} = \{N(\frac{p}{2}\delta + is) : s \in \mathbb{R}\}$ is uniformly bounded in

$\mathcal{B}(L_p(\Omega;\ell_2^n))$. Note that the same argument applies if one replaces $e^{i\lambda}$ by $e^{-i\lambda}$ in the definition of $N(\lambda)$. Altogether, by the observation made in Remark 4.2.16, we have therefore shown

$$\mathcal{R}\Big\{M(te^{\pm i\delta p/2}) : t \in \mathbb{R}\Big\} < \infty$$

for every $\delta \in (0,\tilde{\delta})$. It follows form Proposition 1.2.3(a) that the set $\{M(z) : z \in \Sigma_{\delta\frac{p}{2}}\}$ is $\mathcal{R}$-bounded. Further, one has $zM'(z) = T(z) - M(z)$. Since $\{zM'(z) : z \in \Sigma_{\delta\frac{p}{2}}\}$ is also $\mathcal{R}$-bounded for all $\delta \in (0,\delta')$ by Proposition 1.2.3(a), the $\mathcal{R}$-boundedness of the set $\{T(z) : z \in \Sigma_{\frac{p}{2}\delta}\}$ follows. This shows that $\omega_R(A) \le \frac{\pi}{2}(1 - \frac{p}{2}) + \frac{p}{2}\omega(A)$. If $p > 2$, it follows by a duality argument from Proposition 1.2.3(b) and the first part of this proof that $\omega_R(A) \le \frac{\pi}{2}(1-\frac{q}{2}) + \frac{q}{2}\omega(A)$, where $\frac{1}{p} + \frac{1}{q} = 1$, or equivalently $q = \frac{p}{p-1}$. If the underlying measure space of $L_p(\Omega)$ is not σ-finite, one can replace $L_p(\Omega)$ by the closed separable sublattice generated by fixed elements $f_1,\ldots,f_n$ and their images under $T(t)$ for $t \ge 0$ which is order isometric to a σ-finite L_p-space. Therefore the first part of the proof applies and we obtain the same uniform estimate on $\omega_R(A)$. The assertion now follows from $\omega_R(A) = \omega_{H^\infty}(A)$ (Lemma 1.3.4). □

Remark 4.2.22. The result in the subspace case is due to C. Le Merdy and A. Simard [LMS01, Corollary 3.2]. Their original proof uses a variant of the transference principle. In our proof, however, we construct directly a dilation in the subspace case which is a stronger property. Although one obtains the $\mathcal{R}_\infty$-boundedness in the subspace case as well (Theorem 4.2.18), we do not know whether a similar interpolation argument as in the proof of Theorem 4.2.21, even in the case $p \in (1,2)$, can be applied. Therefore the question whether every negative generator of an r-contractive analytic C_0-semigroup on a closed subspace of L_p for $p \in (1,\infty) \setminus \{2\}$ has a bounded H^∞-calculus with $\omega_{H^\infty}(A) < \frac{\pi}{2}$ remains open.

4.3 Basic Persistence Properties of the H^∞-Calculus

In this section we present some fundamental operations under which the boundedness of the H^∞-calculus of the negative generator of a C_0-semigroup is preserved. Hence, these operations can be used to construct systematically further classes of sectorial operators with bounded H^∞-calculus, ultimately leading to the structural characterizations of the H^∞-calculus on certain spaces that will be proved in Chapter 5.

The following technical lemma will be useful for our studies.

Lemma 4.3.1. *Let $(T(t))_{t\ge0}$ be a C_0-semigroup on a Banach space X with generator $-A$. Then for a closed subspace M of X the following are equivalent.*

(i) *$T(t)M \subset M$ for all $t \geq 0$.*

(ii) *$R(\lambda, A)M \subset M$ for all $\lambda \in \overline{\Sigma_{\omega(A)+\varepsilon}}^c$ for all $\varepsilon > 0$.*

Moreover, if $\omega(A) < \frac{\pi}{2}$, both are equivalent to

(iii) *$T(z)M \subset M$ for all $z \in \Sigma_{\frac{\pi}{2}-\omega(A)-\varepsilon}$ for all $\varepsilon > 0$.*

Proof. (i) $\Rightarrow$ (ii): Recall that for $x \in X$ the resolvent for $\operatorname{Re}\lambda > 0$ is given by

$$R(\lambda, -A)x = \int_0^\infty e^{-\lambda t} T(t)x \, dt.$$

This shows that $x \in M$ implies $R(\lambda, A)x \in M$ for $\operatorname{Re}\lambda > 0$. Since the resolvent mapping $\lambda \mapsto R(\lambda, A)$ is analytic, this implies $R(\lambda, A)M \subset M$ for all $\varepsilon > 0$ and all $\lambda \in \overline{\Sigma_{\omega(A)+\varepsilon}}^c$. Indeed, assume that $y = \lambda R(\lambda, A)x \notin M$ for some $\varepsilon > 0$, some $\lambda \in \overline{\Sigma_{\omega(A)+\varepsilon}}^c$ and some $x \in X$. By the Hahn–Banach theorem there exists an $x^* \in X^*$ that vanishes on M and satisfies $\langle x^*, y\rangle \neq 0$. The identity theorem for analytic functions shows that $\lambda \mapsto \langle x^*, \lambda R(\lambda, A)x\rangle$ vanishes on the whole complement $\overline{\Sigma_{\omega(A)+\varepsilon}}^c$, contradictory to our assumption.

(ii) $\Rightarrow$ (i): This follows from $T(t)x = \lim_{n\to\infty} \left(\frac{n}{t} R\left(\frac{n}{t}, -A\right)\right)^n x$ for all $x \in X$.

(ii) $\Leftrightarrow$ (iii): The non-trivial implication follows from the analyticity of $z \mapsto T(z)$ by the same argument as in the first part of the proof. □

Suppose we have given a C_0-semigroup $(T(t))_{t\geq 0}$ on some Banach space X together with invariant closed subspaces $N \subset M \subset X$. It is then easy to show that the induced semigroups $(T_{|M}(t))_{t\geq 0}$ and $(T_{M/N}(t))_{t\geq 0}$ are C_0-semigroups on M respectively M/N. Let $-A_M$ and $-A_{M/N}$ denote their generators. We now show that the boundedness of the H^∞-calculus passes as well to subspace-quotients.

Lemma 4.3.2. *Let $-A$ be the generator of a bounded C_0-semigroup $(T(t))_{t\geq 0}$ on some Banach space X such that A has a bounded $H^\infty(\Sigma_\theta)$-calculus. Suppose there exist closed subspaces $N \subset M \subset X$ which are invariant under $(T(t))_{t\geq 0}$. Then A_M on M and $A_{M/N}$ on M/N have a bounded $H^\infty(\Sigma_\theta)$-calculus.*

Proof. Lemma 4.3.1 shows that M and N are invariant under the resolvent $R(\lambda, A)$ as well. This shows that the H^∞-functional calculus naturally restricts to subspaces. It remains to show that it also passes to quotients. Again, Lemma 4.3.1 shows that the functional calculus homomorphism factorizes through M/N by the universal property of quotients. More concretely, for $f \in H_0^\infty(\Sigma_\theta)$ and $y \in M/N$ the value of $f(A)y$ is independent of the representant and therefore well-defined. For $\varepsilon > 0$ choose an $x \in y + N$ such that $\|y\| \leq (1+\varepsilon)\|x\|$. We have

$$\|f(A)x\|_X \leq C\|f\|_{H^\infty(\Sigma_\theta)} \|x\|_X \leq (1+\varepsilon)C\|f\|_{H^\infty(\Sigma_\theta)} \|y\|_{M/N}.$$

Hence, $\|f(A)y\| \le (1+\varepsilon)C\|f\|_{H^\infty(\Sigma_\theta)}\|y\|_{M/N}$. Since $\varepsilon > 0$ is arbitrary, this shows $\|f(A)y\| \le C\|f\|_{H^\infty(\Sigma_\theta)}\|y\|_{M/N}$. □

Besides subspace-quotients similarity transforms are a second class of elementary operations which preserve the boundedness of the H^∞-calculus.

Lemma 4.3.3. *Let X, Y be Banach spaces and $-A$ the generator of a bounded C_0-semigroup $(T(t))_{t\ge 0}$ on X such that A has a bounded H^∞-calculus. If $S \in \mathcal{B}(Y, X)$ is an isomorphism and $-B$ generates the induced C_0-semigroup $(S^{-1}T(t)S)_{t\ge 0}$ on Y, then B has a bounded H^∞-calculus with $\omega_{H^\infty}(A) = \omega_{H^\infty}(B)$.*

Proof. Clearly, $\omega(A) = \omega(B)$. Further one has

$$\lambda R(\lambda, -B) = S^{-1}\left(\int_0^\infty e^{-\lambda t}T(t)\,dt\right)S = S^{-1}\lambda R(\lambda, -A)S$$

for all $\lambda > 0$ and therefore for all $\lambda \in \overline{\Sigma_{\omega(A)+\varepsilon}}^c$ and all $\varepsilon > 0$ by Lemma 4.3.1. Thus $f(B) = S^{-1}f(A)S$ for all $f \in H^\infty(\Sigma_\theta)$ which implies the assertion. □

We summarize the above findings in the following proposition.

Proposition 4.3.4. *The boundedness of an $H^\infty(\Sigma_\theta)$-calculus for a sectorial operator is preserved under similarity transforms and by passing to invariant subspace-quotients.*

4.4 Notes & Open Problems

For some $p \in (1,\infty)\setminus\{2\}$ Theorem 4.2.13 gives a characterization of those semigroups on L_p which have a strict dilation in the class of all L_p-spaces. However, a characterization for the more general case of loose dilations is not known.

Problem 4.4.1. Let $p \in (1,\infty)\setminus\{2\}$. Characterize those bounded semigroups on L_p that have a loose dilation in the class of all L_p-spaces.

For more details on (the discrete analogue of) the problem we refer to [AM14].

The problem of whether one can generalize Fendler's dilation theorem to other classes of Banach spaces is largely open. For example, we do not know whether the following analogue holds on UMD-Banach lattices.

Problem 4.4.2. Let $(T(t))_{t\ge 0}$ be a positive and contractive C_0-semigroup on a UMD-Banach lattice. Does $(T(t))_{t\ge 0}$ have a strict or loose dilation in the class of all UMD-Banach lattices?

This would in particular imply that the negative generator of such a semigroup has a bounded H^∞-calculus, which is also not known. In the negative direction one knows the following: there exists a completely positive contraction, in other terms a discrete semigroup, on a noncommutative L_p-space which does not have a strict dilation in the class of all noncommutative L_p-spaces [JLM07, Corollary 4.4]. Hence, the natural noncommutative analogue of Fendler's dilation theorem does not hold. Furthermore, in [GR88, Contre exemple 6.1] it is shown that there exists a positive contraction T on some $L_p(L_q)$-space for some $p \neq q \in (1,\infty) \setminus \{2\}$ such that there do not exist a positive isometry U on some bigger $L_p(L_q)$-space and a positive contractive projection Q from the bigger onto the smaller $L_p(L_q)$-space such that $TQ = QU$.

Even if such dilations exist, it is not clear how to reduce the angle of the H^∞-calculus as it was done in the proof of Theorem 4.2.21. For example, in the case of r-contractive semigroups on closed subspaces of L_p-spaces for which we constructed dilations in Theorem 4.2.11 we do not know whether we can improve the angle.

Problem 4.4.3. Let $-A$ be the generator of an r-contractive C_0-semigroup on a closed subspace of some L_p-space for $p \in (1,\infty)$. Does then its negative generator A have a bounded H^∞-calculus with $\omega_{H^\infty}(A) < \frac{\pi}{2}$?

For the boundedness of the H^∞-calculus similar questions arise. There may of course be other general methods to establish the boundedness of the H^∞-calculus except for constructing dilations, although none are known at the moment. A particular interesting question is the following.

Problem 4.4.4. Let $-A$ be the generator of a contractive C_0-semigroup on some L_p-space for $p \in (1,\infty)$. Does then A have a bounded H^∞-calculus with $\omega_{H^\infty}(A) \le \frac{\pi}{2}$?

Even in the more general case of uniformly convex Banach spaces no counterexample to this question is known, so one even may ask the following.

Problem 4.4.5. Let $-A$ be the generator of a contractive C_0-semigroup on a uniformly convex Banach space. Does then A have a bounded H^∞-calculus with $\omega_{H^\infty}(A) \le \frac{\pi}{2}$?

Further note that we will show in Theorem 5.5.14 that for a C_0-semigroup $(T(t))_{t\ge 0}$ on a uniformly convex space whose negative infinitesimal generator has a bounded H^∞-calculus of angle smaller than $\frac{\pi}{2}$ there exists an equivalent uniformly convex norm for which the semigroup is contractive (we only prove the result for UMD-spaces but it can be generalized to super-reflexive spaces as discussed in Section 5.6). So a positive answer to this problem would give a (too?) beautiful characterization of the boundedness of the H^∞-calculus on uniformly convex spaces.

A Connection with Matsaev's Conjecture Problem 4.4.4 has an interesting connection with *Matsaev's conjecture*. Let $S \in \mathcal{B}(\ell_p(\mathbb{Z}))$ be the bilateral shift. Then Matsaev's conjecture asks whether every contraction $T \in \mathcal{B}(L_p)$ on some L_p-space for $p \in [1,\infty]$ satisfies for all polynomials $p \in \mathcal{P}$

$$\|p(T)\|_{\mathcal{B}(L_p)} \le \|p(S)\|_{\mathcal{B}(\ell_p(\mathbb{Z}))}.$$

For $p = 1,\infty$ this can be easily verified, whereas for $p = 2$ the inequality reduces to the well-known von Neumann inequality. Recently, for the case $p = 4$ S.W. Drury has found a 2×2 matrix counterexample which relies heavily on the use of computers [Dru11]. Until now, all other cases remain open and no analytical approach to a counterexample is known.

However, it was observed by C. Le Merdy that a negative answer to Problem 4.4.4 for some $p \in (1,\infty)$ would also provide a negative answer to Matsaev's conjecture for the same p [LM99b, p. 33].

5

Generic Classes for Bounded H^∞-Calculus

We have seen in the previous chapter that the negative generator A of a bounded analytic semigroup on L_p for some $p \in (1,\infty)$ that is positive and contractive on the real line has a bounded H^∞-calculus with $\omega_{H^\infty}(A) < \frac{\pi}{2}$ (Theorem 4.2.21). The aim of this chapter is to show a converse to this assertion, namely that each semigroup on L_p for $p \in (1,\infty)$ whose negative generator B has a bounded H^∞-calculus with $\omega_{H^\infty}(B) < \frac{\pi}{2}$ can be obtained from a positive contractive analytic semigroup on some L_p-space after

(a) restricting to invariant subspaces,

(b) factoring through invariant subspaces and

(c) taking similarity transforms,

which all preserve the boundedness of the H^∞-calculus by Proposition 4.3.4. Together with Theorem 4.2.21 this new result gives an L_p-space generalization of the following characterization by C. Le Merdy [LM98, Theorem 4.3].

Theorem 5.0.6. *Let $-A$ be the generator of a bounded analytic C_0-semigroup $(T(z))$ on some Hilbert space H. Then A has a bounded H^∞-calculus if and only if $(T(t))_{t\geq 0}$ is similar to a contraction semigroup, i.e. there exists an invertible $S \in \mathcal{B}(H)$ such that*

$$\|S^{-1}T(t)S\| \leq 1 \qquad \text{for all } t \geq 0.$$

In other words, one may say that the class of all positive contractive analytic semigroups on L_p is generic for those semigroups whose negative generators have a bounded H^∞-calculus for some angle strictly smaller than $\frac{\pi}{2}$. This gives a very satisfying description of the structure of the H^∞-calculus on L_p-spaces. In fact, the genericity result can be established for a broader class of Banach lattices and has an analogue for UMD-Banach spaces.

The proof is based on results and techniques from the theory of p-matrix normed spaces, a generalization of the theory of operator spaces which is introduced in Appendix B. We will develop this theory along the way in the depth necessary for our purposes. The presentation and the content of this chapter is closely based on the accepted manuscript [Faca].

5.1 p-Completely Bounded Maps and p-Matrix Normed Spaces

In this section we present the necessary operator space theoretic background on p-completely bounded maps and p-matrix normed spaces. As we do not want to delve deeply in this branch of functional analysis, we give ad-hoc definitions only introducing the concepts necessary to understand our approach following the terminology in [LM96]. See also Appendix B for the closely related case of classical operator spaces.

Let E, F be two Banach spaces which are embedded into the algebras of bounded operators $\mathcal{B}(X)$ and $\mathcal{B}(Y)$ of two Banach spaces X and Y. A linear map $u\colon E \to F$ induces (for $n \in \mathbb{N}$) a linear map

$$\begin{aligned} u_n\colon M_n(\mathcal{B}(X)) \supset M_n(E) &\to M_n(F) \subset M_n(\mathcal{B}(Y)) \\ [a_{ij}] &\mapsto [u(a_{ij})] \end{aligned}$$

between the matrix algebras. For a fixed $p \in [1,\infty]$ we identify the algebra $M_n(\mathcal{B}(X))$ with $\mathcal{B}(\ell_p^n(X))$. For $p < \infty$ the norm of a matrix element $[a_{ij}] \in M_n(E)$ is then given as

$$\left\| [a_{ij}] \right\|_{M_n(E)}^p = \sup \left\{ \sum_{i=1}^n \left\| \sum_{j=1}^n a_{ij}(x_j) \right\|^p : \sum_{j=1}^n \|x_j\|_X^p \le 1 \right\}.$$

Definition 5.1.1 (p-Complete Boundedness). A map $u\colon E \to F$ as above is *p-completely bounded* ($p \in [1,\infty]$) if the induced maps $u_n\colon M_n(E) \to M_n(F)$ seen as linear maps between $\mathcal{B}(\ell_p^n(X))$ and $\mathcal{B}(\ell_p^n(Y))$ are uniformly bounded in $n \in \mathbb{N}$.

Notice that the p-complete boundedness depends on the choice of the embeddings $E \hookrightarrow \mathcal{B}(X)$ and $F \hookrightarrow \mathcal{B}(Y)$. We use the following terminology.

Definition 5.1.2 (p-Matrix Normed Spaces). We call the datum of a Banach space E with an embedding into $\mathcal{B}(X)$ (and the identification of $M_n(\mathcal{B}(X))$ with $\mathcal{B}(\ell_p^n(X))$) for all $n \in \mathbb{N}$ a *p-matrix normed space structure* for E.

We will always consider $\mathcal{B}(Z)$ for a Banach space Z with its natural p-matrix normed space structure given by the embedding $\iota\colon \mathcal{B}(Z) \to \mathcal{B}(Z)$.

5.2 The p-Matrix Normed Space Structure for $H^\infty(\Sigma_\theta)$

In this section we introduce two related p-matrix normed space structures for the algebra $H^\infty(\Sigma_\theta)$ which will later be used for the study of the H^∞-calculus.

Let Y be a Banach space and let $V(t)g(s) = g(s-t)$ be the right shift group on $L_p(\mathbb{R};Y)$ for $p \in (1,\infty)$ and B its negative infinitesimal generator. Then for $\theta > \frac{\pi}{2}$ one has $f(B)g = b * g$ for $f \in H_0^\infty(\Sigma_\theta)$ (see Example 1.3.8), where b is the unique element in $L_1(\mathbb{R}_+)$ such that $f = \mathcal{L}b$, the Laplace transform of b. We have seen in Example 1.2.12 that the Banach space valued variant of the Mikhlin multiplier theorem shows that B has a bounded H^∞-calculus with $\omega_{H^\infty}(B) = \frac{\pi}{2}$ if Y is a UMD-space, which we from now on assume.

Using the boundedness of the $H^\infty(\Sigma_\theta)$-calculus of B on $L_p(\mathbb{R};Y)$ for each $\theta > \frac{\pi}{2}$ one can define the following embedding of $H^\infty(\Sigma_{\frac{\pi}{2}+}) := \bigcup_{\theta>\frac{\pi}{2}} H^\infty(\Sigma_\theta)$ (as vector spaces)

$$\begin{aligned} H^\infty(\Sigma_{\frac{\pi}{2}+}) &\hookrightarrow \mathcal{B}(L_p(\mathbb{R};Y)) \\ f &\mapsto f(B). \end{aligned} \tag{p-MNS1}$$

The above map is indeed injective: First let $f_1(B) = f_2(B)$ with $f_i \in H_0^\infty(\Sigma_\theta)$ for some $\theta > \frac{\pi}{2}$. This implies for the inverse Laplace transforms b_i of f_i that $b_1 * g = b_2 * g$ for all $g \in L_p(\mathbb{R};Y)$ and therefore $b_1 = b_2$, which yields $f_1 = f_2$. For the general case we use the fact that $H_0^\infty(\Sigma_{\frac{\pi}{2}+})$ is an ideal in $H^\infty(\Sigma_{\frac{\pi}{2}+})$: $f_1(B) = f_2(B)$ implies $(f_1\rho)(B) = (f_2\rho)(B)$, where $\rho(\lambda) = \frac{\lambda}{(1+\lambda)^2}$, which in turn shows $f_1\rho = f_2\rho$ and therefore $f_1 = f_2$.

We endow $H^\infty(\Sigma_{\frac{\pi}{2}+})$ with the norm induced from $\mathcal{B}(L_p(\mathbb{R};Y))$ when seen as a subspace via the above embedding. Notice that this also gives $H^\infty(\Sigma_{\frac{\pi}{2}+})$ the structure of a p-matrix normed space. We will call this p-matrix normed space structure the *p-matrix normed space structure* (p-MNS1) *with respect to* Y.

The above choice of a matrix normed space structure is natural in view of transference techniques, but has the disadvantage that it does not make use of the angle ω_{H^∞} and therefore loses information on the strength of the functional calculus. We will now solve this issue by using the fractional powers of B to define refined versions of the above embedding.

In the setting as above choose α with $\alpha \in [1,\infty)$. Then $B^{1/\alpha}$ has a bounded $H^\infty(\Sigma_{\frac{\pi}{2\alpha}+})$-calculus by Proposition 1.3.13. Now consider the embedding

$$\begin{aligned} H^\infty(\Sigma_{\frac{\pi}{2\alpha}+}) &\hookrightarrow \mathcal{B}(L_p(\mathbb{R};Y)) \\ f &\mapsto f(B^{\frac{1}{\alpha}}). \end{aligned} \tag{p-MNS2}$$

Notice that one has $f(B^{\frac{1}{\alpha}}) = (f \circ \cdot^{1/\alpha})(B)$ by the composition formula of the functional calculus and that for $\theta \in (0,\pi)$ one has an isomorphism

$$\begin{aligned} H^\infty(\Sigma_\theta) &\to H^\infty(\Sigma_{\theta/\alpha}) \\ f &\mapsto [\lambda \mapsto f(\lambda^\alpha)]. \end{aligned}$$

Therefore $f \circ \cdot^{1/\alpha} \in H^\infty(\Sigma_{\frac{\pi}{2}+})$ and injectivity follows from the case $\alpha = 1$ considered above. This again endows $H^\infty(\Sigma_{\frac{\pi}{2\alpha}+})$ with the structure of a p-matrix normed space. We will call this p-matrix normed space structure the *p-matrix normed space structure* (p-MNS2) *with respect to Y* (this structure depends on α although not explicitly mentioned).

We want to point out that the norms on H^∞ induced by the p-matrix normed space structures (p-MNS1) and (p-MNS2) respectively do not agree with the usual norms on these algebras as introduced in Definition 1.3.1. We will always consider the spaces $H^\infty(\theta)$ with these non-standard norms.

5.3 A p-Completely Bounded H^∞-Calculus

In this section we show that the H^∞-calculus of a sectorial operator is p-completely bounded for the p-matrix normed space structures just defined. The proof uses vector-valued transference techniques and therefore we start with the case of a bounded group. We need the following vector-valued generalization of the transference result stated in Theorem 1.3.9 which goes back to [LM99a, Theorem 4.1] (compare also with the order-theoretic variant shown in Theorem 4.2.17) and for which we give a new proof. Here $(V(t))_{t\in\mathbb{R}}$ is the shift group defined by $V(t)g(s) = g(s-t)$ over scalar-valued functions.

Theorem 5.3.1 (Vector-Valued Transference Principle). *Let $(U(t))_{t\in\mathbb{R}}$ be a bounded C_0-group on a Banach space X with $M = \sup_{t\in\mathbb{R}} \|U(t)\| < \infty$. Then for all $p \in [1,\infty)$ and all $b \in L_1(\mathbb{R};\mathcal{B}(X))$ for which for almost all $t \in \mathbb{R}$ the operator $b(t)$ commutes with $U(s)$ for all $s \in \mathbb{R}$ one has*

$$\left\| \int_{\mathbb{R}} b(t)U(t)\,dt \right\|_{\mathcal{B}(X)} \le M^2 \left\| \int_{\mathbb{R}} V(t) \otimes b(t)\,dt \right\|_{\mathcal{B}(L_p(\mathbb{R};X))}.$$

Proof. We first assume that b is supported on some compact subset $C \subset \mathbb{R}$. Note that for $x \in X$ the assumption $\|U(-t)x\| \le M\|x\|$ implies $\|x\| \le M\|U(t)x\|$ for all $t \in \mathbb{R}$. By averaging over an arbitrary compact subset $K \subset \mathbb{R}$ with positive measure we obtain

$$\begin{aligned} \left\| \int_{\mathbb{R}} b(t)U(t)x\,dt \right\|_X^p &= \frac{1}{|K|} \int_K \left\| \int_{\mathbb{R}} b(t)U(t)x\,dt \right\|_X^p ds \\ &\le \frac{M^p}{|K|} \int_K \left\| U(s) \int_{\mathbb{R}} b(t)U(t)x\,dt \right\|_X^p ds \end{aligned}$$

By assumption, we have $U(s)b(t) = b(t)U(s)$ for almost everywhere. Hence, the right hand side equals

$$\frac{M^p}{|K|} \int_K \left\| \int_{\mathbb{R}} b(t)U(s+t)x\,dt \right\|_X^p ds \le \frac{M^p}{|K|} \int_{\mathbb{R}} \left\| \int_{\mathbb{R}} b(t)U(s+t)x \mathbb{1}_{K+C}(s+t)\,dt \right\|_X^p ds$$

Note that the integrand on the right hand side is $(V(-t)\otimes b(t))(U(\cdot)x\mathbb{1}_{KC}(\cdot))(s)$. Hence, the right hand side can also be written as

$$\begin{aligned}
&\frac{M^p}{|K|}\left\|\left(\int_{\mathbb{R}} V(-t)\otimes b(t)\,dt\right)(U(\cdot)x\mathbb{1}_{K+C}(\cdot))\,dt\right\|^p_{L_p(\mathbb{R};X)} \\
&\quad\le \frac{M^p}{|K|}\left\|\int_{\mathbb{R}} V(-t)\otimes b(t)\,dt\right\|^p_{\mathcal{B}(L_p(\mathbb{R};X))}\|U(\cdot)x\mathbb{1}_{K+C}(\cdot)\|^p_{L_p(\mathbb{R};X)} \\
&\quad\le \frac{M^{2p}}{|K|}\|x\|^p_X\int_{\mathbb{R}}|\mathbb{1}_{K+C}(t)|\,dt\cdot\left\|\int_{\mathbb{R}} V(-t)\otimes b(t)\,dt\right\|^p_{\mathcal{B}(L_p(\mathbb{R};X))} \\
&\quad= \frac{|K+C|}{|K|}M^{2p}\|x\|^p_X\left\|\int_{\mathbb{R}} V(-t)\otimes b(t)\,dt\right\|^p_{\mathcal{B}(L_p(\mathbb{R};X))}.
\end{aligned}$$

Since we can choose for K arbitrarily large intervals K, we obtain in the limit

$$\left\|\int_{\mathbb{R}} b(t)U(t)x\,dt\right\|_X \le M^2\left\|\int_{\mathbb{R}} V(-t)\otimes b(t)\,dt\right\|_{\mathcal{B}(L_p(\mathbb{R};X))}\|x\|_X$$

as desired. Note that as in the proof of Theorem 4.2.17 one sees by change of variables that

$$\left\|\int_{\mathbb{R}} V(-t)\otimes b(t)\,dt\right\|_{\mathcal{B}(L_p(\mathbb{R};X))} = \left\|\int_{\mathbb{R}} V(t)\otimes b(t)\,dt\right\|_{\mathcal{B}(L_p(\mathbb{R};X))}.$$

Further, a general $b\in L_1(\mathbb{R};\mathcal{B}(X))$ as in the assertion can be approximated by functions of the form $b\mathbb{1}_C$ for $C\subset\mathbb{R}$ compact for which the above estimate holds. The general assertion then follows by taking limits. □

Recall that by Corollary 1.3.10 the negative generator of a bounded group on a UMD-space always has a bounded $H^\infty(\Sigma_\theta)$-calculus for all $\theta>\frac{\pi}{2}$. Further, notice that in the next two propositions we consider $H^\infty_0(\Sigma_{\frac{\pi}{2}+})$ with the norm induced by the p-matrix normed space structure (p-MNS1).

Proposition 5.3.2. *Let $-C$ be the generator of a bounded C_0-group $(U(t))_{t\in\mathbb{R}}$ on a subspace-quotient SQ_X of a UMD-space X. Then the H^∞-calculus homomorphism*

$$u\colon H^\infty_0(\Sigma_{\frac{\pi}{2}+})\to\mathcal{B}(SQ_X)$$

is p-completely bounded for $p\in(1,\infty)$ and the p-matrix normed space structure (p-MNS1) *with respect to X.*

Proof. Let us first assume that the subspace-quotient is X itself. Further let $M:=\sup_{t\in\mathbb{R}}\|U(t)\|$. One has for $[f_{ij}]\in M_n(H^\infty_0(\Sigma_{\frac{\pi}{2}+}))$, $n\in\mathbb{N}$ and the inverse Laplace transforms $[b_{ij}]\in M_n(L_1(\mathbb{R}_+))$ determined by $f_{ij}=\mathcal{L}b_{ij}$

$$\left\|[f_{ij}(C)]\right\|_{M_n(\mathcal{B}(X))} = \left\|[f_{ij}(C)]\right\|_{\mathcal{B}(\ell^n_p(X))} = \left\|\left[\int_0^\infty b_{ij}(t)U(t)\,dt\right]\right\|_{\mathcal{B}(\ell^n_p(X))}$$

$$= \left\| \int_0^\infty ([b_{ij}(t)] \otimes \mathrm{Id}) \mathcal{U}(t)\, dt \right\|_{\mathcal{B}(\ell_p^n(X))},$$

where $\mathcal{U}(t)$ is the diagonal operator $\operatorname{diag}(U(t), \ldots, U(t))$ in $\mathcal{B}(\ell_p^n(X))$. Since the matrix $[b_{ij}(s)] = [b_{ij}(s)\mathrm{Id}]$ commutes with the group $(\mathcal{U}(t))_{t \in \mathbb{R}}$, one has by the vector-valued transference principle (Theorem 5.3.1)

$$\left\| [f_{ij}(C)] \right\|_{M_n(\mathcal{B}(X))} \le M^2 \left\| \int_0^\infty V(t) \otimes [b_{ij}(t)] \otimes \mathrm{Id}\, dt \right\|_{\mathcal{B}(L_p(\mathbb{R}; \ell_p^n(X)))}.$$

Let $\mathcal{V}(t)$ be the diagonal operator $\operatorname{diag}(V(t), \ldots, V(t))$ on $\ell_p^n(L_p(\mathbb{R}; X))$. Then after interchanging the order of the L_p-spaces we obtain

$$\begin{aligned} \left\| [f_{ij}(C)] \right\|_{M_n(\mathcal{B}(X))} &\le M^2 \left\| \int_0^\infty ([b_{ij}(t)] \otimes \mathrm{Id}) \mathcal{V}(t)\, dt \right\|_{\mathcal{B}(\ell_p^n(L_p(\mathbb{R};X)))} \\ &= M^2 \left\| \left[\int_0^\infty b_{ij}(t) V(t)\, dt \right] \right\|_{\mathcal{B}(\ell_p^n(L_p(\mathbb{R};X)))} = M^2 \left\| [f_{ij}(B)] \right\|_{M_n(L_p(\mathbb{R};X))}, \end{aligned}$$

which is the p-complete boundedness of u. Now, if SQ_X is a general subspace-quotient of X, one can repeat all the above arguments replacing X by SQ_X. Further, note that the shift $(V(t))_{t \in \mathbb{R}}$ in $L_p(\mathbb{R}; X)$ naturally restricts to the shift $(V_{SQ_X}(t))_{t \in \mathbb{R}}$ on $L_p(\mathbb{R}; SQ_X)$ with negative generator B_{SQ_X}. From this and the definition of the functional calculus it then follows immediately that for all $[f_{ij}] \in M_n(H_0^\infty(\Sigma_{\frac{\pi}{2}+}))$

$$\left\| [f_{ij}(C)] \right\|_{M_n(\mathcal{B}(SQ_X))} \le M^2 \left\| [f_{ij}(B_{SQ_X})] \right\|_{M_n(L_p(\mathbb{R}; SQ_X))} \le M^2 \left\| [f_{ij}(B)] \right\|_{M_n(L_p(\mathbb{R};X))}.$$

□

We now need the following more precise formulation of the Fröhlich–Weis theorem already stated in Theorem 1.4.3 [FW06, Corollary 5.4].

Theorem 5.3.3. *Let $-A$ be the generator of a bounded analytic C_0-semigroup $(T(t))_{t \ge 0}$ on a UMD-space X such that A has a bounded H^∞-calculus with $\omega_{H^\infty}(A) < \frac{\pi}{2}$. Then for all $p \in (1, \infty)$ there exists an isometric embedding $J \colon X \to L_p([0,1]; X)$, a bounded projection $P \colon L_p([0,1]; X) \to L_p([0,1]; X)$ onto* $\operatorname{Rg} J$ *and a bounded C_0-group $(U(t))_{t \in \mathbb{R}}$ on $L_p([0,1]; X)$ such that*

$$JT(t) = PU(t)J \qquad \text{for all } t \ge 0.$$

Let now $-A$ be the generator of a bounded analytic C_0-semigroup $(T(t))_{t \ge 0}$ with sectorial generator $-A$ that has dense range on a subspace-quotient SQ_X of a UMD-space X. If A has a bounded H^∞-calculus with $\omega_{H^\infty}(A) < \frac{\pi}{2}$, the Fröhlich–Weis theorem (Theorem 5.3.3) yields that the semigroup $(T(t))_{t \ge 0}$ dilates to a bounded C_0-group $(U(t))_{t \in \mathbb{R}}$ with generator $-C$ on the UMD-space

$SQ_Y := L_p([0,1];SQ_X)$ which is itself a subspace-quotient of $Y := L_p([0,1];X)$ (one can choose $p \in (1,\infty)$), i.e.

$$JT(t) = PU(t)J,$$

where $J\colon SQ_X \to SQ_Y$ is an isometric embedding and $P\colon SQ_Y \to SQ_Y$ is a bounded projection onto $\operatorname{Im}(J)$. Then by Proposition 5.3.2 one has for $M := \sup_{t\in\mathbb{R}} \|U(t)\|$

$$\begin{aligned}\left\|[f_{ij}(A)]\right\|_{M_n(\mathcal{B}(SQ_X))} &= \left\|[Jf_{ij}(A)]\right\|_{M_n(\mathcal{B}(SQ_X,SQ_Y))}\\ &= \left\|[Pf_{ij}(C)J]\right\|_{M_n(\mathcal{B}(SQ_X,SQ_Y))} \le \|P\| \left\|[f_{ij}(C)]\right\|_{M_n(\mathcal{B}(SQ_Y))}\\ &\le \|P\| M^2 \left\|[f_{ij}(B)]\right\|_{M_n(L_p(\mathbb{R};Y))},\end{aligned}$$

where $-B$ is the generator of the right shift group on $L_p(\mathbb{R};Y)$. We have shown the following proposition.

Proposition 5.3.4. *Let A be a sectorial operator with dense range on a subspace-quotient SQ_X of a UMD-space X with a bounded $H^\infty(\Sigma_\theta)$-calculus for some $\theta < \frac{\pi}{2}$. Then the H^∞-calculus homomorphism*

$$u\colon H_0^\infty(\Sigma_{\frac{\pi}{2}+}) \to \mathcal{B}(SQ_X)$$

is p-completely bounded for $p \in (1,\infty)$ and the p-matrix normed space structure (p-MNS1) *with respect to* $L_p([0,1];X)$.

Using the p-matrix normed space structure (p-MNS2) instead, we obtain our main result of this section. In what follows, we consider $H_0^\infty(\Sigma_{\frac{\pi}{2\alpha}+})$ with the norm induced by (p-MNS2).

Theorem 5.3.5. *Let A be a sectorial operator with dense range on a subspace-quotient SQ_X of a UMD-space X with a bounded $H^\infty(\Sigma_\theta)$-calculus for some $\theta < \frac{\pi}{2}$. Then for $\alpha \in [1, \frac{\pi}{2\theta})$ the H^∞-calculus homomorphism*

$$u : H_0^\infty(\Sigma_{\frac{\pi}{2\alpha}+}) \to \mathcal{B}(SQ_X)$$

is p-completely bounded for $p \in (1,\infty)$ and the p-matrix normed space structure (p-MNS2) *with respect to* $L_p([0,1];X)$.

Proof. Let $\alpha \in [1, \frac{\pi}{2\theta})$. Then A^α has a bounded $H^\infty(\Sigma_{\alpha\theta})$-calculus with $\alpha\theta < \frac{\pi}{2}$ by Proposition 1.3.13. We apply Proposition 5.3.4 to A^α and obtain that for $C = C(\alpha) > 0$

$$\left\|[\tilde{f}_{ij}(A^\alpha)]\right\| \le C \left\|[\tilde{f}_{ij}(B)]\right\|$$

for all $[\tilde{f}_{ij}] \in M_n(H_0^\infty(\Sigma_{\frac{\pi}{2}+}))$. Now let $[f_{ij}] \in M_n(H_0^\infty(\Sigma_{\frac{\pi}{2\alpha}+}))$. Then there exist $\tilde{f}_{ij} \in H_0^\infty(\Sigma_{\frac{\pi}{2}+})$ such that $\tilde{f}_{ij}(\lambda^\alpha) = f_{ij}(\lambda)$. Now,

$$\left\|[f_{ij}(A)]\right\| = \left\|[(\tilde{f}_{ij} \circ \cdot^\alpha)(A)]\right\| = \left\|[\tilde{f}_{ij}(A^\alpha)]\right\| \le C \left\|[\tilde{f}_{ij}(B)]\right\| = C\|[f_{ij}(B^{\frac{1}{\alpha}})]\|. \square$$

The p-complete boundedness of the functional calculus homomorphism extends from $H_0^\infty(\Sigma_{\frac{\pi}{2\alpha}+})$ to $H^\infty(\Sigma_{\frac{\pi}{2\alpha}+})$.

Corollary 5.3.6. *Let A be a sectorial operator with dense range on a subspace-quotient SQ_X of a UMD-space X with a bounded $H^\infty(\Sigma_\theta)$-calculus for some $\theta < \frac{\pi}{2}$. Then for $\alpha \in [1, \frac{\pi}{2\theta})$ the H^∞-calculus homomorphism*

$$u\colon H^\infty(\Sigma_{\frac{\pi}{2\alpha}+}) \to \mathcal{B}(SQ_X)$$

is p-completely bounded for $p \in (1,\infty)$ and the p-matrix normed space structure (p-MNS2) *with respect to $L_p([0,1];X)$.*

Proof. Let $Y := L_p([0,1];X)$. The case of general $[f_{ij}] \in M_n(H^\infty(\Sigma_{\frac{\pi}{2\alpha}+}))$ follows from Theorem 5.3.5 by the following approximation argument. One has $[\rho_k f_{ij}] \in M_n(H_0^\infty(\Sigma_{\frac{\pi}{2\alpha}+}))$ for all $k \in \mathbb{N}$ (remarks after Definition 1.3.1) and therefore obtains for some constant $C \geq 0$ that for all $n \in \mathbb{N}$ and all $x_1,\dots,x_n \in X$

$$\begin{aligned}
\sum_{i=1}^n \left\| \sum_{j=1}^n f_{ij}(A)x_j \right\|^p &= \lim_{k\to\infty} \sum_{i=1}^n \left\| \sum_{j=1}^n (\rho_k f_{ij})(A)x_j \right\|^p \\
&\leq C^p \sum_{i=1}^n \|x_i\|^p \liminf_{k\to\infty} \left\| [\rho_k(B^{1/\alpha}) f_{ij}(B^{1/\alpha})] \right\|^p_{M_n(\mathcal{B}(L_p(\mathbb{R};Y)))} \\
&\leq C^p \sup_{k\in\mathbb{N}} \left\| \rho_k(B^{1/\alpha}) \right\|^p \left\| [f_{ij}(B^{1/\alpha})] \right\|^p_{M_n(\mathcal{B}(L_p(\mathbb{R};Y)))} \sum_{i=1}^n \|x_i\|^p .
\end{aligned}$$

Since $\sup_{k\in\mathbb{N}} \left\|\rho_k(B^{1/\alpha})\right\| < \infty$, we showed $\left\|[f_{ij}(A)]\right\| \leq \tilde{C} \left\|[f_{ij}(B^{1/\alpha})]\right\|$ for some constant $\tilde{C} > 0$. $\square$

5.4 Pisier's Factorization Theorem Applied to the Functional Calculus

We now apply Pisier's factorization theorem for p-completely bounded maps to the homomorphism obtained from the bounded H^∞-calculus. We start with some terminology that is needed for the formulation of Pisier's factorization result.

Let X be a Banach space and $Y = L_p([0,1];X)$ for some $p \in (1,\infty)$. Further let $(\Omega_j, \mu_j)_{j\in J}$ be a family of measure spaces, $\mathcal{U}$ an ultrafilter on J and $\alpha \in [1,\infty)$. Then for each $j \in J$ and $f \in H^\infty(\Sigma_{\frac{\pi}{2\alpha}+})$ using the notation from the last section we have canonical maps

$$\begin{aligned}
\pi_j(f)\colon L_p(\Omega_j; L_p(\mathbb{R};Y)) &\to L_p(\Omega_j; L_p(\mathbb{R};Y)) \\
g &\mapsto [\omega \mapsto f(B^{1/\alpha})(g(\omega))].
\end{aligned}$$

These mappings induce for all $\varepsilon > 0$ and all $f \in H^\infty(\Sigma_{\frac{\pi}{2\alpha}+\varepsilon})$ a map $\pi(f) \in \mathcal{B}(\hat{X})$ in the ultraproduct $\hat{X} := \prod_{j\in J} L_p(\Omega_j; L_p(\mathbb{R}; Y))/\mathcal{U}$ that satisfies $\|\pi(f)\| \le \sup_{j\in J}\|\pi_j(f)\| \le \|f(B^{1/\alpha})\| \le C_\varepsilon \|f\|_{H^\infty(\Sigma_{\frac{\pi}{2\alpha}+\varepsilon})}$ by the boundedness of the H^∞-calculus. More generally, each $T \in \mathcal{B}(L_p(\mathbb{R}; Y))$ induces a map on the ultraproduct $\hat{X}$ with operator norm at most $\|T\|$. Notice that this in particular implies that an analytic mapping $z \mapsto T_z$ in $\mathcal{B}(L_p(\mathbb{R}; Y))$ induces an analytic mapping in $\hat{X}$.

We now formulate a special case of Pisier's factorization theorem [Pis90].

Theorem 5.4.1 (Pisier's Factorization Theorem for Completely Bounded Mappings). *Let Z, X be Banach spaces and $\mathcal{A} \subset \mathcal{B}(Z)$ a unital subalgebra and $u\colon \mathcal{A} \to \mathcal{B}(X)$ a p-completely bounded unital algebra homomorphism for some $p \in (1,\infty)$. Then there exists a family of measure spaces $(\Omega_j, \mu_j)_{j\in J}$ and an ultrafilter $\mathcal{U}$ on J such that for the ultraproduct $\hat{X} := \prod_{j\in J} L_p(\Omega_j; Z)/\mathcal{U}$ there are closed subspaces $N \subset M \subset \hat{X}$ and an isomorphism $S\colon X \to M/N$ such that for $a \in \mathcal{A}$ the operators $\pi(a)$ defined as the ultraproducts $\prod_{j\in J} \pi_j(a)/\mathcal{U}$, where*

$$\pi_j(a)\colon L_p(\Omega_j; Z) \to L_p(\Omega_j; Z), \qquad (\pi_j(a)f)(\omega) := a(f(\omega)),$$

satisfy $\pi(a)M \subset M$ and $\pi(a)N \subset N$ for all $a \in \mathcal{A}$ and the induced mappings $\hat{\pi}(a)\colon M/N \to M/N$ satisfy

$$u(a) = S^{-1}\hat{\pi}(a)S \qquad \text{for all } a \in \mathcal{A}.$$

This follows from [Pis90, Theorem 3.2]. In the following we will need some details on the proof of the theorem. By checking the reference, one sees that the theorem is first proved for p-completely bounded maps of the form $u\colon \mathcal{A} \to \mathcal{B}(\ell_1(\Gamma), \ell_\infty(\Gamma'))$ for some sets Γ and Γ', where one can choose $M = \hat{X}$ and $N = 0$ in the assertion of the factorization theorem. In the general case of Theorem 5.4.1 one then chooses a metric surjection $Q\colon \ell_1(\Gamma) \to X$ (i.e. Q^* is an isometric embedding) and an isometric embedding $J\colon X \to \ell_\infty(\Gamma')$ and then deduces the theorem from the special case above applied to the p-completely bounded map $a \mapsto Ju(a)Q$.

In our concrete case of the functional calculus we get the following.

Theorem 5.4.2. *Let A be a sectorial operator with dense range on a subspace-quotient SQ_X of a UMD-space X with a bounded $H^\infty(\Sigma_\theta)$-calculus for some $\theta < \frac{\pi}{2}$. Then for each $\psi \in (\theta, \frac{\pi}{2})$ there exist $(\Omega_j, \mu_j)_{j\in J}$, $\mathcal{U}$ and π as in Theorem 5.4.1 together with subspaces $N \subset M$ of the ultraproduct*

$$\prod_{j\in J} L_p(\Omega_j; L_p(\mathbb{R}; L_p([0,1]; X)))/\mathcal{U}$$

such that

$$\pi(f)M \subset M \quad \text{and} \quad \pi(f)N \subset N \qquad \text{for all } f \in H^\infty(\Sigma_\psi).$$

Moreover, if $\hat{\pi}(f)\colon M/N \to M/N$ denotes the induced mapping, there exists an isomorphism $S\colon SQ_X \to M/N$ such that

$$u(f) = S^{-1}\hat{\pi}(f)S \qquad \text{for all } f \in H^\infty(\Sigma_\psi).$$

Proof. Let $\psi > \theta$. We have shown in Theorem 5.3.6 that for a suitably chosen $\alpha > 1$ the functional calculus homomorphism $u\colon H^\infty(\Sigma_\psi) \to \mathcal{B}(SQ_X)$ is p-completely bounded for the p-matrix normed space structure (p-MNS2) with respect to $L_p([0,1];X)$. We now apply Theorem 5.4.1 to u. ☐

5.5 Properties and Regularization of the Constructed Semigroup

We now study the properties of the semigroup obtained from the application of Pisier's factorization theorem to the functional calculus mapping. Note that, in particular, in Theorem 5.4.2 one has, for $\psi < \frac{\pi}{2}$, that $e_z \in H^\infty(\Sigma_\psi)$ for $z \in \Sigma_{\pi/2-\psi}$ and one obtains $T(z) = u(e_z) = S^{-1}\hat{\pi}(e_z)S$, where $(T(z))_{z\in\Sigma_{\pi/2-\omega(A)}}$ is the analytic semigroup generated by $-A$.

Therefore we are interested in the properties of the constructed semigroup $(\pi(e_z))_{z\in\Sigma_{\frac{\pi}{2}-\psi}}$ which in turn leads us to the study of the semigroup generated by $-B^{1/\alpha}$. Here one has the following general result for fractional powers of semigroup generators as defined before Proposition 1.3.13.

Theorem 5.5.1. *Let $-A$ be the generator of a C_0-semigroup $(T(t))_{t\geq 0}$. Then for $\alpha \in (0,1)$ the bounded analytic C_0-semigroup $(T_\alpha(z))_{z\in\Sigma_{\frac{\pi}{2}-\omega(A^\alpha)}}$ generated by $-A^\alpha$ has the following properties:*

- *One has $\|T_\alpha(t)\| \leq 1$ for all $t \geq 0$ if $\|T(t)\| \leq 1$ for all $t \geq 0$.*

Moreover, if X is a Banach lattice, one has

- *$T_\alpha(t) \geq 0$ if $T(t) \geq 0$.*

Proof. The assertions follow from the explicit representation of the semigroup $(T_\alpha(t))_{t\geq 0}$ [Yos80, IX, 11]

$$T_\alpha(t)x = \int_0^\infty f_{t,\alpha}(s)T(s)x\,ds \qquad \text{for all } t > 0 \text{ and } x \in X,$$

where $f_{t,\alpha}$ is a function with $f_{t,\alpha} \geq 0$ and $\int_0^\infty f_{t,\alpha}(s)\,ds = 1$. ☐

In our case we apply the above theorem to the generator $-B$ of the contractive vector-valued shift semigroups. Moreover, if X and therefore $L_p([0,1];X)$ are Banach lattices, the shift semigroup is positive with respect to the natural

Banach lattice structure on $L_p(\mathbb{R}; L_p([0,1]; X))$. Hence, $(\pi(e_t))_{t\geq 0}$ is a (positive if X is a Banach lattice) contractive semigroup. Here we used the fact that the ultraproduct of positive operators is positive on ultraproducts of Banach lattices (see Appendix A.2.1).

Remark 5.5.2. One can now ask if the analytic semigroup generated by $-B^\alpha$ is contractive on the whole sector where the semigroup is defined. Notice that the Fourier transform diagonalizes B to the multiplication operator with ix. The analytic semigroup $(T_\alpha(z))$ generated by $-B^\alpha$ is then given by the Fourier multipliers $\exp(-z(ix)^\alpha)$ for $|\arg z| < \frac{\pi}{2}(1-\alpha) =: \varphi$. Now, if X is a Hilbert space Plancherel's theorem yields that $(T_\alpha(z))$ is a sectorial contractive analytic semigroup on $L_2(\mathbb{R}; X)$. However, even in the case $X = \mathbb{C}$ one does not have contractivity on the whole sector for $p \in (1,\infty)\setminus\{2\}$. Indeed, $\|T(z)\|$ is even unbounded on $\Sigma_\varphi \cap \{z : |z| \leq 1\}$. For this, by [ABHN11, Proposition 3.9.1], it suffices to show that $-e^{i\varphi}A^\alpha$ does not generate a C_0-semigroup on L_p.

Assume that this would be the case. Then in particular $\exp(-e^{i\varphi}(ix)^\alpha)$ is a bounded Fourier multiplier on L_p. Using the boundedness of the Hilbert transform for $p \in (1,\infty)$ one sees that then

$$m(\xi) := \begin{cases} e^{-i\xi^\alpha} & (\xi \geq 2), \\ 0 & (\xi < 2) \end{cases}$$

is also a bounded Fourier multiplier. However, this contradicts [ABHN11, Theorem E.4b) (i)].

Before giving the proof of the next main proposition, we need some preparatory results before. The first is a similarity theorem which can be proved for arbitrary semigroup representations.

Definition 5.5.3. A *semigroup* $\mathcal{S}$ is a set together with an associative binary operation $\cdot\colon \mathcal{S}\times\mathcal{S} \to \mathcal{S}$ for which there exists an element $e \in \mathcal{S}$, the identity element, such that for every element $g \in \mathcal{S}$ the equation $e\cdot g = g\cdot e = g$ holds.

Note that the above definition differs from the common mathematical terminology. Indeed, a mathematical structure satisfying Definition 5.5.3 is usually called a monoid and for a semigroup one usually does not require the existence of an identity element. We make this abuse of terminology because we want to identify C_0-semigroups with strongly continuous semigroup representations of the additive semigroup $\mathbb{R}_{\geq 0}$.

Definition 5.5.4. Let $\mathcal{S}$ be a semigroup and X a Banach space. A mapping $\pi\colon \mathcal{S} \to \mathcal{B}(X)$ is called a *semigroup representation* on X if π satisfies

(i) $\pi(gh) = \pi(g)\pi(h)$ for all $g, h \in \mathcal{S}$,

(ii) $\pi(e) = \mathrm{Id}_X$.

The similarity theorem uses the following short-hand terminology.

Definition 5.5.5. Let X be a Banach space and $u\colon A \to \mathcal{B}(X)$ an arbitrary mapping for some set A. Suppose that there exists closed subspaces $E_2 \subset E_1$ such that

$$u(a)E_1 \subset E_1 \quad \text{and} \quad u(a)E_2 \subset E_2 \qquad \text{for all } a \in A.$$

Then u induces a map $\tilde{u}\colon A \to \mathcal{B}(E_1/E_2)$ which we call the *compression of u to E_1/E_2*.

The following similarity result is a variant of [Pis01, Proposition 4.2] for semigroup representations on Banach spaces. In fact, one can use the same proof without any modifications.

Proposition 5.5.6. *Let $\mathcal{S}$ be a semigroup and X, Z two Banach spaces. Further let $\pi\colon \mathcal{S} \to \mathcal{B}(Z)$ and $\sigma\colon \mathcal{S} \to \mathcal{B}(X)$ be semigroup representations and let $w_1\colon X \to Z$ and $w_2\colon Z \to X$ be bounded linear operators such that*

$$\sigma(g) = w_2 \pi(g) w_1 \qquad \text{for all } g \in \mathcal{S}.$$

Then σ is similar to a compression of π. More precisely, there are π-invariant closed subspaces $E_2 \subset E_1 \subset Z$ and an isomorphism $S\colon X \to E_1/E_2$ such that

$$\|S\| \|S^{-1}\| \le \|w_1\| \|w_2\|$$

and such that the compression $\tilde{\pi}$ of π to E_1/E_2 satisfies

$$\sigma(g) = S^{-1} \tilde{\pi}(g) S \qquad \text{for all } g \in \mathcal{S}.$$

Proof. We first notice that one has $\mathrm{Id}_X = \sigma(e) = w_2 w_1$. Let E_1 be the smallest π-invariant closed subspace of Z containing $w_1(X)$, that is

$$E_1 = \overline{\operatorname{span}}\Big\{ \bigcup_{g \in \mathcal{S}} \pi(g) w_1(X) \Big\}.$$

Further, let $E_2 = E_1 \cap \operatorname{Ker}(w_2)$. We now show that E_2 is π-invariant as well. For this let $z \in E_2$, i.e. $z \in E_1$ and $w_2(z) = 0$. Then z can be written as the limit of finite sums of the form

$$z = \lim_{n \to \infty} \sum_i \pi(g_n^i) w_1(x_n^i)$$

for some $x_n^i \in X$ and $g_n^i \in \mathcal{S}$. Hence, for all $g \in \mathcal{S}$ we have

$$\pi(g) z = \lim_{n \to \infty} \sum_i \pi(g g_n^i) w_1(x_n^i). \tag{5.1}$$

Moreover, because of $w_2 w_1 = \mathrm{Id}_X$ and $w_2(z) = 0$, we obtain

$$0 = \lim_{n\to\infty} \sum_i \sigma(g_n^i) x_n^i.$$

Applying w_2 to both sides of (5.1) yields

$$w_2 \pi(g) z = \lim_{n\to\infty} \sum_i \sigma(g g_n^i) x_n^i = \lim_{n\to\infty} \sigma(g) \left(\sum_i \sigma(g_n^i) x_n^i \right) = 0.$$

Since $\pi(g)z \in E_1$, this yields $\pi(g)z \in E_2$. Now, let $Q\colon E_1 \to E_1/E_2$ be the canonical surjection and define $S\colon X \to E_1/E_2$ by $x \mapsto Qw_1(x)$. Since $w_2 w_1 = \mathrm{Id}_X$, the restriction of w_2 to the image of w_1 and a fortiori $w_{2|E_1}\colon E_1 \to X$ are surjective. By the universal property of quotients, this induces a uniquely determined isomorphism $R\colon E_1/E_2 \to X$ with $\|R\| \le \|w_2\|$ such that $RQ = w_{2|E_1}$. In particular, we have $RS = RQw_1 = w_2 w_1 = \mathrm{Id}_X$. Since R is an isomorphism, this shows that S is invertible and $S = R^{-1}$. Moreover, we have

$$\|S\| \|S^{-1}\| = \|Qw_1\| \|R\| \le \|w_1\| \|w_2\|$$

as desired. Further, the compression $\tilde{\pi}$ of π to E_1/E_2 satisfies for all $g \in \mathcal{S}$

$$S^{-1} \tilde{\pi}(g) S = S^{-1} \tilde{\pi}(g) Q w_1 = S^{-1} Q \pi(g) w_1 = RQ\pi(g) w_1 = w_2 \pi(g) w_1 = \sigma(g).$$

This finishes the proof. □

The second preparatory result deals with the finite representability of vector-valued L_p-spaces. In the following we will need some basic notions from the local theory of Banach spaces which we have summarized in Appendix A.2.2. The following lemma is well-known in the scalar-valued case and can be proved completely analogously in the vector-valued case.

Lemma 5.5.7. *Let $p \in [1,\infty)$ and X be a Banach space. Then for an arbitrary measure space Ω the vector-valued Lebesgue space $L_p(\Omega,\mu;X)$ is finitely representable in $\ell_p(X)$.*

Proof. Let F be a n-dimensional subspace of $L_p(\Omega;X)$. Choose a normalized vector space basis $f_1,\dots,f_n$ of F. Since the simple functions are dense in $L_p(\Omega;X)$, we can choose for every $\varepsilon > 0$ simple functions $g_1,\dots,g_n$ such that

$$\|g_i - f_i\| \le \varepsilon \qquad \text{for all } i = 1,\dots,n.$$

Now let E be the subspace spanned by the vectors $g_1,\dots,g_n$ and let

$$T\colon F \to E$$

$$\sum_{i=1}^{n} \alpha_i f_i \mapsto \sum_{i=1}^{n} \alpha_i g_i.$$

Since $(\alpha_i)_{i=1}^n \mapsto \left\| \sum_{i=1}^n \alpha_i f_i \right\|$ defines a norm on $\mathbb{C}^n$ and all norms on a finite dimensional vector space are equivalent, there exists a constant $C > 0$ such that

$$\sum_{i=1}^{n} |\alpha_i| \le C \left\| \sum_{i=1}^{n} \alpha_i f_i \right\| \le C \sum_{i=1}^{n} |\alpha_i|.$$

Moreover,

$$\begin{aligned} \left\| T\left(\sum_{i=1}^{n} \alpha_i f_i \right) \right\| &= \left\| \sum_{i=1}^{n} \alpha_i g_i \right\| \le \left\| \sum_{i=1}^{n} \alpha_i f_i \right\| + \left\| \sum_{i=1}^{n} \alpha_i (g_i - f_i) \right\| \\ &\le \left\| \sum_{i=1}^{n} \alpha_i f_i \right\| + \varepsilon \sum_{i=1}^{n} |\alpha_i| \le \left\| \sum_{i=1}^{n} \alpha_i f_i \right\| + C\varepsilon \left\| \sum_{i=1}^{n} \alpha_i f_i \right\| \\ &= (1 + C\varepsilon) \left\| \sum_{i=1}^{n} \alpha_i f_i \right\|. \end{aligned}$$

Hence, $\|T\| \le 1 + C\varepsilon$. Similarly, one gets

$$\begin{aligned} \left\| T\left(\sum_{i=1}^{n} \alpha_i f_i \right) \right\| &= \left\| \sum_{i=1}^{n} \alpha_i g_i \right\| \ge \left\| \sum_{i=1}^{n} \alpha_i f_i \right\| - \left\| \sum_{i=1}^{n} \alpha_i (g_i - f_i) \right\| \\ &\ge \left\| \sum_{i=1}^{n} \alpha_i f_i \right\| - \varepsilon \sum_{i=1}^{n} |\alpha_i| \ge \left\| \sum_{i=1}^{n} \alpha_i f_i \right\| - \varepsilon C \left\| \sum_{i=1}^{n} \alpha_i f_i \right\| \\ &= (1 - \varepsilon C) \left\| \sum_{i=1}^{n} \alpha_i f_i \right\|. \end{aligned}$$

Hence, T is an isomorphism between F and G with $\|T^{-1}\| \le (1 - \varepsilon C)^{-1}$. Hence, one has the estimate

$$\|T\| \|T^{-1}\| \le \frac{1 + \varepsilon C}{1 - \varepsilon C}.$$

The functions g_i $(i = 1, \ldots, n)$ are simple, so we can find – changing g_i on a set of measure zero if necessary – disjoint measurable sets $A_1, \ldots, A_N$ such that $\mu(A_k) > 0$ for all $k = 1, \ldots, N$ and such that for each fixed k the function $g_i|_{A_k}$ is constant for all $i = 1, \ldots, n$. Now let

$$\begin{aligned} \iota \colon \mathcal{T} := \left\{ \sum_{k=1}^{N} \mathbb{1}_{A_k} x_k : x_k \in X \right\} &\to \ell_p^N(X) \\ \sum_{k=1}^{N} \mathbb{1}_{A_k} x_k &\mapsto (\mu(A_1)^{1/p} x_1, \ldots, \mu(A_N)^{1/p} x_N). \end{aligned}$$

Observe that ι is an isometric isomorphism. Indeed, one has

$$\left\|\sum_{k=1}^{N} \mathbb{1}_{A_k} x_k\right\| = \left(\sum_{k=1}^{N} \|x_k\|^p \mu(\mathbb{1}_{A_k})\right)^{1/p} = \left\|(\mu(A_1)^{1/p} x_1, \dots, \mu(A_N)^{1/p} x_N)\right\|_{\ell_p^N(X)}.$$

Since G is a subspace of $\mathcal{T}$, G is isometrically isomorphic to the closed subspace $\iota(G)$ of $\ell_p^N(X)$. One then obtains an isomorphism $S\colon F \to \iota(G)$ with

$$\|S\|\|S^{-1}\| \le \frac{1+\varepsilon C}{1-\varepsilon C}.$$

Since ε can be chosen arbitrarily small, this finishes the proof. $\square$

We now prove the following similarity result for semigroups.

Proposition 5.5.8. *Let A be a sectorial operator on a subspace-quotient SQ_X of a UMD-Banach lattice X with a bounded $H^\infty(\Sigma_\theta)$-calculus for some $\theta < \frac{\pi}{2}$ and let $(T(z))$ be the bounded analytic C_0-semigroup generated by $-A$. Then for each $\psi \in (\theta, \frac{\pi}{2})$ there exists a UMD-Banach lattice $\hat{X}$ and a bounded analytic semigroup $(\Pi(z))_{z \in \Sigma_{\pi/2-\psi}}$ on $\hat{X}$, which is positive and contractive on the real line, together with subspaces $N \subset M \subset \hat{X}$ which are invariant under $(\Pi(z))$ and an isomorphism $S\colon SQ_X \to M/N$ such that the induced semigroup $(\hat{\Pi}(z))_{z\in\Sigma_{\pi/2-\psi}}$ on M/N satisfies*

$$T(z) = S^{-1}\hat{\Pi}(z)S \qquad \text{for all } z \in \Sigma_{\pi/2-\psi}.$$

Moreover, if X is separable, $\hat{X}$ can be chosen separable as well.

Proof. Let $\psi \in (\theta, \frac{\pi}{2})$. We first assume that A additionally has dense range. For $z \in \Sigma_{\frac{\pi}{2}-\psi}$ we set $\Pi(z) := \pi(e_z)$, where π is obtained from Theorem 5.4.2. We denote the ultraproduct constructed there by $\hat{X}$. Notice that the analyticity of the semigroup $(\Pi(z))$ follows from the remarks at the beginning of Section 5.4. We now verify that $\hat{X}$ is a UMD-space. Notice that the UMD-property is a super-property, i.e. a property which is inherited by spaces which are finitely representable in a space having this property (for an introduction to these notions we refer to Appendix A.2.2). Again let $Y = L_p([0,1];X)$ for $p \in (1,\infty)$. Now, for all $j \in J$ the space $L_p(\Omega_j; L_p(\mathbb{R};Y))$ is finitely representable in $\ell_p(L_p(\mathbb{R};Y))$ by Lemma 5.5.7 and therefore $\hat{X} = \prod_{j\in J} L_p(\Omega_j; L_p(\mathbb{R};Y))/\mathcal{U}$ is finitely representable in $\ell_p(L_p(\mathbb{R};Y))$ by Proposition A.2.5. This shows that $\hat{X}$ is a UMD-space because X and therefore $\ell_p(L_p(\mathbb{R};Y))$ are UMD-spaces.

Now assume that SQ_X is separable. Then there exist a metric surjection $Q\colon \ell_1 \to SQ_X$ and an isometric embedding $J\colon SQ_X \to \ell_\infty$. By the short description of the proof of Pisier's factorization theorem given after Theorem 5.4.1, the ultraproduct $\hat{X}$ is constructed via a reduction to the p-completely bounded map $\ell_1 \xrightarrow{Q} SQ_X \xrightarrow{u(f)} SQ_X \xrightarrow{J} \ell_\infty$ which factorizes as

$$\ell_1 \xrightarrow{v_1} \hat{X} \xrightarrow{\pi(f)} \hat{X} \xrightarrow{v_2} \ell_\infty$$

for two maps $v_1\colon \ell_1 \to \hat{X}$ and $v_2\colon \hat{X} \to \ell_\infty$. We can now replace $\hat{X}$ by a separable closed vector sublattice constructed as follows: Let X_0 be the closed vector sublattice generated by $v_1(\ell_1)$. Then for $n \geq 1$ define X_n inductively as the closed vector sublattice generated by elements of the form $\Pi(q)X_{n-1}$, where q runs through $(\mathbb{Q} + i\mathbb{Q}) \cap \Sigma_{\frac{\pi}{2}-\psi}$. Then for all $n \in \mathbb{N}_0$ the lattice X_n is separable and by the continuity of the semigroup one has $\Pi(z)X_n \subset X_{n+1}$ for all $z \in \Sigma_{\frac{\pi}{2}-\psi}$. Hence, the closure of $\hat{Y} := \cup_{n\in\mathbb{N}_0} X_n$ is a closed separable vector sublattice of $\hat{X}$ which is invariant under the semigroup $(\Pi(z))$. We may now finish the proof with $\hat{X}$ replaced by $\hat{Y}$.

Now let A be an arbitrary sectorial operator. Then, by the remarks after Definition 1.3.2, one can decompose SQ_X into $SQ_X = \overline{R(A)} \oplus N(A)$ such that A is of the form $A = \begin{pmatrix} A_{00} & 0 \\ 0 & 0 \end{pmatrix}$. Then $\overline{R(A)}$ is a subspace-quotient of X as well and therefore, by Corollary 5.3.6, the sectorial operator A_{00} on $\overline{R(A)}$ – which has dense range – has a p-completely bounded $H^\infty(\Sigma_\psi)$-calculus for $p \in (1,\infty)$ and the p-matrix normed space structure (p-MNS2) with respect to $L_p([0,1];X)$. Hence, by Theorem 5.4.2 and the first part of the proof there exists a UMD-Banach lattice $\hat{X}$ (which can be chosen separable if SQ_X is separable), a semigroup $(\Pi(z))_{z\in\Sigma_{\frac{\pi}{2}-\psi}}$ on $\hat{X}$, $(\Pi(z))$-invariant subspaces $N \subset M \subset \hat{X}$ and an isomorphism $S\colon \overline{R(A)} \to M/N$ such that the induced semigroup $(\hat{\Pi}(z))_{z\in\Sigma_{\frac{\pi}{2}-\psi}}$ on M/N satisfies

$$T_{|\overline{R(A)}}(z) = S^{-1}\hat{\Pi}(z)S \qquad \text{for all } z \in \Sigma_{\frac{\pi}{2}-\psi}.$$

Notice that with respect to the decomposition $X = \overline{R(A)} \oplus N(A)$ one has $T(z) = T_{|\overline{R(A)}}(z) \oplus \mathrm{Id}$. Let now P be the projection onto $N(A)$ and let $\hat{Z}$ be the direct sum $\hat{X} \oplus X$ with its natural Banach lattice structure. Clearly, $\hat{Z}$ is UMD and separable if X is separable. Then $M/N \oplus SQ_X$ is a subspace-quotient of $\hat{Z}$. Now define $V_1\colon SQ_X \to M/N \oplus SQ_X$ as the matrix $V_1 = \begin{pmatrix} S & 0 \\ 0 & \iota \end{pmatrix}$, where ι is the inclusion of $N(A)$ into SQ_X, and $V_2\colon M/N \oplus SQ_X \to SQ_X$ as the matrix $V_2 = \begin{pmatrix} S^{-1} & 0 \\ 0 & P \end{pmatrix}$. Let $\tilde{\pi}(f)$ be the extension of π to $\hat{Z}$ by the identity on the second component for all $f \in H^\infty(\Sigma_\psi)$ and $\hat{\tilde{\pi}}$ its compression to the subspace-quotient $M/N \oplus SQ_X$. Note that $\tilde{\pi}$ is constructed in a way such that one has

$$T(z) = \tilde{u}(e_z) = V_2\hat{\tilde{\pi}}(e_z)V_1 \qquad \text{for all } z \in \Sigma_{\frac{\pi}{2}-\psi}.$$

We can now apply Proposition 5.3.1 which shows that there exist subspaces $E_2 \subset E_1 \subset M/N \oplus SQ_X$ that are invariant under $\hat{\tilde{\pi}}(e_z)$ for all $z \in \Sigma_{\frac{\pi}{2}-\psi}$ and an isomorphism $\hat{S}\colon SQ_X \to E_1/E_2$ such that the compression π_C of $\hat{\tilde{\pi}}$ satisfies

$$T(z) = \hat{S}^{-1}\pi_C(e_z)\hat{S} \qquad \text{for all } z \in \Sigma_{\frac{\pi}{2}-\psi}.$$

Note that E_1 and E_2 can be seen as subspaces of $\hat{Z}$. This finishes the proof. □

Remark 5.5.9. Notice that in the special case where X is an L_p-space the semigroup $(\Pi(z))$ lives on (a closed vector sublattice of) an ultraproduct of L_p-spaces which, by Kakutani's theorem A.4.7, is order isometric to an L_p-space. In this case one can therefore realize $\hat{X}$ as an L_p-space as well.

Strong Continuity Notice that there is one drawback of the ultraproduct construction just employed. In general, the semigroup $(\Pi(z))_{z\in\Sigma}$ obtained in Theorem 5.5.8 is not strongly continuous. However, the semigroup is a bounded analytic semigroup, that is a bounded analytic mapping from the sector Σ to the Banach space $\hat{X}$ satisfying the semigroup law. For such semigroups one has the following decomposition theorem [Are01, beginning of §5].

Lemma 5.5.10. *Let $(T(z))$ be a locally bounded analytic semigroup on some reflexive Banach space X. Then $P := \lim_{t\downarrow 0} T(t)$ exists in the strong operator topology and is a projection onto $X_1 := PX$. Further, the restriction $T(z)_{|X_1}$ defines a strongly continuous semigroup on X_1, whereas $T(z)_{|X_0} = 0$, where $X_0 := (\mathrm{Id} - P)(X)$.*

Since the space $\hat{X}$ obtained in Proposition 5.5.8 is reflexive, we can apply Lemma 5.5.10 in the situation of this proposition. In the next theorem we show that we can always reduce to the obtained strongly continuous part.

Theorem 5.5.11. *Let $-A$ be the generator of a bounded analytic C_0-semigroup $(T(z))$ on a subspace-quotient SQ_X of a UMD-Banach lattice X such hat $\theta := \omega_{H^\infty}(A) < \frac{\pi}{2}$. Then for each $\psi \in (\theta, \frac{\pi}{2})$ there exists a UMD-Banach lattice $\hat{X}$ and a bounded analytic C_0-semigroup $(R(z))_{z\in\Sigma_{\frac{\pi}{2}-\psi}}$ on $\hat{X}$ which is positive and contractive on the real line together with closed subspaces $N \subset M \subset \hat{X}$ which are invariant under $(R(z))$ and an isomorphism $U\colon SQ_X \to M/N$ such that the induced semigroup $(\hat{R}(z))_{z\in\Sigma_{\frac{\pi}{2}-\psi}}$ on M/N satisfies*

$$T(z) = U^{-1}\hat{R}(z)U \qquad \text{for all } z \in \Sigma_{\pi/2-\psi}.$$

Moreover, if X is separable, $\hat{X}$ can be chosen separable as well.

Proof. Let $\psi \in (\theta, \frac{\pi}{2})$ and let $S\colon SQ_X \to M/N$ and $(\Pi(z))_{z\in\Sigma_{\frac{\pi}{2}-\psi}}$ be as in the assertion of Theorem 5.5.8. Further, let $P := \lim_{t\downarrow 0}\Pi(t)$ be the projection onto X_1 from Lemma 5.5.10. It follows directly from the definition that P is a positive contractive projection. Hence, X_1 is a lattice subspace and therefore itself a Banach lattice with the induced order structure [AA02a, Theorem 5.59]. Since the UMD-property passes to subspaces, X_1 is a UMD-Banach lattice. Let $(R(z))_{z\in\Sigma_{\frac{\pi}{2}-\psi}}$ be the restriction of $(\Pi(z))_{z\in\Sigma_{\frac{\pi}{2}-\psi}}$ to X_1. By construction $(R(z))_{z\in\Sigma_{\frac{\pi}{2}-\psi}}$ is a bounded analytic C_0-semigroup on X_1 and positive and contractive on the real line. It follows from the discussion above that $\Pi(z)X_1 \subset X_1$ for all $z \in \Sigma_{\frac{\pi}{2}-\psi}$. Furthermore notice that $\Pi(z)$ leaves $PM \subset X_1$ and $PN \subset X_1$ invariant for all $z \in \Sigma_{\frac{\pi}{2}-\psi}$ and that P restricts to projections defined on M and N by the invariance under the semigroup. This allows us to define maps

$$V_1\colon SQ_X \underset{S}{\to} M/N \underset{P}{\to} PM/PN, \quad V_2\colon PM/PN \hookrightarrow M/PN \to M/N \underset{S^{-1}}{\longrightarrow} SQ_X.$$

Since for all $z \in \Sigma_{\frac{\pi}{2}-\psi}$ the operator $\Pi(z)$ acts only non-trivially on X_1, one obtains

$$T(z) = V_2\tilde{R}(z)V_1 \qquad \text{for all } z \in \Sigma_{\frac{\pi}{2}-\psi},$$

where the tilde indicates the compression of $(R(z))_{z\in\Sigma_{\frac{\pi}{2}-\psi}}$ to PM/PN. We now apply Proposition 5.5.6 which shows that there exist $(\tilde{R}(z))$-invariant subspaces $E_2 \subset E_1 \subset PM/PN$ and an isomorphism $U\colon SQ_X \xrightarrow{\sim} E_1/E_2$ such that the compression $\hat{R}$ of $\tilde{R}$ to E_1/E_2 satisfies

$$T(z) = U^{-1}\hat{R}(z)U \qquad \text{for all } z \in \Sigma_{\frac{\pi}{2}-\psi}.$$

Notice that the proof is finished since E_1/E_2 is a subspace-quotient of X_1. $\square$

Remark 5.5.12. Notice that if the semigroup is defined on a Hilbert space the used ultraproduct construction also yields a Hilbert space. The subspace-quotient M/N in Theorem 5.5.11 then is a Hilbert space as well. So in this case S is an isomorphism between two Hilbert spaces. Moreover, the constructed semigroup is even contractive on the whole sector by Remark 5.5.2. Therefore our general construction recovers Le Merdy's result in the Hilbert space case (Theorem 5.0.6).

Remark 5.5.13. The assumption $\omega_{H^\infty}(A) < \frac{\pi}{2}$ is crucial. Indeed, we have seen in Theorem 2.2.1 that there exists a generator $-A$ of a bounded C_0-semigroup $(T(t))_{t\geq 0}$ on a Hilbert space with $\omega_{H^\infty}(A) = \frac{\pi}{2}$ such that $(T(t))_{t\geq 0}$ is not similar to a contractive semigroup.

In the same spirit a classical counterexample by P.R. Chernoff [Che76] shows that there exists a semigroup on some Hilbert space with a bounded generator which is bounded on the real line but not similar to a contractive semigroup.

Note that the above arguments also work for general UMD-Banach spaces instead of UMD-Banach lattices. The underlying Banach space $\hat{X}$ of the constructed semigroup $(R(z))$ inherits properties from the Banach space X if they are stable under the constructed ultraproduct as well as under quotients and subspaces. For example, if X is uniformly convex the ultraproduct $\hat{X}$

(for $p = 2$) is uniformly convex as well (with the same modulus of uniform convexity) because it is finitely representable in the space $\ell_p(L_p(\mathbb{R}; Y))$ which is uniformly convex by [Day41b, Corollary 1]. As uniform convexity also passes to subspaces and quotients and every UMD-space has an equivalent uniformly convex norm by Theorem A.3.10 and Theorem A.3.8, we obtain the following new theorem.

Theorem 5.5.14. *Let $-A$ be the generator of a bounded analytic C_0-semigroup $(T(z))_{z\in\Sigma}$ on a UMD-space X such that A has a bounded H^∞-calculus with $\omega_{H^\infty}(A) < \frac{\pi}{2}$. Then there exists an equivalent uniformly convex norm on X for which the semigroup $(T(t))_{t\geq 0}$ is contractive.*

Moreover, in the L_p-space setting we obtain the following corollary.

Corollary 5.5.15. *Let $-A$ be the generator of a bounded analytic C_0-semigroup $(T(z))$ on a subspace-quotient SQ_{L_p} of some L_p-space $L_p(\Omega)$ for $p \in (1,\infty)$ such that $\theta := \omega_{H^\infty}(A) < \frac{\pi}{2}$. Then for each $\psi \in (\theta, \frac{\pi}{2})$ there exists an L_p-space $L_p(\tilde{\Omega})$ and a bounded analytic C_0-semigroup $(R(z))_{z\in\Sigma_{\frac{\pi}{2}-\psi}}$ on $L_p(\tilde{\Omega})$ which is contractive and positive on the real line together with closed subspaces $N \subset M \subset L_p(\tilde{\Omega})$, which are invariant under $(R(z))$, and an isomorphism $S\colon SQ_{L_p} \to M/N$ such that the induced semigroup $(\hat{R}(z))_{z\in\Sigma_{\frac{\pi}{2}-\psi}}$ on M/N satisfies*

$$T(z) = S^{-1}\hat{R}(z)S \qquad \text{for all } z \in \Sigma_{\pi/2-\psi}.$$

Moreover, if $L_p(\Omega)$ is separable, $L_p(\tilde{\Omega})$ can be chosen separable as well.

Proof. Note that in this special case the projection P onto X_1 is defined on some L_p-space and positive and contractive. It is known that in this case $\operatorname{Im} P$ is a closed vector sublattice [AA02b, Problem 5.3.12] and therefore order isometric to an L_p-space by Kakutani's theorem (Theorem A.4.7). □

In the case where the semigroup is defined on a subspace-quotient of an L_p-space one obtains the following new particularly nice equivalence, the main result of [Faca].

Corollary 5.5.16. *Let $-A$ be the generator of a bounded analytic C_0-semigroup $(T(z))$ on a subspace-quotient SQ_{L_p} of an L_p-space $L_p(\Omega)$ for some $p \in (1,\infty)$. Then the following are equivalent.*

(i) *A has a bounded H^∞-calculus with $\omega_{H^\infty}(A) < \frac{\pi}{2}$.*

(ii) *There exists a bounded analytic C_0-semigroup $(R(z))_{z\in\Sigma}$ on some L_p-space $L_p(\tilde{\Omega})$ which is contractive and positive on the real line together with $(R(z))$-invariant closed subspaces N, M with $N \subset M$ and an isomorphism*

$S\colon SQ_{L_p} \to M/N$ such that the induced semigroup $(\hat{R}(z))_{z\in\Sigma}$ on M/N satisfies

$$T(z) = S^{-1}\hat{R}(z)S \qquad \text{for all } z \in \Sigma.$$

Moreover, if $L_p(\Omega)$ is separable, then so is $L_p(\tilde{\Omega})$.

Proof. Note that the boundedness of the $H^\infty(\Sigma_\theta)$-calculus passes through invariant subspace-quotients and is preserved by similarity transforms by Proposition 4.3.4. Hence, (ii) implies (i) because the negative generator of every bounded analytic semigroup on some L_p-space which is contractive and positive on the real line has a bounded H^∞-calculus of angle lesser than $\frac{\pi}{2}$ by Weis' result (Theorem 4.2.21).

Conversely, (i) implies (ii) by Corollary 5.5.15. □

5.6 Notes & Open Problems

Corollary 5.5.16 leaves open some natural questions concerning the validity of stronger forms of the obtained result. In particular, we do not know whether in the case $SQ_{L_p} = L_p(\Omega)$ the result remains true without passing to a subspace-quotient.

Problem 5.6.1. Let $-A$ be the generator of a bounded analytic C_0-semigroup $(T(z))_{z\in\Sigma}$ on $L_p(\Omega)$ for $p \in (1,\infty)$ such that A has a bounded H^∞-calculus with $\omega_{H^\infty}(A) < \frac{\pi}{2}$. Does then exist a $\psi > 0$, an L_p-space $L_p(\tilde{\Omega})$, an invertible operator $S\colon L_p(\Omega) \to L_p(\tilde{\Omega})$ such that

$$T(z) = S^{-1}R(z)S \qquad \text{for all } z \in \Sigma_{\pi/2-\psi}$$

for a bounded analytic C_0-semigroup $(R(z))_{z\in\Sigma_{\pi/2-\psi}}$ which is positive and contractive on the real line?

This question may have a negative answer for spectral theoretic reasons. A second question deals with the regularity of the obtained semigroup after the similarity transform. Recall that we have seen in the Hilbert space case that one finds a semigroup $(R(z))$ that is contractive on a sector. Hence, the following question arises naturally.

Problem 5.6.2. Can one choose the semigroup in Corollary 5.5.16 to be contractive in a whole sector of the complex plane and not only on the real line?

Furthermore, in a recent joint work with C. Arhancet and C. Le Merdy that has not yet appeared as a preprint, we could improve some of the results of this chapter. In fact, under the assumptions of Theorem 5.5.11 for a UMD-Banach lattice one can even show that the semigroup $(T(t))_{t\geq 0}$ dilates to a bounded analytic C_0-semigroup that is positive and contractive on the real line. Notice that this directly implies the assertion of Theorem 5.5.11 by Proposition 5.5.6. Moreover, we obtain Theorem 5.5.14 for all super-reflexive spaces, i.e. for the most general case possible. It is interesting to note that the proof uses totally different concepts, namely generalized square functions and the theory of stochastic integration on Banach spaces.

Additionally, we obtain natural analogues of Theorem 5.5.11 for so-called Ritt operators on UMD-Banach lattices which can be seen as the analogue of analytic C_0-semigroups for discrete semigroups and for which one also has the concept of a bounded H^∞-calculus.

Appendices

A
Banach Spaces and Lattices

In this part of the appendix we give the basic definitions and results from the theory of Banach spaces and Banach lattices which are used throughout the text. Some more specialized results which are only used once at a particular place of the thesis are contained in the main body of the text exactly where they are used.

A.1 Schauder Bases & Schauder Decompositions

Of central importance is the concept of a Schauder basis, a generalization of Hilbert space bases to general Banach spaces. We now present the necessary background needed in the main text. Further details can for example be found in [AK06], [Sin70], [LT77], [LT79] and [FHH+11].

Definition A.1.1 (Schauder Basis). A sequence $(e_m)_{m\in\mathbb{N}}$ in a Banach space X is called a *Schauder basis* if for every $x \in X$ there is a unique sequence of scalars $(a_m)_{m\in\mathbb{N}}$ such that

$$x = \sum_{m=1}^{\infty} a_m e_m.$$

A sequence $(e_m)_{m\in\mathbb{N}}$ is called a *basic sequence* if it is a basis in the closed linear span of $(e_m)_{m\in\mathbb{N}}$. The functional $e_m^* \in X^*$ that maps x to the unique m-th coefficient in the expansion of x is called the *m-th coordinate functional*.

One has the following notion of equivalence for Schauder bases.

Definition A.1.2. Let $(e_m)_{m\in\mathbb{N}}$ be a Schauder basis for a Banach space X and $(f_m)_{m\in\mathbb{N}}$ a Schauder basis for a Banach space Y. Then $(e_m)_{m\in\mathbb{N}}$ and $(f_m)_{m\in\mathbb{N}}$ are *equivalent* if for every sequence of scalars $(a_m)_{m\in\mathbb{N}}$ the expansion $\sum_{m=1}^{\infty} a_m e_m$ converges if and only if $\sum_{m=1}^{\infty} a_m f_m$ converges.

In this case there exists an isomorphism $T\colon X \to Y$ such that $Te_m = f_m$ for each $m \in \mathbb{N}$. If $\|T\| = 1$, one says that $(e_m)_{m\in\mathbb{N}}$ and $(f_m)_{m\in\mathbb{N}}$ are *isometrically equivalent*.

The mere concept of a Schauder basis is sometimes too general to be useful in practice, therefore one often considers special bases with additional properties. The most important example is that of an unconditional basis.

Definition A.1.3 (Unconditional Basis). A Schauder basis $(e_m)_{m\in\mathbb{N}}$ for a Banach space X is called *unconditional* if for each $x \in X$ the unique expansion $x = \sum_{m=1}^{\infty} a_m e_m$ converges unconditionally, i.e. $\sum_{m=1}^{\infty} a_{\pi(m)} e_{\pi(m)} = x$ for each permutation π of the natural numbers.

There are several useful equivalent characterizations of unconditionally convergent series (see [AK06, Lemma 2.4.2] and [AK06, Proposition 2.4.9]).

Proposition A.1.4. *Given a series $\sum_{m=1}^{\infty} x_m$ in a Banach space, the following are equivalent.*

(i) $\sum_{m=1}^{\infty} x_m$ *is unconditionally convergent, i.e.* $\sum_{m=1}^{\infty} x_{\pi(m)}$ *converges for every permutation* $\pi\colon \mathbb{N} \to \mathbb{N}$,

(ii) *For every subsequence* $(x_{m_k})_{k\in\mathbb{N}}$ *the series* $\sum_{k=1}^{\infty} x_{m_k}$ *converges,*

(iii) *The series* $\sum_{m=1}^{\infty} \varepsilon_m x_m$ *converges for every choice* $(\varepsilon_m)_{m\in\mathbb{N}} \in \{-1,1\}^{\mathbb{N}}$.

(iv) *The series* $\sum_{m=1}^{\infty} b_m e_m$ *converges (unconditionally) for all* $(b_m)_{m\in\mathbb{N}} \in \ell_\infty$.

The closed graph theorem and the uniform boundedness principle show that there exists a smallest constant $K \geq 0$ such that

$$\left\|\sum_{m=1}^{\infty} b_m a_m e_m\right\| \leq K \left\|\sum_{m=1}^{\infty} a_m e_m\right\|$$

holds for all $(a_m)_{m\in\mathbb{N}}$ for which the expansion converges and all choices of sequences $(b_m)_{m\in\mathbb{N}}$ with $\|(b_m)\|_\infty \leq 1$. This constant is called the *unconditional constant* of $(e_m)_{m\in\mathbb{N}}$.

There are several methods to construct new Schauder bases / basic sequences out of given ones. Block basic sequences are an elementary and important example.

Definition A.1.5 (Block Basic Sequence). Let $(e_m)_{m\in\mathbb{N}}$ be a basis for a Banach space X. Let $(p_m)_{m\in\mathbb{N}_0}$ be a strictly increasing sequence of integers with $p_0 = 0$ and let $(a_m)_{m\in\mathbb{N}}$ be a sequence of scalars. Then the sequence $(u_m)_{m\in\mathbb{N}}$ defined by

$$u_m = \sum_{k=p_{m-1}+1}^{p_m} a_k e_k$$

is a basic sequence called a *block basic sequence of* $(e_m)_{m\in\mathbb{N}}$.

The structure of the normalized block basic sequences of ℓ_p is very simple [AK06, Lemma 2.1.1].

Theorem A.1.6. *Every normalized block basic sequence of the standard unit basis of* ℓ_p *is isometrically equivalent to the standard unit basis.*

The constant coefficient block basic sequences are a technical concept which helps to simply the proofs of some results.

Definition A.1.7 (Constant Coefficient Block Basic Sequence). A block basic sequence $(u_m)_{m\in\mathbb{N}}$ of a basis $(e_m)_{m\in\mathbb{N}}$

$$u_m = \sum_{k=p_{m-1}+1}^{p_m} a_k e_k$$

is a *constant coefficient block basic sequence* if for each $m \in \mathbb{N}$ there is a constant c_m such that $a_k \in \{0, c_m\}$ for $p_{m-1} + 1 \leq k \leq p_m$, that is

$$u_m = c_m \sum_{k\in A_m} e_k$$

for some subset A_m of $(p_{m-1}, p_m] \cap \mathbb{N}$.

The concept of a perfectly homogeneous block basic sequence was originally introduced to give an abstract description of bases that have similar structural properties like the canonical unit vector bases of ℓ_p and c_0. Nowadays, they still play an important role in the development of the theory of Schauder bases.

Definition A.1.8. A basis $(e_m)_{m\in\mathbb{N}}$ of a Banach space X is *perfectly homogeneous* if every normalized constant coefficient block basic sequence of $(e_m)_{m\in\mathbb{N}}$ is equivalent to $(e_m)_{m\in\mathbb{N}}$.

One can show that a perfectly homogeneous basis is already equivalent to all of its normalized block basic sequences. In particular, this follows from the following celebrated theorem by M. Zippin that characterizes all Banach spaces which admit a perfectly homogeneous basis. We omit the proof of this result, although it plays a central role in the development of our approach to the maximal regularity problem, thereby for a single time violating our leitmotif to be self-contained in the presentation of our main results, as the proof both needs some technical effort and is extremely well-covered in the literature (for a proof among our line of presentation see [AK06, Theorem 9.1.8]).

Theorem A.1.9. *Let $(e_m)_{m\in\mathbb{N}}$ be a normalized perfectly homogeneous basis of a Banach space X. Then $(e_m)_{m\in\mathbb{N}}$ is equivalent either to the canonical basis of c_0 or the canonical basis of ℓ_p for some $1 \leq p < \infty$.*

Block perturbations are a second method to construct new Schauder bases out of given Schauder bases.

Definition A.1.10. Let $(e_m)_{m\in\mathbb{N}}$ be a Schauder basis for a Banach space X that is bounded from below, i.e. $\inf_{m\in\mathbb{N}} \|e_m\| > 0$. Let $(n_m)_{m\in\mathbb{N}_0}$ and $(p_m)_{m\in\mathbb{N}}$ be strictly increasing sequences of scalars with $n_0 = 0$ and $n_{m-1} \leq p_m \leq n_m$ for all $m \in \mathbb{N}$. Then a sequence $(z_k)_{k\in\mathbb{N}}$ in X of the form

$$z_k = \begin{cases} e_k & \text{for } k \neq p_m \\ e_{p_m} + x_m & \text{for } k = p_m \end{cases}$$

with $x_m = \sum_{\substack{i=n_{m-1}+1 \\ i\neq p_m}}^{n_m} \alpha_i e_i$ for a scalar-valued sequence $(\alpha_i)_{i\in\mathbb{N}}$ such that $\|x_m\| \leq M$ for all $m \in \mathbb{N}$ for some $M \geq 0$ is called a *block perturbation* of $(e_m)_{m\in\mathbb{N}}$.

One has the following perturbation result for Schauder bases [Sin70, Ch. I, Proposition 4.4].

Proposition A.1.11. *Let $(e_m)_{m\in\mathbb{N}}$ be a Schauder basis for a Banach space X which is bounded from below, i.e.* $\inf_{m\in\mathbb{N}} \|e_m\| > 0$. *Then every block perturbation of $(e_m)_{m\in\mathbb{N}}$ is a Schauder basis of X.*

Proof. Let $(z_k)_{k\in\mathbb{N}}$ and $(x_m)_{m\in\mathbb{N}}$ be as in Definition A.1.10. Then the orthogonal sequence $(z_k^*)_{k\in\mathbb{N}}$ associated to $(z_k)_{k\in\mathbb{N}}$ is given by

$$z_k^* = \begin{cases} e_k^* - \alpha_k e_{p_m}^* & \text{for } k \neq p_m \text{ with } n_{m-1}+1 \leq k \leq n_m, \\ e_{p_m}^* & \text{for } k = p_m. \end{cases}$$

Let $x \in X$. We now calculate the expansion of x with respect to the orthogonal sequence $(z_k^*)_{k\in\mathbb{N}}$. We have for $l \in \mathbb{N}$

$$\begin{aligned}
\sum_{k=1}^{l} z_k^*(x) z_k &= \sum_{k=1}^{l} e_k^*(x) z_k - \sum_{m:n_{m-1}<l} e_{p_m}^*(x) \sum_{\substack{k=n_{m-1}+1 \\ k\neq p_m}}^{\min(n_m,l)} \alpha_k z_k \\
&= \sum_{k=1}^{l} e_k^*(x) e_k + \sum_{m:p_m\leq l} e_{p_m}^*(x) x_m - \sum_{m:n_m\leq l} e_{p_m}^*(x) x_m - \text{remaining terms} \\
&= \begin{cases} \displaystyle\sum_{k=1}^{l} e_k^*(x) e_k - e_{p_m}^*(x) \sum_{i=n_{m-1}+1}^{l} \alpha_i e_i & \text{for } n_{m-1}+1 \leq l \leq p_m - 1 \\ \displaystyle\sum_{k=1}^{l} e_k^*(x) e_k + e_{p_m}^*(x) \sum_{i=l+1}^{n_m} \alpha_i e_i & \text{for } p_m \leq l \leq n_m. \end{cases}
\end{aligned}$$

Recall that the projections $P_N = \sum_{k=1}^{N} e_k^*$ for $N \in \mathbb{N}$ are uniformly bounded by some constant $C \geq 0$. Hence, we obtain the inequalities

$$\left\| \sum_{i=n_{m-1}+1}^{l} \alpha_i e_i \right\| \leq C \|x_m\| \leq CM,$$

$$\left\| \sum_{i=l+1}^{n_m} \alpha_i e_i \right\| \le (C+1)\|x_m\| \le (C+1)M.$$

Further notice that it follows from $e_m^*(x)e_m \to 0$ for $m \to \infty$ and the assumption $\inf_{m\in\mathbb{N}} \|e_m\| > 0$ that the coefficients satisfy $e_m^*(x) \to 0$ for $m \to \infty$ as well. Altogether we obtain that for all $x \in X$ and all $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that

$$\left\| \sum_{k=1}^{l} z_k^*(x) z_k - \sum_{k=1}^{l} e_k^*(x) e_k \right\| \le \varepsilon \qquad \text{for all } l \le N.$$

Form this it follows that one has

$$x = \sum_{k=1}^{\infty} z_k^*(x) z_k$$

for all $x \in X$. This shows that $(z_k)_{k\in\mathbb{N}}$ is a Schauder basis for the space X. $\square$

Schauder decompositions naturally generalize the concept of a Schauder basis.

Definition A.1.12 (Schauder Decomposition). A sequence $(\Delta_m)_{m\in\mathbb{N}} \subset \mathcal{B}(X)$ for a Banach space X is called a *Schauder decomposition* if

$$\Delta_n \Delta_m = 0 \quad \text{for all } m \neq n \quad \text{and} \quad \sum_{m=1}^{\infty} \Delta_m x = x \quad \text{for all } x \in X.$$

The decomposition $(\Delta_m)_{m\in\mathbb{N}}$ is called *unconditional* if the expansion $\sum_{m=1}^{\infty} \Delta_m x$ converges unconditionally for all $m \in \mathbb{N}$.

Almost all comments concerning Schauder bases have natural analogues for Schauder decompositions. In fact, it follows from the compactness of $\{-1,1\}^{\mathbb{N}}$ and the uniform boundedness principle that for an unconditional Schauder decomposition $(\Delta_m)_{m\in\mathbb{N}}$ the family of bounded operators given by $S_{(\varepsilon_n)_n}(\sum_{m=1}^{\infty} \Delta_m x) := \sum_{m=1}^{\infty} \varepsilon_m \Delta_m x$ for $(\varepsilon_n)_{n\in\mathbb{N}} \in \{-1,1\}^{\mathbb{N}}$ is uniformly bounded in operator norm. One calls $K = \sup_{(\varepsilon_n)_{n\in\mathbb{N}} \in \{1,1\}^{\mathbb{N}}} \|S_{(\varepsilon_n)_n}\|$ the *unconditional constant* of the decomposition $(\Delta_m)_{m\in\mathbb{N}}$. Moreover, the projections $(P_N)_{N\in\mathbb{N}}$ defined by $P_N = \sum_{m=1}^{N} \Delta_m$ are uniformly bounded for an arbitrary Schauder decomposition $(\Delta_m)_{m\in\mathbb{N}}$ by the uniform boundedness principle.

A variant of the block perturbation result shown in Proposition A.1.11 can also be proved for general Schauder decompositions.

Proposition A.1.13. *Let $(\Delta_m)_{m\in\mathbb{N}}$ be a Schauder decomposition for a Banach space X. Further let $(e_m)_{m\in\mathbb{N}}$ be a sequence with $e_m \in \operatorname{Rg} \Delta_m$ for all $m \in \mathbb{N}$ and $\inf_{m\in\mathbb{N}} \|e_m\| > 0$ and let $(e_m^*)_{m\in\mathbb{N}}$ be associated contractive rank-one projections.*

Then for strictly increasing sequences $(n_m)_{m\in\mathbb{N}_0}$ and $(p_m)_{m\in\mathbb{N}}$ with $n_0 = 0$ and $n_{m-1} \le p_m \le n_m$ for all $m \in \mathbb{N}$ consider a sequence $(x_m)_{m\in\mathbb{N}}$ of the form $x_m = \sum_{\substack{i=n_{m-1}+1 \\ i\neq p_m}}^{n_m} \alpha_i e_i$ for a scalar-valued sequence $(\alpha_i)_{i\in\mathbb{N}}$ such that $\sup_{m\in\mathbb{N}} \|x_m\| < \infty$. Then the sequence $(Q_l)_{l\in\mathbb{N}}$ given by

$$Q_l = \begin{cases} P_l - e^*_{p_m} \otimes \displaystyle\sum_{i=n_{m-1}+1}^{l} \alpha_i e_i & \text{for } n_{m-1}+1 \le l \le p_m - 1 \\ P_l + e^*_{p_m} \otimes \displaystyle\sum_{i=l+1}^{n_m} \alpha_i e_i & \text{for } p_m \le l \le n_m \end{cases}$$

induces a Schauder decomposition $(\tilde{\Delta}_m)_{m\in\mathbb{N}}$ for X with $\tilde{\Delta}_m = Q_m - Q_{m-1}$ (where $Q_0 := 0$).

Proof. It suffices to show that $(Q_m)_{m\in\mathbb{N}}$ is a sequence consisting of projections satisfying $Q_nQ_m = Q_{\min\{n,m\}}$ for all $m,n \in \mathbb{N}$ and $\lim_{m\to\infty} Q_m x = x$ for all $x \in X$. Notice that for x in the closed span of $\{e_m : m \in \mathbb{N}\}$ all properties follow immediately from Proposition A.1.11 as $(e_m)_{m\in\mathbb{N}}$ is a basic sequence. Moreover, if $x \in X$ has only trivial intersection with this closed subspace, one has $Q_l x = P_l x$ and the first two properties are satisfied because they are satisfied for $(P_l)_{l\in\mathbb{N}}$. One may now finish the proof as in Proposition A.1.11. □

A.2 Geometry of Banach Spaces – General Methods and Techniques

In this section we summarize some basic methods and techniques attributed to the geometry of Banach spaces which are used throughout the text. The following method is often used to construct new Banach spaces.

Definition A.2.1. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of Banach spaces. Then for $p \in [1,\infty)$ the ℓ_p-sum $\oplus^n_{\ell_p} X_n$ of $(X_n)_{n\in\mathbb{N}}$ is given by

$$\oplus^n_{\ell_p} X_n := \left\{ (x_n) \in \prod_{n=1}^{\infty} X_n : \sum_{n=1}^{\infty} \|x_n\|^p_{X_n} < \infty \right\}, \qquad \|(x_n)\| := \left(\sum_{n=1}^{\infty} \|x_n\|^p_{X_n} \right)^{1/p}.$$

We now formulate some instances of the Pełczyński decomposition technique which can be seen as a variant of the Cantor–Schröder–Bernstein theorem for Banach spaces.

Theorem A.2.2 (The Pełczyński Decomposition Technique). *Let X and Y be Banach spaces so that X is isomorphic to a complemented subspace of Y and Y is isomorphic to a complemented subspace of X. Suppose further that either*

(a) $X \simeq X \oplus X$ *and* $Y \simeq Y \oplus Y$ *or*

(b) $X \simeq \ell_p(X)$ *for some* $1 \leq p < \infty$.

Then X *is isomorphic to* Y.

Proof. First, assume that (a) holds. We can write $X \simeq Y \oplus E$ and $Y \simeq X \oplus F$. Then one has

$$X \simeq Y \oplus Y \oplus E \simeq Y \oplus X \simeq X \oplus Y \simeq X \oplus X \oplus F \simeq Y.$$

Now, assume that (b) holds. Then, a fortiori, one has $X \oplus X \simeq \ell_p(X) \oplus \ell_p(X) \simeq \ell_p(X) \simeq X$. Hence, by the proof of part (a), one has $Y \simeq X \oplus Y$. On the other hand

$$\ell_p(X) \simeq \ell_p(Y \oplus E) \simeq \ell_p(Y) \oplus \ell_p(E).$$

Hence, $X \simeq \ell_p(X)$ implies

$$X \simeq \ell_p(Y) \oplus \ell_p(E) \simeq Y \oplus \ell_p(Y) \oplus \ell_p(E) \simeq Y \oplus \ell_p(X) \simeq Y \oplus X.$$

Altogether, we have shown the assertion $Y \simeq X$. □

A.2.1 Ultraproducts of Banach Spaces

Ultraproducts are a powerful method in Banach space theory that trades between local and global results. We discuss their definitions and basic properties.

Let $(X_i)_{i \in I}$ be a family of Banach spaces. We consider the space of bounded sequences $\ell_\infty(I; X_i)$ endowed with the norm

$$\|(x_i)\| := \sup_{i \in I} \|x_i\|.$$

Let $\mathcal{U}$ be an ultrafilter on I. Then it follows by compactness that the image of $\mathcal{U}$ under the map $i \mapsto \|x_i\|$ – which is again an ultrafilter – converges, i.e. for some $c \geq 0$ is finer than the neighbourhood filter of c. Now, consider the closed subspace

$$\mathcal{N}_{\mathcal{U}} := \left\{ (x_i)_{i \in I} \in \ell_\infty(I; X_i) : \lim_{\mathcal{U}} \|x_i\| = 0 \right\}.$$

Definition A.2.3. The *ultraproduct* $\prod X_i/\mathcal{U}$ of the spaces $(X_i)_{i \in I}$ is the quotient Banach space

$$\ell_\infty(I; X_i)/\mathcal{N}_{\mathcal{U}}.$$

Notice that a sequence of bounded operators $T_i \in \mathcal{B}(X_i, Y_i)$ such that $C = \sup_{i\in I} \|T_i\| < \infty$ gives rise to a bounded operator $T\colon (x_i)_{i\in I} \mapsto (Tx_i)_{i\in I}$ in $\mathcal{B}(\prod X_i/\mathcal{U}, \prod Y_i/\mathcal{U})$ with $\|T\| \le C$.

The main usage of ultraproducts of Banach spaces lies in their permanence properties. For example, the ultraproduct of Banach lattices is a Banach lattice when endowed with the order structure induced by the pointwise order modulo null sequences. For further details and results on ultraproducts we refer to [DJT95, Chapter 8].

A.2.2 Local Theory of Banach Spaces

We present the basic terminology and methods from the local theory of Banach spaces, i.e. the study of Banach spaces via the structure of its finite dimensional subspaces. Here the following concept is of fundamental importance.

Definition A.2.4 (Finitely Representable). Let X and Y be two Banach spaces. We say that Y is *finitely representable* in X if for every $\varepsilon > 0$ and every finite dimensional subspace F of Y there is a finite dimensional subspace E of X and an isomorphism $u\colon F \to E$ such that

$$\|u\| \cdot \|u^{-1}\| \le 1 + \varepsilon.$$

Finite representability behaves well with respect to ultraproducts [Pis11, Lemma 3.48].

Proposition A.2.5. *Let $(X_i)_{i\in I}$ be a family of Banach spaces such that each space X_i is finitely representable in a fixed Banach space X. Then every ultraproduct $\prod X_i/\mathcal{U}$ is finitely representable in X.*

We now define a special class of properties of Banach spaces which only depend on the structure of the finite dimensional subspaces.

Definition A.2.6 (Super-Property). Let (P) be a property for Banach spaces. We say that a Banach space Y has *super-(P)* if every Banach space X that is finitely representable in Y has (P). A property (P) is called a *super-property* if (P) is the same as super-(P).

In this way one obtains for example the definition of a *super-reflexive* space. Note that reflexivity is not a super-property, i.e. there are Banach spaces which are reflexive but not super-reflexive.

A.3 Geometric Properties of Banach Spaces

In this section we introduce some geometric properties of Banach spaces that are used throughout the text. For further details we recommend [DJT95].

A.3.1 Type and Cotype

As a starting point to the notions of type and cotype we recall the following classical inequality from probability theory. In the following let $r_k(t) := \operatorname{sign}\sin(2^k\pi t)$ be the k-th *Rademacher function*.

Theorem A.3.1 (Khintchine inequality). *For $p \in [1,\infty)$ there exist constants $A_p, B_p > 0$ such that for any finite sequence $a_1,\dots,a_n$ of scalars and any $n \in \mathbb{N}$ we have*

$$A_p\Big(\sum_{k=1}^n |a_k|^2\Big)^{1/2} \le \Big\|\sum_{k=1}^n r_k a_k\Big\|_{L_p[0,1]} \le \Big(\sum_{k=1}^n |a_k|^2\Big)^{1/2} \qquad \text{for } p \in [1,2]$$

and

$$\Big(\sum_{k=1}^n |a_k|^2\Big)^{1/2} \le \Big\|\sum_{k=1}^n r_k a_k\Big\|_{L_p[0,1]} \le B_p\Big(\sum_{k=1}^n |a_k|^2\Big)^{1/2} \qquad \text{for } p \in [2,\infty).$$

From the geometrical point of view the Khintchine inequality shows that for each $p \in [1,\infty)$ the space $L_p[0,1]$ contains a copy of ℓ_2. Note that the Khintchine inequality in particular implies that on the span of the Rademacher functions all L_p-norms on $[0,1]$ are equivalent for $p \in [1,\infty)$. The following so-called Kahane–Khintchine inequality shows that this particular consequence remains true if $\mathbb{C}$ is replaced by an arbitrary Banach space X.

Theorem A.3.2 (Kahane–Khintchine Inequality). *For each $p \in [1,\infty)$ there exists a constant $C_p \ge 0$ such that for every Banach space X and every finite sequence $x_1,\dots,x_n \in X$ one has*

$$\Big\|\sum_{k=1}^n r_k x_k\Big\|_{L_1([0,1];X)} \le \Big\|\sum_{k=1}^n r_k x_k\Big\|_{L_p([0,1];X)} \le C_p\Big\|\sum_{k=1}^n r_k x_k\Big\|_{L_1([0,1];X)}.$$

Rademacher averages of the above type play a central role in the modern theory of Banach spaces.

Definition A.3.3. For $p \in [1,\infty)$ and a Banach space X let $\operatorname{Rad}_p(X)$ be the closed subspace of $L_p([0,1];X)$ of elements of the form $\sum_{k=1}^\infty r_k x_k$ for some $x_k \in X$ ($k \in \mathbb{N}$). By the Kahane–Khintchine inequality (Theorem A.3.2) the spaces $\operatorname{Rad}_p(X)$ are isomorphic for different values of p and one therefore simply uses the notation $\operatorname{Rad}(X)$.

In particular, one often omits the concrete norm if one is only interested in estimates up to constant factors. Furthermore, one can show that $\operatorname{Rad}_p(X)$ is the closure of the finite Rademacher sums in $L_p([0,1];X)$ [AB03, p. 321].

One-sided weaker variants of the Khintchine inequality give rise to important Banach space invariants, the type and cotype of a Banach space.

Definition A.3.4. A Banach space X has *type* $p \in [1,2]$ if there is a constant $C \geq 0$ such that for every finite sequence $x_1, \ldots, x_n \in X$ one has

$$\left\| \sum_{k=1}^{n} r_k x_k \right\| \leq C \left(\sum_{k=1}^{n} \|x_k\|^p \right)^{1/p}.$$

The space X has *cotype* $q \in [2,\infty]$ if there is a constant $C \geq 0$ such that for every finite sequence $x_1, \ldots, x_n \in X$ one has

$$\left(\sum_{k=1}^{n} \|x_k\|^q \right)^{1/q} \leq C \left\| \sum_{k=1}^{n} r_k x_k \right\|$$

for $q < \infty$, whereas for $q = \infty$ one requires

$$\max_{k=1,\ldots,n} \|x_k\| \leq C \left\| \sum_{k=1}^{n} r_k x_k \right\|.$$

By the triangle inequality and the Hahn–Banach theorem every Banach space X has type 1 and cotype ∞. One therefore says that X has *non-trivial type* if X has type p for some $p > 1$. Analogously, one says that X has *non-trivial cotype* if X has cotype q for some $q < \infty$. Moreover, if X has type p_1 and cotype q_2, then X has type p for all $p \in [1, p_1]$ and cotype q for all $q \in [q_2, \infty]$. It follows from the parallelogram identity that a Banach space isomorphic to a Hilbert space has both type and cotype 2. The converse is a celebrated result by Kwapień [Kwa72].

Theorem A.3.5. *A Banach space X has both type and cotype* 2 *if and only if X is isomorphic to a Hilbert space.*

A.3.2 K-Convexity

In the duality theory of type and cotype the complementability of the space $\operatorname{Rad}(X)$ plays a crucial role.

Definition A.3.6. A Banach space X is called *K-convex* if the sequence of projections $(P_n)_{n \in \mathbb{N}}$ with

$$\begin{aligned} P_n \colon L_p([0,1];X) &\to \operatorname{Rad}_n(X) \\ f &\mapsto \sum_{k=1}^{n} r_k \int_0^1 r_k(t) f(t) \, dt \end{aligned}$$

onto the first n Rademacher functions $\operatorname{Rad}_n(X)$ is uniformly bounded in operator norm and therefore induces a bounded projection from $L_p([0,1];X)$ onto $\operatorname{Rad}_p(X)$ for one (equivalently all) $p \in (1,\infty)$.

The following deep theorem by G. Pisier gives a complete description of K-convex spaces.

Theorem A.3.7. *A Banach space is K-convex if and only if it has non-trivial type.*

For a detailed proof see [DJT95, Chapter 13]. One can show, for example via the passage through B-convexity [DJT95, Chapter 13], that every uniformly convex Banach space is K-convex. In particular, every subspace and quotient of an L_p-space for $p \in (1,\infty)$ is K-convex. However, there is no connection between K-convexity and reflexivity, in particular there are K-convex space which are not reflexive, for a particularly nice approach and further references see [PX87].

A.3.3 Super-Reflexive Spaces

There is a deep connection between uniform convexity and reflexivity. It is well-known that every uniformly convex Banach space is reflexive. However, there exist reflexive spaces which do not have an equivalent uniformly convex norm, a result which goes back to [Day41a]. In the language of the local theory of Banach spaces the two properties cannot be equivalent as the first is a super-property whereas the second is not. In fact, this is the only obstruction by the following celebrated result by P. Enflo [Enf72].

Theorem A.3.8. *A Banach space is super-reflexive if and only if the space has an equivalent uniformly convex norm.*

Notice that it is clear from the above result that every super-reflexive space is K-convex.

A.3.4 UMD-Spaces

The class of UMD-spaces is by now known to be the right class to study vector-valued stochastic integration and harmonic analysis and therefore plays an important role in the development of these theories.

Definition A.3.9 (UMD-Space). A Banach space X has the *UMD-property* if the Fourier multiplier operator for the multiplier $m(x) = \mathbb{1}_{[0,\infty)}(t)$ can be extended to a bounded operator on $L_p(\mathbb{R};X)$ for one (equivalently all) $p \in (1,\infty)$.

Notice that the above definition is clearly equivalent to the requirement that the vector-valued Hilbert transform defines a bounded operator on the space $L_p(\mathbb{R};X)$. The term UMD is an abbreviation for unconditional martingale differences and comes from an equivalent definition of those spaces. For the precise and other equivalent definitions of UMD-spaces as well as proofs and references for the following properties of UMD-spaces we refer to the surveys [RdF86] and [Bur01]. For us it will be important that the UMD-property is stable under closed subspaces and quotients and that $L_p(\Omega;X)$ is a UMD-space for all measure spaces Ω and all $p \in (1,\infty)$ whenever X is a UMD-space. In particular, since $\mathbb{C}$ is a UMD-space, the Lebesgue spaces $L_p(\Omega)$ are UMD-spaces for $p \in (1,\infty)$. Moreover, we make use of the following implications throughout the text.

Theorem A.3.10. *Every UMD-space is super-reflexive. In particular, every UMD-space is K-convex or equivalently has non-trivial type.*

A.3.5 Pisier's Property (α) and Property (Δ)

In the study of Rademacher averages the following structural properties of Banach spaces dealing with multiple sums of Rademacher averages have turned out to be of conceptual importance.

The first property goes back to G. Pisier and was introduced in [Pis78b].

Definition A.3.11. A Banach space X is said to have *Pisier's property* (α) if there is a constant $C \geq 0$ such that for all $n \in \mathbb{N}$, all $n \times n$-matrices $[x_{ij}] \in M_n(X)$ of elements in X and all choices of scalars $[\alpha_{ij}] \in M_n(\mathbb{C})$ one has

$$\int_{[0,1]^2} \left\| \sum_{i,j=1}^{n} \alpha_{ij} r_i(s) r_j(t) x_{ij} \right\| ds\, dt \leq C \sup_{i,j} \left|\alpha_{ij}\right| \int_{[0,1]^2} \left\| \sum_{i,j=1}^{n} r_i(s) r_j(t) x_{ij} \right\| ds\, dt.$$

We remark that L_p-spaces have Pisier's property (α) for $p \in [1,\infty)$ and that Pisier's property (α) passes to subspaces and is stable under the formation of L_p-spaces for $p \in [1,\infty)$. Moreover, it is shown in [Pis78b] that a Banach lattice X has Pisier's property (α) if and only if X has finite cotype. However, there are UMD-spaces, for example the Schatten classes $\mathcal{S}_p$ for $p \in [1,\infty) \setminus \{2\}$, that do not have Pisier's property (α).

The following weaker property originates from the work of N.J. Kalton and L. Weis.

Definition A.3.12. A Banach space X *has property* (Δ) if there is a constant $C \geq 0$ such that for all $n \in \mathbb{N}$ and all $n \times n$-matrices $[x_{ij}] \in M_n(X)$ one has

$$\int_{[0,1]^2} \left\| \sum_{i=1}^{n} \sum_{j=1}^{i} r_i(s) r_j(t) x_{ij} \right\| ds\, dt \leq C \int_{[0,1]^2} \left\| \sum_{i,j=1}^{n} r_i(s) r_j(t) x_{ij} \right\| ds\, dt.$$

The usefulness of property (Δ) partially comes from the validity of the following result proved in [KW01, Proposition 3.2].

Proposition A.3.13. *Every UMD-space has property* (Δ).

A.4 Banach Lattices

We only collect in the following for the sake of the reader some basic concepts from the theory of Banach lattices which are used throughout the text. For further details we refer to the monographs [AB06], [MN91] and [Sch74] and [LT79].

Definition A.4.1. A partially ordered Banach space $(X, \leq)$ over the real numbers is called a *Banach lattice* if

(i) $x \leq y$ implies $x + z \leq y + z$ for all $x, y, z \in X$,

(ii) $ax \geq 0$ for all $x \geq 0$ and all $a \geq 0$,

(iii) for all $x, y \in X$ there exists a least upper bound $x \vee y$ and a greatest lower bound $x \wedge y$,

(iv) $\|x\| \leq \|y\|$ whenever $|x| \leq |y|$, where the absolute value $|x|$ of $x \in X$ is defined by $|x| = x \vee (-x)$.

It follows from the definition that the lattice operations are norm continuous. This implies that the *positive cone* $C := \{x \in X : x \geq 0\}$ is a closed convex subset.

Definition A.4.2. Let $T : X \to Y$ be a linear operator between two Banach lattices X and Y.

(i) T is called *positive* if $Tx \geq 0$ for all $x \geq 0$ in X.

(ii) T is called *order preserving* if T preserves the lattice structure for which it suffices to check that

$$T(x_1 \wedge x_2) = Tx_1 \wedge Tx_2 \qquad \text{for all } x_1, x_2 \in X.$$

(iii) T is called an *order isomorphism* if T is one-to-one and onto and both T and T^{-1} are order preserving.

X and Y are called *order isomorphic* if there exists an order isomorphism between X and Y. Further, X and Y are called *order isometric* if there exists an isometry $T : X \to Y$ which is also an order isomorphism.

In general, not every set in a Banach lattice that is order bounded, i.e. bounded for the order $\leq$, has a least upper bound.

Definition A.4.3. A Banach lattice X is said to be *complete* (σ*-complete*) if every order bounded set (sequence) has a least upper bound.

One defines several mathematical subobjects for Banach lattices.

Definition A.4.4. Let Y be a closed subspace of a Banach lattice X. Then

(i) Y is called a *(vector) sublattice* if $x \wedge y \in Y$ whenever $x, y \in Y$.

(ii) Y is called an *ideal* if $y \in Y$ whenever $|y| \leq |x|$ for some $x \in Y$.

(iii) Y is called a *band* if Y is an ideal and if for every family $(y_i)_{i \in I}$ in Y such that $\bigvee_{i \in I} y_i$ exists in X, this element already belongs to Y.

A further continuity property is the following.

Definition A.4.5. A Banach lattice X is called *order continuous* (*σ-order continuous*) if, for every decreasing net (sequence) (x_α) in X, i.e. $x_\beta \leq x_\alpha$ for $\beta \geq \alpha$, with $\bigwedge_\alpha x_\alpha = 0$, one has $\lim_\alpha \|x_\alpha\| = 0$.

Of fundamental importance in the theory of Banach lattices are so-called representation theorems which give an abstract description of concrete Banach function lattices. We need the following concept.

Definition A.4.6. Let $p \in [1, \infty)$. A Banach lattice is called an *abstract L_p-space* if $\|x + y\|^p = \|x\|^p + \|y\|^p$ holds for all $x, y \in X$ with $x \wedge y = 0$.

For an abstract L_p-space one has the following representation theorem by S. Kakutani [LT79, Theorem 1.b.2].

Theorem A.4.7. *Let $p \in [1, \infty)$. An abstract L_p-space is order isometric to some concrete L_p-space $L_p(\Omega)$ over some measure space (Ω, Σ, μ).*

In particular, every ultraproduct of L_p-spaces is order isometric to some concrete L_p-spaces, i.e. the class of L_p-spaces is stable under ultraproducts.

A.5 Interpolation Theory

In this section we present the basics of interpolation theory. For further details we refer to [Tri78], [BK91] and [BL76].

A.5.1 Interpolation Spaces and Interpolation Functors

We now introduce the general notions from interpolation theory without presenting concrete interpolation methods for which we refer to the aforementioned literature.

We first define the category of interpolation couples.

Definition A.5.1. An *interpolation couple* of Banach spaces is a pair (X_1, X_2) of Banach spaces together with a Hausdorff topological vector space $\mathcal{X}$ such that X_1 and X_2 are continuously embedded in $\mathcal{X}$. A *morphism* between two such couples (X_1, X_2) and (Y_1, Y_2) is a linear operator $T\colon X_1 + X_2 \to Y_1 + Y_2$ whose restrictions to X_i define bounded linear operators from X_i to Y_i for $i = 1, 2$.

By $\operatorname{Mor}((X_1, X_2), (Y_1, Y_2))$ we denote the set of all morphisms from (X_1, X_2) to (Y_1, Y_2). One can show that if (X_1, X_2) is an interpolation couple, then the spaces $X_1 \cap X_2$ and $X_1 + X_2$ are Banach spaces when endowed with the norms $\|x\|_{X_1 \cap X_2} := \max(\|x\|_{X_1}, \|x\|_{X_2})$ and $\|x\|_{X_1 + X_2} := \inf_{\substack{x = x_1 + x_2 \\ x_i \in X_i}} \|x_1\|_{X_1} + \|x_2\|_{X_2}$.

Of special interest are the following immediate spaces lying between both ends of a Banach couple.

Definition A.5.2. Let (X_1, X_2) be an interpolation couple of Banach spaces. A Banach space X is called an *interpolation space of exponent* $\theta \in [0, 1]$ between X_1 and X_2 (or with respect to the interpolation couple (X_1, X_2)) if it satisfies

(i) One has continuous embeddings $X_1 \cap X_2 \hookrightarrow X \hookrightarrow X_1 + X_2$;

(ii) If $T \in \operatorname{Mor}((X_1, X_2), (X_1, X_2))$, then $T(X) \subset X$ and $T_{|X} \in \mathcal{B}(X)$;

(iii) For some constant $C \geq 0$ one has for all $T \in \operatorname{Mor}((X_1, X_2), (X_1, X_2))$

$$\left\|T_{|X}\right\|_{\mathcal{B}(X)} \leq C \left\|T_{|X_1}\right\|_{\mathcal{B}(X_1)}^{1-\theta} \left\|T_{|X_2}\right\|_{\mathcal{B}(X_2)}^{\theta};$$

(iv) For some constant $c \geq 0$ one has

$$\|x\|_X \leq c \|x\|_{X_1}^{1-\theta} \|x\|_{X_2}^{\theta} \qquad \text{for all } x \in X_1 \cap X_2.$$

An interpolation space X between X_1 and X_2 is called *regular* if $X_1 \cap X_2$ is dense in X.

Note that in the literature an interpolation space of exponent θ is usually defined as a Banach space satisfying only (i)-(iii). We add (iv) for technical reasons. General methods to construct interpolation spaces can be formalized with the help of interpolation functors.

Definition A.5.3. An *interpolation functor* $\mathcal{F}$ *of exponent* $\theta \in [0, 1]$ is a functor from the category of interpolation couples into the category of Banach spaces such that

(i) for each interpolation couple (X_1, X_2) the space $\mathcal{F}((X_1, X_2))$ is an interpolation space of exponent θ with respect to (X_1, X_2) and

(ii) one has $\mathcal{F}(T) = T_{|X} \in \mathcal{B}(X, Y)$ for all $T \in \operatorname{Mor}((X_1, X_2), (Y_1, Y_2))$, where $X = \mathcal{F}((X_1, X_2))$ and $Y = \mathcal{F}((Y_1, Y_2))$.

Moreover, an interpolation functor $\mathcal{F}$ is called *regular* if $\mathcal{F}((X_1,X_2))$ is a regular interpolation space for all interpolation couples (X_1,X_2).

Notice that for a space $X = \mathcal{F}((X_1,X_2))$ constructed by an interpolation functor $\mathcal{F}$ of exponent θ property (iv) holds automatically. Indeed, for $x \in X_1 \cap X_2$ apply the functor to the morphism $T\colon \mathbb{C} \to X_1 + X_2$ given by $\lambda \mapsto \lambda x$.

We note without going into further details that the well-known real interpolation (if the interpolation parameter usually denoted by q satisfies $q < \infty$) and complex interpolation methods all define interpolation functors of exponent $\theta \in (0,1)$.

In general, determining interpolation spaces for concrete interpolation functors is a difficult task. However, there is an abstract method which sometimes can be quite useful.

Definition A.5.4. Let X and Y be Banach spaces. An operator $R \in \mathcal{B}(X,Y)$ is said to be a *retraction* if there exists an operator $S \in \mathcal{B}(Y,X)$ such that

$$RS = \mathrm{Id}_Y .$$

In this case one says that S is the *coretraction* belonging to R.

To illustrate this abstract definition, let X be a Banach space and Y a complemented subspace of X. Then one can take $S\colon Y \hookrightarrow X$ as the inclusion mapping and $R\colon X \to Y$ as a projection onto Y.

Retractions may allow to reduce the problem of determining interpolation spaces to known cases via the following theorem [Tri78, 1.2.4].

Theorem A.5.5. *Let (X_1,X_2) and (Y_1,Y_2) be interpolation couples. Further assume that there are*

$$S \in \mathcal{B}((Y_1,Y_2),(X_1,X_2)) \qquad \text{and} \qquad R \in \mathcal{B}((X_1,X_2),(Y_1,Y_2))$$

such that the restrictions $S_i\colon Y_i \to X_i$ and $R_i\colon X_i \to Y_i$ satisfy

$$R_i S_i = \mathrm{Id}_{Y_i} \qquad \text{for } i = 1,2,$$

i.e. R_i are retractions and S_i are coretractions respectively. Then for an interpolation functor $\mathcal{F}$ one has the identification (as Banach spaces)

$$\mathcal{F}((Y_1,Y_2)) \simeq \operatorname{Rg} S R_{|\mathcal{F}((X_1,X_2))} \subset \mathcal{F}((X_1,X_2)).$$

This theorem in particular applies for the interpolation of Rademacher sequences. We are interested in those interpolation functors $\mathcal{F}$ which commute with Rad, i.e. which satisfy $\mathcal{F}((\mathrm{Rad}(X_1),\mathrm{Rad}(X_2))) \simeq \mathrm{Rad}(\mathcal{F}((X_1,X_2)))$ for a sufficiently regular Banach couple (X_1,X_2). This is indeed satisfied if the interpolation functor commutes with the formation of L_p-spaces. We follow the terminology in [KS12].

Definition A.5.6. An interpolation functor $\mathcal{F}$ is called *L_p-compatible* if over every σ-finite measure space one has for all interpolation couples (X_1, X_2) and all $p \in (1, \infty)$ that

$$\mathcal{F}((L_p(X_1), L_p(X_2))) = L_p(\mathcal{F}((X_1, X_2)))$$

as sets with equivalent norms.

It is a well-known fact that both the complex and real interpolation methods define L_p-compatible interpolation functors for fixed interpolation parameters.

For L_p-compatible interpolation functors one obtains the following compatibility with Rademacher spaces [KS12, Proposition 3.14].

Corollary A.5.7. *Let (X_1, X_2) be an interpolation couple of two Banach spaces with non-trivial type and $\mathcal{F}$ an L_p-compatible interpolation functor. Then one has*

$$\mathcal{F}((\operatorname{Rad}(X_1), \operatorname{Rad}(X_2))) \simeq \operatorname{Rad}(\mathcal{F}((X_1, X_2))).$$

Proof. Since X_i have non-trivial type, the Rademacher spaces $\operatorname{Rad}(X_i)$ are complemented in $L_2([0,1]; X)$ for $i = 1, 2$ as a consequence of Pisier's characterization of K-convex spaces (Theorem A.3.7). Hence, if $S\colon \operatorname{Rad}(X_1 + X_2) \hookrightarrow L_2([0,1]; X_1 + X_2)$ is the inclusion mapping and $R\colon L_2([0,1]; X_1 + X_2) \to \operatorname{Rad}(X_1 + X_2)$ the Rademacher projection, the pair (R, S) satisfies the assumptions of Theorem A.5.5. Hence, we obtain the identifications

$$\mathcal{F}((\operatorname{Rad}(X_1), \operatorname{Rad}(X_2))) \simeq SR(L_2([0,1]; \mathcal{F}((X_1, X_2)))) \simeq \operatorname{Rad}(\mathcal{F}((X_1, X_2))),$$

where we have used the L_p-compatibility of the interpolation functor in the first identification. □

A.5.2 Stein Interpolation Theorems

The methods used in the proofs of the Riesz–Thorin interpolation theorem and the Stein interpolation theorem already suggest that there should be a generalization of the Stein Interpolation theorem to the setting of the complex interpolation method. We present the version proved in [Voi92]. Let S denote the strip $S \coloneqq \{z \in \mathbb{C} : 0 \le \operatorname{Re} z \le 1\}$ in the complex plane.

Theorem A.5.8 (Abstract Stein Interpolation). *Let (X_1, X_2) and (Y_1, Y_2) be two interpolation couples of Banach spaces and let Z be a dense subspace of $(X_1 \cap X_2, \|\cdot\|_{X_1 \cap X_2})$. Moreover, let $(T(z))_{z \in S}$ be a family of linear mappings with $T(z)\colon Z \to Y_1 + Y_2$ for all $z \in S$ and the following properties:*

(a) *For all $z \in Z$ the function $T(\cdot)z\colon S \to Y_1 + Y_2$ is continuous, bounded and analytic on the interior of S.*

(b) *For all $z \in Z$ the functions*

$$s \mapsto T(is)z \qquad \text{and} \qquad s \mapsto T(1+is)z$$

are continuous as mappings from $\mathbb{R}$ to Y_1 respectively Y_2 and satisfy for $j = 1, 2$

$$M_j := \sup\{\|T(j-1+is)x\|_{Y_j} : s \in \mathbb{R}, x \in Z, \|x\|_{X_j} \leq 1\} < \infty.$$

For $t \in [0,1]$ let $X_t = (X_1, X_2)_t$ and $Y_t = (Y_1, Y_2)_t$ denote the complex interpolation spaces. Then for all $t \in [0,1]$ one has $T(t)Z \subset Y_t$ and

$$\|T(t)x\|_{Y_t} \leq M_1^{1-t} M_2^t \|x\|_{X_t} \qquad \text{for all } x \in Z.$$

B

Operator Spaces

In this appendix we give a very short introduction to the theory of operator spaces and completely bounded maps which is used in some parts of the thesis. For detailed references the reader can consult [ER00], [Pis01], [Pau02], [Pis03] and [Hel10].

We now define the category of operator spaces.

Definition B.0.9. An *operator space* is a Banach space X that is isometrically embedded into $\mathcal{B}(H)$ for some Hilbert space H.

Note that the datum of an operator space therefore consists of a Banach space together with a fixed embedding. There may be different embeddings for the same Banach space which give rise to different operator space structures. Further we will always see X as a subspace of $\mathcal{B}(H)$.

Note that if $X \subset \mathcal{B}(H)$ is an operator space, one can define for all $n \in \mathbb{N}$ the matrix algebras $M_n(X) \subset M_n(\mathcal{B}(H))$ of all $[a_{ij}] \in M_n(\mathcal{B}(H))$ for which $a_{ij} \in X$ for all $i, j = 1, \ldots, n$. Here one identifies $M_n(\mathcal{B}(H))$ with $\mathcal{B}(\ell_2^n(H))$ for all $n \in \mathbb{N}$. Moreover, a map $u \colon X \to Y$ induces mappings $u_n \colon M_n(X) \to M_n(Y)$ via $[x_{ij}] \mapsto [u(x_{ij})]$ for all $n \in \mathbb{N}$. We now define the morphisms between operator spaces.

Definition B.0.10. Let $X \subset \mathcal{B}(H)$ and $Y \subset \mathcal{B}(K)$ be two operator spaces. A map $u \colon X \to Y$ is called *completely bounded* if $\|u\|_{\mathrm{cb}} := \sup_{n \in \mathbb{N}} \|u_n\| < \infty$, i.e. the induced maps $u_n \colon M_n(X) \to M_n(Y)$ are uniformly bounded.

There is a close connection between completely bounded maps and similarity problems which is in detailed covered in [Pis01] and [Pau02]. We need only the following result by V.I. Paulsen [Pau02, Theorem 9.1] going back to the work [Pau84].

Theorem B.0.11. *Let H and K be Hilbert spaces. Further let $\mathcal{A} \subset \mathcal{B}(K)$ be a unital operator algebra and let $u \colon \mathcal{A} \to \mathcal{B}(H)$ be a completely bounded algebra homomorphism. Then there exists an invertible $S \in \mathcal{B}(H)$ such that $a \mapsto S^{-1} u(a) S$ is a contractive homomorphism.*

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List of Mathematical Symbols

$(T(t))_{t\geq 0}$	One-parameter semigroup of bounded linear operators (not necessarily strongly continuous)
$\lvert T\rvert$	The modulus of an operator T between two Banach lattices
$\arg z$	The argument of a complex number $z \in \mathbb{C}\setminus\{0\}$ with $\arg z \in (-\pi, \pi]$
$\mathcal{A}_p(\mathbb{R})$	Muckenhoupt or A_p-weight
$\mathcal{A}_p(\mathbb{T})$	Periodic Muckenhoupt or A_p-weight
$\mathcal{B}(X,Y)$	The space of all bounded linear operators between two Banach spaces X and Y
$\mathcal{P}$	The space of all complex polynomials
$\mathcal{P}_1$	The set of all polynomials $p \in \mathcal{P}$ with $\lvert p(1)\rvert < \sup_{\lvert z\rvert\leq 1} \lvert p(z)\rvert$
$\mathcal{R}$	The $\mathcal{R}$-bound of a set
$\lVert T\rVert_r$	The regular norm of an operator T between two (subspaces) of Banach lattices
$\omega(A)$	The sectorial angle of a sectorial operator A
$\omega_R(A)$	The $\mathcal{R}$-sectorial angle of a sectorial operator A
$\oplus_{\ell_p}^m X_m$	The ℓ_p-sum of the sequence of Banach spaces $(X_m)_{m\in\mathbb{N}}$, i.e. for $p \in [1,\infty)$ the space endowed with the norm $\lVert (x_m)\rVert^p := \sum_{m=1}^{\infty} \lVert x_m\rVert_{X_m}^p$
$\mathrm{Rad}(X)$	The Rademacher space over a Banach space X, i.e. the closure in $L_p([0,1];X)$ of the space of all finite sums of the form $\sum_{k=1}^{N} r_k x_k$ for $x_1,\ldots,x_N \in X$
$\rho(A)$	Resolvent set of a closed operator A
$\sigma(A)$	Spectrum of a closed operator A
Σ_θ	Sector in the complex plane with opening angle θ defined as $\Sigma_\theta := \{z \in \mathbb{C}\setminus\{0\} : \lvert\arg(z)\rvert < \theta\}$
$A(\mathbb{D})$	The Disc algebra, i.e. the closure of the set of complex polynomials with respect to the uniform convergence on the closed disc in the complex plane

BV	The space of all sequences with bounded variation
$H^\infty(\Sigma_{\theta+})$	The vector space $H^\infty(\Sigma_{\theta+}) := \cup_{\theta'>\theta} H^\infty(\Sigma_{\theta'})$
$H^\infty(\Sigma_\theta)$	The space of all bounded analytic functions on the sector Σ_θ
$H_0^\infty(\Sigma_\theta)$	The space of all $f \in H^\infty(\Sigma_\theta)$ which satisfy $\lvert f(z)\rvert \le C\frac{\lvert z\rvert^\varepsilon}{\lvert 1+z\rvert^{2\varepsilon}}$ for some $C, \varepsilon > 0$ and all $z \in \Sigma_\theta$
$L_p(\Omega)$	Lebesgue space over some measure space Ω
$M_n(\mathcal{A})$	The algebra of $n \times n$-matrices over an algebra $\mathcal{A}$
$R(\lambda, A)$	The resolvent $R(\lambda, A) = (\lambda - A)^{-1}$ of a closed operator A defined for $\lambda \in \rho(A)$
r_k	The k-th Rademacher function
SQ_X	A subspace-quotient of a Banach space X
X, Y, Z	Banach spaces which are assumed to be complex unless otherwise stated
X^*	The topological dual space of the Banach space X
X_p	The space $\oplus_{\ell_p}^n \ell_2^n$ for $p \in [1, \infty)$

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Zusammenfassung in deutscher Sprache

In dieser Arbeit untersuchen wir Regularitätseigenschaften sektorieller Operatoren und deren gegenseitiges Wechselspiel. Im Zentrum stehen die Regularitätseigenschaften *maximale Regularität* oder im wesentlichen äquivalent dazu die *$\mathcal{R}$-Sektorialität* und der Begriff des *beschränkten H^∞-Kalküls*. Das Konzept der maximalen Regularität spielt mittlerweile eine zentrale Rolle im Studium nichtlinearer partieller Differenzialgleichungen und der eng verwandte H^∞-Kalkül hat sich seit seiner Einführung zu einem zentralen Werkzeug für das Studium von Halbgruppen entwickelt. Die vorliegende Arbeit enthält drei Hauptresultate.

Das erste Hauptresultat gibt einen wichtigen Beitrag zu dem *maximalen Regularitätsproblem*. Es ist klassisch bekannt, dass der Generator einer stark stetigen Halbgruppe mit maximaler Regularität auf einem Banachraum notwendigerweise eine analytische Halbgruppe erzeugt. Auf Hilberträumen hat umgekehrt jeder Generator einer analytischen stark stetigen Halbgruppe maximale Regularität. Das maximale Regularitätsproblem fragt, welche Banachräume diese Eigenschaft besitzen. Für L_p- oder allgemeiner UMD-Räume geht diese Frage auf H. Brézis zurück. Dieses Problem blieb länger offen, bis es von N.J. Kalton und G. Lancien negativ beantwortet wurde [KL00]. Sie zeigten, dass ein Banachraum mit einer unbedingten Basis die maximale Regularitätseigenschaft genau dann besitzt, wenn dieser bereits isomorph zu einem Hilbertraum ist. Der Ansatz von Kalton und Lancien basiert auf abstrakten Resultaten aus der Theorie der Banachräume und liefert insbesondere kein explizites Beispiel eines Generators einer analytischen Halbgruppe ohne maximale Regularität. Tatsächlich war bis jetzt kein explizites Beispiel auf einem UMD-Raum bekannt. In unserem ersten Hauptresultat geben wir einen neuen expliziten Beweis für das Resultat von Kalton und Lancien. Insbesondere können wir die ersten expliziten Beispiele von analytischen Halbgruppen ohne maximale Regularität auf UMD-Räumen – auch auf L_p-Räumen – konstruieren. Zudem können wir mit unseren Methoden zeigen, dass es positive Halbgruppen auf UMD-Banachverbänden ohne maximale Regularität gibt und ein offenes Problem über die Struktur von Schauderbasen auf L_p-Räumen negativ beantworten.

Das zweite Hauptresultat beschäftigt sich mit der Extrapolation von maximaler Regularität. Konkreter beschäftigen wir uns mit folgendem aus den Anwendungen motivierten bisher offenem Problem: Gegeben sei eine Familie von konsistenten Halbgruppen $(T_p(t))_{t\geq 0}$ auf L_p für $p \in (1,\infty)$. Ferner sei

$(T_2(t))_{t\geq 0}$ analytisch und besitze damit maximale Regularität. Folgt dann, dass $(T_p(t))_{t\geq 0}$ für alle $p \in (1,\infty)$ maximale Regularität besitzt? Wir geben Gegenbeispiele für diese Vermutung in der stärkstmöglichen Form. Konkret zeigen wir, dass für jedes Teilintervall $I \subset (1,\infty)$ mit $2 \in I$ eine Familie von konsistenten analytischen Halbgruppen $(T_p(z))_{z\in\Sigma_{\frac{\pi}{2}}}$ auf $L_p(\mathbb{R})$ für $p \in (1,\infty)$ existiert so, dass $(T_p(z))_{z\in\Sigma_{\frac{\pi}{2}}}$ genau dann maximale Regularität besitzt, wenn $p \in I$ gilt. Ersetzt man maximale Regularität durch die Analyzität der Halbgruppe, so besitzt die Frage als eine klassische Anwendung des Steinschen Interpolationssatzes eine positive Antwort. Für dieses Resultat geben wir einen neuen Beweis, der die Aussage auf eine breite Klasse von Interpolationsräumen verallgemeinert. Zudem erlaubt dieselbe Methode, ein Extrapolationsresultat für maximale Regularität für eine breite Klasse von Interpolationsfunktoren zu zeigen.

Das dritte Hauptresultat beschreibt die Struktur von sektoriellen Operatoren mit einem beschränkten H^∞-Kalkül auf L_p-Räumen. Wir zeigen, dass ein sektorieller Operator genau dann einen H^∞-Kalkül von Winkel kleiner als $\frac{\pi}{2}$ auf L_p für $p \in (1,\infty)$ besitzt, wenn dieser nach der Bildung von invarianten Quotientenunterräumen und Ähnlichkeitstransformationen aus einem sektoriellen Operator A auf einem anderen L_p-Raum hervorgeht derart, dass $-A$ eine analytische C_0-Halbgruppe erzeugt, die positiv und kontraktiv auf der reellen Achse ist. Auf dem Weg zu diesem Resultat verallgemeinern wir Fendlers Dilatationsresultat auf r-kontraktive Halbgruppen auf abgeschlossenen Unterräumen von L_p-Räumen, geben einen vollständigen Beweis von Weis' Resultat über den beschränkten H^∞-Kalkül von negativen Generatoren von positiven kontraktiven Halbgruppen auf L_p-Räumen für $p \in (1,\infty)$ und zeigen einen punktweisen Ergodensatz für allgemeine Unterräume von L_p-Räumen für $p \in (1,\infty)$.

Ehrenwörtliche Erklärung

Ich versichere hiermit, dass ich die Arbeit selbstständig angefertigt habe und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt sowie die wörtlich oder inhaltlich übernommenen Stellen als solche kenntlich gemacht habe.

Ulm, den 9. Juli 2014

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Stephan Fackler

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