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Infinitely divisible and related distributions and Lévy driven stochastic partial differential equations

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Abstract

This thesis deals with different topics in probability theory. We are interested in infinitely divisible distributions and their densities, the class of quasi-infinitely divisible distributions and Lévy driven stochastic partial differential equations.

In Chapter 2 we deal with infinitely divisible distributions and their densities. We obtain bounds of the integral modulus of continuity in terms of the characteristic triplet. We then apply our results to stochastic integrals.

In Chapter 3 we study the class of probability measures $\mu(dx) = \mu_{ld}(dx) + \mu_{ac}(dx)$, where μ_{ld} is a discrete lattice distribution and μ_{ac} is absolutely continuous. We prove that if $\hat{\mu}_{ld}(z) \neq 0$ for all $z \in \mathbb{R}$ then μ is quasi-infinitely divisible if and only if $\hat{\mu}(z) \neq 0$ for all $z \in \mathbb{R}$. As an application of this result we study certain variance mixtures and prove that they are quasi-infinitely divisible.

In Chapter 4 we give sufficient conditions for the existence of a generalized solution s in the space of distributions of the stochastic partial differential equation $p(D)s = q(D)\dot{L}$, where p and q are polynomials in $\mathbb{C}^d$ and $\dot{L}$ is a so called Lévy white noise. Furthermore, we give sufficient conditions for the existence of a mild solution and provide a sufficient condition when the mild solution can be identified with a generalized solution.

Chapter 5 deals with linear and semilinear Lévy driven stochastic partial differential equations. In the linear case we work with different distributional spaces and show existence and uniqueness results under different assumptions. As a next step we analyze a semilinear partial differential equation driven by Lévy white noise in weighted Besov spaces.

In Chapter 6 we prove central limit theorems for the sample mean and autocovariance of a moving average random field. We use a sampling scheme on a grid, which can be deterministic or random.

Zusammenfassung

Diese Arbeit beschäftigt sich mit verschiedenen Themen in der Wahrscheinlichkeitstheorie. Wir sind interessiert an unendlich teilbaren Verteilungen und deren Dichten, quasi-unendlich teilbaren Verteilungen und Lévy getriebenen stochastisch partiellen Differentialgleichungen.

In Kapitel 2 beschäftigen wir uns mit unendlich teilbaren Verteilungen und deren Dichten. Wir finden Schranken für den Integralmodulus der Stetigkeit in Abhängigkeit vom charakteristischen Triplet. Wir wenden diese Resultate auf stochastische Integrale an.

In Kapitel 3 untersuchen wir die Menge aller Wahrscheinlichkeitsmaße gegeben durch $\mu(dx) = \mu_{ld}(dx) + \mu_{ac}(dx)$, wo μ_{lc} ein Maß auf einem diskreten Gitter ist und μ_{ac} absolut stetig ist. Wir zeigen, dass, falls $\hat{\mu}_{ld}(z) \neq 0$ für alle $z \in \mathbb{R}$, μ genau dann quasi-unendlich teilbar ist, wenn $\hat{\mu}(z) \neq 0$ für alle $z \in \mathbb{R}$. Als Anwendung davon untersuchen wir bestimmte Varianzmischungen und zeigen, dass diese quasi-unendlich teilbar sind.

In Kapitel 4 geben wir hinreichende Bedingungen für die Existenz generalisierter Lösungen s im Raum der Distributionen der stochastisch partiellen Differentialgleichungen $p(D)s = q(D)\dot{L}$, wobei p und q Polynome in $\mathbb{C}^d$ und $\dot{L}$ ein Lévy weißes Rauschen sind. Weiterhin finden wir hinreichende Bedingungen für die Existenz milder Lösungen und zeigen hinreichende Bedingungen, dass die milde Lösung identifiziert werden kann mit der generalisierten Lösung.

Kapitel 5 beschäftigt sich mit linearen und semilinearen Lévy getriebenen stochastisch partiellen Differentialgleichungen. Im linearen Fall arbeiten wir mit unterschiedlichen Distributionsräumen und zeigen Existenz- und Eindeutigkeitsresultate. Als nächsten Schritt analysieren wir eine von Lévy weißem Rauschen getriebene semilineare partielle Differentialgleichung in gewichteten Besovräumen.

In Kapitel 6 beweisen wir zentrale Grenzwertsätze für das Stichprobenmittel und die Stichprobenautokovarianz für Moving Average Zufallsfelder. Wir benutzen Stichproben auf einem Gitter, wobei die Stichprobe deterministisch oder zufällig sein kann.

Contents

Abstract V

Zusammenfassung VII

1 Introduction 1

1.1 Infinitely divisible distributions and the integral modulus of continuity (Chapter 2) . . . 2
1.2 Quasi-infinitely divisible distributions (Chapter 3) . . . 4
1.3 Lévy driven CARMA SPDEs on $\mathcal{D}'(\mathbb{R}^d)$ (Chapter 4) . . . 8
1.4 CARMA SPDEs in the spaces of tempered (ultra-)distributions and Fourier hyperfunctions and semilinear SPDEs (Chapter 5) . . . 13
1.5 Central limit theorems for moving average random fields (Chapter 6) . . . 16
1.6 Structure of the thesis . . . 19

2 On the integral modulus of continuity of infinitely divisible distributions, especially of stochastic integrals 21

2.1 Introduction . . . 21
2.2 Notation and preliminaries . . . 23
2.3 Densities of infinitely divisible distributions . . . 23
2.4 Densities of stochastic integrals . . . 28
2.4.1 Stochastic integrals over compact intervals . . . 29
2.4.2 Stochastic integrals over $[0, \infty)$ . . . 34

3 On Quasi-Infinitely Divisible Distributions with a Point Mass 39

3.1 Introduction . . . 39
3.2 Notation and Preliminaries . . . 41
3.3 Complex quasi-Lévy Type measures . . . 42
3.4 Some new quasi-infinitely divisible distributions . . . 45
3.4.1 Absolutely continuous distributions plus a point mass . . . 45
3.4.2 Absolutely continuous distributions plus a lattice distribution . . . 51
3.5 Topological Properties of quasi-infinitely divisible distributions . . . 54

4 Lévy driven CARMA generalized processes and stochastic partial differential equations 57

4.1 Introduction . . . 57

4.2 Notation and Preliminaries . . . 59
4.3 SPDEs and generalized solutions . . . 61
4.3.1 The concept of generalized solutions . . . 61
4.3.2 Generalized stochastic processes constructed from Lévy white noise . . 63
4.4 CARMA generalized processes . . . 71
4.4.1 Homogeneous Elliptic SPDEs . . . 71
4.4.2 General CARMA setting . . . 73
4.5 CARMA random fields . . . 75
4.6 Moment properties . . . 79
4.7 CARMA random fields in the sense of Brockwell and Matsuda . . . 81

5 Lévy driven linear and semilinear stochastic partial differential equations 85
5.1 Introduction . . . 85
5.2 Notation and Preliminaries . . . 86
5.3 Linear stochastic partial differential equations in the spaces of tempered distributions, tempered ultradistributions, Fourier hyperfunctions and Besov spaces with polynomial weights . . . 88
5.4 Semilinear stochastic partial differential equations . . . 97

6 Central Limit Theorems for Moving Average Random Fields with Non-Random and Random Sampling On Lattices 101
6.1 Introduction . . . 101
6.2 Notation and Preliminaries . . . 103
6.3 Main results . . . 105
6.3.1 Central limit theorems for the sample mean . . . 105
6.3.2 From Non-Random Sampling to Random Sampling . . . 108
6.3.3 Non-Random Sampling of the Autocovariance . . . 109
6.3.4 Random Sampling of the Autocovariance . . . 110
6.4 Applications . . . 110
6.5 Proof of Theorems 6.2 and 6.7 . . . 112
6.6 Proof of Theorems 6.9 and 6.10 . . . 114

Bibliography 119

Acknowledgements 125

Erklärung 127

Curriculum Vitae 129

1 Introduction

Infinitely divisible distributions are well-studied probability measures and can be used in different applications, see [58] for a detailed introduction. An infinitely divisible distribution μ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that its characteristic function $\hat{\mu}$ is given by

$$\hat{\mu}(z) := \int_{\mathbb{R}} e^{ixz} \mu(dx) = \exp(\psi(z)), \tag{1.1}$$

where $\psi : \mathbb{R}^d \to \mathbb{C}$ is given by

$$\psi(z) = i\gamma z - \frac{1}{2} a z^2 + \int_{\mathbb{R}} (e^{iyz} - 1 - iyz \mathbf{1}_{|y| \leq 1}) \nu(dy), \tag{1.2}$$

where $\gamma \in \mathbb{R}$, $a \geq 0$ and ν is a Lévy measure, i.e. a measure such that $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}} \min(1, y^2) \nu(dy) < \infty$. We call (a, γ, ν) the *characteristic triplet* of the infinitely divisible distribution μ.
One defining property of an infinitely divisible distribuiton μ is that $\hat{\mu}^t$ defines the characteristic function for some probability measure μ_t for every $t \in [0, \infty)$. This is equivalent to the fact that for every $n \in \mathbb{N}$ there exists a probability measure μ_n on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that

$$\mu_n^{*n} = \mu, \tag{1.3}$$

where μ_n^{*n} is the n-fold convolution of μ_n with itself.
Therefore, it is also possible to define an infinitely divisible distribution μ by (1.3).
In this thesis it is more suitable to define an infinitely divisible distribution by (1.1) and (1.2), as the characteristic triplet (a, γ, ν) plays a central role in all following chapters.
The thesis can be divided into two parts. The first part deals with infinitely divisible and related distributions, namely in Chapter 2 we study the integral modulus of continuity of a density of an infinitely divisible distribution and in Chapter 3 we deal with quasi-infinitely divisible distributions, an extension of the infinitely divisible distributions.
The second part deals with stochastic partial differential equations driven by Lévy white noise and random fields. Chapter 4 deals with linear stochastic partial differential equations in the space of infinitely differentiable functions with compact support. In the next chapter we use different function spaces, e.g. the space of tempered distributions and the space of Fourier hyperfunctions, to obtain existence and uniqueness of certain linear and seminlinear stochastic partial differential equations. In the last chapter we

obtain for moving average random fields central limit theorems for the sample mean and autocovariance.

1.1 Infinitely divisible distributions and the integral modulus of continuity (Chapter 2)

Chapter 2 deals with densities of infinitely divisible distributions and their integral modulus of continuity.

Every probability measure μ can be decomposed into $\mu_d + \mu_{ac} + \mu_{cs}$, where μ_d is a discrete, μ_{ac} is an absolutely continuous and μ_{cs} is a continuous singular measure, where the continuity is with respect to the Lebesgue measure. It is interesting to know that there exist probability measures μ_1, μ_2 and μ_3 which are infinitely divisible and μ_1 is discrete, μ_2 is absolutely continuous and μ_3 is continuous singular. For each class (discrete, absolutely continuous and continuous singular) there are sufficient conditions in terms of the characteristic triplet. In this section, we are mainly interested in the absolutely continuous case. Let therefore μ be an infinitely divisible distribution which is absolutely continuous with respect to the Lebesgue measure λ, i.e. there exists a function $f_\mu \in L^1(\mathbb{R}, [0,\infty))$ such that $\mu(dx) = f_\mu(x)\lambda(dx)$. Until now there exists no complete characterization, in terms of the characteristic triplet, when an infinitely divisible probability measure μ is absolutely continuous, but at least sufficient conditions are known. Even more, several properties of the density can be proven by the characteristic triplet, e.g. (Hölder)-continuity, differentiability, integrability and many other properties. We are interested in the integral modulus of continuity. The integral modulus of continuity of a function $g : \mathbb{R} \to \mathbb{R}$ is defined by

$$I(g) : \mathbb{R} \to \mathbb{R}^+,\ z \mapsto \int_{\mathbb{R}} |g(z+x) - g(x)|\lambda(dx),$$

and we are interested in estimates of the form

$$I(g)(z) \leq C|z|^\alpha \text{ for all } z \in \mathbb{R} \tag{1.4}$$

for some fixed $\alpha \in (0,1]$ and $C > 0$. In the special case that $\alpha = 1$, (1.4) implies that g is of bounded variation. So we are looking for an $\alpha \in (0,1]$ such that $I(f_\mu)$ satisfies an estimate of the form (1.4). We give sufficient conditions in terms of the Lévy measure ν by assuming that it has a Lebesgue density around 0 and obtain in Theorem 2.2 the following results:

> Let μ be an infinitely divisible distribution with characteristic triplet (a, γ, ν) where $a \geq 0$, $\gamma \in \mathbb{R}$ and ν a Lévy measure such that $|x|\nu(dx)$ has a Lebesgue density k in a neighbourhood around zero with $\liminf_{x\to 0+} k(x) + \liminf_{x \to 0-} k(x) =: c_{\inf}$.

i) If $c_{\inf} > 1/p$ for $1 < p \le 2$, then μ has a Lebesgue density $f_\mu \in L^1(\mathbb{R}, \mathbb{R}^+) \cap L^{p/(p-1)}(\mathbb{R}, \mathbb{R}^+)$ and there exists a constant $C > 0$ such that

$$\int_{\mathbb{R}} |f_\mu(x-z) - f_\mu(x)| \, \lambda(dx) \le C|z|^{\frac{1}{p}}$$

for every $z \in \mathbb{R}$.
ii) If $c_{\inf} > 1$, then f_μ is continuous on $\mathbb{R}$ and there exists a constant $C > 0$ such that

$$\int_{\mathbb{R}} |f_\mu(x-z) - f_\mu(x)| \, \lambda(dx) \le C|z|$$

for every $z \in \mathbb{R}$.
iii) Now let $c_{\sup} := \limsup_{x \to 0+} k(x) + \limsup_{x \to 0-} k(x) < \frac{1}{p}$ with $p \in (0, \infty)$ and let $a = 0$. Then, if μ has a Lebesgue density f_μ, it satisfies

$$\sup_{0 \le h \le |z|} \int_{\mathbb{R}} |f_\mu(x-h) - f_\mu(x)| \, \lambda(dx) \ge C|z|^{\frac{1}{p}} \tag{1.5}$$

for some constant $C > 0$ and $z \in (-1, 1)$.

As an application of the above result we will analyze the density of the stochastic integral $\int_{[0,t)} g(x) dL(x)$, where $t \in [0, \infty]$, $g : [0, t) \to \mathbb{R}$ is a deterministic function and L is a non-deterministic Lévy process with characteristic triplet (a, γ, ν). The existence of the integrals $\int_0^t g(s) dL(s)$ or $\int_0^\infty g(s) dL(s)$ can be completely characterized by the characteristic triplet (a, γ, ν) of L and g, see [56, Theorem 2.7, p. 461]. Moreover, the integrals are infinitely divisible with characteristic triplet (a_g, γ_g, ν_g) where

$$\begin{aligned}
\gamma_g &= \int_{[0,t)} \left(\gamma g(s) + \int_{\mathbb{R}} g(s) r (\mathbf{1}_{[-1,1]}(g(s)r) - \mathbf{1}_{[-1,1]}(r)) \, \nu(dr) \right) \lambda(ds), \\
a_g &= \int_{[0,t)} a g(s)^2 \, \lambda(ds) \text{ and} \\
\nu_g(B) &= \int_{[0,t)} \int_{\mathbb{R}} \mathbf{1}_{B \setminus \{0\}}(g(s)r) \, \nu(dr) \lambda(ds), \quad B \in \mathcal{B}
\end{aligned}$$

with $t \in [0, \infty]$. In the case that t is finite, we will obtain the following lemma, see Lemma 2.5:

Let $g : [0, t] \to \mathbb{R}$ be a $\mathcal{C}^1$-Diffeomorphism onto its range.
i) Then $|x|\nu_g(dx)$ is absolutely continuous with Lebesgue density k given by

$$k(x) = \int_{\mathbb{R}} \mathbf{1}_{g([0,t])}(x/r) \frac{|x|}{|r|} \left| (g^{-1})'(x/r) \right| \nu(dr) < \infty$$

for all $x \in \mathbb{R} \setminus \{0\}$.
ii) Let $g > 0$ in $[0,t]$. If $\liminf_{x\to 0+} \nu(\frac{x}{g([0,t])}) = \lambda_1 > 0$, then $\liminf_{x\to 0+} k(x) \geq \inf_{y\in g([0,t])} |y||(g^{-1})'(y)|\lambda_1$ and if $\limsup_{x\to 0+} \nu(\frac{x}{g([0,t])}) = \lambda_2 < \infty$, then $\limsup_{x\to 0+} k(x) \leq \sup_{y\in g([0,t])} |y||(g^{-1})'(y)|\lambda_2$.

In the case that $t = \infty$ we only cite some special cases, the more general result can be found in Chapter 2. We assume that g is a strictly positive, continuous function which attains its maximum

$$c := \max_{t\in[0,\infty)} g(t)$$

and that there exists a decomposition $(t_i)_{i\in\mathbb{N}_0}$ with $0 = t_0 < t_1 < \ldots$ and $t_i \to \infty$ for $i \to \infty$ such that g restricted to (t_i, t_{i+1}) is a $\mathcal{C}^1-$diffeomorphism onto its range for every $i \in \mathbb{N}_0$. In Corollary 2.14 we will prove:

> Let $g : [0,\infty) \to (0,\infty)$ have the same properties as above, denote $T := \{t_i : i \in \mathbb{N}\}$ and assume that
>
> $$\liminf_{x\to\infty, x\notin T} \left|\frac{g(x)}{g'(x)}\right| = \alpha$$
>
> for some $\alpha \in (0,\infty]$. Then $\int_{[0,\infty)} g(t)\,dL(t)$ has a density of bounded variation, if $\nu(\mathbb{R}) > \frac{1}{\alpha}$.

For $g(x) = e^{-x^2}$ we cannot use Corollary 2.14 to find a sufficient condition when the density is of bounded variation. Therefore, we obain by a similar technique Corollary 2.17:

> Let $g(x) = e^{-\psi(x)}$ with $\psi : [0,\infty) \to \mathbb{R}$ continuous such that $\psi : (0,\infty) \to (0,\infty)$ is a strictly increasing $\mathcal{C}^1$-diffeomorphism and such that $\psi(0) = 0$ and $(\psi^{-1})'$ is decreasing. Then the Lebesgue density of $\int_{[0,\infty)} g(t)dL(t)$ is of bounded variation if
>
> $$\liminf_{x\to 0+} \frac{\nu((x,1))}{\psi'(\psi^{-1}(-\log(x)))} + \liminf_{x\to 0-} \frac{\nu((-1,x))}{\psi'(\psi^{-1}(-\log|x|))} > 1.$$

1.2 Quasi-infinitely divisible distributions (Chapter 3)

An infinitely divisible distribution μ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is characterized by its characteristic triplet (a,γ,ν), where $a \geq 0$, $\gamma \in \mathbb{R}$ and ν is a Lévy measure. A natural extension of this concept would be to allow more complex characteristic triplets, like $a \in \mathbb{R}$ and ν a "signed" Lévy measure. It is clear that not for every "signed" Lévy measure the function

$$\exp\left(i\gamma z - \frac{1}{2}az^2 + \int_{\mathbb{R}} (e^{iyz} - 1 - iyz\mathbf{1}_{|y|\leq 1})\nu(dy)\right)$$

is a characteristic function of a probability measure. Therefore, we need to differentiate between "good" and "bad" "signed" Lévy measures. The formal definition is then given by (see Definition 3.1):

i) Let $\mathcal{B}_r := \{B \in \mathcal{B} |\, B \cap (-r,r) = \emptyset\}$ for $r > 0$ and $\mathcal{B}_0 := \cup_{r>0}\mathcal{B}_r$ be the class of all Borel sets that are bounded away from zero. Let $\nu : \mathcal{B}_0 \to \mathbb{R}$ be a function such that $\nu|_{\mathcal{B}_r}$ is a finite signed measure for each $r > 0$ and denote the total variation, positive and negative part of $\nu_{|\mathcal{B}_R}$ by $|\nu_{|\mathcal{B}_r}|, \nu^+_{|\mathcal{B}_r}$ and $\nu^-_{|\mathcal{B}_r}$, respectively. Then the total variation $|\nu|$, the positive part ν^+ and the negative part ν^- of ν are defined to be the unique measures on $(\mathbb{R}, \mathcal{B})$ satisfying

$$|\nu|(\{0\}) = \nu^+(\{0\}) = \nu^-(\{0\}) = 0$$

and

$$|\nu|(A) = |\nu_{|\mathcal{B}_r}|(A),\ \nu^+(A) = \nu^+_{|\mathcal{B}_r}(A),\ \nu^-(A) = \nu^-_{|\mathcal{B}_r}(A)$$

when $A \in \mathcal{B}_r$ for some $r > 0$.
ii) A *quasi-Lévy type measure* is a function satisfying the condition of *a)* such that its total variation $|\nu|$ satisfies $\int_{\mathbb{R}}(1 \wedge x^2)|\nu|(dx) < \infty$.
iii) Let μ be a probability distribution on $\mathbb{R}$. We say that μ is *quasi-infinitely divisible* if its characteristic function has a representation

$$\hat{\mu}(z) = \exp\left(-\frac{1}{2}az^2 + i\gamma z + \int_{\mathbb{R}} (e^{ixz} - 1 - ixz\mathbf{1}_{[-1,1]}(x))\nu(dx)\right),$$

where $a, \gamma \in \mathbb{R}$ and ν is a quasi-Lévy-type measure. The *characteristic triplet* (a, γ, ν) of μ is unique (see [58, Exercise 12.2]), and a is called the *Gaussian variance* of μ.
iv) A quasi-Lévy type measure ν is called *quasi-Lévy measure*, if additionally there exist a quasi-infinitely divisible distribution μ and some $a, \gamma \in \mathbb{R}$ such that (a, γ, ν) is the characteristic triplet of μ. We call ν the *quasi-Lévy measure of μ*.
v) Let μ be quasi-infinitely divisible with characteristic triplet (a, γ, ν). If $\int_{[-1,1]} |z|\, |\nu|(dx) < \infty$, then we call $\gamma_0 := \gamma - \int_{[-1,1]} z\, \nu(dz)$ the *drift* of μ. In that case, the characteristic function of μ allows the representation

$$\hat{\mu}(z) = \exp\left(-\frac{1}{2}az^2 + i\gamma_0 z + \int_{\mathbb{R}} (e^{ixz} - 1)\nu(dx)\right).$$

[51] started with a detailed study of this subject and obtained a complete characterization of quasi-infinitely divisible distributions on every affine lattice, i.e. a probability measure μ on a lattice $r + h\mathbb{Z}$, $r \in \mathbb{R}$ and $h > 0$, is quasi-infinitely divisible if and only if $\hat{\mu}(z) \neq 0$

for all $z \in \mathbb{R}$. This result has several consequences, and we discuss some of them here. At first, the quasi-infinitely divisible distributions are dense in the space of probability measures, which implies that the quasi-infinitely divisible distributions are not closed in the topology of the weak convergence. An easy example can be constructed by the Bernoulli probability measures $(\mu_p)_{p\in[0,1]}$ defined by

$$\mu_p(dx) = p\delta_0(dx) + (1-p)\delta_1(dx)$$

for $p \in [0,1]$. We see from above that μ_p is quasi-infinitely divisible if and only if $p \neq \frac{1}{2}$. As for every sequence $(p_n)_{n\in\mathbb{N}} \subset [0,1] \setminus \{\frac{1}{2}\}$ converging to $\frac{1}{2}$ the sequence $(\mu_{p_n})_{n\in\mathbb{N}}$ converges weakly to $\mu_{\frac{1}{2}}$ as $n \to \infty$, we see that the space of quasi-infinitely divisible distributions is not closed in the topology generated by the weak convergence. Moreover, the counterexample shows even that the quasi-infinitely divisible distribution intersected with the probability measures with existing q-moment, $q \geq 1$, is not closed in the topology generated by the q-Wasserstein metric.
In this chapter we characterize the quasi-infinitely divisible distributions of the form

$$\mu(dx) = p\delta_0(dx) + (1-p)f(x)\lambda(dx), \tag{1.6}$$

with

$$\hat{\mu}(z) = p + (1-p)\int_{\mathbb{R}} e^{ixz} f(x)\lambda(dx), \tag{1.7}$$

where $f(x)\lambda(dx)$ is an absolutely continuous probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and $p \in (0,1]$.
A keytool of proving that every lattice distribution without vanishing characteristic function is quasi-infinitely divisible was the so called Wiener-Lévy Lemma. The Wiener-Lévy Lemma basically gives at first sight a complex quasi-Lévy measure, i.e. the real and imaginary part are quasi-Lévy type measures, and it was shown that in this case the quasi-Lévy measure is indeed a real (signed) measure. So it is only natural to ask if there exists a probability measure such that its characteristic function has a Lévy-Khintchine-type representation with a complex quasi-Lévy measure. We prove that in this case, the complex quasi-Lévy measure must indeed be signed, implying that the measare μ is infact quasi-infinitely divisible, see Theorem 3.5.

> Let μ be a distribution on $\mathbb{R}$ whose characteristic function allows a representation of the form
>
> $$\hat{\mu}(z) = \exp\left(i\gamma z - \frac{1}{2}az^2 + \int_{\mathbb{R}} (e^{ixz} - 1 - izx\mathbf{1}_{[-1,1]}(x))\nu(dx)\right) \tag{1.8}$$
>
> where $\gamma \in \mathbb{C}$, $a \in \mathbb{C}$ and ν is a complex quasi-Lévy-type measure. Then $a, \gamma \in \mathbb{R}$, $a \geq 0$ and $\operatorname{Im} \nu = 0$, i.e. ν is a quasi-Lévy measure and μ is quasi-infinitely divisible.

By using a Wiener-Lévy Lemma for functions like in (1.7) and the above theorem, we give a similar characterization of quasi-infinitely divisible distributions of type (1.6) as in the case of lattice distributions. But first we need the definition of the index of $\hat{\mu}$ given in (1.7):

> Let $F(x) = p + \int_{\mathbb{R}} e^{ixz} f(z)\lambda(dz) \neq 0$ for every $x \in \mathbb{R}$ and $p \in \mathbb{C} \setminus \{0\}$. Then we can interpret F as a closed curve in $\mathbb{C}$. By the property of the distinguished logarithm, there exists a continuous function $g : \mathbb{R} \to \mathbb{R}$ such that
>
> $$\frac{F(x)}{|F(x)|} = \exp\left(ig(x)\right) \qquad \text{for all } x \in \mathbb{R}.$$
>
> Then the *index* $\operatorname{ind}(F)$ of F is defined as
>
> $$\operatorname{ind}(F) := \frac{1}{2\pi}\left(\lim_{z\to+\infty} g(z) - \lim_{z\to-\infty} g(z)\right) =: \frac{1}{2\pi}(g(\infty) - g(-\infty)).$$

We obtain the following result, see Theorem 3.11:

> Let μ be a probability distribution of the form (1.6). Then μ is quasi-infinitely divisible if and only if $\hat{\mu}(z) \neq 0$ for every $z \in \mathbb{R}$. In that case, the quasi-Lévy measure ν of μ is given by
>
> $$\left(g(x) + \frac{me^{-|x|}}{|x|}\left(\mathbf{1}_{(0,\infty)}(x) - \mathbf{1}_{(-\infty,0)}(x)\right) \right) \lambda(dx),$$
>
> where $g \in L^1(\mathbb{R}, \mathbb{R})$ and m is the index of $\widehat{\mu}$. Furthermore, $\int_{-1}^{1} |x||\nu|(dx) < \infty$, μ has drift 0 and Gaussian variance 0. Finally, $|\nu|$ is finite if and only if $m = 0$, and if $m \neq 0$, then $\nu^-(\mathbb{R}) = \nu^+(\mathbb{R}) = \infty$.

Using the above result, in Example 3.12 we construct a non-continuous quasi-infinitely divisible distribution with infinite quasi-Lévy measure, thus answering an open question posed in [51, Open Question 7.2, p. 8510] in the negative. The theorem above can be used to find sufficient conditions such that a distribution

$$\mu(dx) = p\mu_d(dx) + (1-p)\mu_a c,$$

where μ_d is a lattice distribution and μ_{ac} an absolutely continuous probability measure, is quasi-infinitely divisible. Namely, if $\hat{\mu_d}(z) \neq 0 \neq \hat{\mu}(z)$ for all $z \in \mathbb{R}$, then μ is quasi-infinitely divisible.

Furthermore, we obtain some topological properties of the space of the quasi-infinitely divisible distributions (QID). We prove that QID is not open, but path-connected, and therefore even connected, see Propositions 3.21 and 3.22.

1.3 Lévy driven CARMA SPDEs on $\mathcal{D}'(\mathbb{R}^d)$ (Chapter 4)

In this chapter we will deal with a stochastic partial differential equation of the form

$$p(D)s = q(D)\dot{L}, \tag{1.9}$$

where $p(z) = \sum_{|\alpha|\le m} p_\alpha z^\alpha$, $q(z) = \sum_{|\beta|\le n} q_\beta z^\beta$, $n, m \in \mathbb{N}_0$, are real polynomials in d variables, $p(D) = \sum_{|\alpha|\le m} p_\alpha \partial_1^{\alpha_1}\cdots\partial_d^{\alpha_d}$ and $\dot{L}$ is a Lévy white noise on a suitable function space. We will later give the definitions and results, but at first we give a small introduction to (C)ARMA processes and multidimensional extensions.
A CARMA(p, q) process $(X_t)_{t\in\mathbb{R}}$, where $p > q$, is given by

$$X_t = b' Y_t,\ t \in \mathbb{R}, \tag{1.10}$$

where $Y = (Y_t)_{t\in\mathbb{R}}$ is a $\mathbb{C}^p$-valued process satisfying the stochastic differential equation

$$dY_t = AY_t dt + e_p dL_t \tag{1.11}$$

with

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_p & -a_{p-1} & -a_{p-2} & \dots & -a_1 \end{pmatrix}, e_p = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \in \mathbb{C}^p \text{ and } b = \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_{p-2} \\ b_{p-1} \end{pmatrix},$$

where $a_1, \dots, a_p, b_0, \dots, b_{p-1} \in \mathbb{C}$ are determinstic coefficients such that $b_q \neq 0$ and $b_j = 0$ for every $j > q$, b' denotes the transpose of b and $L = (L_t)_{t\in\mathbb{R}}$ is a two-sided Lévy process. The equations (1.10) and (1.11) are the so called state-space representation of the formal stochastic differential equation

$$a(D)Y_t = b(D)DL_t, \tag{1.12}$$

with D the differential operator and $a(z) = z^p + a_1 z^{p-1} + \ldots + a_p$ and $b(z) = b_0 + b_1 z + \ldots + b_q z^q$ are polynomials. It is easy to see that the CARMA process is a continuous extension of the so called ARMA processes, i.e. an ARMA process $(X_k)_{k\in\mathbb{Z}}$, $p, q \in \mathbb{N}_0$, is the stationary solution of the difference equation

$$X_k - \sum_{i=1}^{p} a_i X_{k-i} = W_k + \sum_{j=1}^{q} b_j W_{k-j},$$

where $a_1, \dots, a_p, b_1, \dots, b_q \in \mathbb{C}$ are deterministic coefficients and $(W_k)_{k\in\mathbb{Z}}$ is white noise or even an independent and identically distributed (iid) sequence of random variables. In short form we can also write

$$a(B)X_k = b(B)W_k,$$

where $a(z) = 1 - \sum_{i=1}^{p} a_i z^k$, $b(z) = 1 + \sum_{j=1}^{q} b_j z^j$ are polynomials and B is the shift operator defined by $B^l Y_k = Y_{k-l}$ for $l \in \mathbb{N}$. ARMA and CARMA processes have many applications in different areas, e.g. in finance, astrophysics, engineering and traffic data, see [35], [29], [67] and [46]. In [15] necessary and sufficient conditions on L and A were given such that there exists a strictly stationary solution of (1.10) and (1.11), namely it was shown that it is sufficient and necessary that $\mathbb{E}\log^+(|L_1|) < \infty$.
The interpretation of the stochastic differential equation (1.12) through the state space representation (1.10), (1.11) makes only sense when the degree of a is greater than the degree of b. [14] studied generalized solutions of (1.12) under the assumption that the degree of b is greater than the degree of a, and introduced for Wiener white noise the CARMA(a, b) generalized processes.
The goal of this chapter is to find a mulitdimensional extension of (1.10) and (1.11). Until now there exist at least two other definitions of a multidimensional CARMA random field, here we recall the definition of Brockwell and Matsuda [16] and Pham [55].
In [16] the new CARMA random field was given by

$$S_d(t) := \int_{\mathbb{R}^d} \sum_{r=1}^{p} \frac{b(\lambda_r)}{a'(\lambda_r)} e^{\lambda_r \|t-u\|} dL(u), \tag{1.13}$$

where dL denotes the integration over a Lévy bases, a and b are polynomials such that $a(z) = \prod_{i=1}^{p}(z^2 - \lambda_i^2)$ and some further restrictions.
Pham [55] defines a CARMA random field Y as a mild solution of the system of SPDEs given by

$$Y(t) = b'X(t),\, t \in \mathbb{R}^d, \tag{1.14}$$

$$(I_p\partial_d - A_d)\cdots(I_p\partial_1 - A_1)X(t) = c\dot{L}(t),\, t \in \mathbb{R}^d, \tag{1.15}$$

where $\dot{L}$ is a Lévy basis, $A_1, \dots, A_d \in \mathbb{R}^{p\times p}$ are matrices and I_p is the identity matrix.
Both models have their advantages, for example the models have a well understood second order behaviour, can be used for statistical estimation and the solution of the system (1.14) depends only on the past in the sense that the solution at point x depends solely on the behaviour of $\dot{L}$ on $(-\infty, x_1] \times \cdots \times (-\infty, x_d]$ (therefore in [55] he speaks about causal CARMA random fields).
Our definition of a CARMA "random field" (more generalized stochastic process, see the later definitions) is motivated by (1.12), as both models give not a complete picture in terms of (1.12). In order to give a clear definition of our concept of a CARMA random field, we need the definition of a generalized process. We denote by $\mathcal{D}(\mathbb{R}^d)$ the space of infinitely differentiable functions with compact support and equip it with its usual

topology. Here, $\mathcal{D}'(\mathbb{R}^d)$ denotes the space of distributions on $\mathbb{R}^d$, i.e. the topological dual of $\mathcal{D}(\mathbb{R}^d)$.

A generalized random process is a linear and continuous function $s : \mathcal{D}(\mathbb{R}^d) \to L^0(\Omega)$. The linearity means that, for every $\varphi_1, \varphi_2 \in \mathcal{D}(\mathbb{R}^d)$ and $\gamma \in \mathbb{R}$,

$$s(\varphi_1 + \gamma\varphi_2) = s(\varphi_1) + \gamma s(\varphi_2) \text{ almost surely.}$$

The continuity means that if $\varphi_n \to \varphi$ in $\mathcal{D}(\mathbb{R}^d)$, then $s(\varphi_n) \to s(\varphi)$ in $L^0(\Omega)$.

As shown in [65, Corollary 4.2], there exists a measurable version from $(\Omega, \mathcal{F})$ to $(\mathcal{D}'(\mathbb{R}^d), \mathcal{C})$ with respect to the cylindrical σ-field $\mathcal{C}$ generated by the sets

$$\{u \in \mathcal{D}'(\mathbb{R}^d) | \, (\langle u, \varphi_1 \rangle, \dots , \langle u, \varphi_N \rangle) \in B\}$$

with $N \in \mathbb{N}$, $\varphi_1, \dots , \varphi_N \in \mathcal{D}(\mathbb{R}^d)$ and $B \in \mathcal{B}(\mathbb{R}^N)$.
The probability law of a generalized random process s is given by

$$\mathcal{P}_s(B) := \mathcal{P}(s \in B)$$

for $B \in \mathcal{C}$. The characteristic functional $\widehat{\mathcal{P}}_s$ is then defined by

$$\widehat{\mathcal{P}}_s(\varphi) = \int\limits_{\mathcal{D}'(\mathbb{R}^d)} \exp(i\langle u, \varphi \rangle) d\mathcal{P}_s(u), \; \varphi \in \mathcal{D}(\mathbb{R}^d).$$

We will work with Lévy white noise, which is a generalized random process where the characteristic functional satisfies a Lévy-Khintchine representation.

A Lévy white noise $\dot{L}$ is a generalized random process, where the characteristic functional is given by

$$\widehat{\mathcal{P}}_{\dot{L}}(\varphi) = \exp\left(\int\limits_{\mathbb{R}^d} \psi(\varphi(x)) \lambda^d(dx) \right)$$

for every $\varphi \in \mathcal{D}(\mathbb{R}^d)$, where $\psi : \mathbb{R} \to \mathbb{C}$ is given by

$$\psi(z) = i\gamma z - \frac{1}{2} a z^2 + \int\limits_{\mathbb{R}} (e^{ixz} - 1 - ixz\mathbf{1}_{|x| \le 1}) \nu(dx)$$

where $a \in \mathbb{R}^+$, $\gamma \in \mathbb{R}$ and ν is a Lévy measure, i.e. a measure such that $\nu(\{0\}) = 0$ and

$$\int\limits_{\mathbb{R}} \min(1, x^2) \nu(dx) < \infty.$$

We say that $\dot{L}$ has the characteristic triplet (a, γ, ν).

The existence of the Lévy-white noise was proven in [34]. The domain of the Lévy white noise can also be extended to indicator functions $\mathbf{1}_A$ for Borel sets A with finite Lebesgue measure by using the construction in [32, Proposition 3.4]. For a more general function f we say that f is in the domain of $\dot{L}$ if there exists a sequence of elementary functions f_n converging almost everywhere to f such that $\langle \dot{L}, f_n \mathbf{1}_A \rangle$ converges in probability for $n \to \infty$ for every Borel set A and set $\langle \dot{L}, f \rangle$ as the limit in probability of $\langle \dot{L}, f_n \rangle$ for $n \to \infty$, where for an elementary function $f := \sum_{j=1}^m a_j \mathbf{1}_{A_j}$, $\langle \dot{L}, f \rangle$ is defined by $\sum_{j=1}^m a_j \langle \dot{L}, \mathbf{1}_{A_j} \rangle$, see also [32, Definition 3.6]. For the maximal domain of the Lévy white noise $\dot{L}$ we write $L(\dot{L})$. By setting $L(A) := \langle \dot{L}, \mathbf{1}_A \rangle$ for bounded Borel sets A, the extention of a Lévy white noise $\dot{L}$ can be identified with a Lévy basis L in the sense of Rajput and Rosinski [56], see [32, Theorem 3.5 and Theorem 3.7]. As a Lévy basis can be identified with a Lévy white noise in a canonical way, i.e. $\langle \dot{L}, \varphi \rangle := \int_{\mathbb{R}^d} \varphi(x) dL(x)$ for $\varphi \in \mathcal{D}(\mathbb{R}^d)$, we do not differ between a Lévy basis and Lévy-white noise. In particular, a Borel-measurable function $f : \mathbb{R}^d \to \mathbb{R}$ is in $L(\dot{L})$ if and only if f is integrable with respect to the Lévy basis L in the sense of Rajput and Rosinski [56], see [32, Def. 3.6].
The Lévy white noise is stationary in the following sense.

A generalized process s is called *stationary* if for every $t \in \mathbb{R}^d$, $s(\cdot + t)$ has the same law as s. Here, $s(\cdot + t)$ is defined by

$$\langle s(\cdot + t), \varphi \rangle := \langle s, \varphi(\cdot - t) \rangle \text{ for every } \varphi \in \mathcal{D}(\mathbb{R}^d).$$

Now we can state our definition, see Definition 4.12, of our CARMA generalized process:

Let $\dot{L}$ be a Lévy white noise, $n, m \in \mathbb{N}_0$ and $p, q : \mathbb{R}^d \to \mathbb{R}$ be polynomials of the form

$$p(x) = \sum_{|\alpha| \le n} p_\alpha x^\alpha$$

and

$$q(x) = \sum_{|\alpha| \le m} q_\alpha x^\alpha.$$

A generalized process $s : \mathcal{D}(\mathbb{R}^d) \to L^0(\Omega)$ is called a *CARMA*(p, q) *generalized process* if s solves the equation

$$p(D)s = q(D)\dot{L},$$

which means that

$$\langle s, p(D)^* \varphi \rangle = \langle \dot{L}, q(D)^* \varphi \rangle \text{ a.s. for every } \varphi \in \mathcal{D}(\mathbb{R}^d). \tag{1.16}$$

Here $p(D)^*$ and $q(D)^*$ are the formal adjoint operators of $p(D)$ and $q(D)$, given by $p(D)^* =$

$p(-D)$ and $q(D)^* = q(-D)$, respectively. In order to prove the existence of s for suitable polynomials p and q, we prove a more general theorem, see Theorem 4.5:

> Let $\dot{L}$ be a Lévy white noise with characteristic triplet (a, γ, ν) and $G : \mathbb{R}^d \to \mathbb{R}$ be a measurable function such that $G \in L^1(\mathbb{R}^d)$. Define
>
> $$G_R(x) := \int\limits_{B_R(x)} |G(y)| \lambda^d(dy)$$
>
> for every $x \in \mathbb{R}^d$ and $R > 0$ and
>
> $$h_R(x) = x \int\limits_0^{1/x} \lambda^d(\{x \in \mathbb{R}^d : G_R(x) > \alpha\}) \lambda^1(d\alpha) \text{ for } x > 0.$$
>
> Assume that
>
> $$\int\limits_{\mathbb{R}} \mathbf{1}_{|r|>1} h_R(|r|) \nu(dr) < \infty$$
>
> for every $R > 0$. Then
>
> $$s(\varphi) := \langle \dot{L}, G * \varphi \rangle, \quad \varphi \in \mathcal{D}(\mathbb{R}^d)$$
>
> defines a stationary generalized random process.

Our main theorem states the following (see Theorem 4.16):

> Let p, q be real polynomials in d variables and assume that the rational function $q(i\cdot)/p(i\cdot)$ has a holomorphic extension in a strip $\{z \in \mathbb{C}^d : \|\Im z\| < \varepsilon\}$ for some $\varepsilon > 0$. Furthermore, let $\dot{L}$ be a Lévy white noise with characteristic triplet (a, γ, ν) such that
>
> $$\int\limits_{\mathbb{R}} \mathbf{1}_{|r|>1} \log(|r|)^d \nu(dr).$$
>
> Then there exists a stationary CARMA(p, q) generalized process.

Furthermore, we provide in this chapter sufficient conditions for an existing mild solution of (1.9) and show that under some conditions this mild solution defines by a suitable identification a CARMA(p, q) generalized process.
Independent of this discussion before we give also a similar result to Theorem 1.3 for functions $G \in L^1_{loc}(\mathbb{R}^d)$ such that $\|G * \varphi_n\|_{L^2(\mathbb{R}^d)} \to 0$ for $n \to \infty$ for every sequence $(\varphi_n)_{n\in\mathbb{N}} \subset \mathcal{D}(\mathbb{R}^d)$ converging to 0. Therefore, we will prove the existence of solutions of homogeneous elliptic partial differential equations driven by Lévy white, namely we show (see Proposition 4.13):

> Let $p(D) = \sum_{|\alpha|=m} a_\alpha D^\alpha$, $m \in \mathbb{N}$, be an elliptic homogeneous partial dif-

ferential operator. If $d > 2m$ and the Lévy white noise $\dot{L}$ with characteristic triplet (a, γ, ν) satisfies

$$\int_{\mathbb{R}} \mathbf{1}_{|r|>1} |r|^{\frac{d}{d-m}+\varepsilon} \nu(dr) < \infty$$

for some $\varepsilon > 0$ and the first moment of $\dot{L}$ vanishes, then there exists a generalized process s which solves the SPDE

$$p(D)s = \dot{L}.$$

1.4 CARMA SPDEs in the spaces of tempered (ultra-)distributions and Fourier hyperfunctions and semilinear SPDEs (Chapter 5)

Until now we only considered the existence of the solution of (1.9) but did not show uniqueness of this solution in the space $\mathcal{D}(\mathbb{R}^d)$ (more precisely in its dual). In order to give some uniqueness results we prove the existence and uniqueness in different function spaces. At first, we assume a Lévy white noise in the space $\mathcal{S}'$, the space of tempered distributions. The space of tempered distributions is the topological dual of the Schwartz space with its usual topology. It is known that a Lévy white noise $\dot{L}$ with characteristic triplet (a, γ, ν) exists on $\mathcal{S}'$ if and only if $\int_{|r|>1} |r|^{\varepsilon} \nu(dr) < \infty$, see [21, Theorem 3.13, p. 4412]. We obtain on the space of tempered distributions the following result, see Proposition 5.5:

> Let $\dot{L}$ be a Lévy white noise on the space of tempered distributions $\mathcal{S}'$. Let p and q be two polynomials such that there exists two polynomials h and l such that $\frac{q(i\cdot)}{p(i\cdot)} = \frac{h(i\cdot)}{l(i\cdot)}$ on $\mathbb{R}^d$ and l has no zeroes on $i\mathbb{R}^d$. Then there exists a generalized process s on the space of tempered distributions solving (1.9), i.e. it holds that
>
> $$\langle s, p(D)^* \varphi \rangle = \langle \dot{L}, q(D)^* \varphi \rangle \tag{1.17}$$
>
> for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$, which is stationary. If $p(iz) \neq 0$ for all $z \in \mathbb{R}^d$, then the solution s is unique.

In the next step we will look at the space $\mathcal{S}'_\omega$, the space of tempered ultradistributions. We give here the definition, which can be also founded in Definition 5.6:

> Let $\omega : \mathbb{R}^d \to \mathbb{R}$ be a real-valued function such that $\omega(x) = \sigma(\|x\|)$, where $\sigma(t)$ is an increasing continuous concave function on $[0, \infty)$ with
>
> $$\sigma(0) = 0,$$

$$\int_0^\infty \frac{\sigma(t)}{1+t^2}\lambda^1(dt) < \infty,$$
$$\sigma(t) \geq c + m\log(1+t) \text{ if } t \geq 0$$

for some $c \in \mathbb{R}$ and $m > 0$. Then the space $\mathcal{S}_\omega$ is the set of all infinitely differentiable functions $\varphi : \mathbb{R}^d \to \mathbb{C}$ such that

$$p_{\alpha,\eta}(\varphi) := \sup_{x\in\mathbb{R}^d} e^{\eta\omega(x)}\|D^\alpha\varphi(x)\| < \infty,$$
$$\pi_{\alpha,\eta}(\varphi) := \sup_{x\in\mathbb{R}^d} e^{\eta\omega(x)}\|D^\alpha(\mathcal{F}\varphi)(x)\| < \infty,$$

for every multi-index α and every $\eta > 0$. The space is equipped with its seminorms given above and its topological dual $\mathcal{S}'_\omega$ is called the *space of tempered ultradistributions*.

At first we need to construct under sufficient conditions the existence of the Lévy white noise on $\mathcal{S}'_\omega$, see Theorem 5.7:

Let (a, γ, ν) be a characteristic triplet and ω be a suitable weight function. If

$$\int_{|r|>1} |r| \int_0^{1/|r|} \omega^{\to}(c\log(|\alpha|^{-1}))^d\lambda^1(d\alpha)\nu(dr) < \infty$$

for some $c \in (0,\infty)$, where $\omega^{\to}(\alpha) := \sup\{x \in [0,\infty) : \sigma(x) < \alpha\}$ for $\alpha \in (0,\infty)$, then there exists a Lévy white noise $\dot{L} : (\Omega, \mathcal{F}) \to (\mathcal{S}'_\omega, C(\mathcal{S}'_\omega))$ with characteristic triplet (a, γ, ν).

The existence and uniqueness of a solution s of (1.9) driven by a Lévy white noise on $\mathcal{S}'_\omega$ follows by a similar condition as in Theorem 5.10:

Let p, q be two real polynomials and assume that the rational function $q(i\cdot)/p(i\cdot)$ has a holomorphic extension in a strip $\{z \in \mathbb{C}^d : \|\Im z\| < \varepsilon\}$ for some $\varepsilon > 0$. Furthermore, let ω be a suitable weight function and $\dot{L}$ be a Lévy white noise on the space of tempered ultradistribution $\mathcal{S}'_\omega$ under the conditions above. Then there exists a generalized stationary process s in the space of tempered ultradistributions $\mathcal{S}'_\omega$ such that

$$p(D)s = q(D)\dot{L}.$$

Moreover, if $p(i\cdot)$ has no zeroes in the strip, then the solution is unique.

In the next step we want to find a distributional space such that there exists a Lévy white noise with characteristic triplet (a, γ, ν) on this space under the very weak assumption that

$$\int_{|r|>1} \log(|r|)^d \nu(dr) < \infty,$$

as this was the assumption that guaranteed the existence of a solution to the CARMA SPDE on the space $\mathcal{D}'(\mathbb{R}^d)$ of distributions. The suitable space is the space of Fourier Hyperfunctions:

> The space $\mathcal{P}_*$ consists of all functions $\varphi \in C^\infty(\mathbb{R}^d, \mathbb{C})$ which have an analytic continuation on a strip
>
> $$A_\delta := \{z \in \mathbb{C}^d : \|\Im z\| < \delta\}$$
>
> for some $\delta > 0$ and it holds that
>
> $$\sup_{z \in A_l} |\exp((\delta - \varepsilon)\|z\|)\varphi(z)| < \infty \tag{1.18}$$
>
> for every $0, \varepsilon, l < \delta$. The space $\mathcal{P}_*$ is nuclear with its inductive topology, i.e. a sequence $(\varphi_n)_{n\in\mathbb{N}} \subset \mathcal{P}_*$ converges to 0 if and only if there exists a $\delta > 0$ such that φ_n has an analytic continuation in A_δ for every $n \in \mathbb{N}$ and
>
> $$\sup_{z \in A_{\delta/2}} |\exp(\delta/2\|z\|)\varphi_n(z)| \to 0 \text{ for } n \to \infty,$$
>
> see [43, p. 408]. We denote by $\mathcal{Q}$ its topological dual and call it the *space of Fourier hyperfunctions.*

We obtain the following corresponding results, see Proposition 5.12:

> Let (a, γ, ν) be a characteristic triplet such that
>
> $$\int_{|r|>1} \log(|r|)^d \nu(dr) < \infty.$$
>
> Then there exists a Lévy white noise on $(\mathcal{Q}, C(\mathcal{Q}))$.

Using this we therefore obtain the following result for CARMA(p, q) generalized processes, see Theorem 5.13:

> Let $\dot{L}$ be a Lévy white noise on $\mathcal{Q}$. Let p, q be two real polynomials such that the rational function $q(i\cdot)/p(i\cdot)$ has a holomorphic extension in a strip $\{z \in \mathbb{C}^d : \|\Im z\| < \varepsilon\}$ for some $\varepsilon > 0$. Then there exists a generalized stationary process s in $\mathcal{Q}$ such that
>
> $$p(D)s = q(D)\dot{L}.$$
>
> Moreover, if p has no zeroes in the strip, than the solution is unique.

When we want to solve more sophisticated equations like

$$p(D)s = g(\cdot, s) + \dot{L}, \tag{1.19}$$

where $g : \mathbb{R}^d \times \mathbb{C} \to \mathbb{R}$ satisfies some Lipschitz condition, we need to work on a more regular distributional space. The weighted Besov spaces $B^l_{r,r}(\mathbb{R}^d, \rho)$, where $r \geq 2$, $l \in \mathbb{R}$ and $\rho < 0$, defined for example in Section 5.2, seem to be the right context to study equation (1.19). We obtain the following result, see Proposition 5.16:

> Let $r \in [2, \infty]$, $\rho < -\frac{d}{\min\{2,\varepsilon\}}$, $\kappa > d(1 - 1/r) + \beta$ for some $\beta > 0$ and $p(D)$ be a partial differential operator satisfying
>
> $$\left| D^\gamma \frac{1}{p(i\xi)} \right| \leq c_\gamma (1 + \|\xi\|^2)^{(-\kappa - |\gamma|)/2} \tag{1.20}$$
>
> for every $\gamma \in \mathbb{N}_0^d$, where $c_\gamma \geq 0$. Furthermore, let $g : \mathbb{R}^d \times \mathbb{C} \to \mathbb{R}$ be a Lipschitz function such that
>
> $$|g(x, y)| \leq C(1 + |y|)$$
>
> for some constant $C > 0$ for all $x \in \mathbb{R}^d$ and $y \in \mathbb{C}$ and assume that
>
> $$\begin{aligned} \|g\|_{Lip} := &\sup_{x \in \mathbb{R}^d} \sup_{z,y \in \mathbb{C}} \frac{|g(x, y) - g(x, z)|}{|y - z|} \\ < &(\|p(D)^{-1}\|_{L^r(\mathbb{R}^d,\rho) \to B^\beta_{r,r}(\mathbb{R}^d,\rho)} \|id\|_{B^\beta_{r,r}(\mathbb{R}^d,\rho) \to L^r(\mathbb{R}^d,\rho)})^{-1} < \infty. \end{aligned} \tag{1.21}$$
>
> Let $\dot{L}$ be a Lévy white noise on $\mathcal{S}'$ with characteristic triplet (a, γ, ν) such that $\int_{|r|>1} |r|^\varepsilon \nu(dr) < \infty$. Let $l = \beta - \kappa + d\left(\frac{1}{2} - \frac{1}{r}\right) < -\frac{d}{2}$ and choose a version of $\dot{L}$ in the Sobolev space $B^l_{2,2}(\mathbb{R}^d, \rho)$.
> Then there exists a unique measurable mapping $s : (\Omega, \mathcal{F}) \to (B^\beta_{r,r}(\mathbb{R}^d, \rho), \mathcal{B}(B^\beta_{r,r}(\mathbb{R}^d, \rho)))$, which solves the equation (1.19). Especially, it holds that $s \in L^r(\Omega, B^\beta_{r,r}(\mathbb{R}^d, \rho))$ if $\varepsilon > r \geq 2$.

We see from the proof of this theorem that s can be even identified with a random field, as the Besov regularity of the solution is positive. Using the mapping from the Besov space to a certain weighted L^r space gives us the identification of s with a random field.

1.5 Central limit theorems for moving average random fields (Chapter 6)

In the last chapter of this thesis we deal with central limit theorems for a moving average random field, i.e. a random field $(X_t)_{t\in\mathbb{R}^d}$ such that there exists a function $f : \mathbb{R}^d \to \mathbb{R}$

with

$$X_t := \int_{\mathbb{R}^d} f(t-x)dL(x),$$

where dL denotes a Lévy basis with characteristic triplet (a, γ, ν). For example, the mild solution of (1.9) can be written as a moving average random field.
In [18] the authors studied central limit theorems with deterministic sampling for moving average processes driven by a Lévy process. In this sense our results are a generalization to random fields and even more, we also allow that we have a random sampling method. We start with central limit theorem for the sample mean, see Theorem 6.2

Let L be a Lévy basis with $\mathbb{E}(L([0,1]^d)^2 < \infty$ and $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, and let

$$X_t := \int_{\mathbb{R}^d} f(t-u)dL(u), \quad t \in \mathbb{R}^d.$$

Let $\Delta > 0$, $A \in O(d)$, and $(\Gamma_n)_{n\in\mathbb{N}}$ be a sequence of finite subsets of $\Delta A \mathbb{Z}^d$ such that

a) $\Gamma_n \subset \Gamma_{n+1}$ for every $n \in \mathbb{N}$,

b) $|\Gamma_n| \to \infty$ as $n \to \infty$, and

c) $a_l^n := \frac{|\{(t,s)\in\Gamma_n\times\Gamma_n : t-s=l\}|}{|\Gamma_n|}$ converges as $n \to \infty$ to some a_l for each $l \in \Delta A \mathbb{Z}^d$.

Assume that

$$\sum_{t\in\Delta A\mathbb{Z}^d} \sup_{n\in\mathbb{N}} a_t^n \int_{\mathbb{R}^d} |f(-u)f(t-u)|\lambda^d(du) < \infty. \tag{1.22}$$

Then

$$\sum_{t\in\Delta A\mathbb{Z}^d} a_t |\mathrm{cov}\,(X_t, X_0)| < \infty,$$

and

$$\frac{1}{\sqrt{|\Gamma_n|}} \sum_{t\in\Gamma_n} \left(X_t - \mathbb{E}L([0,1]^d) \int_{\mathbb{R}^d} f(u)\lambda^d(du) \right) \xrightarrow{d} N\left(0, \sum_{t\in\Delta A\mathbb{Z}^d} a_t \mathrm{cov}\,(X_t, X_0)\right).$$

Under some stricter moment assumptions on the random field, we find also central limit theorems when assuming that our sampling is in some sense random, see Theorem 6.7:

Let $(Y_t)_{t\in\Delta A\mathbb{Z}^d}$ be a $\{0,1\}-$valued $\alpha-$mixing random field with mixing coefficients $\alpha_Y(k; u, v)$, which is independent of the Lévy basis L and satisfies $P(Y_0 = 1) > 0$. Moreover, assume there exists a $\delta > 0$ such that Y satisfies

i) for every $u, v \in \mathbb{N}$ it holds $\alpha_Y(k; u, v)k^d \to 0$ for $k \to \infty$,

ii) for every $u, v \in \mathbb{N}$ such that $u + v \leq 4$ it holds $\sum_{k=0}^{\infty} k^{d-1}\alpha_Y(k; u, v) < \infty$ and especially $\sum_{k=0}^{\infty} k^{d-1}\alpha_Y(k; 1, 1)^{\delta/(2+\delta)} < \infty$.

Let Γ_n be as above and $X = (X_t)_{t\in\mathbb{R}^d}$ be a moving average random field with $X_t = \int_{\mathbb{R}^d} f(t-u)\, dL(u)$ with $\mathbb{E}|L([0,1]^d)|^{2+\delta} < \infty$ and $f \in L^1(\mathbb{R}^d) \cap L^{2+\delta}(\mathbb{R}^d)$. If

$$\sum_{t\in\Delta A\mathbb{Z}^d} \mathbb{E}Y_0Y_t \int_{\mathbb{R}^d} |f(-u)|\, |f(t-u)|\, \lambda^d(du) < \infty,$$

then we have that

$$\frac{1}{\sqrt{|\Gamma_n|}} \sum_{t\in\Gamma_n} \left(X_t - \mathbb{E}L([0,1]^d) \int_{\mathbb{R}^d} f(u)\lambda^d(du) \right) \xrightarrow{d} N\left(0, \sum_{t\in\Delta A\mathbb{Z}^d} \frac{1}{\mathbb{E}Y_0} \operatorname{cov}\left(Y_tX_t, Y_0X_0\right)\right).$$

In the special case that Y is $h-$dependent for some finite $h > 0$, it is enough to assume that $\mathbb{E}|L([0,1]^d)|^2 < \infty$ and $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$.

We set

$$\mathbb{E}\, L([0,1]^d)^4 < \infty,\ \mathbb{E}\, L([0,1]^d) = 0,\ \sigma^2 := \mathbb{E}\, L([0,1]^d)^2 > 0 \tag{1.23}$$

We obtain also central limit theorems for the sample autocovariance, see Theorem 6.9:

Let $m \in \mathbb{N}$ and $\Delta_1, \ldots, \Delta_m \in \Delta A\mathbb{Z}^d$, Γ_n as in Theorem 1.5, and let $(X_t)_{t\in\mathbb{R}^d} = (\int_{\mathbb{R}^d} f(t-s)dL(s))_{t\in\mathbb{R}^d}$ be a moving average random field such that it satisfying some assumptions, $f \in L^2(\mathbb{R}^d) \cap L^4(\mathbb{R}^d)$ and

$$\sum_{l\in\Delta A\mathbb{Z}^d} \int_{\mathbb{R}^d} \sup_{n\in\mathbb{N}} a_l^n |f(u)f(u+l)f(u+\Delta_p)f(u+l+\Delta_d)|\lambda^d(du) < \infty$$

for every $p, d \in \{1, \ldots, m\}$ and

$$\sum_{l\in\Delta A\mathbb{Z}^d} \sup_{n\in\mathbb{N}} a_l^n \gamma_X(l)^2 < \infty.$$

Then

$$\sqrt{|\Gamma_n|}(\gamma_n^*(\Delta_1) - \gamma_X(\Delta_1), \ldots, \gamma_n^*(\Delta_m) - \gamma_X(\Delta_m)) \xrightarrow{d} N(0, V), \tag{1.24}$$

the multivariate normal distribution with mean 0 and covariance matrix $V =$

$(v_{pq})_{p,q\in\{1,\dots,m\}}$ given by

$$v_{pq} = \sum_{l\in\Delta A\mathbb{Z}^d} a_l \Bigg((\eta-3)\sigma^4 \int_{\mathbb{R}^d} f(u)f(u+\Delta_p)f(u+l)f(u+l+\Delta_q)\,\lambda^d(du)$$
$$+ \gamma_X(l)\gamma_X(l+\Delta_q-\Delta_p) + \gamma_X(l+\Delta_q)\gamma_X(l-\Delta_p)\Bigg).$$

In the case of the random sampling we obtain in Theorem 6.10 the following result:

Let $(Y_t)_{t\in\Delta A\mathbb{Z}^d}$ be a $\{0,1\}$-valued $\alpha-$mixing random field with mixing rates as in Theorem 1.5 ($\delta > 0$), which is independent of the Lévy basis L. Let $X = (X_t)_{t\in\mathbb{R}^d}$ be a moving average random field with $X_t = \int_{\mathbb{R}^d} f(t-u)\,dL(u)$ such that (1.23) holds with $\mathbb{E}|L([0,1]^d)|^{4+\delta} < \infty$ and $f \in L^2(\mathbb{R}^d) \cap L^{4+\delta}(\mathbb{R}^d)$. Let $\Delta_1,\dots,\Delta_m \in \Delta A\mathbb{Z}^d$ and for every $p,d \in \{1,\dots,m\}$ assume that

$$\sum_{t\in\Delta A\mathbb{Z}^d} \mathbb{E}Y_0Y_t \int_{\mathbb{R}^d} |f(u)f(u+t)f(u+\Delta_p)f(u+t+\Delta_d)|\lambda^d(du) < \infty$$

and

$$\sum_{l\in\Delta A\mathbb{Z}^d} \mathbb{E}Y_0Y_l\gamma_X(l)^2 < \infty.$$

Then for $\Gamma_n := \{t \in \Delta A[-n,n)^d \cap \Delta A\mathbb{Z}^d \,:\, Y_t = 1\}$ we have

$$\sqrt{|\Gamma_n|}(\gamma_n^*(\Delta_1)-\gamma_X(\Delta_1),\dots,\gamma_n^*(\Delta_m)-\gamma_X(\Delta_m)) \xrightarrow{d} N(0,V), \tag{1.25}$$

with covariance matrix $V = (v_{pq})_{p,q\in\{1,\dots,m\}}$ given by

$$v_{pq} = \sum_{l\in\Delta A\mathbb{Z}^d} \frac{\mathbb{E}Y_0Y_l}{\mathbb{E}Y_0} \Bigg((\eta-3)\sigma^4 \int_{\mathbb{R}^d} f(u)f(u+\Delta_p)f(u+l)f(u+l+\Delta_q)\,\lambda^d(du)$$
$$+ \gamma_X(l)\gamma_X(l+\Delta_p-\Delta_q) + \gamma_X(l+\Delta_p)\gamma_X(l+\Delta_q)\Bigg). \tag{1.26}$$

1.6 Structure of the thesis

Apart from this introduction, the thesis consists of 5 chapters, which in turn are all based on published or submitted research articles. In detail:

- Chapter 2 is based on the submitted article:
  D. Berger, On the integral modulus of infinitely divisible distributions, Arxiv 1805.01641.
- Chapter 3 is based on the published article:

D. Berger, On quasi-infinitely divisible distributions with a point mass, Math. Nachrichten, Published online on 23 April 2019, DOI: 10.1002/mana.201800073.

- Chapter 4 is based on the submitted article:
  D. Berger, Lévy driven CARMA generalized processes and stochastic partial differential equations, Arxiv 1904.02928.
- Chapter 5 is based on the submitted article:
  D. Berger, Lévy driven linear and semilinear stochastic partial differential equations, Arxiv 1907.01926.
- Chapter 6 is based on the submitted article:
  D. Berger, Central Limit Theorems for Moving Average Random Fields with Non-Random and Random Sampling On Lattices, Arxiv 1902.01255.

We have decided to follow the structure of the corresponding articles closely, so that the chapters can be read individually. In particular, each chapter contains an introduction of its own and also clarifies the notation used therein.

2 On the integral modulus of continuity of infinitely divisible distributions, especially of stochastic integrals

We derive estimates for the integral modulus of continuity of probability densities of infinitely divisible distributions. The chapter is splitted into two parts. The first part deals with general infinitely divisible distributions. The second part is concerned with densities of random integrals with respect to a Lévy process. We will see major differences between integrals over compact and non-compact intervals.

2.1 Introduction

The modulus of continuity $||f(z-\cdot)-f(\cdot)||_{L^p(\mathbb{R})}$ of a function $f:\mathbb{R}\to\mathbb{R}$, for $z\in\mathbb{R}$, has a deep connection to Fourier series and also to the Fourier transform. The decaying rate of the Fourier transform (or weighted versions, see [10] and cited articles) can be estimated by the modulus of continuity and vice versa. For $1\leq p\leq 2$ it is hard to obtain estimates for the modulus of continuity in terms of the Fourier transform, but especially the case $p=1$ is very interesting as $\int_{\mathbb{R}}|f(x-z)-f(x)|\,dx\leq C|z|$ for all z is equivalent to the fact that f is of bounded variation (see [1, Exercise 3.3, p. 208]).
In statistics it is also interesting to know if a probability density is of bounded variation if one wants to estimate the density, see [23, Theorem 3]. Moreover, if one has a linear process $X=(X_t)_{t\in\mathbb{Z}}$ with $X_t=\sum_{i=0}^{\infty}a_iZ_{t-i}$ with $a_i\in\mathbb{R}$ and $(Z_i)_{i\in\mathbb{Z}}$ are iid random variables with Lebesgue density f, the strong mixing rate of the process X depends on the L^1-modulus of continuity $||f(z-\cdot)-f(\cdot)||_{L^1(\mathbb{R})}$, see [36, Theorem and proof].
In this chapter we are mostly interested in special classes of infinitely divisible distributions and the L^1-modulus of continuity of their densities, if existent. The chapter is separated into two parts. The first part is interested in general infinitely divisible distributions and their densities. A probability measure μ on $\mathbb{R}$ is *infinitely divisible*, if there exist constants $\gamma\in\mathbb{R}$, $a\geq 0$ and a Lévy measure ν on $\mathbb{R}$ (i.e. a measure ν satisfying $\nu(\{0\})=0$ and $\int_{\mathbb{R}}\min\{1,x^2\}\nu(dx)<\infty$) such that the Fourier transform $\hat{\mu}$ satisfies

$$\hat{\mu}(z)=\exp\left(-\frac{1}{2}az^2+i\gamma z+\int\limits_{\mathbb{R}}(e^{ixz}-1-ixz\mathbf{1}_{[-1,1]}(x))\nu(dx)\right)\tag{2.1}$$

for every $z \in \mathbb{R}$. It can be shown that the triplet (a, γ, ν) is unique, and that for every such triplet (a, γ, ν) the right-hand side of (2.1) defines the Fourier transform of an infinitely divisible distribution, see [58, Theorem 8.1, p. 37].
The question whether an infinitely divisible distribution is absolutely continuous or not is a difficult one, and although many sufficient and many necessary conditions are known, a complete characterization in terms of the characteristic triplet is not known. See [58, Section 27] for an overview. Moreover, there are also many results regarding the continuity and differentiability properties of their densities, see [58, Section 28] and cited articles. We give an extension of these results and study the integral modulus of continuity of densities of infinitely divisible distributions and especially of stochastic integrals driven by a Lévy process, where the integrand is a deterministic function g.
The normal distribution is itself an infinitely divisible distribution with characteristic triplet $(a, \gamma, 0)$. If $a > 0$ it has of course a Lebesgue density with very nice properties so it is not very suprising that we find bounds for the modulus and as a consequence we obtain for the larger class of distributions with characteristic triplet (a, γ, ν) with $a > 0$ similiar estimates for the integral modulus.
In the more complicated case $a = 0$, we will give sufficient conditions on the characteristic triplet $(0, \gamma, \nu)$ to have Hölder bounds for the modulus $||f(z - \cdot) - f(\cdot)||_{L^1(\mathbb{R})}$ if $\nu(dx)$ has a Lebesgue density in a neighborhood of zero, where f is the Lebesgue density of μ, the probability measure with characteristic triplet $(0, \gamma, \nu)$.
An important subclass of infinitely divisible distributions is the class of self-decomposable distributions. They are infinitely divisible distributions for which the Lévy measure has a density of the form $\frac{k(x)}{|x|}$, such that k is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$, see [58, Theorem 15.10, p. 95]. They have a Lebesgue density if they are non-degenerate. Furthermore, explicit bounds for the decay of their Fourier transform are known, so it seems natural to start the search for bounds with this class. An important property of these distributions is the unimodality. We will use this property in our proof for the main result. By using known estimates for the modulus and the decay of their Fourier transform it is possible to find upper bounds for the integral modulus and we will see that most of our results are in some sense optimal.
The second part of the chapter deals with stochastic integrals of deterministic functions with respect to Lévy processes and their corresponding densities, where we consider integration over compact and non-compact intervals. For the compact support we will deal with kernels which are $\mathcal{C}^1-$diffeomorphisms on their support. We will see that every stochastic integral with such a kernel has a Lebesgue density when the underlying Lévy process has infinite Lévy measure and derive necessary and sufficient conditions on the Lévy process and the kernel such that the density is of bounded variation. Based on this we will consider for the non-compact support $[0, \infty)$ kernels such that there exists a sequence $0 = t_0 < t_1 < \ldots < t_n \to \infty$ for $n \to \infty$ such that the kernel is a $\mathcal{C}^1$-diffeomorphism in every (t_i, t_{i+1}) for every $i \in \mathbb{N}_0$. We will find sufficient conditions for the existence of a Lebesgue density of bounded variation.

2.2 Notation and preliminaries

To fix notation, by a distribution on $\mathbb{R}$ we mean a probability measure on $(\mathbb{R}, \mathcal{B})$ with $\mathcal{B}$ being the Borel $\sigma-$algebra on $\mathbb{R}$, and similarly, by a signed measure on $\mathbb{R}$ we mean it to be defined on $(\mathbb{R}, \mathcal{B})$. By a measure on $\mathbb{R}$ we always mean a positive measure on $(\mathbb{R}, \mathcal{B})$, i.e. a $[0, \infty]$-valued $\sigma-$additive set function on $\mathcal{B}$ that assigns the value 0 to the empty set. The Dirac measure at a point $b \in \mathbb{R}$ will be denoted by δ_b, the Gaussian distribution with mean $a \in \mathbb{R}$ and variance $b \geq 0$ by $N(a, b)$ and the Lebesgue measure by dx. For a Lebesgue measure of a Borel set $A \in \mathcal{B}$ we write $|A|$. The Fourier transform at $z \in \mathbb{R}$ of a finite positive measure μ on $\mathbb{R}$ will be denoted by $\hat{\mu}(z) = \int_{\mathbb{R}} e^{ixz}\, \mu(dx)$. The convolution of two positive measures μ_1 and μ_2 on $\mathbb{R}$ is defined by $\mu_1 * \mu_2(B) = \int_{\mathbb{R}} \mu_1(B - x)\, \mu_2(dx)$, $B \in \mathcal{B}$, where $B - x = \{y - x | \, y \in B\}$. The law of a random variable X will be dentoted by $\mathcal{L}(X)$. The imaginary unit will be denoted by i. We write $\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ for the set of integers, real numbers and complex numbers, respectively. The indicator function of a set $A \subset \mathbb{R}$ is denoted by $\mathbf{1}_A$. By $L^1(\mathbb{R}, A)$ for $A \subset \mathbb{C}$ we denote the set of all Borel-measurable functions $f : \mathbb{R} \to A$ such that $\int_{\mathbb{R}} |f(x)|\, dx < \infty$. By $BV(\mathbb{R}, \mathbb{R})$ we denote the set of functions $f : \mathbb{R} \to \mathbb{R}$ of bounded variation, which means for every decomposition $-\infty < a_1 < \ldots < a_n < \infty$ it holds $\sum_{i=1}^{n-1} |f(a_i) - f(a_{i+1})| \leq C < \infty$ for some $C > 0$ independent of the decomposition. By $TV_f([a, b])$ we denote the total variation of the function $f \in BV(\mathbb{R}, \mathbb{R})$ in the interval $[a, b]$.

2.3 Densities of infinitely divisible distributions

Our goal of this section is to prove some aspects of the integral modulus of continuity of densities from infinitely divisible distributions. We will specialize on infinitely divisible distributions with Lévy measure ν such that $|x|\nu(dx)$ has a Lebesgue density around a neighborhood of 0.
As stated in the introduction the class of self-decomposable distributions is a subclass of such distributions. All self-decomposable distributions are unimodal (see [58, Theorem 53.1, p. 404]), which will play a major rule in the proof of the main theorem. We will derive the main result by minorizing the Lévy measure by a Lévy measure corresponding to a self-decomposable distribution.
We start with an easy example and derive some bounds for the integral modulus of continuity of normal distributions and infinitely divisible distributions with a non-vanishing Gaussian variance.

Remark 2.1.

i) If μ_1 is an absolutely continuous distribution with Lebesgue density f such that $\int_{\mathbb{R}} |f(x) - f(x - z)|\, dx \leq h(z)$ for some $z \in \mathbb{R}$ and μ_2 is a probability measure, then the Lebesgue density g of $\mu_1 * \mu_2$ satisfies $\int_{\mathbb{R}} |g(x) - g(x - z)|\, dx \leq h(z)$.

ii) If μ is an infinitely divisible distribution with characteristic triplet (a, γ, ν) such that $a > 0$, then the Lebesgue density f_μ of μ satifies

$$\int_{\mathbb{R}} |f_\mu(x) - f_\mu(x-z)|\, dx \leq C|z|$$

for some constant C and every $z \in \mathbb{R}$ and especially for μ with characterstic triplet $(a, \gamma, 0)$, $a > 0$, it holds

$$\lim_{z \to 0} |z|^{-1} \int_{\mathbb{R}} |f_\mu(x) - f_\mu(x-z)|\, dx = \sqrt{\frac{2}{\pi a}}.$$

Proof. i) We know that $\mu_1 * \mu_2$ is absolutely continuous with Lebesgue density $g(x) = \int_{\mathbb{R}} f(x-y)\mu_2(dy)$. Then

$$\int_{\mathbb{R}} |g(x) - g(x-z)|\, dx \leq \int_{\mathbb{R}} \int_{\mathbb{R}} |f(x-y) - f(x-z-y)|\, dx\, \mu_2(dy) \leq h(z).$$

ii) Let $\mu_1 = N(0, a)$ be a normal distribution with mean 0 and variance $a > 0$. We have that $f_{\mu_1}(x) = 1/\sqrt{2\pi a} \exp(-x^2/(2a))$ and find by a simple calculation that

$$\int_{\mathbb{R}} |f_{\mu_1}(x) - f_{\mu_1}(x-z)|\, dx = \sqrt{\frac{2}{\pi a}} \int_{(-|z|/2, |z|/2)} \exp(-x^2/(2a))\, dx,$$

which is $O(|z|)$ for $|z| \to 0$. The rest follows from i), since $\mu = \mu_1 * \mu_2$, where μ_2 is infinitely divisible with characteristic triplet $(0, \gamma, \nu)$.

□

Now we will state and prove our main result. There are many consequences of this result and we will later show some applications to obtain further infinitely divisible distributions with a density of bounded variation.

Theorem 2.2. *Let μ be an infinitely divisible distribution with characteristic triplet (a, γ, ν) where $a \geq 0$, $\gamma \in \mathbb{R}$ and ν a Lévy measure such that $|x|\nu(dx)$ has a Lebesgue density k in a neighborhood around zero with* $\liminf_{x \to 0+} k(x) + \liminf_{x \to 0-} k(x) =: c_{\inf}$.

i) If $c_{\inf} > 1/p$ for $1 < p \leq 2$, then μ has a Lebesgue density $f_\mu \in L^1(\mathbb{R}, \mathbb{R}^+) \cap L^{p/(p-1)}(\mathbb{R}, \mathbb{R}^+)$ and there exists a constant $C > 0$ such that

$$\int_{\mathbb{R}} |f_\mu(x-z) - f_\mu(x)|\, dx \leq C|z|^{\frac{1}{p}} \tag{2.2}$$

for every $z \in \mathbb{R}$.

ii) If $c_{\inf} > 1$, then f is continuous on $\mathbb{R}$ and there exists a constant $C > 0$ such that

$$\int_{\mathbb{R}} |f_\mu(x-z) - f_\mu(x)|\, dx \le C|z| \tag{2.3}$$

for every $z \in \mathbb{R}$.

iii) Now let $c_{\sup} := \limsup_{x\to 0+} k(x) + \limsup_{x\to 0-} k(x) < \frac{1}{p}$ with $p \in (0,\infty)$ and let $a = 0$. Then, if μ has a Lebesgue density f_μ, it satisfies

$$\sup_{0\le h\le |z|} \int_{\mathbb{R}} |f_\mu(x-h) - f_\mu(x)|\, dx \ge C|z|^{\frac{1}{p}} \tag{2.4}$$

for some constant $C > 0$ and all $z \in (-1,1)$.

Proof. For the proof assume that $a = 0$ as otherwise (2.2) and (2.3) would be implied by Remark 2.1 ii). For the proof of i) and ii) we assume first that k is increasing on $(-\delta, 0)$ and decreasing on $(0,\delta)$ for some $\delta > 0$ and else 0 such that $(0, \gamma, \frac{k(x)}{|x|}\, dx)$ is the characteristic triplet of a self-decomposable distribution μ, see [58, Theorem 15.10].
i) We then know that $c = c_{\inf} = c_{\sup} = k(0+) + k(0-) > 0$. Then it holds true that $|\hat{\mu}(z)| = o(|z|^{-\alpha})$ as $|z| \to \infty$ with $0 < \alpha < c$, see [58, Lemma 28.5, p. 191]. If $c > \frac{1}{p}$, it follows that $\hat{\mu} \in L^p(\mathbb{R}, \mathbb{C})$ and we conclude that $f_\mu \in L^{p^*}(\mathbb{R}, [0,\infty))$, see [37, Proposition 2.2.16, p. 104], where $p^* = \frac{p}{p-1}$. As μ is unimodal (with mode m), we get for z positive

$$\begin{aligned}
&\int_{\mathbb{R}} |f_\mu(x-z) - f_\mu(x)|\, dx \\
&= \int_{(-\infty,m)} f_\mu(x) - f_\mu(x-z)\, dx + \int_{(m,m+z)} |f_\mu(x-z) - f_\mu(x)|\, dx \\
&\quad + \int_{(m+z,\infty)} f_\mu(x-z) - f_\mu(x)\, dx \\
&= \int_{(-\infty,m)} f_\mu(x)\, dx + \int_{(m+z,\infty)} f_\mu(x-z)\, dx + \int_{(m,m+z)} |f_\mu(x-z) - f_\mu(x)|\, dx \\
&\quad + \int_{(m-z,m+z)} f_\mu(x)\, dx - \left(\int_{(-\infty,m)} f_\mu(x-z)\, dx + \int_{(m-z,m+z)} f_\mu(x)\, dx + \int_{(m+z,\infty)} f_\mu(x)\, dx \right) \\
&= \int_{(m,m+z)} |f_\mu(x-z) - f_\mu(x)|\, dx + \int_{(m-z,m+z)} f_\mu(x)\, dx.
\end{aligned} \tag{2.5}$$

Now as $f_\mu \in L^{p^*}(\mathbb{R})$, we conclude from (2.5) and the triangle inequality that

$$\int_{\mathbb{R}} |f_\mu(x-z) - f_\mu(x)|\, dx \le ||f_\mu||_{L^{p^*}} z^{\frac{1}{p}} + ||f_\mu||_{L^{p^*}} z^{\frac{1}{p}} + 2^{\frac{1}{p}} ||f_\mu||_{L^{p^*}} z^{\frac{1}{p}} \le (2 + 2^{\frac{1}{p}}) ||f_\mu||_{L^{p^*}}\, z^{\frac{1}{p}}.$$

The assumption for $z < 0$ follows by symmetry.
ii) Since $c = k(0+) + k(0-) > 1$, it follows from [58, Theorem 28.4] that f_μ is continuous on $\mathbb{R}$, and since μ is also unimodal, f_μ must be bounded. Hence we can bound the modulus by (2.5) (for $z > 0$) by

$$\begin{aligned}\int_{\mathbb{R}} |f_\mu(x-z) - f_\mu(x)|\, dx \le & \int_{(m,m+z)} |f_\mu(x-z) - f_\mu(x)|\, dx + \int_{(m-z,m+z)} f_\mu(x)\, dx \\ \le & 4 \sup_{x\in\mathbb{R}} |f_\mu(x)| z.\end{aligned}$$

Now we assume that μ is infinitely divisible with characteristic triplet $(0, \gamma, \nu)$ such that there exists $\delta > 0$ such that $|x|\nu(dx)$ has a Lebesgue density k in $(-\delta, \delta)$. We know that there exists for small $\varepsilon > 0$ a $\rho > 0$ such that $k(x) \ge \liminf_{y\to 0+} k(y) - \frac{\varepsilon}{2} > 0$ for every $x \in (0, \rho)$ and $k(x) \ge \liminf_{y\to 0-} k(y) - \frac{\varepsilon}{2} > 0$ for every $x \in (-\rho, 0)$. So we can find a minorizing Lévy measure $l(x)/|x|\, dx$ for ν by setting

$$l(x) = \mathbf{1}_{(0,\rho)}(x) \left(\liminf_{y\to 0+} k(y) - \frac{\varepsilon}{2} \right) + \mathbf{1}_{(-\rho,0)}(x) \left(\liminf_{y\to 0-} k(y) - \frac{\varepsilon}{2} \right),$$

(if $\liminf\limits_{y\to 0+} k(y) = 0$, but $\liminf\limits_{y\to 0-} k(x) = c_{\inf}$, set $l(x) = \mathbf{1}_{-\rho,0}(x) \left(c_{\inf} - \frac{\varepsilon}{2} \right)$). Let μ_1 be the self-decomposable distribution with triplet $(0, \gamma, \frac{l(x)}{|x|}\, dx)$ and μ_2 be the infinitely divisible distribution wtih triplet $(0, 0, \nu - \frac{l(x)}{|x|} dx)$. Then $\mu = \mu_1 * \mu_2$ and since μ_1 satisfies i) and ii), respectively, if ε is chosen small enough, so does μ by Remark 2.1 i).

iii) First assume that μ is such that $|x|\nu(dx)$ has a (bounded) Lebesgue density k in $(-\delta, \delta)$ such that k is monotone on $(-\delta, 0)$ and on $(0, \delta)$. Observe that for every $\varepsilon > 0$ there exists a constant $C > 0$ such that $|\hat{\mu}(z)| > C(1 + |z|)^{-c-\varepsilon}$, $c = c_{\sup}$, for every $z \in \mathbb{R}$, see [62, Proposition 1]. Moreover, we know by [10, Corollary 3] that

$$\sup_{|x| \ge \frac{1}{|z|}} |\hat{\mu}(x)| \le C' \sup_{0 \le h \le |z|} \int_{\mathbb{R}} |f_\mu(x-h) - f_\mu(x)|\, dx$$

for some constant C' and all $z > 0$. So we see that

$$\tilde{C}|z|^{c+\varepsilon} \le C \left(1 + \frac{1}{|z|} \right)^{-c-\varepsilon} \le C'' \sup_{0 \le h \le |z|} \int_{\mathbb{R}} |f_\mu(x-h) - f_\mu(x)|\, dx$$

for some constants $C, \tilde{C}, C'' > 0$ and all $|z| < 1$. Choosing $\varepsilon = \frac{1}{p} - c$ gives the claim in

this special case.
For general μ we set

$$l(x) = \mathbf{1}_{(0,\rho)}(x)\left(\limsup_{y\to 0+} k(y) + \frac{\varepsilon}{2}\right) + \mathbf{1}_{(-\rho,0)}(x)\left(\limsup_{y\to 0-} k(y) + \frac{\varepsilon}{2}\right)$$

for small enough ρ and ε and majorize ν by $l(x)/|x||_{(-\delta,\delta)}dx + \nu|_{(-\delta,\delta)^c}(dx)$ which gives us our assertion by Remark 2.1 i), as otherwise the majorizing distribution would not satisfy iii). □

Remark 2.3.

i) For Theorem 2.2 i) and ii) it is sufficient that $|x|\nu(dx)$ can be minorized by a measure with the sufficient conditions. Similarly, for Theorem 2.2 iii) it is sufficient that $|x|\nu(dx)$ can be majorized by a measure with the sufficient conditions and that $a = 0$. This follows from Remark 2.1.

ii) For Theorem 2.2 iii) one can give further conditions on k such that $c_{\sup} = 1/p$ is sufficient for (2.4) to hold, see for example [62, Proposition 1].

Another example where we can apply the same techniques is a symmetric infinitely divisible distribution μ with characteristic triplet $(0, 0, \nu)$ such that ν is unimodal and has mode 0. Then also μ is unimodal with mode 0, see [58, Theorem 54.2].

Corollary 2.4. *Let μ be an infinitely divisible distribution with characteristic triplet $(0, 0, \nu)$. Assume that*

$$\liminf_{r\to 0} \frac{\int_{[-r,r]} x^2\,\nu(dx)}{r^2 \log(\frac{1}{r})} =: C > \frac{1}{2p}$$

for some $1 < p \leq 2$. Then μ has a Lebesgue density $f_\mu \in L^1(\mathbb{R}, [0,\infty)) \cap L^{p/(p-1)}(\mathbb{R}, [0,\infty))$. Furthermore, if ν is unimodal with mode 0 and μ is symmetric, then there exists a constant $C > 0$ such that

$$\int_{\mathbb{R}} |f_\mu(x - z) - f_\mu(x)|\,dx \leq C|z|^{\frac{1}{p}}$$

for every $z \in \mathbb{R}$.
If additionally the condition

$$\liminf_{r\to 0} \frac{\int_{[-r,r]} x^2\nu(dx)}{r^{2-\alpha}} > 0,$$

is satisfied for some $\alpha \in (0, 2)$, we can bound the modulus by $|z|$ times a constant.

Proof. Assume that

$$\liminf_{r\to 0} \frac{\int_{[-r,r]} x^2\,\nu(dx)}{r^2\log(\frac{1}{r})} := C > \frac{1}{2p}.$$

Then there exists a constant $\varepsilon > 0$ such that $\int_{[-r,r]} x^2\nu(dx) \geq (C-\varepsilon)r^2\log\frac{1}{r}$ for small enough r and $C-\varepsilon > \frac{1}{2p}$. As $1-\cos(u) \geq 2\left(\frac{u}{\pi}\right)^2$ for $|u| \leq \pi$, we see that

$$\begin{aligned}
|\hat{\mu}(z)| &= \exp\left(\int_{\mathbb{R}} (\cos(xz)-1)\nu(dx)\right)\\
&\leq \exp\left(-\frac{2}{\pi^2}\int_{|x|\leq\pi/|z|} z^2x^2\nu(dx)\right)\\
&\leq \exp\left(-\frac{2}{\pi^2}z^2(C-\varepsilon)\frac{\pi^2}{z^2}\log\left|\frac{z}{\pi}\right|\right)\\
&= \exp\left(-\log\left|\frac{z}{\pi}\right|^{2(C-\varepsilon)}\right) = \frac{\pi^{2(C-\varepsilon)}}{|z|^{2(C-\varepsilon)}} \leq \frac{\pi^{2(C-\varepsilon)}}{|z|^{\frac{1}{p}+\delta}}
\end{aligned}$$

for some $\delta > 0$ and $|z|$ large enough. It follows that $\hat{\mu} \in L^p(\mathbb{R},\mathbb{C})$ and from that we conclude that there exists a density, which is $p/(p-1)$-integrable. Now if ν is additionally unimodal with mode 0 then so is μ, see [58, Theorem 54.2]. By the same proof as in Theorem 2.2 i) we conclude that the modulus of continuity can be bounded by $|z|^{\frac{1}{p}}$ times a constant. If the Lévy-measure especially satisfies the condition

$$\liminf_{r\to 0} \frac{\int_{[-r,r]} x^2\nu(dx)}{r^{2-\alpha}} > 0,$$

the Lebesgue density is continuous, see [58, Proposition 28.3] and the modulus of continuity is bounded by $|z|$ times a constant by the same proof as in Theorem 2.2 ii).

□

2.4 Densities of stochastic integrals

In this section we look at distributions arising as stochastic integrals $\int_0^t g(s)dL(s)$ or $\int_0^\infty g(s)dL(s)$, when g is a deterministic function and L a Lévy process. A Lévy process is a real-valued stochastic process $L = (L_t)_{t\geq 0}$ with stationary and independent increments, such that $L_0 = 0$ almost surely and such that the paths of L are right-continuous with finite left-limits. There exists a one-to-one correspondence between infinitely divisible distributions and Lévy processes (in law). In particular, the distribution of a Lévy process L

at time 1 is infinitely divisible and characterizes the distribution of L. The characteristic triplet of $\mathcal{L}(L_1)$ is then also called the characteristic triplet of L.
The existence of the integrals $\int_0^t g(s)dL(s)$ or $\int_0^\infty g(s)dL(s)$ can be completely characterized by the characteristic triplet (a,γ,ν) of L and g, see [56, Theorem 2.7, p. 461]. In the case that t is finite, it is sufficient that g is bounded for the existence of the stochastic integral. If $t=\infty$ and $\mathbb{E}L_1^2<\infty$, then $g\in L^1(\mathbb{R})\cap L^2(\mathbb{R})$ is sufficient for the existence of the integral and if g is exponentially decreasing, the finiteness of $\mathbb{E}\log^+(L_1)$ is sufficient. For a more detailed overview see [56, Theorem 2.7, p. 461] or [58, Supplements 57].
The integrals are infinitely divisible with characteristic triplet (a_g,γ_g,ν_g) where

$$\gamma_g = \int_{[0,t)} \left(\gamma g(s) + \int_{\mathbb{R}} g(s)r(\mathbf{1}_{[-1,1]}(g(s)r) - \mathbf{1}_{[-1,1]}(r))\,\nu(dr)\right) ds,$$

$$a_g = \int_{[0,t)} a g(s)^2\,ds \text{ and} \tag{2.6}$$

$$\nu_g(B) = \int_{[0,t)}\int_{\mathbb{R}} \mathbf{1}_{B\setminus\{0\}}(g(s)r)\,\nu(dr)\,ds, \quad B\in\mathcal{B} \tag{2.7}$$

with $t\in[0,\infty]$.

2.4.1 Stochastic integrals over compact intervals

Now look at distributions of the form $Z=\int_{[0,t]} g(s)\,dL(s)$, where $t\in[0,\infty)$ and $L=(L_s)_{s\geq 0}$ is a Lévy process with characteristic triplet $(0,\gamma,\nu)$ with $\nu(\mathbb{R})>0$. We give sufficient conditions depending on L and g such that Z satisfies the assumptions of Theorem 2.2. We immediately restrict to the case when the Gaussian variance $a=0$, for otherwise $a_g>0$ by (2.6) (unless $\int_0^t g(s)^2\,ds=0$) and hence Remark 2.1 ii) can be applied. We start with the following lemma, where we write $\frac{x}{B}:=\{\frac{x}{b}:b\in B\}$ for $x\in\mathbb{R}$ and $B\subset\mathbb{R}\setminus\{0\}$.

Lemma 2.5. *Let $g:[0,t]\to\mathbb{R}$ be a $\mathcal{C}^1$-diffeomorphism onto its range.*

i) Then $|x|\nu_g(dx)$ is absolutely continuous with Lebesgue density k given by

$$k(x)=\int_{\mathbb{R}} \mathbf{1}_{g([0,t])}(x/r)\frac{|x|}{|r|}\left|(g^{-1})'(x/r)\right|\nu(dr)<\infty$$

for all $x\in\mathbb{R}\setminus\{0\}$.

ii) Let $g>0$ in $[0,t]$. If $\liminf_{x\to 0+}\nu(\frac{x}{g([0,t])})=\lambda_1>0$, then $\liminf_{x\to 0+}k(x)\geq \inf_{y\in g([0,t])}|y||(g^{-1})'(y)|\lambda_1$ and if $\limsup_{x\to 0+}\nu(\frac{x}{g([0,t])})=\lambda_2<\infty$, then $\limsup_{x\to 0+}k(x)\leq \sup_{y\in g([0,t])}|y||(g^{-1})'(y)|\lambda_2$.

Proof. i) We know from (2.7) that for every $A \in \mathcal{B}(\mathbb{R})$

$$
\begin{aligned}
(|x|\nu_g)(A) &= \int_{\mathbb{R}} \int_{[0,t]} |g(s)r| \mathbf{1}_A(g(s)r)\, ds\, \nu(dr) \\
&= \int_{\mathbb{R}} \int_{rg([0,t])} \frac{|x|}{|r|} \left|(g^{-1})'(x/r)\right| \mathbf{1}_A(x)\, dx\, \nu(dr) \\
&= \int_{\mathbb{R}} \mathbf{1}_A(x) \int_{\mathbb{R}} \mathbf{1}_{g([0,t])}(x/r) \frac{|x|}{|r|} \left|(g^{-1})'(x/r)\right| \nu(dr)\, dx.
\end{aligned}
$$

So we see that the density is given by $\int_{\mathbb{R}} \mathbf{1}_{g([0,t])}(x/r) \frac{|x|}{|r|} \left|(g^{-1})'(x/r)\right| \nu(dr)$. Observe that the integral is taken for every $x \neq 0$ in a set away from zero, so boundedness is enough for the finiteness of the integral.
ii) Now assume that $g > 0$. We see that for $x \in \mathbb{R} \setminus \{0\}$

$$
\int_{\mathbb{R}} \mathbf{1}_{g([0,t])}(x/r) \frac{|x|}{|r|} \left|(g^{-1})'(x/r)\right| \nu(dr) \leq \sup_{y \in g([0,t])} |y||(g^{-1})'(y)|(\nu(x/g([0,t]))
$$

and

$$
\int_{\mathbb{R}} \mathbf{1}_{g([0,t])}(x/r) \frac{|x|}{|r|} \left|(g^{-1})'(x/r)\right| \nu(dr) \geq \inf_{y \in g([0,t])} |y||(g^{-1})'(y)|\nu\left(\frac{x}{g([0,t])}\right).
$$

The rest follows by taking the limits.

□

Remark 2.6. For the existence of a Lebesgue density of ν_g it is enough to assume that preimages of Lebesgue null sets under g are again Lebesgue null sets, a condition called Lusin (N^{-1})-condition. To see this, let $B \in \mathcal{B}$ be a Lebesgue null set. Then so is $\frac{1}{r}(B \setminus \{0\})$ for every $r \neq 0$ and hence by (2.7) and the Lusin (N^{-1})-condition we obtain

$$
\begin{aligned}
\nu_g(B) &= \int_{\mathbb{R}} \int_{[0,t]} \mathbf{1}_{g^{-1}\left(\frac{1}{r}(B \setminus \{0\})\right)}(s)\, ds\, \nu(dr) \\
&= \int_{\mathbb{R}} \left| g^{-1}\left(\frac{1}{r}\left(B \setminus \{0\}\right)\right)\right| \nu(dr) = 0
\end{aligned}
$$

(recall that $|\cdot|$ denotes here the Lebesgue measure). This shows that ν_g is absolutely continuous and hence has a density. Sufficient conditions for the Lusin $(N)^{-1}$-conditions to hold can be found in [39, Theorem 4.13, p. 74].

As a consequence of Lemma 2.5 and Theorem 2.2 we find sufficient conditions for the existence of a Lebesgue density of bounded variation.

Corollary 2.7. *Let $g : [0,t] \to \mathbb{R}$ be a $\mathcal{C}^1-$diffeomorpism onto its range and L be a Lévy process with characteristic triplet $(0,\gamma,\nu)$ with $\nu(\mathbb{R}) > 0$. Let $Z = \int_{[0,t]} g(t)\,dL(t)$.*
i) Let $\nu(\mathbb{R}) = \infty$. Then the distribution of Z is absolutely continuous.
ii) Let $g > 0$ on $[0,t]$. If

$$\left(\liminf_{x\to 0+} \nu\left(\frac{x}{g([0,t])}\right) + \liminf_{x\to 0-} \nu\left(\frac{x}{g([0,t])}\right)\right) \inf_{y\in g([0,t])} |y||(g^{-1})'(y)| > 1,$$

then Z has a density which is of bounded variation.
iii) Let $g > 0$ on $[0,t]$. If

$$\left(\limsup_{x\to 0+} \nu\left(\frac{x}{g([0,t])}\right) + \limsup_{x\to 0-} \nu\left(\frac{x}{g([0,t])}\right)\right) \sup_{y\in g([0,t])} |y||(g^{-1})'(y)| < 1,$$

then the density of the random variable Z (if existent) cannot be of bounded variation.

Proof. i) This follows by (2.7), Lemma 2.5 i) and [58, Theorem 27.7, p. 177].
ii) + iii) Clear by Theorem 2.2 and Lemma 2.5 ii). Observe that the condition in ii) implies $\nu(\mathbb{R}) = \infty$ such that Z has a density by i). □

Example 2.8. Let $g(s) = 1+s$ on $[0,t]$ and L be a Lévy process with characteristic triplet $(0,\gamma,\nu)$ such that ν has a Lebesgue density f with $f(x) = 0$ for $x < 0$ and $f(x) = cx^{-1}$ for small $x > 0$ and some $c > 0$. Then $\inf_{y\in g([0,t])} |y|\,|(g^{-1})'(y)| = 1$ and

$$\nu\left(\frac{x}{g([0,t])}\right) = \int_{x/(1+t)}^{x} cs^{-1}ds = c\log(1+t).$$

By Corollary 2.7, $\int_0^t (1+s)dL_s$ has a Lebesgue density and this is of bounded variation if $c > (\log(1+t))^{-1}$ and not of bounded variation if $c < (\log(1+t))^{-1}$. In contrast, by Theorem 2.2, the density of L_1 is of bounded variation if $c > 1$ and not of bounded variation if $c < 1$. This shows that the integral has a smoothing effect on L.

Example 2.9. Let us look at the Lévy-measure $\nu(dx) = \sum_{n=0}^{\infty} k_n \delta_{b^{-n}}(dx)$ for some integer $b \in \mathbb{N}\setminus\{1\}$ such that $\sum_{n=0}^{\infty} k_n = \infty$ and $\sup_{n\in\mathbb{N}} k_n \le C < \infty$ for some positive $C > 0$. It is indeed a Lévy measure as even

$$\int_{\mathbb{R}} \min\{1,x\}\nu(dx) = \sum_{n=0}^{\infty} k_n b^{-n} \le C\sum_{n=0}^{\infty} b^{-n} = \frac{C}{1-b^{-1}} < \infty.$$

It is known that the one-dimensional distribution of the Lévy process L with characteristic triplet $(0,0,\nu)$ is continuous singular, see [58, Theorem 27.19]. Let $g : [0,1] \to \mathbb{R}$ be a positive, increasing $\mathcal{C}^1$ diffeomorphism onto its range with $\frac{g(1)}{g(0)} \geq b^l$ for some $l \in \mathbb{N}$. Let $x \in [0,1]$. We know that there exists an $n \in \mathbb{N}$ such that $b^{-n} < x \leq b^{-n+1}$. We have that

$$\nu\left(\left[x, \frac{g(1)}{g(0)}x\right]\right) \geq \nu\left((b^{-n}, b^{-n+1+l}]\right) = \sum_{r=n-l-1}^{n-1} k_r$$

and see by Corollary 2.7 ii) that if there exist $\varepsilon > 0$ and $m \in \mathbb{N}$, $m \geq l+1$, such that $\sum\limits_{r=i-l-1}^{i-1} k_r \geq \frac{1+\varepsilon}{\inf_{y \in g([0,1])} |y||(g^{-1})'(y)}$ for every $i \geq m$ then the density of the random variable $Z = \int\limits_0^1 g(t)dL(t)$ is of bounded variation (observe that $\liminf_{x\to 0+} \nu\left(\frac{x}{g([0,t])}\right) = \liminf_{x\to 0+} \nu\left(\left[x, \frac{g(1)}{g(0)}x\right]\right)$). Examples of such sequences $(k_n)_{n\in\mathbb{N}}$ are easily constructed. Now let g be an increasing positive $\mathcal{C}^1$-diffeomorphism onto its range with $\frac{g(1)}{g(0)} \leq b^l$ for some $l \in \mathbb{N}$. Then we have $\nu\left([x, \frac{g(1)}{g(0)}x]\right) \leq \nu\left([b^{-n+1}, b^{-n+1+l}]\right) = \sum\limits_{r=n-l-1}^{n-1} k_r$ and we see that if there exist $\varepsilon > 0$ and an $m \in \mathbb{N}$, $m \geq l+1$, such that $\sum\limits_{r=i-l-1}^{i-1} k_r \leq \frac{1-\varepsilon}{\sup_{y\in g([0,1])} |y||(g^{-1})'(y)}$ for every $i > m$, then by Corollary 2.7 iii) the density of Z is not of bounded variation (the density exists by Corollary 2.7 i)). It is easy to construct such examples. Observe that they satisfy $\nu(\mathbb{R}) = \infty$, hence positive $\mathcal{C}^1$-diffeomorphisms and $\nu(\mathbb{R}) = +\infty$ do not imply bounded variation of the density of Z.

Example 2.10. Let $\nu(dx) = \sum\limits_{n=0}^{\infty} k_n \delta_{b^{-2^n}}$ with $b > 1$, $\sum_{n=0}^{\infty} k_n = \infty$ and $\sup_{n\in\mathbb{N}} k_n \leq C < \infty$. Let $g : [0,1] \to \mathbb{R}^+$ be a positive increasing $\mathcal{C}^1$-diffeomorphism onto its range and $m > 0$ such that $\frac{g(1)}{g(0)} = b^m$. Then it is relatively easy to see that

$$\nu\left(\left[\frac{x}{g(1)}, \frac{x}{g(0)}\right]\right) = k_{n-1}$$

if there exists an $n \in \mathbb{N}$ such that $x \in [g(1)b^{-2^{(n-1)}-m}, g(1)b^{-2^{(n-1)}}]$, otherwise the term is equal to 0 for x small enough. We see directly that we cannot use Corollary 2.7 ii) anymore to give a sufficient condition for the density to be of bounded variation since $\liminf_{x\to 0+} \nu\left(\frac{x}{g([0,t])}\right) = 0$, but if $k_n \leq \frac{1-\varepsilon}{\sup_{y\in g([0,1])} |y||(g^{-1})'(y)}$ for some $\varepsilon > 0$ for every $n > n_0 \in \mathbb{N}$ then the density is not of bounded variation by Corollary 2.7 iii).

Now assume that we have a non-deterministic Lévy-process $L = (L_t)_{t\geq 0}$, $\mathcal{L}(L_1)$ being self-decomposable, with characteristic triplet $(0, \gamma, \nu)$ with $\frac{l(x)}{|x|}$ the Lebesgue-density of the Lévy-measure, and that we have a bounded strictly positive function $g > 0$ on an interval $[0,t]$. This is as in Corollary 2.7, but observe that we no longer assume that g is a $\mathcal{C}^1$-diffeomorphism on the cost of more restrictive conditions on L. It follows from (2.7) that the Lévy measure of Z and hence also Z has a density f_g by [58, Theorem 27.7].

Corollary 2.11. *Let Z be as above with density $f_g \in L^1(\mathbb{R}, [0, \infty))$.*

i) If $l(0+) + l(0-) > 1/(pt)$ with $p \in (1, 2]$, then there exists a constant $C > 0$ such that

$$\int_{\mathbb{R}} |f_g(x - z) - f_g(x)|\, dx \le C|z|^{\frac{1}{p}}$$

for every $z \in \mathbb{R}$.

ii) If $l(0+) + l(0-) > 1/t$, then there exists a constant $C > 0$ such that

$$\int_{\mathbb{R}} |f_g(x - z) - f_g(x)|\, dx \le C|z|$$

for every $z \in \mathbb{R}$.

iii) If $l(0+) + l(0-) < \frac{1}{pt}$ with $p \in (0, \infty)$ and $a = 0$, then

$$\sup_{0 \le h \le |z|} \int_{\mathbb{R}} |f_g(x - h) - f_g(x)|\, dx \ge C|z|^{\frac{1}{p}}$$

for some constant $C > 0$ and all $z \in (-1, 1)$.

Proof. The characteristic triplet of $\widehat{\mathcal{L}(Z)}$ is given by $(0, \gamma_g, \nu_g)$ as before, where

$$\nu_g(B) = \int_{[0,t]} \int_{\mathbb{R}} \mathbf{1}_B(g(s)r) \frac{l(r)}{|r|}\, dr\, ds$$

by (2.7). By easy calculations we find that

$$k(r)/|r| := \int_{[0,t]} l(r/g(s))\, ds/|r| \tag{2.8}$$

is the Lebesgue density of ν_g. Then

$$\begin{aligned} c_g &:= k(0+) + k(0-) \\ &= \lim_{r \to 0+} \int_{[0,t]} l(r/g(s))\, ds + \lim_{r \to 0-} \int_{[0,t]} l(r/g(s))g(s)\, ds \\ &= \int_{[0,t]} l(0+)\, ds + \int_{[0,t]} l(0-)\, ds \\ &= (l(0+) + l(0-))t, \end{aligned}$$

and the assertions follow by Theorem 2.2. □

Remark 2.12. It follows from (2.9) that in the situation of Corollary 2.11 the distribution of Z is also self-decomposable. By Corollary 2.11 iii) we see that its probability density is not of bounded variation if the Lévy measure $\frac{l(x)}{|x|}\,dx$ satisfies $l(0+) + l(0-) < 1/t$. As this property is independent of g, we see that for fixed t we cannot find a positive $\mathcal{C}^1$-diffeomorphism for every characteristic triplet such that the stochastic integral has a density of bounded variation.

2.4.2 Stochastic integrals over $[0, \infty)$

Now we want to prove some aspects of the densities of distributions of the form $\int_{[0,\infty)} g(t)dL(t)$, whenever such an integral exists. As before we assume that L has characteristic triplet $(0, \gamma, \nu)$ with $\nu(\mathbb{R}) > 0$. We assume that g is a strictly positive, continuous function which attains its maximum

$$c := \max_{t\in[0,\infty)} g(t)$$

and that there exists a decomposition $(t_i)_{i\in\mathbb{N}_0}$ with $0 = t_0 < t_1 < \ldots$ and $t_i \to \infty$ for $i \to \infty$ such that g restricted to (t_i, t_{i+1}) is a $\mathcal{C}^1-$diffeomorphism onto its range for every $i \in \mathbb{N}_0$. Then we can write

$$\int_{[0,\infty)} g(t)\, dL(t) = \sum_{i=0}^{\infty} \int_{t_i}^{t_{i+1}} g(t)\, dL(t)$$

where the limit is taken in probability and from Lemma 2.5 i) we see that $\int_{[0,\infty)} g(t)dL(t)$ has a Lévy ν_g measure with Lebesgue density

$$\frac{k(x)}{|x|} := \frac{1}{|x|} \sum_{i\in\mathbb{N}_0} \int_{\mathbb{R}} \mathbf{1}_{g((t_i,t_{i+1}))}(x/r) \frac{|x|}{|r|} |(g^{-1})'(x/r)| \nu(dr).$$

From (2.7) we further see that $\nu_g(\mathbb{R}) = +\infty$, so that $\int_{[0,\infty)} g(t)dL(t)$ has a Lebesgue density by [58, Theorem 27.7, p. 177]. Now we can write the density of the Lévy measure for $x > 0$ as

$$\begin{aligned}
\frac{k(x)}{|x|} =& \frac{1}{|x|} \int\limits_{\mathbb{R}} \sum_{i\in\mathbb{N}_0} \mathbf{1}_{g((t_i,t_{i+1}))}(x/r) \frac{|x|}{|r|} |(g^{-1})'(x/r)| \nu(dr) \\
=& \frac{1}{|x|} \int_{\frac{x}{g((0,\infty))}} h(x/r)\nu(dr) \\
=& \frac{1}{|x|} \int\limits_{\left[\frac{x}{c},\infty\right)} h(x/r)\nu(dr) \qquad a.e.,
\end{aligned} \tag{2.9}$$

with

$$h(s) := \sum_{i\in I_s} |s| |(g^{-1}|_{(t_i,t_{i+1})})'(s)|,$$

where $I_s = \{i \in \mathbb{N}_0 : s \in g((t_i, t_{i+1}))\}$. Similarly, $\frac{k(x)}{|x|} = \frac{1}{|x|} \int_{(-\infty, \frac{x}{c}]} h\left(\frac{x}{r}\right) \nu(dr)$ for $x < 0$. Now we obtain immediately from Theorem 2.2:

Proposition 2.13. *Let g have the same properties as above.*
i) The random variable $\int_{[0,\infty)} g(t)\, dL(t)$, if existent, has a density of bounded variation, if

$$\liminf_{x \to 0+} \int_{\left[\frac{x}{c}, \infty\right)} h(x/r)\nu(dr) + \liminf_{x \to 0-} \int_{\left(-\infty, \frac{x}{c}\right]} h(x/r)\nu(dr) > 1.$$

ii) The random variable $\int_{[0,\infty)} g(t)\, dL(t)$, if existent, has not a density of bounded variation, if

$$\limsup_{x \to 0+} \int_{\left[\frac{x}{c}, \infty\right)} h(x/r)\nu(dr) + \limsup_{x \to 0-} \int_{\left(-\infty, \frac{x}{c}\right]} h(x/r)\nu(dr) < 1.$$

If the integral is existent, it is known that there exists a sequence $(z_n)_{n \in \mathbb{N}}$ such that $z_n \to \infty$ and $g(z_n) \to 0$ for $n \to \infty$. We use this simple fact to prove our next corollary.

Corollary 2.14. *Let $g : [0, \infty) \to (0, \infty)$ have the same properties as above, denote $T := \{t_i : i \in \mathbb{N}\}$ and assume that*

$$\liminf_{x \to \infty, x \notin T} \left| \frac{g(x)}{g'(x)} \right| = \alpha$$

for some $\alpha \in (0, \infty]$. Then $\int_{[0,\infty)} g(t)\, dL(t)$ has a density of bounded variation, if $\nu(\mathbb{R}) > \frac{1}{\alpha}$.

Proof. Assume that $\liminf_{x \to \infty, x \notin T} \left| \frac{g(x)}{g'(x)} \right| = \alpha$ for some $\alpha \in (0, \infty]$.
We define the function $\tilde{h} : (0, c] \to \mathbb{R}^+ \cup \{\infty\}$ by $\tilde{h}(x) = h(x)$ for all $x \in (0, c] \setminus g(T)$ and $\tilde{h}(x) = \infty$ otherwise. Then it holds for $x > 0$ that

$$k(x) = \int_{\left[\frac{x}{c}, \infty\right)} h\left(\frac{x}{r}\right) \nu(dr) = \int_{\left[\frac{x}{c}, \infty\right) \setminus \{\frac{x}{r} \in g(T)\}} h\left(\frac{x}{r}\right) \nu(dr) + \int_{\{\frac{x}{r} \in g(T)\}} h\left(\frac{x}{r}\right) \nu(dr).$$

Now as ν has a countable number of points with positive mass we conclude that only in the set $\{\frac{x}{r} \in g(T)\} \cap \{r \in B\}$, where B is the set of points with a positive mass of ν, $\int_{\{\frac{x}{r} \in g(T)\}} h\left(\frac{x}{r}\right) \nu(dr)$ is unequal to 0. So we see that we only differ on a Lebesgue null set by considering $\tilde{k}(x) = \int_{\left[\frac{x}{c}, \infty\right)} \tilde{h}\left(\frac{x}{r}\right) \nu(dr)$ instead of k. Oberserve that the same arguments work for $x < 0$.
Let $x_n \to 0+$ for $n \to \infty$ with $x_n \notin g(T)$ and choose $y_n \to \infty$ with $g(y_n) = x_n$ (existent since g is continuous and by the observation above). We see that

$$\liminf_{n \to \infty} \tilde{h}(x_n) \geq \liminf_{n \to \infty} \frac{|x_n|}{|g'(y_n)|} = \liminf_{n \to \infty} \frac{|g(y_n)|}{|g'(y_n))|} \geq \alpha,$$

as $y_n \to \infty$. Therefore we obtain by the Lemma of Fatou

$$\liminf_{x\to 0+} k(x) \geq \alpha\nu((0,\infty)) \quad \text{and} \quad \liminf_{x\to 0-} k(x) \geq \alpha\nu((-\infty,0)).$$

Proposition 2.13 implies that $\int_{[0,\infty)} g(t)dL(t)$ has a density of bounded variation if $\nu(\mathbb{R}) > \frac{1}{\alpha}$. □

Remark 2.15. We could also use other specifications for g. For example consider a strictly positive and continuous function g on $[0,\infty)$ such that there exist sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ with $0 < a_n < b_n \leq a_{n+1}$ for every $n \in \mathbb{N}$ such that $g|_{(a_n,b_n)}$ is a $\mathcal{C}^1$-diffeomorphism onto its range and $g(\cup_{n=m}^{\infty}[a_n,b_n))$ is a half-open interval with a maximum $c < \infty$ and infimum 0 for an $m \in \mathbb{N}$, i.e. $g(\cup_{n=m}^{\infty}[a_n,b_n)) = (0,c]$. For these kind of functions Proposition 2.13 i) and Corollary 2.14 also hold true, where

$$h(s) := \sum_{i\in I_s} |s||(g^{-1}|_{(a_i,b_i)})'(s)|,$$

with $I_s = \{i \in \mathbb{N}_0 : s \in g((a_i,b_i))\}$.

Example 2.16. Applying Corollary 2.14 to the function $g(x) = e^{-bx}$ with $b > 0$ gives $\alpha = \frac{1}{b}$, hence $\int_{[0,\infty)} e^{-bt}dL(t)$ (if existent) has a density of bounded variation if $\nu(\mathbb{R}) > \frac{1}{b}$. Applying Corollary 2.14 (more precisely, the extension according to Remark 2.15) to the function $g(x) = \min\{x^{-p},1\}$ with $p > 0$ gives $\alpha = \infty$. Hence $\int_{[0,\infty)} \min\{t^{-p},1\}dL(t)$ (if existent) has a density of bounded variation when $\nu(\mathbb{R}) > 0$.

If $g(x) = e^{-x^2}$ we cannot use Corollary 2.14 as $\frac{g(x)}{g'(x)} = 1/(2x) \to 0$ for $x \to \infty$. We will give another condition such that we can obtain sufficient conditions for the existence of a probability density of bounded variation implied by such a kernel function.

Corollary 2.17. *Let* $g(x) = e^{-\psi(x)}$ *with* $\psi : [0,\infty) \to \mathbb{R}$ *continuous such that* $\psi : (0,\infty) \to (0,\infty)$ *is a strictly increasing* $\mathcal{C}^1$*-diffeomorphism and such that* $\psi(0) = 0$ *and* $(\psi^{-1})'$ *is decreasing. Then the Lebesgue density of* $\int_{[0,\infty)} g(t)dL(t)$ *is of bounded variation if*

$$\begin{aligned}
&\liminf_{x\to 0+}(\psi^{-1})'(-\log(x))\nu((x,1)) + \liminf_{x\to 0-}(\psi^{-1})'(-\log|x|)\nu((-1,x)) \\
=&\liminf_{x\to 0+} \frac{\nu((x,1))}{\psi'(\psi^{-1}(-\log(x)))} + \liminf_{x\to 0-} \frac{\nu((-1,x))}{\psi'(\psi^{-1}(-\log|x|))} > 1.
\end{aligned}$$

Proof. For simplicity of notation, we assume that $\nu((-\infty,0)) = 0$. A direct calculation gives us from (2.9) that

$$k(x) = \int_{(x,\infty)} (\psi^{-1})'(\log(r) - \log(x))\nu(dr) = \int_{(x,\infty)} \frac{1}{\psi'(\psi^{-1}(\log(r) - \log(x)))}\nu(dr).$$

As $(\psi^{-1})'$ is decreasing we see that for $0 < x < 1$

$$k(x) \geq \int_{(x,1)} (\psi^{-1})'(-\log(x))\nu(dr) = (\psi^{-1})'(-\log(x))\nu((x,1)).$$

So we see by Proposition 2.13 that if

$$\liminf_{x\to 0+} (\psi^{-1})'(-\log(x))\nu((x,1)) = \liminf_{x\to 0+} \frac{\nu((x,1))}{\psi'(\psi^{-1}(-\log(x)))} > 1$$

the Lebesgue density of $\int_{[0,\infty)} g(t)dL(t)$ is of bounded variation. □

Example 2.18. Let $\psi(x) = x^p$ for $p > 1$. Then we have $\psi^{-1}(x) = x^{1/p}$, $(\psi^{-1})'(x) = \frac{1}{p}x^{1/p-1}$, which is decreasing. We see that if

$$\liminf_{x\to 0+} \frac{\nu((x,1))}{\left(\log\left(\frac{1}{x}\right)\right)^{1/p-1}} + \liminf_{x\to 0-} \frac{\nu((-1,x))}{\left(\log\left(\frac{1}{|x|}\right)\right)^{1/p-1}} > p$$

the Lebesgue density of $\int_{[0,\infty)} e^{-t^p} dL(t)$ is of bounded variation.

3 On Quasi-Infinitely Divisible Distributions with a Point Mass

This chapter is grounded on the published article by Berger [6]. An infinitely divisible distribution on $\mathbb{R}$ is a probability measure μ such that the characteristic function $\hat{\mu}$ has a Lévy-Khintchine representation with characteristic triplet (a, γ, ν), where ν is a Lévy measure, $\gamma \in \mathbb{R}$ and $a \geq 0$. A natural extension of such distributions are quasi-infinitely distributions. Instead of a Lévy measure, we assume that ν is a 'signed Lévy measure', for further information on the definition see [51]. We show that a distribution $\mu = p\delta_{x_0} + (1-p)\mu_{ac}$ with $p > 0$ and $x_0 \in \mathbb{R}$, where μ_{ac} is the absolutely continuous part, is quasi-infinitely divisible if and only if $\hat{\mu}(z) \neq 0$ for every $z \in \mathbb{R}$. We apply this to show that certain variance mixtures of mean zero normal distributions are quasi-infinitely divisible distributions, and we give an example of a quasi-infinitely divisible distribution that is not continuous but has infinite quasi-Lévy measure. Furthermore, it is shown that replacing the signed Lévy measure by a seemingly more general complex Lévy measure does not lead to new distributions. Last but not least it is proven that the class of quasi-infinitely divisible distributions is not open, but path-connected in the space of probability measures with the Prokhorov metric.

3.1 Introduction

The class of infinitely divisible distributions is an important class of distributions, since they correspond in a natural way to Lévy processes. It is well known that infinitely divisible distributions on $\mathbb{R}$ are characterized by the Lévy-Khintchine formula in the sense that a distribution μ is infinitely divisible if and only if there exist $a \geq 0$, $\gamma \in \mathbb{R}$ and a Lévy measure ν such that

$$\hat{\mu}(z) = \exp\left(-\frac{1}{2}az^2 + i\gamma z + \int_{\mathbb{R}} \left(e^{ixz} - 1 - ixz\mathbf{1}_{[-1,1]}(x)\right)\nu(dx)\right) \tag{3.1}$$

for each z, where $\hat{\mu}$ denotes the characteristic function of μ.
The class of quasi-infinitely divisible distributions generalizes the class of infinitely divisible distributions. By definition, a probability distribution μ is quasi-infinitely divisible if and only if its characteristic function admits a Lévy-Khintchine representation (3.1), but

with $a, \gamma \in \mathbb{R}$ and ν being a quasi-Lévy measure, meaning informally that ν is a 'signed Lévy measure'. See [51] and Section 2 below for the precise defintion. It is easily seen that a distribution μ is quasi-infinitely divisible if and only if its characteristic function is the quotient of the characteristic functions of two infinitely divisible distributions, equivalently if there exist two infinitely divisible distributions μ_1, μ_2 such that $\mu_1 * \mu = \mu_2$. Hence, quasi-infinitely divisible distributions appear naturally in the study of factorisation of infinitely divisible distributions.

Applications of quasi-infinitely divisible distributions can be found in physics, see [17 and 25], and in insurance mathematics, see [66].

Although examples of quasi-infinitely divisible distributions have appeared before in the literature (e.g. [20] and [52]), a first step to a systematic treatment of these distributions has only been given recently by Lindner et al. [51]. They showed in particular that the class of quasi-infinitely divisible distributions is dense in the class of probability distributions with respect to weak convergence, and using the Wiener-Lévy theorem they showed that a discrete distribution μ concentrated on a lattice of the form $h\mathbb{Z} + r$ with $h > 0$, $r \in \mathbb{R}$, is quasi-infinitely divisible if and only if its characteristic function $\hat{\mu}$ does not have zeroes on the real line. They also gave an example of a distribution whose characteristic function has no real zeroes, but such that the distribution nevertheless was not quasi-infinitely divisible. They also studied various distributional properties of quasi-infinitely divisible distributions in terms of the characteristic triplet. Another important class of quasi-infinitely divsible distributions was established much earlier by Cuppens [19, Proposition 1; 20 Theorem 4.3.7]. He showed that any probability distribution μ that has an atom of mass greater than 1/2 is quasi-infinitely divisible.

The goal of this chapter is to obtain a further class of quasi-infinitely divisible distributions. A main result in this direction will be that a distribution μ of the form $\mu(dx) = p\delta_{x_0}(dx) + (1-p)f(x)\lambda(dx)$, where $p \in (0,1]$, $x_0 \in \mathbb{R}$, and f being a Lebesgue density, is quasi-infinitely divisible if and only if its characteristic function has no zeroes on the real line. This can then be seen on the one hand as a counter part to the above mentioned result by Cuppens, and on the other to the above mentioned result by Lindner et al. Its proof makes use of a Wiener-Lévy theorem due to Krein [48] for a specific Banach algebra. As a byproduct of our result, we find a quasi-infinitely divisible distribution that is not continuous but has infinite quasi-Lévy measure, thus answering an open question in [51, Open Question 7.2, p. 8510] in the negative. We also show that convex combinations of $N(0, a_i)$-distributions, or more generally variance mixtures of mean zero normal distributions are quasi-infinitely divisible, provided that the lower endpoint t_1 of the mixing distribution ϱ is strictly positive and that $\varrho(\{t_1\}) > 0$. We also treat quasi-infinite divisibility for distributions whose singular part μ_d is supported on a lattice and such that $\hat{\mu}_d$ has no zeroes on the real line.

The results mentioned above can be found in Section 3.4. In Section 3.2, we recall basic notation and the formal definition of quasi-infinitely divisible distributions. In Section 3.3, we address the question if it also makes sense to look at probability distributions μ whose characteristic function have a Lévy-Khintchine type representation with a 'complex

Lévy measure', and show that this does not lead to a new class, i.e. that no probability distribution exists such that the Lévy measure in the Lévy-Khintchine type representation of its characteristic function has a non-zero imaginary part. This result will then be used intensively in the proofs for Section 3.4. Finally, in Section 3.5 we show that the complement of the class of quasi-infinitely divisible distributions is also dense with repect to weak convergence, and that the set of quasi-infinitely divisible distributions is path-connected with respect to the Prokhorov topology. This sheds some further light on the topological properties of this class of distributions.

3.2 Notation and Preliminaries

To fix notation, by a distribution on $\mathbb{R}$ we mean a probability measure on $(\mathbb{R}, \mathcal{B})$ with $\mathcal{B}$ being the Borel $\sigma-$algebra on $\mathbb{R}$, and similarly, by a signed measure on $\mathbb{R}$ we mean it to be defined on $(\mathbb{R}, \mathcal{B})$. By a measure on $\mathbb{R}$ we always mean a positive measure on $(\mathbb{R}, \mathcal{B})$, i.e. an $[0, \infty]$-valued $\sigma-$additive set function on $\mathcal{B}$ that assigns the value 0 to the empty set. The Dirac measure at a point $b \in \mathbb{R}$ will be denoted by δ_b, the Gaussian distribution with mean $a \in \mathbb{R}$ and variance $b \geq 0$ by $N(a, b)$ and the Lebesgue measure by $\lambda(dx)$. Weak convergence of measures will be denoted by '$\xrightarrow{d}$' and the Fourier transform at $z \in \mathbb{R}$ of a finite complex measure μ on $\mathbb{R}$ will be denoted by $\hat{\mu}(z) = \int_{\mathbb{R}} e^{ixz}\, \mu(dx)$. The convolution of two complex measures μ_1 and μ_2 on $\mathbb{R}$ is defined by $\mu_1 * \mu_2(B) = \int_{\mathbb{R}} \mu_1(B - x)\, \mu_2(dx)$, $B \in \mathcal{B}$, where $B - x = \{y - x |\, y \in B\}$. The law of a random variable X will be dentoted by $\mathcal{L}(X)$. The real and imaginary part of a complex number z will be denoted by $\mathrm{Re}\, z$ and $\mathrm{Im}\, z$, respectively, the imaginary unit will be denoted by i. We write $\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ for the set of integers, real numbers and complex numbers, respectively. The indicator function of a set $A \subset \mathbb{R}$ is denoted by $\mathbf{1}_A$. By $L^1(\mathbb{R}, A)$ for $A \subset \mathbb{C}$ we denote the set of all Borel-measurable functions $f : \mathbb{R} \to A$ such that $\int_{\mathbb{R}} |f(x)|\, \lambda(dx) < \infty$.

Informally, a quasi-Lévy type measure is the difference of two Lévy measures. This however is not always a signed measure, because the difference of two Lévy measures ν_1 and ν_2 such that $\nu_1(\mathbb{R}) = \nu_2(\mathbb{R}) = \infty$ is not a signed measure. Therefore, the definition is slighty different. Let us recall the following definitions of [51, Def.2.1-3, pp. 8487-8488]:

Definition 3.1 (6, Definition 2.1, p. 1675)**.**
a) Let $\mathcal{B}_r := \{B \in \mathcal{B} |\, B \cap (-r, r) = \emptyset\}$ for $r > 0$ and $\mathcal{B}_0 := \cup_{r>0} \mathcal{B}_r$ be the class of all Borel sets that are bounded away from zero. Let $\nu : \mathcal{B}_0 \to \mathbb{R}$ be a function such that $\nu|_{\mathcal{B}_r}$ is a finite signed measure for each $r > 0$ and denote the total variation, positive and negative part of $\nu_{|\mathcal{B}_R}$ by $|\nu_{|\mathcal{B}_r}|, \nu^+_{|\mathcal{B}_r}$ and $\nu^-_{|\mathcal{B}_r}$, respectively. Then the total variation $|\nu|$, the positive part ν^+ and the negative part ν^- of ν are defined to be the unique measures on $(\mathbb{R}, \mathcal{B})$ satisfying

$$|\nu|(\{0\}) = \nu^+(\{0\}) = \nu^-(\{0\}) = 0$$

and

$$|\nu|(A) = |\nu_{|\mathcal{B}_r}|(A),\ \nu^+(A) = \nu^+_{|\mathcal{B}_r}(A),\ \nu^-(A) = \nu^-_{|\mathcal{B}_r}(A)$$

when $A \in \mathcal{B}_r$ for some $r > 0$.
b) A *quasi-Lévy type measure* is a function satisfying the condition of *a)* such that its total variation $|\nu|$ satisfies $\int_{\mathbb{R}}(1 \wedge x^2)|\nu|(dx) < \infty$.

Definition 3.2 (6, Definition 2.2, p. 1676)**.** *a)* Let μ be a probability distribution on $\mathbb{R}$. We say that μ is *quasi-infinitely divisible* if its characteristic function has a representation

$$\hat{\mu}(z) = \exp\left(-\frac{1}{2}az^2 + i\gamma z + \int_{\mathbb{R}}(e^{ixz} - 1 - ixz\mathbf{1}_{[-1,1]}(x))\nu(dx)\right),$$

where $a, \gamma \in \mathbb{R}$ and ν a quasi-Lévy-type measure. The *characteristic triplet* (a, γ, ν) of μ is unique (see [58, Exercise 12.2]).
b) A quasi-Lévy type measure ν is called *quasi-Lévy measure*, if additionally there exist a quasi-infinitely divisible distribution μ and some $a, \gamma \in \mathbb{R}$ such that (a, γ, ν) is the characteristic triplet of μ. We call ν the *quasi-Lévy measure of* μ.
c) Let μ be quasi-infinitely divisible with characteristic triplet (a, γ, ν). If $\int_{[-1,1]} |z|\,|\nu|(dx) < \infty$, then we call $\gamma_0 := \gamma - \int_{[-1,1]} z\,\nu(dz)$ the *drift* of μ. In that case, the characteristic function of μ allows the representation

$$\hat{\mu}(z) = \exp\left(-\frac{1}{2}az^2 + i\gamma_0 z + \int_{\mathbb{R}}(e^{ixz} - 1)\nu(dx)\right).$$

We shall see in Theorem 3.5 that the parameter a is necessarily non-negative, and we call it the *Gaussian variance* of μ.

Remark 3.3 (6, Remark 2.3, p. 1676)**.** Not every quasi-Lévy-type measure is a quasi-Lévy measure. For example, the signed measure $\nu(dx) = -\delta_1(dx)$ is a quasi-Lévy-type measure but not a quasi-Lévy measure, see [51, Example 2.9, p. 8491].

3.3 Complex quasi-Lévy Type measures

As stated in Definition 3.2, a quasi-infinitely divisible distribution is a probability distribution μ whose characteristic function admits a Lévy-Khintchine type representation with a quasi-Lévy-type measure. It is natural to ask if there are further distributions whose characteristic function allows a Lévy-Khintchine type representation with a complex quasi-Lévy-type measure. Theorem 3.5 below shows that this is not the case. But before that we need a precise definition:

Definition 3.4 (6, Definition 3.1, p. 1676)**.**
A *complex quasi-Lévy type measure* is a function $\nu : \mathcal{B}_0 \to \mathbb{C}$ such that $\operatorname{Re}\nu$ and $\operatorname{Im}\nu$ are quasi-Lévy type measures.

The integral of a function $f : \mathbb{R} \to \mathbb{C}$ satisfying $|f(x)| \leq C(x^2 \wedge 1)$ for some constant C with respect to a complex quasi-Lévy type measure can be defined in the obvious way as

$$\int_{\mathbb{R}} f(x)\nu(dx) := \lim_{r\downarrow 0} \int_{|x|\geq r} f(x)\nu_{|\mathcal{B}_r}(dx) = \int_{\mathbb{R}} f(x)(\operatorname{Re}\nu)(dx) + i\int_{\mathbb{R}} f(x)(\operatorname{Im}\nu)(dx),$$

which shows in particular that $x \mapsto e^{izx} - 1 - izx\mathbf{1}_{[-1,1]}(x)$ is integrable with respect to ν for every $z \in \mathbb{R}$.
We now come to the aforementioned result:

Theorem 3.5 (6, Theorem 3.2, p. 1676)**.** *Let μ be a distribution on $\mathbb{R}$ whose characteristic function allows a representation of the form*

$$\hat{\mu}(z) = \exp\left(i\gamma z - \frac{1}{2}az^2 + \int_{\mathbb{R}} (e^{ixz} - 1 - izx\mathbf{1}_{[-1,1]}(x))\nu(dx)\right) \tag{3.2}$$

where $\gamma \in \mathbb{C}$, $a \in \mathbb{C}$ and ν is a complex quasi-Lévy-type measure. Then $a, \gamma \in \mathbb{R}$, $a \geq 0$ and $\operatorname{Im}\nu = 0$, i.e. ν is a quasi-Lévy measure and μ is quasi-infinitely divisible.

Proof. We see that

$$\begin{aligned}|\hat{\mu}(z)|^2 = \hat{\mu}(z)\hat{\mu}(-z) &= \exp\left(-az^2 + \int_{\mathbb{R}} (e^{ixz} + e^{-ixz} - 2)\nu(dx)\right)\\ &= \exp\left(-az^2 + \int_{\mathbb{R}} 2(\cos(xz) - 1)\nu(dx)\right)\end{aligned}$$

and

$$\frac{\hat{\mu}(z)}{\hat{\mu}(-z)} = \exp\left(i2\gamma z + i\int_{\mathbb{R}} 2(\sin(xz) - zx\mathbf{1}_{[-1,1]}(x))\nu(dx)\right).$$

As $|\hat{\mu}(z)| > 0$ for every $z \in \mathbb{R}$, we see that

$$|\hat{\mu}(z)|^2 = \exp\left(\log(|\hat{\mu}(z)|^2)\right),$$

where log is the natural logarithm. As the distinguished logarithm is uniquely determined (see [58], Lemma 7.6) and

$$g(z) = -az^2 + \int_{\mathbb{R}} 2(\cos(xz) - 1)\nu(dx)$$

is continuous and $g(0) = 0$, we conclude that

$$-\frac{1}{2}az^2 + \int_{\mathbb{R}} (\cos(xz) - 1)\nu(dx) \in \mathbb{R}$$

for every $z \in \mathbb{R}$ and especially we obtain

$$-\frac{1}{2}\mathrm{Im}\, az^2 + \int_{\mathbb{R}} (\cos(xz) - 1)(\mathrm{Im}\,\nu)(dx) = 0.$$

Furthermore, as $|\hat{\mu}(z)| = |\overline{\hat{\mu}(z)}| = |\hat{\mu}(-z)|$, we conclude that

$$\gamma z + \int_{\mathbb{R}} (\sin(xz) - zx\mathbf{1}_{[-1,1]}(x))\nu(dx) \in \mathbb{R}$$

for every $z \in \mathbb{R}$. It follows that

$$0 = \mathrm{Im}\,\gamma z + \int_{\mathbb{R}} (\sin(xz) - zx\mathbf{1}_{[-1,1]}(x))(\mathrm{Im}\,\nu)(dx)$$

for every $z \in \mathbb{R}$. At last, with the quasi-Lévy measure $\mathrm{Im}\,\nu$ we obtain a Lévy-Khintchine formula for $\delta_0(dx)$, because

$$\begin{aligned} 0 &= i\mathrm{Im}\,\gamma z - \frac{1}{2}\mathrm{Im}\, az^2 + \int_{\mathbb{R}} (\cos(xz) - 1)(\mathrm{Im}\,\nu)(dx) + i\int_{\mathbb{R}} (\sin(xz) - xz\mathbf{1}_{[-1,1]})(\mathrm{Im}\,\nu)(dx) \\ &= i\mathrm{Im}\,\gamma z - \frac{1}{2}\mathrm{Im}\, az^2 + \int_{\mathbb{R}} (e^{ixz} - 1 - ixz\mathbf{1}_{[-1,1]}(x))(\mathrm{Im}\,\nu)(dx). \end{aligned}$$

Hence

$$\hat{\delta}_0(z) = 1 = \exp(0) = \exp\left(i\mathrm{Im}\,\gamma z - \frac{1}{2}\mathrm{Im}\, az^2 + \int_{\mathbb{R}} (e^{ixz} - 1 - ixz\mathbf{1}_{[-1,1]}(x))(\mathrm{Im}\,\nu)(dx)\right).$$

The uniqueness of the characteristic triplet of quasi-infinitely divisible distributions then shows that $\mathrm{Im}\,\gamma = \mathrm{Im}\, a = 0$ and that $\mathrm{Im}\,\nu$ is the null-measure. Hence μ is quasi-infinitely divisible. That $a \geq 0$ follows from Lemma 2.7 in [51]. □

Remark 3.6 (6, Remark 3.3, p. 1677)**.** Theorem 3.5 is very helpful to prove quasi-infinite divisibility of certain distributions, as it is often easier to establish a Lévy-Khintchine type representation with a complex quasi-Lévy-type measure rather than directly with a quasi-Lévy-type measure. An example is the proof of Theorem 8.1 in [51]. There it is shown, using the Lévy-Wiener-Theorem, that a distribution μ on $\mathbb{Z}$ with $\hat{\mu}(z) \neq 0$ for all z allows a Lévy-Khintchine type representation with a complex Lévy-type measure $\nu = \sum_{k\in\mathbb{Z}\setminus\{0\}} b_k\delta_k$ for some summable sequence $b_k \in \mathbb{C}$. There, it is shown using an involved approximation argument that the b_k are actually real and hence μ quasi-infinitely divisible. This step can now be simplified considerably by using Theorem 3.5.

3.4 Some new quasi-infinitely divisible distributions

3.4.1 Absolutely continuous distributions plus a point mass

In this section we will look at distributions of the form

$$\mu(dx) = p\delta_{x_0}(dx) + (1-p)f(x)\lambda(dx), \tag{3.3}$$

where λ is the Lebesgue measure, f a Lebesgue density, $x_0 \in \mathbb{R}$ and $p \in (0,1)$. We first specialize to $x_0 = 0$. The characteristic function is then given by

$$\hat{\mu}(z) = p + (1-p)\hat{f}(z),$$

where $\hat{f}(z) = \int_{\mathbb{R}} e^{ixz} f(x)\,\lambda(dx)$. We want to use a similar argument as in [51] in order to show every distribution μ of the form (3.3) is quasi-infinitely divisible if and only if $\hat{\mu}(z) \neq 0$ for every $z \in \mathbb{R}$. We denote by $\overline{\mathbb{R}}$ the extended real numbers, i.e. $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$. A characteristic function $\hat{\mu}$ of the distribution μ of the form (3.3) is then nonzero on the set $\overline{\mathbb{R}}$ if and only if it is nonzero on $\mathbb{R}$. This follows directly from the Riemann-Lebesgue Lemma.
At first let us fix some notation.

Definition 3.7 (6, Definition 4.1, p. 1678)**.** We denote by $W(\mathbb{R},\mathbb{C})$ the space

$$\begin{aligned} W(\mathbb{R},\mathbb{C}) := \{F : \mathbb{R} \to \mathbb{C}\,|\, &\exists p \in \mathbb{C}, f \in L^1(\mathbb{R},\mathbb{C}) \text{ such that} \\ &F(z) = p + \int_{\mathbb{R}} f(x)e^{ixz}\,\lambda(dx) \text{ for all } z \in \mathbb{R}\}. \end{aligned}$$

With the norm $||F|| = |p| + ||f||_{L^1(\mathbb{R},\mathbb{C})}$ for $F(z) = p + \int_{\mathbb{R}} f(x)e^{ixz}\,\lambda(dx)$ the normed space $(W(\mathbb{R},\mathbb{C}), ||\cdot||)$ becomes a Banach algebra (see [49, Theorem 4.1]).

Definition 3.8 (6, Definition 4.2, p. 1678)**.** Let $W(\mathbb{R},\mathbb{C}) \ni F(z) = p + \int_{\mathbb{R}} e^{ixz} f(x)\lambda(dx) \neq 0$ for every $z \in \mathbb{R}$ and $p \in \mathbb{C}\setminus\{0\}$. Then we can interpret F as a closed curve in $\mathbb{C}$. By the property of the distinguished logarithm, there exists a continuous function $g : \mathbb{R} \to \mathbb{R}$

such that

$$\frac{F(z)}{|F(z)|} = \exp\left(ig(z)\right) \quad \text{for all } z \in \mathbb{R}.$$

Then the *index* $\operatorname{ind}(F)$ of F is defined as

$$\operatorname{ind}(F) := \frac{1}{2\pi}\left(\lim_{z\to+\infty} g(z) - \lim_{z\to-\infty} g(z)\right).$$

Remark 3.9 (6, Remark 4.3, p. 1678)**.** By the Riemann-Lebesgue, it is relatively easy to see that $\operatorname{ind}(F)$ is well-defined and $\operatorname{ind}(F) \in \mathbb{Z}$. Also, for $F(z) = p + \int_{\mathbb{R}} e^{ixz} f(x)\,\lambda(dx) \in W(\mathbb{R},\mathbb{C})$ we have $F(z) \neq 0$ for all $z \in \overline{\mathbb{R}}$ if and only if $p \neq 0$ and $F(z) \neq 0$ for all $z \in \mathbb{R}$.

The key ingredient for identifying further quasi-infinitely divisible distributions is [48, Theorem L, p. 175] which asserts the following: Given a function $F(z) = p + \int_{\mathbb{R}} e^{ixz} f(x)\lambda(dx)$ such that $f \in L^1(\mathbb{R},\mathbb{C})$, $p \in \mathbb{C}\setminus\{0\}$, $F(z) \neq 0$ for every $x \in \mathbb{R}$ and $\operatorname{ind}(F) = 0$ there exist some $q \in \mathbb{C}$ and a function $g \in L^1(\mathbb{R},\mathbb{C})$ such that

$$F(z) = \exp\left(q + \int_{\mathbb{R}} e^{ixz} g(x)\,\lambda(dx)\right) \quad \text{for all } z \in \mathbb{R}.$$

With the aid of Theorem [48, Theorem L, p. 175] we can now give a Lévy-Khintchine type representation for arbitrary $F \in W(\mathbb{R},\mathbb{C})$ that do not vanish on $\overline{\mathbb{R}}$ and are such that $F(0) = 1$.

Theorem 3.10 (6, Theorem 4.4, p. 1678)**.**
Let $F \in W(\mathbb{R},\mathbb{C})$ with $F(z) \neq 0$ for every $z \in \overline{\mathbb{R}}$ and $F(0) = 1$. Denote by m the index of F. Then there is some function $g \in L^1(\mathbb{R},\mathbb{C})$ such that

$$F(z) = \exp\left(\int_{\mathbb{R}} (e^{ixz} - 1)\left(g(x) + \frac{me^{-|x|}}{|x|}(\mathbf{1}_{(0,\infty)}(x) - \mathbf{1}_{(-\infty,0)}(x))\right)\lambda(dx)\right) \tag{3.4}$$

for all $z \in \mathbb{R}$.

Proof. a) Let us first assume that $m = \operatorname{ind}(F) = 0$. By [48, Theorem L, p. 175] as stated above there exist a constant $c \in \mathbb{C}$ and a function $g \in L^1(\mathbb{R},\mathbb{C})$ such that

$$F(z) = \exp\left(c + \int_{\mathbb{R}} e^{izx} g(x)\,\lambda(dx)\right).$$

As $F(0) = 1$, we conclude that

$$c + \int_{\mathbb{R}} g(x)\,\lambda(dx) \in 2\pi i\mathbb{Z}.$$

Hence, we can write

$$\begin{aligned} F(z) &= \exp\left(c + \int_{\mathbb{R}} g(x)\,\lambda(dx)\right) \exp\left(\int_{\mathbb{R}} (e^{ixz} - 1)g(x)\,\lambda(dx)\right) \\ &= \exp\left(\int_{\mathbb{R}} (e^{ixz} - 1)g(x)\,\lambda(dx)\right). \end{aligned}$$

b) Now assume that $0 \neq m = \mathrm{ind}(F) \in \mathbb{N}$. Define the function $Q : \mathbb{R} \to \mathbb{C}$ by

$$Q(z) = \frac{(z-i)^m}{(z+i)^m}, \quad z \in \mathbb{R}.$$

Since

$$\frac{z+i}{z-i} = 1 + \frac{2i}{z-i} = 1 - \frac{2}{1+iz} = 1 - 2\int_{-\infty}^{0} e^x e^{ixz}\,\lambda(dx), \quad z \in \mathbb{R},$$

it follows that $z \mapsto \frac{z+i}{z-i} \in W(\mathbb{R}, \mathbb{C})$ and hence, since $W(\mathbb{R}, \mathbb{C})$ is a Banach algebra, that also $Q^{-1} \in W(\mathbb{R}, \mathbb{C})$ and that $Q^{-1}F \in W(\mathbb{R}, \mathbb{C})$. Then obviously $Q^{-1}(z)F(z) \neq 0$ for all $z \in \overline{\mathbb{R}}$, and by the proof of Theorem 2.2, p. 180 in Krein [48], it follows that $\mathrm{ind}(Q^{-1}F) = 0$ and hence $\mathrm{ind}(Q(0)Q^{-1}F) = 0$. Hence, by part *a)* already proved, there is some $g \in L^1(\mathbb{R}, \mathbb{C})$ such that

$$Q(0)Q^{-1}(z)F(z) = \exp\left(\int_{\mathbb{R}} (e^{ixz} - 1)g(x)\,\lambda(dx)\right), \quad \text{for all } z \in \mathbb{R}. \tag{3.5}$$

But

$$\frac{1}{(z+i)^m} = (-i)^m \frac{1}{(1-iz)^m} = (-i)^m \exp\left(\int_0^\infty (e^{ixz} - 1)\frac{me^{-x}}{x}\lambda(dx)\right) \quad \text{for all } z \in \mathbb{R}$$

(see [58], Example 8.10), hence

$$(z-i)^m = \left((-1)^m \frac{1}{(-z+i)^m}\right)^{-1} = (-1)^m i^m \exp\left(-\int_0^\infty (e^{-izx} - 1)m\frac{e^{-x}}{x}\,\lambda(dx)\right)$$

so that

$$Q(z) = \left(\frac{z-i}{z+i}\right)^m = (-1)^m \exp\left(\int_{\mathbb{R}} (e^{ixz}-1)\left(\frac{me^{-x}}{x}\mathbf{1}_{(0,\infty)}(x) - \frac{me^{-|x|}}{|x|}\mathbf{1}_{(-\infty,0)}(x)\right)\lambda(dx)\right).$$

Observe that $(-1)^m = Q(0)$. Together with (3.5) this gives the desired result when $m \in \mathbb{N}$.
c) Now assume that $m = \operatorname{ind}(F) \in -\mathbb{N}$. Then $x \mapsto F(-z) =: G(z) \in W(\mathbb{R},\mathbb{C})$ with $\operatorname{ind}(G) = -m$. The result then follows from *b)*. ☐

Similarly as in Lindner et al. [51], who showed that a distribution on $\mathbb{Z}$ is quasi-infinitely divisible if and only if its characteristic function has no zeroes, we can now prove that a distribution whose singular part consists of a non-trivial atom is quasi-infinitely divsible if and only if its characteristic function has no zeroes:

Theorem 3.11 (6, Theorem 4.5, p. 1679)**.** *Let μ be a probability distribution of the form* (3.3)*. Then μ is quasi-infinitely divisible if and only if $\hat{\mu}(z) \neq 0$ for every $z \in \mathbb{R}$. In that case, the quasi-Lévy measure ν of μ is given by*

$$\left(g(x) + \frac{me^{-|x|}}{|x|}(\mathbf{1}_{(0,\infty)}(x) - \mathbf{1}_{(-\infty,0)}(x))\right)\lambda(dx),$$

where $g \in L^1(\mathbb{R},\mathbb{R})$ and m is the index of $\widehat{\mu * \delta_{-x_0}}$. Furthermore, $\int_{-1}^1 |x||\nu|(dx) < \infty$, μ has drift x_0 and Gaussian variance 0. Finally, $|\nu|$ is finite if and only if $m = 0$, and if $m \neq 0$, then $\nu^-(\mathbb{R}) = \nu^+(\mathbb{R}) = \infty$.

Proof. That $\hat{\mu}(z) \neq 0$ for all $z \in \mathbb{R}$ is obviously necessary for μ to be quasi-infinitely divisible. To see that it is sufficient, it is sufficient to assume that $x_0 = 0$, since μ is quasi-infinitely divsible if and only if $\tilde{\mu} := \mu * \delta_{-x_0}$ is quasi-infinitely divsible. By Theorem 3.10 we see that $\tilde{\mu}$ has a Lévy-Khintchine representation given by

$$\hat{\tilde{\mu}}(z) = \exp\left(\int_{\mathbb{R}} (e^{ixz}-1)(g(x) + \frac{me^{-|x|}}{|x|}(\mathbf{1}_{(0,\infty)}(x) - \mathbf{1}_{(-\infty,0)}(x)))\lambda(dx)\right)$$

for all $z \in \mathbb{R}$ with some $g \in L^1(\mathbb{R},\mathbb{C})$. Then Theorem 3.5 implies that $g \in L^1(\mathbb{R},\mathbb{R})$ and μ is quasi-infinitely divisible. The remaining assertions are clear. ☐

Example 3.12 (6, Example 4.6, p. 1680)**.** It is worth noting that distributions of the above form with non-zero index can indeed occur. For example, consider the distribution

$$\mu(dx) = \frac{1}{1000}\delta_0(dx) + \frac{999}{1000}\rho(dx),$$

where $\rho = N(1,1)$. Then $\hat{\mu}(z) = \frac{1}{1000} + \frac{999}{1000} e^{iz} e^{-z^2/2} \neq 0$ for all $z \in \mathbb{R}$. Observing that $\hat{\mu}(\pi) < 0$, $\operatorname{Re} \hat{\mu}(z) > 0$ for all $z \geq 2\pi$, $\operatorname{Im} \hat{\mu}(z) = e^{-z^2/2} \sin(z)$ it is easy to see that $\hat{\mu}$ has index 2. Hence μ is quasi-infinitely divisible with quasi-Lévy measure ν satisfying $\nu^-(\mathbb{R}) = \nu^+(\mathbb{R}) = |\nu|(\mathbb{R}) = \infty$ by Theorem 3.11.

Remark 3.13 (6, Remark 4.7, p. 1680)**.** In [51, Open Question 7.2, p. 8510] it was asked that if for a quasi-infinitely divisible distribution μ with triplet (a, γ, ν) continuity of μ is equivalent to $a \neq 0$ or $|\nu|(\mathbb{R}) = \infty$. Example 3.12 answers this question in the negative. Indeed, the distribution μ there is not continuous, but the total variation of the quasi-Lévy measure is infinite.

Remark 3.14 (6, Remark 4.8, p. 1680)**.** It is known that distributions of the form (3.3) with $p \in [0,1)$, $x_0 = 0$ and f vanishing on $(-\infty, 0)$ are infinitely divisible provided $\log f$ is convex on $(0, \infty)$ or f is completely montone on $(0, \infty)$, see [58, Theorem 51.4, 51.6]. Theorem 3.11 shows that quasi-infinite divsibility can be achieved for a much wider class of distributions μ of this type, provided $p > 0$ and $\hat{\mu}$ has no zeroes, but with no other assumptions on f.

As in [51, Corollary 8.3, p. 8513], one can now see that factors of a quasi-infinitely divisible distribution of the form (3.3) are also quasi-infinitely divisible.

Corollary 3.15 (6, Corollary 4.9, p. 1680)**.** *Let μ be a distribution of the form* (3.3) *and μ_1, μ_2 probability distributions such that $\mu = \mu_1 * \mu_2$. Then μ is quasi-infinitely divisible if and only if μ_1 and μ_2 are quasi-infinitely divisible.*

Proof. We write

$$\mu_i = \mu_i^d + \mu_i^{ac} + \mu_i^{cs}$$

for $i = 1, 2$ where μ_i^d is the discrete part, μ_i^{ac} is the absolute continuous part and μ_i^{cs} is the continuous singular part. Hence, we can write

$$\begin{aligned} \mu = &\mu_1^d * \mu_2^d + \mu_1^d * \mu_2^{cs} + \mu_1^d * \mu_2^{ac} \\ &+ \mu_1^{cs} * \mu_2^d + \mu_1^{cs} * \mu_2^{cs} + \mu_1^{cs} * \mu_2^{ac} + \mu_1^{ac} * \mu_2^d + \mu_1^{ac} * \mu_2^{cs} + \mu_1^{ac} * \mu_2^{ac}. \end{aligned}$$

As $\mu_1^d * \mu_2^d$ is the only discrete part, we conclude that μ_1 and μ_2 each have exactly one point mass. Moreover, μ_i^{cs} has to be zero for $i = 1, 2$, as $\mu_i^{cs} * \mu_j^d$ is continuous singular for $j \neq i$. So we can write

$$\mu = (p_1 \delta_{z_1}(dx) + \mu_1^{ac}) * (p_2 \delta_{z_2}(dx) + \mu_2^{ac}),$$

such that $p = p_1 p_2$. It follows from Theorem 3.11 that μ_1 and μ_2 are quasi-infinitely divisible if and only if μ is quasi-infinitely divisible. $\square$

It follows from Theorem 3.11 and the closedness of the class of quasi-infinitely divisible distribution under convolution [51, Remark 2.6.a, p. 8490] that if μ is a distribution of the form (3.3) with $\hat{\mu}(z) \neq 0$ for all $z \in \mathbb{R}$, and if μ' is an infinitely-divisible (or quasi-infinitely divisible) distribution, then $\mu' * \mu$ is again quasi-infinitely divisible. This observation can be used to derive quasi-infinite divisibility of certain variance mixtures of normal distributions or more generally mixtures of distributions of Lévy processes. More precisely, we have:

Corollary 3.16 (6, Corollary 4.10, p. 1681)**.**
a) Let ϱ be a probability distribution on $\mathbb{R}$ with $\varrho((-\infty, t_1)) = 0$ and $\varrho(\{t_1\}) > 0$ for some $t_1 > 0$. Let $L = (L_t)_{t \geq 0}$ be a Lévy process such that $\mathcal{L}(L_t)$ is absolutely continuous for each $t > 0$. Define the mixture $\mu := \int_{[t_1,\infty)} \mathcal{L}(L_t)\, \varrho(dt)$ by

$$\mu(B) := \int\limits_{[t_1,\infty)} \mathcal{L}(L_t)(B)\, \varrho(dt), \quad B \in \mathcal{B}. \tag{3.6}$$

Then μ is quasi-infinitely divisible if and only if $\hat{\mu}(z) \neq 0$ for all $z \in \mathbb{R}$. In particular, if $\varrho = \sum_{i=1}^n p_i \delta_{t_i}$ with $t_1 < t_2 < \ldots < t_n$ and $0 < p_1, \ldots, p_n < 1$, $\sum_{i=1}^n p_i = 1$, then $\mu = \sum_{i=1}^n p_i \mathcal{L}(L_{t_i})$ is quasi-infinitely divisible if and only if $\hat{\mu}(z) \neq 0$ for all $z \in \mathbb{R}$.
b) The assumption $\hat{\mu}(z) \neq 0$ for all $z \in \mathbb{R}$ for μ of the form (3.6) is in particular satisfied when $\mathcal{L}(L_1)$ is symmetric.

Proof. a) Write $\mu_t = \mathcal{L}(L_t)$. Then

$$\begin{aligned} \mu &= \int\limits_{[t_1,\infty)} \mu_t\, \varrho(dt) = \varrho(\{t_1\})\mu_{t_1} + \int\limits_{(t_1,\infty)} \mu_t \varrho(dt) \\ &= \mu_{t_1} * \left(\varrho(\{t_1\})\delta_0 + \int\limits_{(t_1,\infty)} \mu_{t-t_1} \varrho(dt) \right). \end{aligned}$$

Assume that $\hat{\mu}(z) \neq 0$ for all $z \in \mathbb{R}$. Then $(\varrho(\{t_1\})\delta_0 + \int_{(t_1,\infty)} \mu_{t-t_1} \varrho(dt))\hat{}(z) \neq 0$ for all $z \in \mathbb{R}$. Since μ_{t-t_1} is absolutely continuous for all $t > t_1$, so is $\int_{(t_1,\infty)} \mu_{t-t_1} \varrho(dt)$. Hence $\varrho(\{t_1\})\delta_0 + \int_{(t_1,\infty)} \mu_{t-t_1} \varrho(dt)$ is quasi-infinitely divisible by Theorem 3.11. Since μ_1 is infinitely divisible, this shows quasi-infinite divisibility of μ. The converse and the specialization to $\varrho = \sum_{i=1}^n p_i \delta_{t_i}$ are clear.
b) This follows from the fact that

$$\hat{\mu}(z) = \int\limits_{[t_1,\infty)} \hat{\mu}_t\, \varrho(dt)$$

and that $\hat{\mu}_t(z) > 0$ when μ_t is symmetric. □

Corollary 3.16 applies in particular when L is a standard Brownian motion and hence to variance mixtures of the form $\sum_{i=1}^n p_i N(0, a_i)$ or more generally to variance mixtures of the form $\int_{[t_1,\infty)} N(0,t)\varrho(dt)$ when $\varrho(\{t_1\}) > 0$ and $t_1 > 0$. That a variance mixture of the form $pN(0,a) + (1-p)N(0,b)$ with $0 < a < b$ and $p \in (1/2, 1)$ is quasi-infinitely divisible was already observed in [51, Example 3.6]. Corollary 3.16 improves in particular on that result in the sense that it shows that $p \in (1/2, 1)$ is superfluous.
Observe that a distribution of the form $\int_{[0,\infty)} N(0,t)\varrho(dt)$ cannot be infinitely divisible when the support of ϱ is additionally bounded and ϱ is non-degenerate, see [45, Theorem 2], but it is infinitely divisible if ϱ is infinitely divisible (e.g. [60, Example IV, 11.6). Hence Corollary 3.16 sheds some further light onto the behaviour of variance mixtures of normal distributions.

Remark 3.17 (6, Remark 4.11, p. 1681)**.** Corollary 3.16 continues to hold when L is replaced by an additive process for which all increment distribution $\mathcal{L}(L_t - L_s)$ with $0 < s < t$ are absolutely continuous. The proof is exactly the same as in Corollary 3.16. In particular, $\mu = \sum_{i=1}^n p_i N(b_i, a_i)$ is quasi-infinitely divisible for $0 < p_1, \dots, p_n < 1$, $\sum_{i=1}^n p_i = 1$, $0 < a_1 < a_2 < \dots < a_n$ and $b_1, \dots, b_n \in \mathbb{R}$ if and only if $\hat{\mu}(z) \neq 0$ for all $z \in \mathbb{R}$.

3.4.2 Absolutely continuous distributions plus a lattice distribution

Until now we considered distributions of the form $p\delta_{x_0} + \mu_{ac}$, where $p > 0$ and μ_{ac} was absolutely continuous. We will now generalise Theorem 3.11 to distributions of the form

$$\mu = \mu_d + \mu_{ac},$$

where μ_d is a non-zero discrete measure supported on a lattice with non-vanishing characteristic function, and μ_{ac} is absolutely continuous. We use the well known Wiener's Lemma, which states that for a function $f(z) = \sum\limits_{k\in\mathbb{Z}} c_k e^{ikz}$ with $\sum_{k\in\mathbb{Z}} |c_k| < \infty$ such that $f(z) \neq 0$ for every $z \in \mathbb{R}$ there exists a function $g(z) = \sum\limits_{k\in\mathbb{Z}} d_k e^{ikz}$ with $\sum_{k\in\mathbb{Z}} |d_k| < \infty$ such that

$$f(z)g(z) = 1$$

for every $z \in \mathbb{R}$. For a proof see [53, Corollary 4.27]. Now we prove the aforementioned generalisation.

Theorem 3.18 (6, Theorem 4.12, p. 1681)**.** *Let μ be a probability distribution of the form*

$$\mu(dx) = \mu_d(dx) + f(x)\lambda(dx),$$

where μ_d is a non-zero discrete measure supported on a lattice of the form $r + h\mathbb{Z}$ for some $r \in \mathbb{R}$ and $h > 0$, $\hat{\mu}_d(z) \neq 0$ for all $z \in \mathbb{R}$ and $f \in L^1(\mathbb{R}, [0, \infty))$. Then μ is quasi-infinitely divisible if and only if $\hat{\mu}(z) \neq 0$ for all $z \in \mathbb{R}$. In that case, the Gaussian variance of μ is zero and the quasi-Lévy measure ν satisfies $\int_{-1}^{1} |x| \, |\nu|(dx) < \infty$.

Proof. By shifting and scaling the distribution, we assume without loss of generality that $\operatorname{supp} \mu_d \subset \mathbb{Z}$, hence we can write

$$\mu_d(dx) = \sum_{k \in \mathbb{Z}} p_k \delta_k(dx).$$

Its characteristic function is given by

$$\hat{\mu}_d(z) = \sum_{k \in \mathbb{Z}} p_k e^{ikz}.$$

Now by Wiener's Lemma [53, Corollary 4.27] as stated above there exists a function g with $g(z)\hat{\mu}_d(z) = 1$ for all $z \in \mathbb{R}$ and

$$g(z) = \sum_{k \in \mathbb{Z}} c_k e^{ikz}$$

and $\sum_{k \in \mathbb{Z}} |c_k| < \infty$. We can associate a complex measure ϱ such that

$$\varrho(dx) = \sum_{k \in \mathbb{Z}} c_k \delta_k(dx),$$

and especially we conclude that

$$\mu_d * \varrho = \delta_0.$$

Now we decompose μ as follows

$$\mu = \mu_d * (\delta_0 + \varrho * \mu_{ac}),$$

where $\mu_{ac}(dx) = f(x)\lambda(dx)$. Since $\hat{\mu}_d(z) \neq 0$ for all $z \in \mathbb{R}$, μ_d is quasi-infinitely divisible with finite quasi-Lévy measure by [51, Theorem 8.1, p. 8512]. Furthermore, $\varrho * \mu_{ac}$ is absolutely continuous, hence there exists some $g \in L^1(\mathbb{R}, \mathbb{C})$ such that $\varrho * \mu_{ac} = g(x)\lambda(dx)$. Theorem 3.10 then shows that $(\delta_0 + \varrho * \mu_{ac})\hat{}$ has a (possibly complex) Lévy-Khintchine type representation, and since μ_d is quasi-infinitely divisible, so does $\hat{\mu}$. But μ is a probability distribution and it follows from Theorem 3.5 that μ is quasi-infinitely divisible. That the Gaussian variance of μ is zero and the quasi-Lévy measure satisfies $\int_{-1}^{1} |x| \, |\nu|(dx) < \infty$ then follows from Theorem 3.11 and its proof. $\square$

As in Corollary 3.15, we can now show that factors of a quasi-infinitely divisible distribution of the form $\mu = \mu_d + \mu_{ac}$ as above are also quasi-infinitely divisible.

Corollary 3.19 (6, Corollary 4.13, p. 1682)**.** *Let μ be of the form $\mu(dx) = \mu_d(dx) + f(x)\lambda(dx)$ as above with $\hat{\mu}_d(z) \neq 0$ for all $z \in \mathbb{R}$ and μ_d being concentrated on a lattice. Let $\mu = \mu_1 * \mu_2$ be a factorisation of μ. Then μ is quasi-infinitely divisible if and only if μ_1 and μ_2 are quasi-infinitely divisible.*

Proof. Assume $\hat{\mu}(z) \neq 0$ for all $z \in \mathbb{R}$. As in Corollary 3.15 we can write μ as

$$\begin{aligned}\mu =&\mu_1^d * \mu_2^d + \mu_1^d * \mu_2^{cs} + \mu_1^d * \mu_2^{ac}\\ &+ \mu_1^{cs} * \mu_2^d + \mu_1^{cs} * \mu_2^{cs} + \mu_1^{cs} * \mu_2^{ac} + \mu_1^{ac} * \mu_2^d + \mu_1^{ac} * \mu_2^{cs} + \mu_1^{ac} * \mu_2^{ac}.\end{aligned}$$

Now we know that $\mu_1^d * \mu_2^d$ is the only discrete part of μ, so we conclude

$$\mu_d = \mu_1^d * \mu_2^d.$$

By [51, Corollary 8.3] we know that μ_1^d and μ_2^d are lattice distributions with non-vanishing characteristic functions. $\mu_i^d * \mu_j^{cs}$ is continuous singular for $i \neq j$, hence $\mu_i^{cs} = 0$ and from Theorem 3.18 we conclude that μ_1 and μ_2 are quasi-infinitely divisible.
The converse is clear. □

In Remark 3.17 we characterized quasi-infinite divisibility of $\sum_{i=1}^n p_i N(b_i, a_i)$ as long as $a_1 < a_2 < \ldots < a_n$. With the aid of Theorem 3.18, we can now also consider the case when $a_1 \leq a_2 \leq \ldots \leq a_n$, provided the b_i satisfy a small restriction:

Example 3.20 (6, Example 4.14, p. 1683)**.** Let $\mu = \sum\limits_{i=1}^n p_i \mu_i$ with $b_i, \ldots, b_n \in \mathbb{R}$, $0 < a_1 \leq a_2 \leq \ldots \leq a_n$, $0 < p_1, \ldots, p_n < 1$, $\sum_{i=1}^n p_i = 1$, $\mu_i \sim N(b_i, a_i)$ and $\hat{\mu}(z) \neq 0$ for every $z \in \mathbb{R}$. We denote by J the set of indices for which $b_i = b_1$ and assume that all b_i, $i \in J$, lie on a lattice and that

$$\sum_{j \in J} p_j e^{ib_j z} \neq 0$$

for $z \in \mathbb{R}$. Then μ is quasi-infinitely divisible.

Proof. We decompose μ as

$$\mu = \tilde{\mu} * (\sum_{i \in \{1,\ldots,n\} \setminus J} p_i \tilde{\mu}_i + \sum_{j \in J} p_j \delta_{b_j})$$

with $\tilde{\mu}_i \sim N(b_i, a_i - a_1)$ and $\tilde{\mu} \sim N(0, a_1)$. We conclude from Theorem 3.18 that μ is quasi-infinitely divisible. □

3.5 Topological Properties of quasi-infinitely divisible distributions

In [51, Theorem 4.1, p. 8499] it was shown that the class of quasi-infinitely divsible distributions on $\mathbb{R}$ is dense in the class of all probability distributions with respect to weak convergence. Since distributions exists that are not quasi-infinitely divisible, the class of quasi-infinitely divisible distributions cannot be closed, unlike the class of infinitely divisible distributions. In this section we show that the class of infinitely divisible distributions is neither open. However, it is path-connected.

Denote by $\mathcal{P}(\mathbb{R})$ the set of all probability measures. Denote by $\pi : \mathcal{P}(\mathbb{R}) \times \mathcal{P}(\mathbb{R}) \to [0, \infty)$ the Prokhorov metric on $\mathcal{P}(\mathbb{R})$. Then it is known that $(\mathcal{P}(\mathbb{R}), \pi)$ is a complete metric space, and that the topology defined by the weak convergence is the same as for π, i.e. for a sequence $(\mu_n)_{n\in\mathbb{N}} \subset \mathcal{P}(\mathbb{R})$ and $\mu \in \mathcal{P}(\mathbb{R})$ weak convergence of μ_n to μ is equivalent to $\pi(\mu_n, \mu) \to 0$ as $n \to \infty$, see [7, Theorem 6.8, p. 73]. Now we can show:

Proposition 3.21 (6, Proposition 5.1, p. 1683)**.** *The set $QID(\mathbb{R})$ of all quasi-infinitely divisible distribution on $\mathbb{R}$ is not open in the space $(\mathcal{P}(\mathbb{R}), \pi)$. Moreover, $\mathcal{P}(\mathbb{R}) \setminus QID(\mathbb{R})$ is dense in $(\mathcal{P}(\mathbb{R}), \pi)$.*

Proof. Let $\mu, \nu \in \mathcal{P}(\mathbb{R})$ be such that the characteristic function $\hat{\nu}$ has zeroes on $\mathbb{R}$. We define the sequence of measures

$$\mu_n(dx) = \mu(dx) * \nu(n\, dx)$$

for every $n \in \mathbb{N}$. Then μ_n is clearly not quasi-infinitely divisible as its characteristic function has zeroes on $\mathbb{R}$, since

$$\hat{\mu_n}(z) = \hat{\mu}(z)\hat{\nu}(\frac{z}{n}).$$

Moreover, $\mu_n \xrightarrow{d} \mu$. This shows that $\mathcal{P}(\mathbb{R}) \setminus QID(\mathbb{R})$ is dense in $(\mathcal{P}(\mathbb{R}), \pi)$. In particular, $QID(\mathbb{R})$ cannot be open. $\square$

Now we show that $QID(\mathbb{R})$ is path-connected. Recall that for a metric space (X, d), a subset $Y \subseteq X$ is called path-connected if for every $x, y \in Y$ there exists a continuous function $p : [0, 1] \to Y$ such that $p(0) = x$ and $p(1) = y$.

Proposition 3.22 (6, Proposition 5.1, p. 1683)**.** *The space of quasi-infinitely divisible distributions is path-connected, especially connected.*

Proof. Let μ_0 and μ_1 be two quasi-infinitely divisible distributions. Then it holds that

$$\mu_t(dx) := \mu_0(\frac{1}{1-t}dx) * \mu_1(\frac{1}{t}dx)$$

is also quasi-infinitely divisible for $t \in (0,1)$. This holds because

$$h_t(dx) := \mu_1\left(\frac{1}{t}dx\right)$$

has the characteristic function

$$\hat{h}_t(z) := \hat{\mu}_1(zt) = \exp\left(-\frac{1}{2}at^2z^2 + i\gamma tz + \int_{\mathbb{R}} \left(e^{izx} - 1 - izx\mathbf{1}_{[-1,1]}\right)\nu(\frac{1}{t}dx)\right)$$

where μ_0 is quasi-infinitely divisible with characteristic triplet (a, γ, ν). Similarly

$$\mu_0\left(\frac{1}{1-t}dx\right)$$

is also quasi-infinitely divisible. We conclude that $\mu_t(dx)$ is quasi-infinitely divisible for every $t \in (0,1)$ with $\hat{\mu}_t(z) = \hat{\mu}_0((1-t)z)\hat{\mu}_1(tz)$. Moreover $p : [0,1] \to \mathcal{P}(\mathbb{R})$ with $p(0) = \mu_0$, $p(1) = \mu_1$ and $p(t) = \mu_t$ for $t \in (0,1)$ is continuous, because it holds for every $t_0 \in [0,1]$ $\hat{\mu}_t(z) \to \hat{\mu}_{t_0}(z)$ for $t \to t_0$ for every $z \in \mathbb{R}$. Hence $QID(\mathbb{R})$ is path connected. Finally, observe that path-connectness implies connectness, see [2, Theorem 3.29, p. 61]. $\square$

4 Lévy driven CARMA generalized processes and stochastic partial differential equations

We define a Lévy driven CARMA random field as a generalized solution of a stochastic partial differential equation (SPDE) and provide a sufficient criterion for the existence of this generalized processes. Furthermore, we give sufficient conditions for the existence of a mild solution of our SPDE. Our model finds a connection between all known definitions of CARMA random fields.

4.1 Introduction

Autoregressive moving average (ARMA) processes are very well known processes in time series analysis. An ARMA(p, q) process $(X_k)_{k\in\mathbb{Z}}$, $p, q \in \mathbb{N}_0$, is given by

$$X_k - \sum_{i=1}^{p} a_i X_{k-i} = W_k + \sum_{j=1}^{q} b_j W_{k-j}, \tag{4.1}$$

where $a_1, \dots, a_p, b_1, \dots, b_q \in \mathbb{C}$ are deterministic coefficients and $(W_k)_{k\in\mathbb{Z}}$ is white noise or even an independent and identically distributed (iid) sequence of random variables. In short form we can also write

$$a(B)X_k = b(B)W_k,$$

where $a(z) = 1 - \sum\limits_{i=1}^{p} a_i z^k$, $b(z) = 1 + \sum\limits_{j=1}^{q} b_j z^j$ are polynomials and B is the shift operator defined by $B^l Y_k = Y_{k-l}$ for $l \in \mathbb{N}$. ARMA(p, q) processes were generalized in various ways and have many applications, e.g. in finance, astrophysics, engineering and traffic data, see [35], [29], [67] and [46].

As the solution of (4.1) is a discrete process on a lattice, a possible way to generalize the concept is to study a continous version of (4.1), which is called continuous ARMA (CARMA) process. A CARMA(p, q) process $(X_t)_{t\in\mathbb{R}}$, where $p > q$, is given by

$$X_t = b' Y_t,\ t \in \mathbb{R}, \tag{4.2}$$

where $Y = (Y_t)_{t\in\mathbb{R}}$ is a $\mathbb{C}^p$-valued process satisfying the stochastic differential equation

$$dY_t = AY_t dt + e_p dL_t \tag{4.3}$$

with

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_p & -a_{p-1} & -a_{p-2} & \dots & -a_1 \end{pmatrix}, e_p = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \in \mathbb{C}^p \text{ and } b = \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_{p-2} \\ b_{p-1} \end{pmatrix},$$

where $a_1, \dots, a_p, b_0, \dots, b_{p-1} \in \mathbb{C}$ are determinstic coefficients such that $b_q \neq 0$ and $b_j = 0$ for every $j > q$, b' denotes the transpose of b and $L = (L_t)_{t\in\mathbb{R}}$ is a two-sided Lévy process. The equations (4.2) and (4.3) are the so called state-space representation of the formal stochastic differential equation

$$a(D)Y_t = b(D)DL_t,$$

with D the differential operator and $a(z) = z^p + a_1 z^{p-1} + \ldots + a_p$ and $b(z) = b_0 + b_1 z + \ldots + b_q z^q$ are polynomials. In [15] necessary and sufficient conditions on L and A were given such that there exists a strictly stationary solution of (4.2) and (4.3), namely it was shown that it is sufficient and necessary that $\mathbb{E}\log^+(|L_1|) < \infty$. CARMA processes have many applications, see [33] and [13].
As the CARMA process is defined on $\mathbb{R}$, spatial problems cannot be easily transferred. Our starting point to tackle this problem is the equation

$$p(D)s = q(D)\dot{L}, \tag{4.4}$$

where p, q are polynomials in d variables, D denotes the differential operator and $\dot{L}$ denotes Lévy white noise. Our solution s is defined as a generalized solution, see Section 4.3. We will define s to be the CARMA(p, q) generalized process.
There are already some extensions of the CARMA process to the multidimensional setting, which can partially be seen as special cases of our definition. Lately, there were the two papers of Brockwell and Matsuda [16] and Pham [55], who introduce different concepts of CARMA processes in the multidimensional setting. In [16] the new CARMA random field was given by

$$S_d(t) := \int_{\mathbb{R}^d} \sum_{r=1}^{p} \frac{b(\lambda_r)}{a'(\lambda_r)} e^{\lambda_r \|t-u\|} dL(u), \tag{4.5}$$

where dL denotes the integration over a Lévy bases, a and b are polynomials such that $a(z) = \prod_{i=1}^{p}(z^2 - \lambda_i^2)$ and some further restrictions. The model has a well understood second order behaviour and can be used for statistical estimation. However, the authors

do not deal with a dynamical description.
Pham [55] follows another way and defines a CARMA random field Y as a mild solution of the system of SPDEs given by

$$Y(t) = b'X(t),\ t \in \mathbb{R}^d, \tag{4.6}$$
$$(I_p\partial_d - A_d)\cdots(I_p\partial_1 - A_1)X(t) = c\dot{L}(t),\ t \in \mathbb{R}^d, \tag{4.7}$$

where $\dot{L}$ is a Lévy basis, $A_1, \dots, A_d \in \mathbb{R}^{p\times p}$ are matrices and I_p is the identity matrix. Pham speaks of causal CARMA random fields, as the solution of the system (4.6) depends only on the past in the sense that the solution at point x depends solely on the behavior of $\dot{L}$ on $(-\infty, x_1] \times \cdots \times (-\infty, x_d]$. So we can see directly that there is a big difference between these two definitions.
We find a connection between these two models and our proposed definition.
We will start with an abstract analysis of generalized processes and prove for a far more general class than (4.4) the existence of a generalized solution under mild conditions on the Lévy white noise. Our solution is similar to the definition of generalized CARMA(p, q) process in [14] and as there, we do not assume that the degree of the polynomial p is higher than the degree of the polynomial q. We will discuss two examples, which are related to the processes of Brockwell and Matsuda [16] and Pham [55]. We will also give certain conditions on p and q that guarantee that the obtained generalized solutions are random fields.
The above mentioned results can be found in Section 4.3 and Section 4.4, where our main results are Theorem 4.5 and Theorem 4.16. In Section 4.2 we recall some basic notation. In Section 4.3 we recall the definitions of Lévy white noise and generalized random processes. Moreover, we prove that a convolution operator with certain properties regarding its integrability defines a generalized random process and as an application we will study stochastic homogeneous elliptic partial differential equations. In Section 4.4 we use this theorem to show the existence of a CARMA generalized processes. Moreover, we study the concept of mild solutions in Section 4.5, prove existence of mild CARMA random fields and show some connections between the mild and generalized solutions. In Section 4.6 we study the moment properties of a CARMA random fields and show that if the Lévy white noise has existing α-moment for some $0 < \alpha \leq 2$, then the CARMA random field has also finite α-moment, see Proposition 4.27. In Section 4.7 we will study the connection between our model and the CARMA random field of Brockwell and Matsuda [16].

4.2 Notation and Preliminaries

To fix notation, by $(\Omega, \mathcal{F})$ we denote a measurable space, where Ω is a set and $\mathcal{F}$ is a σ-algebra and by $L^0(\Omega, \mathcal{F}, \mathbb{K})$ we denote all measurable functions $f : \Omega \to \mathbb{K}$ with respect to $\mathcal{F}$ where $\mathbb{K} = \mathbb{R}, \mathbb{C}$. In the case that $\mathcal{F}$ and $\mathbb{K}$ are clear from the context

we set $L^0(\Omega) = L^0(\Omega, \mathcal{F}, \mathbb{K})$. If we consider a probability space $(\Omega, \mathcal{F}, \mathcal{P})$, where $\mathcal{P}$ is a probability measure on $(\Omega, \mathcal{F})$, we say that a sequence $(f_n)_{n\in\mathbb{N}} \subset L^0(\Omega)$ converges to f in $L^0(\Omega)$ if f_n converges in probability to f with respect to the measure $\mathcal{P}$. In the case of $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ we denote by $\mathcal{B}(\mathbb{R}^d)$ the Borel-σ-algebra on $\mathbb{R}^d$. $\mathcal{B}_b(\mathbb{R}^d)$ is the set of all Borel sets, which are bounded.
We write $\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ for the set of integers, real numbers and complex numbers, respectively. If $z \in \mathbb{C}$, we denote by $\Im z$ and $\Re z$ the imaginary and the real part of z. $\|\cdot\|$ denotes the Euclidean norm and $r^+ := \max\{0, r\}$ for every $r \in \mathbb{R}$. The indicator function of a set $A \subset \mathbb{R}^d$, $d \in \mathbb{N}$, is denoted by $\mathbf{1}_A$. By $L^p(\mathbb{R}^d, A)$ for $A \subseteq \mathbb{C}$ and $0 < p \leq \infty$ we denote the set of all Borel-measurable functions $f : \mathbb{R}^d \to A$ such that $\int_{\mathbb{R}^d} |f(x)|^p \, \lambda^d(dx) < \infty$ for $0 < p < \infty$ and $\operatorname{ess\,sup}_{x\in\mathbb{R}^d} |f(x)| < \infty$ for $p = \infty$, where λ^d is the $d-$dimensional Lebesgue measure. We denote by $||f||_{L^p} = (\int_{\mathbb{R}} |f(x)|^p \, \lambda(dx))^{1/p}$ for $0 < p < \infty$ and $\|f\|_{L^\infty} = \operatorname{ess\,sup}_{\mathbb{R}^d} |f|$ the L^p-(quasi-)norm for a measurable function f. By d_f we denote the distribution function of f, which means that

$$d_f(\alpha) := \lambda^d(\{x \in \mathbb{R}^d : |f(x)| > \alpha\}), \, \alpha \geq 0. \tag{4.8}$$

We denote by $B_R(x)$ the ball $\{y \in \mathbb{R}^d : \|x - y\| < R\}$ and $x \wedge y := \min\{x, y\}$ for two real numbers x and y. For a set $A \subset \mathbb{R}^d$ and an element $x \in \mathbb{R}^d$ we set $dist(x, A) := \inf\{\|x - y\| : y \in A\}$. The space $\mathcal{D}(\mathbb{R}^d)$ denotes the set of all infinitely differentiable functions $f : \mathbb{R}^d \to \mathbb{R}$ with compact support, where we denote the support of f by supp f. The topological dual space of $\mathcal{D}(\mathbb{R}^d)$ will be denoted by $\mathcal{D}'(\mathbb{R}^d)$, where an element $u \in \mathcal{D}'(\mathbb{R}^d)$ is called a distribution. We will write $\langle u, \varphi\rangle := u(\varphi)$ for $\varphi \in \mathcal{D}(\mathbb{R}^d)$. For a function $f \in L^1(\mathbb{R}^d, \mathbb{C}^d)$ we set $\mathcal{F}f(x) = \int_{\mathbb{R}^d} e^{-i\langle z,x\rangle} f(z)\lambda^d(dz)$ and the L^2-Fourier transform likewise. Let $p(z) = \sum_{|\alpha|\leq m} p_\alpha z^\alpha$, $\alpha \in \mathbb{N}_0^d$ and $z^\alpha = z_1^{\alpha_1} \dots z_d^{\alpha_d}$, such that $p_\beta \neq 0$ for some β with $|\beta| := \beta_1 + \dots + \beta_d = m$. Then we define $\deg(p) := m$, the degree of p. We set $D^\alpha = \partial_{x_1}^{\alpha_1} \dots \partial_{x_d}^{\alpha_d}$ for $\alpha \in \mathbb{N}_0^d$. We denote by A^* the adjoint of the operator A.
We recall here the definition of a Lévy basis, as we explain some connection between a Lévy basis and generalized stochastic process, which will be defined later.

Definition 4.1 (see [56, p. 455]). A *Lévy basis* is a family $(L(A))_{A\in\mathcal{B}_b(\mathbb{R}^d)}$ of real valued random variables such that

i) $L(\bigcup_{n=0}^\infty A_n) = \sum_{n=0}^\infty L(A_n)$ *a.s.* for pairwise disjoint sets $(A_n)_{n\in\mathbb{N}_0} \subset \mathcal{B}_b(\mathbb{R}^d)$ with $\bigcup_{n\in\mathbb{N}_0} A_n \in \mathcal{B}_b(\mathbb{R}^d)$,

ii) $L(A_i)$ are independent for pairwise disjoint sets $A_1, \dots, A_n \in \mathcal{B}_b(\mathbb{R}^d)$ for every $n \in \mathbb{N}$,

iii) there exist $a \in [0, \infty)$, $\gamma \in \mathbb{R}$ and a Lévy measure ν on $\mathbb{R}$ (i.e. a measure ν on $\mathbb{R}$ such that $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}} \min\{1, x^2\}\nu(dx) < \infty$) such that

$$\mathbb{E}e^{izL(A)} = \exp\left(\psi(z)\lambda^d(A)\right)$$

for every $A \in \mathcal{B}_b(\mathbb{R}^d)$, where

$$\psi(z) := i\gamma z - \frac{1}{2}az^2 + \int_{\mathbb{R}} (e^{ixz} - 1 - ixz\mathbf{1}_{[-1,1]}(x))\nu(dx), \quad z \in \mathbb{R}.$$

The triplet (a, γ, ν) is called the *characteristic triplet* of L and ψ its *characteristic exponent*. By the Lévy-Khintchine formula, $L(A)$ is then infinitely divisible.

4.3 SPDEs and generalized solutions

4.3.1 The concept of generalized solutions

This section deals with Lévy white noise and the definition of solutions of the SPDEs given in (4.4). We will prove a multiplier theorem for general Lévy white noise and use this theorem to prove the existence of our CARMA random process. We will follow mainly [32, Section 2].

As already mentioned, we denote by $\mathcal{D}(\mathbb{R}^d)$ the space of infinitely differentiable functions with compact support, where we assume that the space is equipped with the usual topology, i.e. we say that a sequence $(\varphi_n)_{n\in\mathbb{N}} \subset \mathcal{D}(\mathbb{R}^d)$ converges to φ in $\mathcal{D}(\mathbb{R}^d)$ if there exists a compact subset $K \subset \mathbb{R}^d$ such that $\operatorname{supp} \varphi_n, \operatorname{supp} \varphi \subset K$ for every $n \in \mathbb{N}$ and $\sup_{x\in\mathbb{R}^d} |D^\alpha(\varphi_n(x) - \varphi(x))| \to 0$ for $n \to \infty$ for every multiindex $\alpha \in \mathbb{N}_0^d$.

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space. We recall the definition of a generalized random process.

Definition 4.2 (see [32, Definition 2.1])**.** A generalized random process is a linear and continuous function $s : \mathcal{D}(\mathbb{R}^d) \to L^0(\Omega)$. The linearity means that, for every $\varphi_1, \varphi_2 \in \mathcal{D}(\mathbb{R}^d)$ and $\gamma \in \mathbb{R}$,

$$s(\varphi_1 + \gamma\varphi_2) = s(\varphi_1) + \gamma s(\varphi_2) \text{ almost surely.}$$

The continuity means that if $\varphi_n \to \varphi$ in $\mathcal{D}(\mathbb{R}^d)$, then $s(\varphi_n) \to s(\varphi)$ in $L^0(\Omega)$.

As shown in [65, Corollary 4.2], there exists a measurable version from $(\Omega, \mathcal{F})$ to $(\mathcal{D}'(\mathbb{R}^d), \mathcal{C})$ with respect to the cylindrical σ-field $\mathcal{C}$ generated by the sets

$$\{u \in \mathcal{D}'(\mathbb{R}^d) | \, (\langle u, \varphi_1 \rangle, \dots, \langle u, \varphi_N \rangle) \in B\}$$

with $N \in \mathbb{N}$, $\varphi_1, \dots, \varphi_N \in \mathcal{D}(\mathbb{R}^d)$ and $B \in \mathcal{B}(\mathbb{R}^N)$. From now on we will always work with such a version.

The probability law of a generalized random process s is given by

$$\mathcal{P}_s(B) := \mathcal{P}(s \in B)$$

for $B \in \mathcal{C}$. The characteristic functional $\widehat{\mathcal{P}}_s$ is then defined by

$$\widehat{\mathcal{P}}_s(\varphi) = \int\limits_{\mathcal{D}'(\mathbb{R}^d)} \exp(i\langle u, \varphi\rangle) d\mathcal{P}_s(u),\ \varphi \in \mathcal{D}(\mathbb{R}^d).$$

We will work with Lévy white noise, which is a generalized random process where the characteristic functional satisfies a Lévy-Khintchine representation.

Definition 4.3. A Lévy white noise $\dot{L}$ is a generalized random process, where the characteristic functional is given by

$$\widehat{\mathcal{P}}_{\dot{L}}(\varphi) = \exp\left(\int\limits_{\mathbb{R}^d} \psi(\varphi(x))\lambda^d(dx) \right)$$

for every $\varphi \in \mathcal{D}(\mathbb{R}^d)$, where $\psi : \mathbb{R} \to \mathbb{C}$ is given by

$$\psi(z) = i\gamma z - \frac{1}{2}az^2 + \int\limits_{\mathbb{R}} (e^{ixz} - 1 - ixz\mathbf{1}_{|x|\leq 1})\nu(dx)$$

where $a \in \mathbb{R}^+$, $\gamma \in \mathbb{R}$ and ν is a Lévy-measure, i.e. a measure such that $\nu(\{0\}) = 0$ and

$$\int\limits_{\mathbb{R}} \min(1, x^2)\nu(dx) < \infty.$$

We say that $\dot{L}$ has the characteristic triplet (a, γ, ν).

The existence of the Lévy-white noise was proven in [34]. The domain of the Lévy white noise can also be extended to indicator functions $\mathbf{1}_A$ for A be a Borel set with finite Lebesgue measure by using the construction in [32, Proposition 3.4]. For a more general function f we say that f is in the domain $\dot{L}$ if there exists a sequence of elementary functions f_n converging almost everywhere to f such that $\langle \dot{L}, f_n\mathbf{1}_A\rangle$ convergens in probability for $n \to \infty$ for every Borel set A and set $\langle \dot{L}, f\rangle$ as the limit in probability of $\langle \dot{L}, f_n\rangle$ for $n \to \infty$, where for an elementary function $f := \sum_{j=1}^m a_j\mathbf{1}_{A_j}$, $\langle \dot{L}, f\rangle$ is defined by $\sum_{j=1}^m a_j\langle \dot{L}, \mathbf{1}_{A_j}\rangle$, see also [32, Definition 3.6]. For the maximal domain of the Lévy white noise $\dot{L}$ we write $L(\dot{L})$. By setting $L(A) := \langle \dot{L}, \mathbf{1}_A\rangle$ for bounded Borel sets A, the extention of a Lévy white noise $\dot{L}$ can be identified with a Lévy basis L in the sense of Rajput and Rosinski [56], see [32, Theorem 3.5 and Theorem 3.7]. As a Lévy basis can be identified with a Lévy white noise in a canonical way, i.e. $\langle \dot{L}, \varphi\rangle := \int_{\mathbb{R}^d} \varphi(x)dL(x)$ for $\varphi \in \mathcal{D}(\mathbb{R}^d)$, we do not differ between a Lévy basis and Lévy-white noise. In particular, a Borel-measurable function $f : \mathbb{R}^d \to \mathbb{R}$ is in $L(\dot{L})$ if and only if f is integrable with respect to the Lévy basis L in the sense of Rajput and Rosinski [56], see [32, Def. 3.6]. The Lévy white noise is stationary in the following sense.

Definition 4.4. A generalized process s is called stationary if for every $t \in \mathbb{R}^d$, $s(\cdot + t)$ has the same law as s. Here, $s(\cdot + t)$ is defined by

$$\langle s(\cdot + t), \varphi \rangle := \langle s, \varphi(\cdot - t) \rangle \text{ for every } \varphi \in \mathcal{D}(\mathbb{R}^d).$$

4.3.2 Generalized stochastic processes constructed from Lévy white noise

We now state and prove our first theorem which asserts that a large class of SPDEs has a generalized solution by only assuming weak moment conditions on the Lévy white noise.

Theorem 4.5. *Let $\dot{L}$ be a Lévy white noise with characteristic triplet (a, γ, ν) and $G : \mathbb{R}^d \to \mathbb{R}$ be a measurable function such that $G \in L^1(\mathbb{R}^d)$. Define*

$$G_R(x) := \int\limits_{B_R(x)} |G(y)| \lambda^d(dy) \tag{4.9}$$

for every $x \in \mathbb{R}^d$ and $R > 0$ and

$$h_R(x) = x \int\limits_0^{1/x} d_{G_R}(\alpha) \lambda^1(d\alpha) \text{ for } x > 0. \tag{4.10}$$

Assume that

$$\int\limits_{\mathbb{R}} \mathbf{1}_{|r|>1} h_R(|r|) \nu(dr) < \infty \tag{4.11}$$

for every $R > 0$. Then

$$s(\varphi) := \langle \dot{L}, G * \varphi \rangle, \quad \varphi \in \mathcal{D}(\mathbb{R}^d) \tag{4.12}$$

defines a stationary generalized random process.

Observe that although $\varphi \in \mathcal{D}(\mathbb{R}^d)$, $G * \varphi$ is in general not in $\mathcal{D}(\mathbb{R}^d)$ unless G has compact support. The point is that nevertheless, s defined by (4.12) gives a generalized random process. Sufficient conditions for (4.11) to hold will be treated in Example 4.6.

Proof. We need to show that $G * \varphi \in L(\dot{L})$ and $\langle \dot{L}, G * \varphi_n \rangle \to \langle \dot{L}, G * \varphi \rangle$ as $n \to \infty$ in $L^0(\Omega)$ for a sequence $(\varphi_n)_{n \in \mathbb{N}}$ converging to φ in $\mathcal{D}(\mathbb{R}^d)$. As $\langle \dot{L}, G * \cdot \rangle$ is linear, this is equivalent to check that $\langle \dot{L}, G * (\varphi_n - \varphi) \rangle \to 0$ as $n \to \infty$ in $L^0(\Omega)$, which is implied by

$$\int\limits_{\mathbb{R}^d} \left| \gamma \varphi_n * G(x) + \int\limits_{\mathbb{R}} r(\varphi_n * G)(x) (\mathbf{1}_{|r(\varphi_n * G)(x)| \leq 1} - \mathbf{1}_{|r| \leq 1}) \nu(dr) \right| \lambda^d(dx) \to 0, \tag{4.13}$$

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}} 1 \wedge (r(\varphi_n * G)(x))^2 \nu(dr)\lambda^d(dx) \to 0 \text{ and} \tag{4.14}$$

$$a^2 \int_{\mathbb{R}^d} |G * \varphi_n(x)|^2 \lambda^d(dx) \to 0 \tag{4.15}$$

for $n \to \infty$ if $\varphi_n \to 0$ for $n \to \infty$ in $\mathcal{D}(\mathbb{R}^d)$, see [56, Theorem 2.7] (that $G * \varphi \in L(\dot{L})$ follows if the above quantities are finite).
Since $G \in L^1(\mathbb{R}^d)$ it is easily seen that

$$\int_{\mathbb{R}^d} |\gamma\varphi_n * G(x)|\, \lambda^d(dx) \leq |\gamma|\, ||\varphi_n||_{L^1}||G||_{L^1} \to 0$$

for $n \to \infty$. The other term in (4.13) will be splitted up into

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\mathbb{R}} |r(\varphi_n * G)(x)| \cdot |\mathbf{1}_{|r(\varphi_n * G)(x)| \leq 1} - \mathbf{1}_{|r|\leq 1}|\nu(dr)\lambda^d(dx) \\
=&\int_{\mathbb{R}^d}\int_{\mathbb{R}} |r(\varphi_n * G)(x)|\mathbf{1}_{|r(\varphi_n * G)(x)| \leq 1, |r|>1}\nu(dr)\lambda^d(dx) \\
&+\int_{\mathbb{R}^d}\int_{\mathbb{R}} |r(\varphi_n * G)(x)|\mathbf{1}_{|r(\varphi_n * G)(x)| > 1, |r|\leq 1}\nu(dr)\lambda^d(dx) \\
=&\int_{\mathbb{R}} |r|\mathbf{1}_{|r|>1}\int_{\mathbb{R}^d} |(\varphi_n * G)(x)|\mathbf{1}_{|(\varphi_n * G)(x)| \leq \frac{1}{|r|}}\lambda^d(dx)\nu(dr) \\
&+\int_{\mathbb{R}} |r|\mathbf{1}_{|r|\leq 1}\int_{\mathbb{R}^d} |(\varphi_n * G)(x)|\mathbf{1}_{|(\varphi_n * G)(x)| > \frac{1}{|r|}}\lambda^d(dx)\nu(dr) = I_1(n) + I_2(n).
\end{aligned}$$

We give a pointwise upper bound for the convolution. Let $R > 0$ be such that supp $(\varphi_n) \subset B_r(0)$ for some $r < R$. We then see that for every $x \in \mathbb{R}^d$

$$(\varphi_n * G)(x) = \int_{\mathbb{R}^d} G(y)\varphi_n(x-y)\lambda^d(dy) = \int_{B_R(x)} G(y)\varphi_n(x-y)\lambda^d(dy) \leq G_R(x)||\varphi_n||_\infty.$$

We then obtain

$$\begin{aligned}
d_{\varphi_n * G}(\alpha) =&\lambda^d\left(\{x \in \mathbb{R}^d : |\varphi_n * G(x)| > \alpha\}\right) \\
\leq&\lambda^d\left(\{x \in \mathbb{R}^d : |G_R(x)| > \alpha/||\varphi_n||_{\infty}\}\right) = d_{G_R}(\alpha/||\varphi_n||_\infty).
\end{aligned} \tag{4.16}$$

So we see by [37, Exercise 1.1.10, p. 14] that

$$\int_{\mathbb{R}^d} |(\varphi_n * G)(x)|\mathbf{1}_{|(\varphi_n * G)(x)| \leq \frac{1}{|r|}}\lambda^d(dx) \leq \int_0^{\frac{1}{|r|}} d_{\varphi_n * G}(\alpha)\lambda^1(d\alpha) \leq \int_0^{\frac{1}{|r|}} d_{G_R}(\alpha/||\varphi_n||_\infty)\lambda^1(d\alpha).$$

We see that the right hand side converges to 0 for $n \to \infty$ and for n large enough we have

$$\int_0^{\frac{1}{|r|}} d_{G_R}\left(\frac{\alpha}{||\varphi_n||_\infty}\right)\lambda^1(d\alpha) \le \int_0^{\frac{1}{|r|}} d_{G_R}(\alpha)\,\lambda^1(d\alpha) = \frac{1}{|r|}h_R(|r|).$$

Lebesgue's dominated convergence theorem using (4.11) implies

$$\int_{\mathbb{R}} |r|\mathbf{1}_{|r|>1} \int_{\mathbb{R}^d} |(\varphi_n * G)(x)|\mathbf{1}_{|(\varphi_n*G)(x)|\le\frac{1}{|r|}}\lambda^d(dx)\nu(dr) \to 0$$

for $n \to \infty$.
For $I_2(n)$ we see from Young's inequality that

$$\int_{\mathbb{R}^d} |(\varphi_n * G)(x)|\mathbf{1}_{|(\varphi_n*G)(x)|>\frac{1}{|r|}}\lambda^d(dx) \le |r|\cdot||\varphi_n * G||^2_{L^2(\mathbb{R}^d)} \le |r|\|G\|^2_{L^1(\mathbb{R}^d)}\|\varphi_n\|^2_{L^2(\mathbb{R}^d)}$$

and again from Lebesgue's dominated convergence theorem (since $\int_{|r|\le1} r^2\nu(dr) < \infty$)

$$\int_{\mathbb{R}} |r|\mathbf{1}_{|r|\le1} \int_{\mathbb{R}^d} |(\varphi_n * G)(x)|\mathbf{1}_{|(\varphi_n*G)(x)|>\frac{1}{|r|}}\lambda^d(dx)\nu(dr) \to 0$$

for $n \to \infty$. This gives (4.13).
Now we check (4.14). We first note that

$$\begin{aligned}1 \wedge (r^2(\varphi_n * G)(x)^2) \le &\mathbf{1}_{|r(\varphi_n*G)(x)|>1}\mathbf{1}_{|r|>1} + |\varphi_n * G(x)||r|\mathbf{1}_{|r(\varphi_n*G)(x)|>1}\mathbf{1}_{|r|\le1}\\ &+ (\varphi_n * G(x)r)^2\mathbf{1}_{|r(\varphi_n*G)(x)|\le1}\mathbf{1}_{|r|\le1} + |\varphi_n * G(x)||r|\mathbf{1}_{|r(\varphi_n*G)(x)|\le1}\mathbf{1}_{|r|>1}.\end{aligned}$$

From the calculations that led to (4.13) we conclude that the second and fourth term (when integrated with respect to $\nu(dr)\lambda^d(dx)$) converge to 0 for $n \to \infty$ and for the first term we note that

$$\int_{\mathbb{R}^d} \mathbf{1}_{|r(\varphi_n*G)(x)|>1}\lambda^d(dx) = d_{\varphi_n*G}\left(\frac{1}{|r|}\right) \le d_{G_R}\left(\frac{1}{|r|||\varphi_n||_\infty}\right)$$

and by Lebesgue's dominated convergence theorem we conclude that

$$\int_{\mathbb{R}} \mathbf{1}_{|r|>1}d_{G_R}\left(\frac{1}{|r|||\varphi_n||_\infty}\right)\nu(dr) \to 0$$

for $n \to \infty$, as $h_R(|r|) \ge d_{G_R}(1/|r|)$. For the third term we easily see that

$$\int_{\mathbb{R}}\int_{\mathbb{R}^d}(\varphi_n * G(x)r)^2\mathbf{1}_{|r(\varphi_n*G)(x)|\le1}\mathbf{1}_{|r|\le1}\lambda^d(dx)\nu(dr)$$

$$\leq ||\varphi_n * G(x)||_{L^2}^2 \int_{\mathbb{R}} \mathbf{1}_{|r|\leq 1} |r|^2 \nu(dr) \to 0$$

for $n \to \infty$. This gives (4.14). Finally, (4.15) follows from Young's inequality since

$$\|G * \varphi_n\|_{L^2(\mathbb{R}^d)}^2 \leq \|G\|_{L^1(\mathbb{R}^d)}^2 \|\varphi_n\|_{L^2(\mathbb{R}^d)}^2 \to 0 \text{ for } n \to \infty.$$

The stationarity of the Lévy white noise implies the stationarity of the generalized process s, as

$$\langle s(\cdot + t), \varphi\rangle = \langle s, \varphi(\cdot - t)\rangle = \langle \dot{L}, G * \varphi(\cdot + t)\rangle = \langle \dot{L}(\cdot + (-t)), G * \varphi\rangle.$$

□

Example 4.6. We assume that

$$e^{c||x||} G(x) \in L^2(\mathbb{R}^d)$$

for some constant $c > 0$. By the Hölder inequality we conclude

$$\int_{\mathbb{R}^d} |G(x)| \lambda^d(dx) \leq || \exp(-c|| \cdot ||)||_{L^2} \cdot || \exp(c|| \cdot ||) G(\cdot)||_{L^2} < \infty$$

and

$$\int_{B_R(x)} |G(y)| \lambda^d(dy) \leq ||e^{c||\cdot||} G||_{L_2} \left(\int_{B_R(x)} e^{-2c||y||} \lambda^d(dy) \right)^{1/2} \leq C_R \exp(-c||x||)$$

for some constant $C_R > 0$. Hence,

$$d_{G_R}(\alpha) \leq d_{\exp(-c||\cdot||)} \left(\frac{\alpha}{C_R} \right),$$

for $\alpha > 0$. We conclude that for $r \geq C_R$,

$$\begin{aligned} \int_0^{1/|r|} d_{G_R}(a) \lambda^1(da) &\leq \int_0^{\frac{1}{|r|}} C_d \left(\frac{\log\left(\frac{C_R}{\alpha}\right)}{c} \right)^d \lambda^1(d\alpha) \\ &= \frac{C_d}{c^d} C_R \Gamma(d+1, \log(C_R|r|)) \\ &= \frac{C}{|r|} \sum_{k=0}^{d} \frac{\log(C_R|r|)^k}{k!} \end{aligned}$$

for some finite constants C_d and C, where $\Gamma(d+1, z) = \int_z^\infty t^d e^{-t} \lambda^1(dt)$ denotes the upper incomplete gamma function. Assuming $\int\limits_{|r|>1} \log(|r|)^d \nu(dr) < \infty$, we conclude

$$\int\limits_{\mathbb{R}} \mathbf{1}_{|r|>1/C_R} \left(C \sum_{k=0}^{d} \frac{\log(C_R|r|)^k}{k!} \right) \nu(dr) < \infty$$

and by Theorem 4.5 we obtain that s defined as above defines a generalized process.

The kernel function G has not always such nice integrability properties as assumed in Theorem 4.5. For example, the Green function of the Laplacian is neither integrable nor square integrable. As this is the case, we will prove another theorem, which will assure the existence of the generalized process s under some other, but stronger, conditions.

Theorem 4.7. *Let $\dot{L}$ be a Lévy white noise with characterstic triplet (a, γ, ν) such that the first moment of $\dot{L}$ vanishes, i.e. $\mathbb{E}|\langle \dot{L}, \varphi \rangle| < \infty$ and $\mathbb{E}\langle \dot{L}, \varphi \rangle = 0$ for every $\varphi \in \mathcal{D}(\mathbb{R}^d)$ and assume $G \in L^1_{loc}(\mathbb{R}^d)$ such that $\|G * \varphi_n\|_{L^2(\mathbb{R}^d)} \to 0$ for $n \to \infty$ for every sequence $(\varphi_n)_{n\in\mathbb{N}} \subset \mathcal{D}(\mathbb{R}^d)$ converging to 0. Then $s : \mathcal{D}(\mathbb{R}^d) \to L^0(\Omega)$ defined by*

$$s(\varphi) := \langle \dot{L}, G * \varphi \rangle$$

defines a stationary generalized process if

$$\int\limits_{\mathbb{R}} \mathbf{1}_{|r|>1} |r| \int\limits_{\frac{1}{|r|}}^{\infty} d_{G_R}(\alpha) \lambda^1(d\alpha) \nu(dr) < \infty \text{ and} \tag{4.17}$$

$$\int\limits_{\mathbb{R}} \mathbf{1}_{|r|>1} |r|^2 \int\limits_{0}^{\frac{1}{|r|}} \alpha d_{G_R}(\alpha) \lambda^1(d\alpha) \nu(dr) < \infty \tag{4.18}$$

for all $R > 0$, where G_R is defined by (4.9).

Observe that (4.11) can be written as $\int_{|r|>1} |r| \int_0^{1/|r|} d_{G_R}(\alpha)\lambda^d(d\alpha)\nu(dr)$, which is slightly stronger than (4.18). However, for Theorem 4.7 we additionally need (4.17) and $\mathbb{E}\langle \dot{L}, \varphi \rangle = 0$ for every $\varphi \in \mathcal{D}(\mathbb{R}^d)$.

Proof. By [58, Example 25.12, p. 163] we need to show similarly to Theorem 4.5 that (4.14), (4.15) and

$$\int\limits_{\mathbb{R}^d} \left| \int\limits_{\mathbb{R}} r(\varphi_n * G)(x) \mathbf{1}_{|r(\varphi_n * G)(x)|>1} \nu(dr) \right| \lambda^d(dx) \to 0, \tag{4.19}$$

are satisfied for all $(\varphi_n)_{n\in\mathbb{N}}$ converging to 0 in $\mathcal{D}(\mathbb{R}^d)$. Let $(\varphi_n)_{n\in\mathbb{N}} \subset \mathcal{D}(\mathbb{R}^d)$ converging to 0 such that $\operatorname{supp} \varphi_n \subset B_R(0)$ for some $R > 0$ and all $n \in \mathbb{N}$. Using that

$\int_{\mathbb{R}^d} |f(x)|\mathbf{1}_{|f(x)|>\beta}\lambda^d(dx) = \int_\beta^\infty d_f(\alpha)\lambda^1(d\alpha) + \beta d_f(\beta)$ for $\beta > 0$ and measurable f (cf. [37, Exercise 1.1.10, p. 14]), we estimate (4.19) by

$$\begin{aligned}
&\int_{\mathbb{R}^d} \left| \int_{\mathbb{R}} r(\varphi_n * G)(x)\mathbf{1}_{|r(\varphi_n*G)(x)|>1}\nu(dr)\right| \lambda^d(dx) \\
\leq &\int_{\mathbb{R}} \mathbf{1}_{|r|\leq 1}|r|^2\nu(dr)\|G*\varphi_n\|^2_{L^2(\mathbb{R}^d)} \\
&+ \int_{\mathbb{R}} \mathbf{1}_{|r|>1}|r| \int_{\mathbb{R}^d} |(\varphi_n * G)(x)|\mathbf{1}_{|r(\varphi_n*G)(x)|>1}\lambda^d(dx)\nu(dr) \\
= &\int_{\mathbb{R}} \mathbf{1}_{|r|\leq 1}|r|^2\nu(dr)\|G*\varphi_n\|^2_{L^2(\mathbb{R}^d)} \\
&+ \int_{\mathbb{R}} \mathbf{1}_{|r|>1}|r| \int_{\frac{1}{|r|}}^{\infty} d_{\varphi_n*G}(\alpha)\lambda^1(d\alpha)\nu(dr) + \int_{\mathbb{R}} \mathbf{1}_{|r|>1} d_{\varphi_n*G}(1/|r|)\nu(dr) \\
&\to 0
\end{aligned}$$

for $n\to\infty$ by Lebesgue's dominated convergence, where we used that by (4.16)

$$\begin{aligned}
&\int_{\mathbb{R}} \mathbf{1}_{|r|>1}|r| \int_{\frac{1}{|r|}}^{\infty} d_{\varphi_n*G}(\alpha)\lambda^1(d\alpha)\nu(dr) + \int_{\mathbb{R}} \mathbf{1}_{|r|>1} d_{\varphi_n*G}(1/|r|)\nu(dr) \\
\leq &\int_{\mathbb{R}} \mathbf{1}_{|r|>1}|r| \int_{\frac{1}{|r|}}^{\infty} d_{G_R}(\alpha/\|\varphi_n\|_\infty)\lambda^1(d\alpha)\nu(dr) + \int_{\mathbb{R}} \mathbf{1}_{|r|>1} d_{G_R}(1/(|r|\|\varphi_n\|_\infty))\nu(dr) \\
\leq &\int_{\mathbb{R}} \mathbf{1}_{|r|>1}|r| \int_{\frac{1}{|r|}}^{\infty} d_{G_R}(\alpha)\lambda^1(d\alpha)\nu(dr) + \int_{\mathbb{R}} \mathbf{1}_{|r|>1} d_{G_R}(1/|r|)\nu(dr)
\end{aligned}$$

for large n and the latter integral is finite by (4.17), (4.18) and

$$\int_0^x \alpha d_{G_R}(\alpha)\lambda^1(d\alpha) \geq d_{G_R}(x)\int_0^x \alpha\lambda^1(d\alpha) = \frac{1}{2}d_{G_R}(x)x^2 \text{ for every } x>0.$$

This gives (4.19). We control (4.14) by

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\mathbb{R}} 1\wedge (r(\varphi_n * G)(x))^2\nu(dr)\lambda^d(dx) \\
\leq &\int_{\mathbb{R}^d}\int_{\mathbb{R}} \mathbf{1}_{|r(\varphi_n*G)(x)|>1}\mathbf{1}_{|r|>1} + |\varphi_n * G(x)|^2|r|^2\mathbf{1}_{|r|\leq 1} + |\varphi_n * G(x)|^2|r|^2\mathbf{1}_{|r(\varphi_n*G)(x)|\leq 1}\mathbf{1}_{|r|>1}\nu(dr)\lambda^d(dx)
\end{aligned}$$

$$=: I_1 + I_2 + I_3.$$

We have already shown in the proof of Theorem 4.5 how to control I_1 and I_2, so we only need to show that I_3 converges to 0 for $n \to \infty$. We conclude by [37, Exercise 1.1.10] that

$$\begin{aligned}
&\int_{\mathbb{R}^d} \int_{\mathbb{R}} |\varphi_n * G(x)|^2 |r|^2 \mathbf{1}_{|r(\varphi_n * G)(x)| \leq 1} \mathbf{1}_{|r|>1} \nu(dr) \lambda^d(dx) \\
\leq &2 \int_{\mathbb{R}} \mathbf{1}_{|r|>1} r^2 \int_0^{\frac{1}{|r|}} \alpha d_{\varphi_n * G}(\alpha) \lambda^1(d\alpha) \nu(dr) \to 0
\end{aligned}$$

since

$$\int_{\mathbb{R}} \mathbf{1}_{|r|>1} r^2 \int_0^{\frac{1}{|r|}} \alpha d_{\varphi_n * G}(\alpha) \lambda^1(d\alpha) \nu(dr) \leq \int_{\mathbb{R}} \mathbf{1}_{|r|>1} r^2 \int_0^{\frac{1}{|r|}} \alpha d_{G_R}(\alpha) \lambda^1(d\alpha) \nu(dr) < \infty$$

for large n by (4.16) and by our assumption (4.18). Hence, we conclude that s defines a generalized process. Stationarity follows by the same arguments as in the proof of Theorem 4.5. □

Remark 4.8. If for every $R > 0$ there exists a bounded Borel set A_R and a constant $C_R > 0$ such that $G_R(x) \leq C_R G(x)$ for all $x \in \mathbb{R}^d \setminus A_R$, then we can replace G_R by G in (4.10), (4.17) and (4.18).This follows from the estimate $d_{G_R}(\alpha) \leq \lambda^d(A_R) + d_G(\alpha/C_R)$ for (4.10) and (4.18), and for (4.17) one can argue similarly to the proof of Example 4.10 below, using the boundedness of G_R on a set A_{2R} related to A_R.

Remark 4.9. Under certain conditions one can replace $h_R(|r|)$ in (4.11) by $d_{G_R}(1/|r|)$, for example if for every $R > 0$, $d_{G_R} \in L^p([0,1])$ for some $p > 1$ and $d_{G_R}(x) x^{1/p} \geq C$ for some constant $C > 0$ independent of x. This follows by

$$\frac{1}{x d_{G_R}(x)} \int_0^x d_{G_R}(\alpha) \lambda^1(d\alpha) \leq \frac{x^{1-1/p}}{x d_{G_R}(x)} \|d_{G_R}\|_{L^p([0,1])} \leq \frac{1}{C} \|d_{G_R}\|_{L^p([0,1])} < \infty \text{ for all } x \in (0,1).$$

Example 4.10. We assume that $G \in L^1_{loc}(\mathbb{R}^d)$ and there exist $\beta > d/2$, $C > 0$ and a bounded, open set A with $0 \in A$ such that $|G(x)| \leq C\|x\|^{-\beta}$ for all $x \in \mathbb{R}^d \setminus A$. We find that

$$d_{G_R}(\alpha) \leq C'(\alpha^{-\frac{d}{\beta}} + \mathbf{1}_{\alpha \leq \|G_R\|_{L^\infty(A_{2R})}}),$$

where $A_{2R} := \{x \in \mathbb{R}^d : dist(x, A) \leq 2R\}$. We conclude

$$\int_{\frac{1}{|r|}}^{\infty} d_{G_R}(\alpha)\lambda^1(d\alpha) \leq \tilde{C}|r|^{\frac{d}{\beta}-1} + C' \max\{\|G_R\|_{L^\infty(A_{2R})} - \frac{1}{|r|}, 0\}$$

and

$$\int_{0}^{\frac{1}{|r|}} \alpha d_{G_R}(\alpha)\lambda^1(d\alpha) \leq \tilde{C}\left(|r|^{\frac{d}{\beta}-2} + |r|^{-2}\right)$$

for some constant $\tilde{C} > 0$ for all $|r| > 1$. Writing $G = G\mathbf{1}_{B_M(0)} + G\mathbf{1}_{\mathbb{R}^d \setminus B_M(0)}$ for large M, we have $G\mathbf{1}_{B_M(0)} \in L^1(\mathbb{R}^d)$ and $G\mathbf{1}_{\mathbb{R}^d \setminus B_M(0)} \in L^2(\mathbb{R}^d)$ and since $\varphi_n \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ we obtain from Young's inequality that $\|G * \varphi_n\|_{L^2(\mathbb{R}^d)} \to 0$, $n \to \infty$. If $\int_{|r|>1} |r|^{\frac{d}{\beta}}\nu(dr) < \infty$, we conclude by Theorem 4.7 (if $\dot{L}$ satisfies the assumptions specified there) that $s(\varphi) := \langle \dot{L}, G * \varphi\rangle$ defines a generalized random process.

Until now we have only given sufficient conditions for the existence of a generalized process s defined by a convolution with a suitable kernel G. We will give a necessary condition if G is positive in $\mathbb{R}^d$.

Corollary 4.11. *Let $G \in L^1_{loc}(\mathbb{R}^d)$ such that $G(x) \geq 0$ λ^d−a.e (or $G(x) \leq 0$ λ^d−a.e.). Let $\dot{L}$ be a Lévy white noise with characteristic triplet (a, γ, ν). If $s : \mathcal{D}(\mathbb{R}^d) \to L^0(\Omega)$ defined by $s(\varphi) := \langle \dot{L}, G * \varphi\rangle$ for $\varphi \in \mathcal{D}(\mathbb{R}^d)$ defines a generalized process, then*

$$\int_{\mathbb{R}} \mathbf{1}_{|r|>1} d_{G_R}\left(\frac{1}{|r|}\right)\nu(dr) < \infty$$

for every $R > 0$ and G_R defined by (4.9).

Proof. We know that for $\varphi \in \mathcal{D}(\mathbb{R}^d)$, it is necessary for $G * \varphi \in L(\dot{L})$ that (cf. [56, Theorem 2.7, p.461-462])

$$\begin{aligned}\infty > \int_{\mathbb{R}}\int_{\mathbb{R}^d} (1 \wedge (r\varphi * G)(x)^2)\lambda^d(dx)\nu(dr) &\geq \int_{\mathbb{R}}\int_{\mathbb{R}^d} \mathbf{1}_{|r|>1}\mathbf{1}_{|r\varphi*G|>1}\lambda^d(dx)\nu(dr) \\ &= \int_{\mathbb{R}} \mathbf{1}_{|r|>1} d_{G*\varphi}\left(1/|r|\right)\nu(dr). \qquad (4.20)\end{aligned}$$

Now let $\varphi \in \mathcal{D}(\mathbb{R}^d)$, $\varphi \geq 0$ such that $\varphi \geq 1$ in $B_R(0)$. We see that

$$\begin{aligned} d_{G*\varphi}(\alpha) &= \lambda^d \left(\left\{ x \in \mathbb{R}^d : \int_{\mathbb{R}^d} G(x-y)\varphi(y)\lambda^d(dy) > \alpha \right\} \right) \\ &\geq \lambda^d \left(\left\{ x \in \mathbb{R}^d : \int_{B_R(x)} G(y)\lambda^d(dy) > \alpha \right\} \right) = d_{G_R}(\alpha). \end{aligned}$$

By assumption we conclude

$$\int_{\mathbb{R}} \mathbf{1}_{|r|>1} d_{G_R}\left(1/|r|\right) \nu(dr) \leq \int_{\mathbb{R}} \int_{\mathbb{R}^d} (1 \wedge (r\varphi * G)(x)^2)\lambda^d(dx)\nu(dr) < \infty.$$

□

4.4 CARMA generalized processes

We construct a generalization of CARMA processes. A CARMA generalized process is a generalized solution of a special SPDE.

Definition 4.12. Let $\dot{L}$ be a Lévy white noise, $n, m \in \mathbb{N}_0$ and $p, q : \mathbb{R}^d \to \mathbb{R}$ be polynomials of the form

$$p(x) = \sum_{|\alpha| \leq n} p_\alpha x^\alpha \text{ and } q(x) = \sum_{|\alpha| \leq m} q_\alpha x^\alpha.$$

A generalized process $s : \mathcal{D}(\mathbb{R}^d) \to L^0(\Omega)$ is called a CARMA(p, q) generalized process if s solves the equation

$$p(D)s = q(D)\dot{L},$$

which means that

$$\langle s, p(D)^*\varphi \rangle = \langle \dot{L}, q(D)^*\varphi \rangle \text{ a.s. for every } \varphi \in \mathcal{D}(\mathbb{R}^d). \tag{4.21}$$

Recall that $p(D)^* = p(-D)$ and $q(D)^* = q(-D)$.

4.4.1 Homogeneous Elliptic SPDEs

Let $p(z) = \sum_{|\alpha| \leq m} a_\alpha z^\alpha$ be a polynomial in d variables and $\dot{L}$ some Lévy noise. We are interested in generalized solutions of the stochastic partial differential equation

$$p(D)s = \dot{L}. \tag{4.22}$$

Let G be a fundamental solution of the operator $p^*(D)$, i.e. a distribution such that $p^*(D)G * \varphi = \varphi$ for every $\varphi \in \mathcal{D}(\mathbb{R}^d)$. By the theorem of Malgrange-Ehrenpreis, such a fundamental solution always exists. Suppose this fundamental solution arises actually from a function G such that the assumptions of Theorem 4.5 or Theorem 4.7 are satisfied. If we then define the generalized process s by

$$\langle s, \varphi \rangle := \langle \dot{L}, G * \varphi \rangle \text{ for } \varphi \in \mathcal{D}(\mathbb{R}^d),$$

then this defines a generalized process that satisfies (4.22). This follows from the simple calculation

$$\langle p(D)s, \varphi \rangle = \langle s, p^*(D)\varphi \rangle = \langle \dot{L}, G * (p^*(D)\varphi) \rangle = \langle \dot{L}, p^*(D)G * \varphi \rangle = \langle \dot{L}, \varphi \rangle.$$

To find conditions when Theorem 4.7 can be applied, we at first specialise to homogeneous elliptic partial differential operators. We say that a polynomial is elliptic homogeneous of degree m if $p(z) = \sum_{|\alpha|=m} a_\alpha z^\alpha$ and $p(z) \neq 0$ for all $z \in \mathbb{R}^d \setminus \{0\}$. We call $p(D)$ then an elliptic homogenous partial differential operator of degree m. Observe that in this case the adjoint operator is given by $p^*(D) = (-1)^m p(D)$. Hence, the fundamental solution of $p^*(D)$ and $p(D)$ differ only by the factor $(-1)^m$. We now have:

Proposition 4.13. *Let $p(D)$ be an elliptic homogeneous partial differential operator of degree $m \in \mathbb{N}$. If $d > 2m$ and the Lévy white noise $\dot{L}$ with characteristic triplet (a, γ, ν) satisfies*

$$\int_{\mathbb{R}} \mathbf{1}_{|r|>1} |r|^{\frac{d}{d-m}+\varepsilon} \nu(dr) < \infty$$

for some $\varepsilon > 0$ and the first moment of $\dot{L}$ vanishes, then there exists a generalized process s which solves the SPDE (4.22).

Proof. It is known that in the case of such a partial differential operator, the fundamental solution arises from a locally integrable function G that satisfies $|G(x)| \leq c\|x\|^{m-d} \log(\|x\|)$ for all $\|x\| \geq 2$ and some constant $c > 0$, see [54, Proposition 2.4.8, p. 155]. The rest follows from Example 4.10. □

Remark 4.14. In the case of the Laplacian Δ, when $d \geq 5$, with methods similar to the proof of Example 4.6 one can show that it is enough that

$$\int_{\mathbb{R}} \mathbf{1}_{|r|>1} |r|^{\frac{d}{d-2}} \nu(dr) < \infty \tag{4.23}$$

for the existence of a generalized solution. Moreover, if we choose for Δ the fundamental solution $G(x) = c_d |x|^{2-d}$, where $c_d \in \mathbb{R} \setminus \{0\}$, then by Corollary 4.11 it is also necessary that (4.23) holds true for $\langle \dot{L}, G * \varphi \rangle$ to define a generalized solution.

4.4.2 General CARMA setting

For classical CARMA processes in dimension 1 the assumptions on the polynomials are that q/p has only removable singularities on the imaginary axis and the degree of the polynomial p is higher than the degree of q, which implies that $\|q/p\|_{L^2(i\mathbb{R})} < \infty$. For a detailed discussion see [15]. In dimension 1 CARMA generalized processes were defined in [14], where the white noise was assumed to be Gaussian and the polynomial p has no zeroes on the imaginary axis, see [14, Proposition 2.5, p. 3616]. All the assumptions on the polynomials above imply even more, namely that q/p has a holomorphic extension on the strip $\{z \in \mathbb{C} : |\Re z| < \varepsilon\}$ for a small $\varepsilon > 0$. We take this as an assumption also for higher dimensions d:

Assumption 4.15. The rational function $q(i\cdot)/p(i\cdot)$ has a holomorphic extension in a strip $\{z \in \mathbb{C}^d : \|\Im z\| < \varepsilon\}$ for some $\varepsilon > 0$.

This assumption implies especially that there exist two polynomials h and l such that $h(i\cdot)/l(i\cdot) = p(i\cdot)/q(i\cdot)$ and $l(i\cdot)$ has no zeroes in the strip $\{z \in \mathbb{C}^d : \|\Im z\| \leq \varepsilon/2\}$. Hence we may and do assume for the rest of this section that $h = p$ and $l = q$.
We prove an existence theorem under mild moment conditions.

Theorem 4.16. *Let p, q be polynomials as in Definition 4.12 such that the Assumption 4.15 holds true. Furthermore, let $\dot{L}$ be a Lévy white noise with characteristic triplet (a, γ, ν) such that*

$$\int_{\mathbb{R}} \mathbf{1}_{|r|>1} \log(|r|)^d \nu(dr).$$

Then there exists a stationary CARMA(p, q) generalized process.

Proof. Under the Assumption 4.15 it follows by arguments similar as in the proof of [40, Lemma 2, p. 557] that there exists an $\alpha \in \mathbb{N}$ and $\delta > 0$ such that

$$\sup_{\|\eta\| \leq \delta} \left\| \frac{q(-i \cdot +\eta)}{p(-i \cdot +\eta)\psi(\cdot + i\eta)} \right\|_{L^2(\mathbb{R}^d)} < \infty,$$

where

$$\psi(z) := \left(1 + \sum_{j=1}^{d} z_i^2\right)^{\alpha}.$$

It follows by a Paley-Wiener theorem (e.g. [57, Theorem XI.13, p.18]) that the inverse Fourier transform G of $\frac{q(-i\cdot)}{\psi(\cdot)p(-i\cdot)}$ satisfies

$$e^{c\|x\|} G(x) \in L^2(\mathbb{R}^d)$$

for some $0 < c < \delta$. Observe that G is indeed real-valued, as $\frac{q(-i\cdot)}{p(-i\cdot)\psi(\cdot)} = \overline{\frac{q(i\cdot)}{p(i\cdot)\psi(-\cdot)}}$. Observe that $\mathcal{F}^{-1}\psi\mathcal{F}\cdot : \mathcal{D}(\mathbb{R}^d) \to \mathcal{D}(\mathbb{R}^d)$ is a continuous mapping, as

$$\mathcal{F}^{-1}(\psi(\cdot)\mathcal{F}\varphi) = (1-\Delta)^{\alpha}\varphi.$$

By Example 4.6 follows that s defined by

$$\langle s, \varphi\rangle := \left\langle \dot{L}, G * \mathcal{F}^{-1}\left(\psi(\cdot)\mathcal{F}\varphi\right)\right\rangle \text{ for every } \varphi \in \mathcal{D}(\mathbb{R}^d) \tag{4.24}$$

is a generalized process and by similar arguments to the proof of Theorem 4.5 it follows that s is stationary. Now let $\varphi \in \mathcal{D}(\mathbb{R}^d)$, we conclude by $\mathcal{F}p(-D)\varphi = p(-i\cdot)\mathcal{F}\varphi$ for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$ that

$$\begin{aligned}
\langle s, p(D)^*\varphi\rangle &= \left\langle \dot{L}, \left(G * \mathcal{F}^{-1}\left(\psi(\cdot)\mathcal{F}(p(D)^*\varphi)\right)\right)\right\rangle \\
&= \left\langle \dot{L}, \mathcal{F}^{-1}\left(\psi(\cdot)\frac{q(-i\cdot)}{\psi(\cdot)p(-i\cdot)}p(-i\cdot)\mathcal{F}\varphi\right)\right\rangle \\
&= \langle \dot{L}, \mathcal{F}^{-1}\left(q(-i\cdot)\mathcal{F}\varphi\right)\rangle = \langle \dot{L}, q(D)^*\varphi\rangle.
\end{aligned}$$

□

Remark 4.17. Under the assumptions of Theorem 4.16 the only solutions of (4.21) are of the form $s + u$, where s is the solution constructed in Theorem 4.16 and u solves the equation $\langle u, p(D)^*\varphi\rangle = 0$ a.s. for every $\varphi \in \mathcal{D}(\mathbb{R}^d)$.

We obtain directly the following corollary, which generalizes [14, Proposition 2.5, p. 3616] from Gaussian noise to Lévy white noise.

Corollary 4.18. *Let $d = 1$ and $p(z) = \prod_{j=1}^{n}(p_j - z)$ and $q(z) = \prod_{j=1}^{m}(q_j - z)$ be two real polynomials, such that p/q has no roots on the imaginary axis. Then there exists a stationary generalized solution $s : \mathcal{D}(\mathbb{R}^d) \to L^0(\Omega)$ of the equation $p(\frac{d}{dx})s = q(\frac{d}{dx})\dot{L}$ for every Lévy white noise $\dot{L}$ with characteristic triplet (a, γ, ν) such that $\int_{|r|>1}\log(|r|)\nu(dr)$.*

Example 4.19. Consider the polynomial $p(iz) := -\lambda - \sum_{j=1}^{d} z_j^2$ for $d \in \mathbb{N}$ with $\lambda > 0$, which corresponds to the partial differential operator $L = -\lambda + \Delta$. The real part is given by $\Re\, p(iz) = -\lambda - \sum_{j=1}^{d}((\Re\, z_j)^2 - (\Im\, z_j)^2)$, from which we conclude that $p(i\cdot)$ has no roots in $\{z \in \mathbb{C}^d : \|\Im\, z\|^2 < \lambda\}$. It follows that for every polynomial q there exists a generalized solution $s : \mathcal{D}(\mathbb{R}^d) \to L^0(\Omega)$ of

$$(-\lambda + \Delta)s = q(D)\dot{L}. \tag{4.25}$$

Example 4.20. Let $p(D) = \prod_{j=1}^{d}(\lambda_j - \partial_{x_j})^{\alpha_j}$, $\alpha_j \in \mathbb{N}_0$ for all $j \in \{1, \cdots, d\}$ and $|\lambda_j| > 0$. Then its corresponding polynomial is given by $p(iz) = \prod_{j=1}^{d}(\lambda_j - iz_j)^{\alpha_j}$ and by Theorem

4.16 we find a generalized solution of the equation $p(D)s = q(D)\dot{L}$ for every partial differential operator $q(D)$, as $1/p(i\cdot)$ is holomorphic in $\{z \in \mathbb{C}^d : \|\Im z\| < \varepsilon\}$ for some $\varepsilon > 0$.

4.5 CARMA random fields

Until now we have studied generalized solutions of the CARMA SPDE (4.4), but in the literature of stochastic partial differential equations driven by Lévy noise the concept of mild solutions seems to be more used, as the mild solution is itself a random field. We show under stronger conditions the existence of a mild solution of (4.4). But first we recall what a mild solution is.

Definition 4.21 (see [65])**.** Let $p(D)$ and $q(D)$ be partial differential operators and let $G : \mathbb{R}^d \to \mathbb{R}$ be a locally integrable fundamental solution of the equation $p(D)u = q(D)\delta_0$, which means that for every $\varphi \in \mathcal{D}(\mathbb{R}^d)$, $p(D)G * \varphi = q(D)\varphi$. We say that $X = (X_t)_{t\in\mathbb{R}^d}$ defined by

$$X_t = \int\limits_{\mathbb{R}^d} G(t-s)\, dL(s),$$

where L denotes a Lévy basis, is the mild solution of the equation $p(D)X = q(D)dL$, provided that the integral exists. Observe that it is necessary that G is a function.

We know already that $\dot{L}$ can be extended to a Lévy basis, see [32]. We state our first result, which follows directly from the proofs of Theorem 4.5 and Corollary 4.11.

Proposition 4.22.

i) Let $G : \mathbb{R}^d \to \mathbb{R}$ be a measurable function with $G \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. We define

$$h(x) := x \int\limits_0^{1/x} d_G(a)\lambda^1(da) \text{ for } x > 0.$$

Let L be a Lévy basis (equivalently $\dot{L}$ a Lévy white noise) with characteristic triplet (a, γ, ν), and assume that

$$\int\limits_{\mathbb{R}} \mathbf{1}_{|r|>1} h(|r|)\nu(dr) < \infty.$$

Then the integral

$$X_t = \int_{\mathbb{R}^d} G(t-s)dL(s)$$

exists and defines a stationary random field $(X_t)_{t\in\mathbb{R}^d}$.

ii) Conversely, if $G : \mathbb{R}^d \to \mathbb{R}$ is measurable and the integral $\int_{\mathbb{R}^d} G(-s)dL(s)$ exists, then necessarily

$$\int_{\mathbb{R}} \mathbf{1}_{|r|>1} d_G(1/|r|)\nu(dr) < \infty.$$

Proof. By [56, Theorem 2.7], the integral $\int_{\mathbb{R}^d} G(t-s)dL(s)$ exists if and only if

$$\int_{\mathbb{R}^d} \left(\gamma G(x) + \int_{\mathbb{R}} rG(x)(\mathbf{1}_{|rG(x)|\leq 1} - \mathbf{1}_{|r|\leq 1})\nu(dr) \right) \lambda^d(dx) < \infty,$$
$$\int_{\mathbb{R}^d} \int_{\mathbb{R}} 1 \wedge (rG(x))^2 \nu(dr)\lambda^d(dx) < \infty \text{ and}$$
$$\int_{\mathbb{R}^d} a|G(x)|^2 \lambda^d(dx) < \infty.$$

That the conditions specified in (i) are sufficient then follows by calculations similar to those in the proof of Theorem 4.5, while necessity of the condition specified in (ii) follows as in (4.20). That X_t as defined in (i) is stationary is clear. □

Now we conclude that there exists a mild solution of the CARMA(p, q) SPDE under some further restrictions.

Theorem 4.23. *Let L be a Lévy basis in $\mathbb{R}^d$ with characteristic triplet (a, γ, ν) such that $\int_{\mathbb{R}} \mathbf{1}_{|r|>1} \log(|r|)^d \nu(dr) < \infty$. Assume furthermore that there exists $\varepsilon > 0$ such that*

$$\sup_{\eta \in B_\varepsilon(0)} \left\| \frac{q(i \cdot +\eta)}{p(i \cdot +\eta)} \right\|_{L^2} < \infty. \tag{4.26}$$

Then there exists a mild solution of the equation

$$p(D)X = q(D)\, dL, \tag{4.27}$$

which is given by

$$X_t = \int_{\mathbb{R}^d} \mathcal{F}^{-1} \left(\frac{q(i\cdot)}{p(i\cdot)} \right) (t-x)dL(x),\ t \in \mathbb{R}^d. \tag{4.28}$$

Proof. Taking Fourier transforms, it is easy to check that $G := \mathcal{F}^{-1} \frac{q(i\cdot)}{p(i\cdot)}$ is a fundamental solution of $p(D)u = q(D)\delta_0$. By [57, Theorem XI.13, p.18] we see that $e^{c\|\cdot\|}G \in L^2(\mathbb{R}^d)$ for all $0 < c < \varepsilon$ and G is real-valued by the same argument as in Theorem 4.16. It follows that

$$h(r) = r \int_0^{1/r} d_G(\alpha)\lambda^1(d\alpha) \leq d_{G\exp(c\|\cdot\|)}(1) + r \int_0^{1/r} d_{\exp(-c\|\cdot\|)}(\alpha)\lambda^1(d\alpha)$$

The rest follows by Proposition 4.22 and similar calculations as in Example 4.6. □

Example 4.24. Let $d = 1, 2, 3$, $\lambda > 0$ and $p(D) = (\lambda - \Delta)$. We see that

$$\sup_{\|\eta\| \leq \lambda/2} \|1/p(i \cdot +\eta)\|_{L^2(\mathbb{R}^d)} < \infty$$

and by Theorem 4.23 we conclude that there exists a mild solution of the equation $(\lambda - \Delta)X = dL$.

Example 4.25. The causal CARMA random field constructed in [55, Definition 3.3] and [47, Definition 2.1] is the mild solution of the equation $P(D)X = Q(D)dL$, where P and Q are given in [47, Proposition 2.5]. We observe that P and Q satisfy the assumption of Theorem 4.23, so that the causal CARMA random field of [55,47] can be seen as a special case of CARMA random fields defined in the present chapter.

In classical analysis, a locally integrable function $f : \mathbb{R}^d \to \mathbb{R}$ specifies a distribution T_f by $T_f(\varphi) := \int_{\mathbb{R}^d} f(x)\varphi(x)\lambda^d(dx)$ for $\varphi \in \mathcal{D}(\mathbb{R}^d)$. It is now natural to ask if a mild solution X of $p(D)X = q(D)dL$ also gives rise to a generalized solution of $p(D)X = q(D)\dot{L}$ via $\langle X, \varphi\rangle := \int_{\mathbb{R}^d} X_s\varphi(s)\lambda^d(ds)$.

That this is indeed the case, at least under some weak conditions which allow the application of a stochastic Fubini theorem, is the contents of the next proposition.

Proposition 4.26. *Let L be a Lévy basis with existing first moment and p and q be as in Theorem 4.23. Let*

$$G := \mathcal{F}^{-1}\left(\frac{q(i\cdot)}{p(i\cdot)}\right).$$

Then the mild solution

$$X_s = \int_{\mathbb{R}^d} G(s-u)dL(u),\ s \in \mathbb{R}^d,$$

of (4.28) gives rise to a generalized solution X of the SPDE $p(D)X = q(D)\dot{L}$ via

$$\langle X, \varphi\rangle := \int_{\mathbb{R}^d} X_s\varphi(s)\lambda^d(ds),\ \varphi \in \mathcal{D}(\mathbb{R}^d).$$

Proof. Observe that $G \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ by the proof of Theorem 4.23. We see that for every $\varphi \in \mathcal{D}(\mathbb{R}^d)$

$$\begin{aligned}
&|r\varphi(t)G(t-s)| \wedge |r\varphi(t)G(t-s)|^2 \\
=&\mathbf{1}_{|r\varphi(t)G(t-s)|>1}|r\varphi(t)G(t-s)| + \mathbf{1}_{|r\varphi(t)G(t-s)|\le 1}|r\varphi(t)G(t-s)|^2 \\
\le&\mathbf{1}_{|r|>1}\mathbf{1}_{|r\varphi(t)G(t-s)|>1}|r\varphi(t)G(t-s)| + \mathbf{1}_{|r|\le 1}\mathbf{1}_{|r\varphi(t)G(t-s)|>1}|r\varphi(t)G(t-s)|^2 \\
&+ \mathbf{1}_{|r|\le 1}\mathbf{1}_{|r\varphi(t)G(t-s)|\le 1}|r\varphi(t)G(t-s)|^2 + \mathbf{1}_{|r|>1}\mathbf{1}_{|r\varphi(t)G(t-s)|\le 1}|r\varphi(t)G(t-s)| \\
=&\mathbf{1}_{|r|>1}|r\varphi(t)G(t-s)| + \mathbf{1}_{|r|\le 1}|r\varphi(t)G(t-s)|^2.
\end{aligned}$$

Since

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}} \mathbf{1}_{|r|>1}|r\varphi(t)G(t-s)|\nu(dr)\lambda^d(ds)\lambda^d(dt) \le \int_{\mathbb{R}} \mathbf{1}_{|r|>1}|r|\nu(dr)\||\varphi| * |G|\|_{L^1} < \infty \text{ and}$$
$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}} \mathbf{1}_{|r|\le 1}|r\varphi(t)G(t-s)|^2\nu(dr)\lambda^d(ds)\lambda^d(dt) \le \int_{\mathbb{R}} \mathbf{1}_{|r|\le 1}|r|^2\nu(dr)\||\varphi|^2 * |G|^2\|_{L^1} < \infty$$

by Young's inequality and by assumption we conclude from a stochastic Fubini result ([4, Theorem 3.1 and Remark 3.2, p. 926]; observe that φ has compact support and that λ^d is finite on the support of φ) that

$$\begin{aligned}
\langle X, \varphi\rangle := \int_{\mathbb{R}^d} X_s\varphi(s)\lambda^d(ds) &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(s-t)\varphi(s)dL(t)\lambda^d(ds) \\
&\overset{a.s.}{=} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(s-t)\varphi(s)\lambda^d(ds)dL(t)
\end{aligned}$$

(from the discussions preceeding Theorem 3.1 in [4] it follows also that a version of X_s can be chosen such that $X_s\varphi(s)$ is integrable with respect to λ^d). Further, $X : \mathcal{D}(\mathbb{R}^d) \to L^0$ is clearly linear and estimates as above show that it is also continuous, hence X is a generalized random process. To see that $p(D)X = q(D)\dot{L}$, observe that

$$\begin{aligned}
\langle X, p(D)^*\varphi\rangle &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(s-t)p(D)^*\varphi(s)\lambda^d(ds)dL(t) \\
&= \int_{\mathbb{R}^d} (G(-\cdot) * p(D)^*\varphi)(t)dL(t) \\
&= \int_{\mathbb{R}^d} (p(D)^*G(-\cdot) * \varphi)(t)dL(t) \\
&= \int_{\mathbb{R}^d} q(D)^*\varphi(t)dL(t) \\
&= \langle \dot{L}, q(D)^*\varphi\rangle,
\end{aligned}$$

where we used in the last equality but one that $G(-\cdot)$ is the fundamental solution of $p(-D)u = q(-D)\delta_0$. It follows that X is a generalized solution of the SPDE $p(D)X = q(D)\dot{L}$. □

4.6 Moment properties

We say that a generalized process $s : \mathcal{D} \to L^0(\Omega)$ has existing β-moment, $\beta > 0$, if $\mathbb{E}|\langle s, \varphi\rangle|^\beta < \infty$ for every $\varphi \in \mathcal{D}(\mathbb{R}^d)$.
Let $\dot{L}$ be a Lévy white noise with characteristic triplet (a, γ, ν). Then it is easy to see (cf. [58, Theorem 25.3, p. 159]) that $\dot{L}$ has existing β-moment if and only if

$$\int\limits_{|z|>1} |z|^\beta \nu(dz) < \infty.$$

Next we show that if $\dot{L}$ has existing β-moment then so has the CARMA generalized process given in Theorem 4.16.

Proposition 4.27. *Let $\dot{L}$ have existing β-moment ($\beta > 0$) and let p and q be polynomials satisfying Assumption 4.15. Then the stationary CARMA(p, q) generalized process s constructed in Theorem 4.16 has existing β-moment, too.*

Proof. Let $\varphi \in \mathcal{D}(\mathbb{R}^d)$. From (4.24) and [56, Theorem 2.7] we see that the Lévy measure of the random variable $s(\varphi)$ is given by

$$\nu_{s(\varphi)}(B) = \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}} \mathbf{1}_{B\setminus\{0\}}(rG * (1-\Delta)^\alpha\varphi(x))\nu(dr)\lambda^d(dx),$$

where G and α are defined as in the proof of Theorem 4.16. We conclude

$$\begin{aligned}\int\limits_{|z|>1} |z|^\beta \nu_{s(\varphi)}(dz) &= \int\limits_{\mathbb{R}} |r|^\beta \int\limits_{|(G*(1-\Delta)^\alpha\varphi)(x)|>\frac{1}{|r|}} |G*(1-\Delta)^\alpha\varphi(x)|^\beta \lambda^d(dx)\nu(dr)\\ &= \int\limits_{|r|\le 1} |r|^\beta \int\limits_{|(G*(1-\Delta)^\alpha\varphi)(x)|>\frac{1}{|r|}} |G*(1-\Delta)^\alpha\varphi(x)|^\beta \lambda^d(dx)\nu(dr) \qquad (4.29)\\ &\quad+ \int\limits_{|r|>1} |r|^\beta \int\limits_{|(G*(1-\Delta)^\alpha\varphi)(x)|>\frac{1}{|r|}} |G*(1-\Delta)^\alpha\varphi(x)|^\beta \lambda^d(dx)\nu(dr).\end{aligned}$$

For $\beta \geq 1$ we see by the Young inequality

$$\|G*(1-\Delta)^\alpha\varphi\|_{L^\beta}^\beta \le \|(1-\Delta)^\alpha\varphi\|_{L^\beta}^\beta \|G\|_{L^1}^\beta \qquad (4.30)$$

and for $0 < \beta < 1$ we note that

$$\begin{aligned}
&\int\limits_{\mathbb{R}^d} |G * (1-\Delta)^\alpha \varphi(x)|^\beta \lambda^d(dx) \\
=&\int\limits_{\mathbb{R}^d} |G * (1-\Delta)^\alpha \varphi(x)|^\beta \exp(-(b/4)\beta\|x\|)\exp((b/4)\beta\|x\|)\lambda^d(dx) \\
\leq& C\left(\int\limits_{\mathbb{R}^d} |G * (1-\Delta)^\alpha \varphi(x)\exp((b/4)\|x\|)|\lambda^d(dx)\right)^\beta \\
\leq& C\left(\int\limits_{\mathbb{R}^d} |(1-\Delta)^\alpha \varphi(y)|\exp((b/4)\|y\|)\int\limits_{\mathbb{R}^d} |G(x)|\exp((b/4)\|x\|)\lambda^d(dx)\lambda^d(dy)\right)^\beta \\
\leq& C'\|G\exp(b\|\cdot\|)\|^\beta_{L^2(\mathbb{R}^d)}\left(\int\limits_{\mathbb{R}^d} |(1-\Delta)^\alpha \varphi(y)|\exp((b/4)\|y\|)\lambda^d(dy)\right)^\beta,
\end{aligned}$$

where $b > 0$ is chosen such that $\|G\exp(b\|\cdot\|)\|_{L^2} < \infty$ and C and C' are finite constants. From the previous calculations it is immediate that the term in (4.29) corresponding to the integral when $|r| > 1$ is finite for all $\beta > 0$, and that the integral corresponding to the term $|r| \leq 1$ is finite when $\beta \geq 2$. When $\beta \in (0, 2]$ we estimate similar to (4.30)

$$\begin{aligned}
&\int\limits_{|r|\leq 1} |r|^\beta \int\limits_{|(G*(1-\Delta)^\alpha\varphi)(x)|>\frac{1}{|r|}} |G * (1-\Delta)^\alpha \varphi(x)|^\beta \lambda^d(dx)\nu(dr) \\
\leq& \int\limits_{|r|\leq 1} |r|^2 \nu(dr)\|G * (1-\Delta)^\alpha \varphi(x)\|^2_{L^2(\mathbb{R}^d)} < \infty.
\end{aligned}$$

We conclude that $\int\limits_{|z|>1} |z|^\beta \nu_{s(\varphi)}(dz)$ is finite for $\beta > 0$. □

By the same means we obtain the following.

Proposition 4.28. *Let X be the mild solution of a CARMA(p,q)-equation constructed in Theorem 4.23. If the $\beta-$moment of the Lévy-white noise exists for $0 < \beta \leq 2$, then $\mathbb{E}|X_x|^\beta < \infty$ for every $x \in \mathbb{R}^d$.*

Proof. Let $G = \mathcal{F}^{-1}\frac{q(i\cdot)}{p(i\cdot)}$ and denote the Lévy measure of $X_x = \int\limits_{\mathbb{R}^d} G(x-t)dL(t)$ by ν_G. Then by [56, Theorem 2.7],

$$\int\limits_{|z|>1} |z|^\beta \nu_G(dz) = \int\limits_{\mathbb{R}} |r|^\beta \int\limits_{|G(x)|>\frac{1}{|r|}} |G(x)|^\beta \lambda^d(dx)\nu(dr)$$

$$
\begin{aligned}
&= \int\limits_{|r|\le 1} |r|^\beta \int\limits_{|G(x)|>\frac{1}{|r|}} |G(x)|^\beta \lambda^d(dx)\nu(dr) + \int\limits_{|r|>1} |r|^\beta \int\limits_{|G(x)|>\frac{1}{|r|}} |G(x)|^\beta \lambda^d(dx)\nu(dr) \\
&\le \int\limits_{|r|\le 1} |r|^2 \int\limits_{|G(x)|>\frac{1}{|r|}} |G(x)|^2 \lambda^d(dx)\nu(dr) + \int\limits_{|r|>1} |r|^\beta \int\limits_{|G(x)|>\frac{1}{|r|}} |G(x)|^\beta \lambda^d(dx)\nu(dr) \\
&= I_1 + I_2.
\end{aligned}
$$

I_1 is clearly finite, and I_2 is finite since $e^{c\|\cdot\|}G \in L^2(\mathbb{R}^d)$ for some $c > 0$ (see the proof of Theorem 4.23) and hence $G \in L^\beta(\mathbb{R}^d)$. □

Remark 4.29. The β considered in Proposition 4.28 has to be smaller or equal than 2, as otherwise there may exist some β for which the Proposition does not hold. Look for example at the fundamental solution of the partial differential operator $\lambda - \Delta$ for some $\lambda > 0$ in dimension 3, which is given by $c\frac{\exp(-\sqrt{\kappa}\|x\|)}{\|x\|}$ with c a constant. The fundamental solution does not live in $L^3_{loc}(\mathbb{R}^3)$, see [41, Section 2.1, Equation (21)], which implies that $\mathbb{E}|X_x|^3 = \infty$ for all $x \in \mathbb{R}^3$.

As a corollary we get the following easy result.

Corollary 4.30. *Let the Lévy basis L have existing second moment σ^2 (i.e., $\mathbb{E}(L([0,1]^d)^2) = \sigma^2$) with vanishing first moment. Then, under the assumptions of Theorem 4.23, the spectral density of the mild solution X of a CARMA(p,q)-SPDE with polynomials p and q is given by*

$$
f(\xi) = \sigma^2 \left| \frac{q(i\xi)}{p(i\xi)} \right|^2. \tag{4.31}
$$

Proof. It is clear that X_x has existing second moment and vanishing first moment. Moreover, we see from the Itô-isometry that

$$
\mathbb{E}X_0X_y = \sigma^2 \int\limits_{\mathbb{R}^d} G(x)G(x-y)\lambda^d(dx).
$$

As G is the inverse Fourier transform of $\frac{q(i\xi)}{p(i\xi)}$ we conclude as in [16, Theorem 2, p. 841] that the spectral density is given by (4.31). □

4.7 CARMA random fields in the sense of Brockwell and Matsuda

We will now analyze the CARMA random fields in the sense of Brockwell and Matsuda defined in [16] and show that the corresponding random field defines a mild solution of a fractional stochastic partial differential equation. In our setting we find for odd dimensions the corresponding CARMA generalized processes with respect to a SPDE of type (4.21). A CARMA random field in the sense of Brockwell and Matsuda is defined as follows: Let $0 \leq q < p$, $a_*(z) = z^p + a_1 z^{p-1} + \ldots + a_p = \prod_{i=1}^{p}(z - \lambda_i)$ be a polynomial with real coefficients and distinct roots λ_i with strictly negative real parts and $b_*(z) = b_0 + b_1 z + \ldots + b_{q-1} z^{q-1} + z^q = \prod_{i=1}^{q}(z - \kappa_i)$ also be a polynomial with real coefficients. Assume that $\lambda_i \neq \kappa_j$ for all i and j. Define the functions

$$a(z) = \prod_{i=1}^{p}(z^2 - \lambda_i^2) \text{ and } b(z) = \prod_{i=1}^{q}(z^2 - \kappa_i^2).$$

Let L be a Lévy basis in $\mathbb{R}^d$ with finite second moment. Then the isotropic CARMA(p, q) field driven by L (in the sense of Brockwell and Matsuda) is given by

$$X_t = \int_{\mathbb{R}^d} \sum_{i=1}^{p} \frac{b(\lambda_i)}{a'(\lambda_i)} e^{\lambda_i \|t-u\|} \, dL(u) \tag{4.32}$$

for every $t \in \mathbb{R}^d$. Here, a' denotes the derivative of the polynomial a. For a more detailed introduction see [16, Definition 3.1, p. 837].

Proposition 4.31. *Let $X = (X_t)_{t \in \mathbb{R}^d}$ be defined by (4.32) and d be odd. Then X is the mild solution of the SPDE*

$$\prod_{i=1}^{p} a'(\lambda_i)(-\Delta + \lambda_i^2)^{\frac{d+1}{2}} X = c_d \sum_{i=1}^{p} 2\lambda_i b(\lambda_i) \prod_{j=1, j\neq i}^{p} a'(\lambda_j)(-\Delta + \lambda_j^2)^{\frac{d+1}{2}} dL \tag{4.33}$$

for some constant c_d depending on the dimension d.

Proof. We know from [16, Theorem 2, p.841] that the Fourier transform of the isotropic CARMA kernel is given by

$$c_d \sum_{i=1}^{p} \frac{2\lambda_i b(\lambda_i)}{a'(\lambda_i)(\|z\|^2 + \lambda_i^2)^{\frac{d+1}{2}}} = c_d \frac{\sum_{i=1}^{p} 2\lambda_i b(\lambda_i) \prod_{j=1, j\neq i}^{p} a'(\lambda_j)(\|z\|^2 + \lambda_j^2)^{\frac{d+1}{2}}}{\prod_{i=1}^{p} a'(\lambda_i)(\|z\|^2 + \lambda_i^2)^{\frac{d+1}{2}}}, \; z \in \mathbb{R}^d,$$

for some constant c_d dependend on the dimension d. We conclude that S_d is the mild solution of the SPDE

$$\prod_{i=1}^{p} a'(\lambda_i)(-\Delta + \lambda_i^2)^{\frac{d+1}{2}} X = c_d \sum_{i=1}^{p} 2\lambda_i b(\lambda_i) \prod_{j=1, j\neq i}^{p} a'(\lambda_j)(-\Delta + \lambda_j^2)^{\frac{d+1}{2}} \dot{L}$$

by comparing our mild solution to the definition in (4.28). □

For even d we see that $\prod_{j=1}^{p}(-\Delta + \lambda_i^2)^{\frac{d+1}{2}}$ defines a fractional Laplace operator, which is defined by

$$\prod_{j=1}^{p}(-\Delta+\lambda_i^2)^{\frac{d+1}{2}}\varphi := \mathcal{F}^{-1}\prod_{j=1}^{p}(\sum_{m=1}^{d} z_m^2+\lambda_j^2)^{\frac{d+1}{2}}\mathcal{F}\varphi. \tag{4.34}$$

A fundamental solution G of $Au = B\delta_0$, where A and B are fractional operators defined by (4.34), is defined by $AG * \varphi = B\varphi$ for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$. Allowing this larger class of solutions we obtain the following.

Proposition 4.32. *Let $X = (X_t)_{t\in\mathbb{R}^d}$ be defined by* (4.32)*. Then X is the mild solution of the (fractional) SPDE*

$$\prod_{i=1}^{p} a'(\lambda_i)(-\Delta+\lambda_i^2)^{\frac{d+1}{2}}X = c_d\sum_{i=1}^{p} 2\lambda_i b(\lambda_i)\prod_{j=1, j\neq i}^{p} a'(\lambda_j)(-\Delta+\lambda_j^2)^{\frac{d+1}{2}}\dot{L} \tag{4.35}$$

for some constant c_d dependend on the dimension d.

Proof. Follows the same arguments as above. □

5 Lévy driven linear and semilinear stochastic partial differential equations

The goal of this chapter is twofold. In the first part we will study Lévy white noise in different distributional spaces and solve equations of the type $p(D)s = q(D)\dot{L}$, where p and q are polynomials. Furthermore, we will study measurability of s in Besov spaces. By using this result we will prove that stochastic partial differential equations of the form

$$p(D)u = g(\cdot, u) + \dot{L}$$

have measurable solutions in weighted Besov spaces, where $p(D)$ is a partial differential operator in a certain class, $g : \mathbb{R}^d \times \mathbb{C} \to \mathbb{R}$ satisfies some Lipschitz condition and $\dot{L}$ is a Lévy white noise.

5.1 Introduction

A stochastic process $X = (X_t)_{t\in\mathbb{R}}$ is called a CARMA process, if X is a solution of the (formal) stochastic differential equation

$$\sum_{j=0}^{m} a_j \frac{d^j X(t)}{dt^j} = \sum_{k=1}^{n} b_k \frac{d^k L(t)}{dt^k} \tag{5.1}$$

where $m, n \in \mathbb{N}$, $a_j, b_k \in \mathbb{R}$ for every $0 \leq j \leq m$ and $0 \leq k \leq n$ and L is a Lévy process. Equation (5.1) can also be written as $a(D)X_t = b(D)L(t)$, where $a(z) = \sum\limits_{j=0}^{m} a_j z^j$ and $b(z) = \sum\limits_{j=1}^{n} b_j z^j$. In [15] necessary and sufficient conditions on L on the polynomials a and b were given such that there exists a strictly stationary solution of (5.1), namely it was shown that it is sufficient and necessary that $\mathbb{E}\log^+(|L_1|) < \infty$. CARMA processes have many applications, see for example [33] and [13].
For dimensions greater than 1, there exist more than one definition of a CARMA random field. Here, we will recall only the definition in the sense of Chapter 4. For the other definitions see the two papers of Brockwell and Matsuda [16] and Pham [55]. In Chapter

4 a CARMA random field s is a stationary generalized stochastic process on the space of test functions $\mathcal{D}(\mathbb{R}^d) := \mathcal{C}_c^\infty(\mathbb{R}^d)$, which solves the equation

$$p(D)s = q(D)\dot{L}, \tag{5.2}$$

where p and q are real polynomials in d-variables and $\dot{L}$ is Lévy white noise with characteristic triplet (a, γ, ν), where (5.2) means that

$$\langle s, p(D)^*\varphi\rangle = \langle \dot{L}, q(D)^*\varphi\rangle$$

for every $\varphi \in \mathcal{D}(\mathbb{R}^d)$, where $p(D)^*$ denotes the (formal) adjoint operator of $p(D)$. For the definition of stationary generalized processes and Lévy white noise see Section 5.3 or Chapter 4, Definition 4.3. It was shown that if the rational function $\frac{p}{q}$ has a holomorphic extension in a certain set and $\int_{|r|>1} \log(|r|)^d \nu(dr) < \infty$, then there exists a stationary solution of (5.2). The problem of the stationary generalized solution s is that it may not have a random field representation and the question of uniqueness is open. Furthermore, as the regularity of s is not well-understood, it is not directly clear if one can solve more complex SPDEs than (5.2). The goal of this chapter is to tackle these problems and give some answers to these questions. We will show the existence of the Lévy white noise in the space of tempered ultradistributions and Fourier hyperfunctions defined as in [63] and [43] and show that (5.2) has solutions in the space of tempered (ultra-)distributions, Fourier hyperfunctions and Besov spaces under specific assumptions. Furthermore, we will analyze the semilinear equation

$$p(D)s = g(\cdot, s) + \dot{L} \tag{5.3}$$

in certain weighted Besov spaces, where $g : \mathbb{R}^d \times \mathbb{C} \to \mathbb{R}$ is a sufficiently regular function. The above mentioned results can be found in Sections 5.3 and 5.4, where our main results are Theorem 5.10, Theorem 5.13 and Proposition 5.16. In detail, in Section 5.3 we recall the definition of generalized stochastic processes and study (5.2) in the three different spaces. In Section 5.4 we study (5.3) in different Besov spaces.

5.2 Notation and Preliminaries

To fix notation, by $(\Omega, \mathcal{F})$ we denote a measurable space, where Ω is a set and $\mathcal{F}$ is a σ-algebra and by $L^0(\Omega, \mathcal{F}, \mathbb{K})$ we denote all measurable functions $f : \Omega \to \mathbb{K}$ with respect to $\mathcal{F}$ where $\mathbb{K} = \mathbb{R}, \mathbb{C}$. In the case that $\mathcal{F}$ and $\mathbb{K}$ are clear from the context we set $L^0(\Omega) = L^0(\Omega, \mathcal{F}, \mathbb{K})$. If we consider a probability space $(\Omega, \mathcal{F}, \mathcal{P})$, where $\mathcal{P}$ is a probability measure on $(\Omega, \mathcal{F})$, we say that a sequence $(f_n)_{n\in\mathbb{N}} \subset L^0(\Omega)$ converges to f in $L^0(\Omega)$ if f_n converges in probability to f with respect to the measure $\mathcal{P}$. In the case of $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ we denote by $\mathcal{B}(\mathbb{R}^d)$ the Borel-σ-set on $\mathbb{R}^d$.
We write $\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ for the set of integers, real numbers

and complex numbers, respectively. If $z \in \mathbb{C}$, we denote by $\Im z$ and $\Re z$ the imaginary and the real part of z. The Euclidean norm is denoted by $\|\cdot\|$ and $r^+ := \max\{0, r\}$ for every $r \in \mathbb{R}$. By $C^\infty(\mathbb{R}^d, \mathbb{C})$ we denote the set of all functions $\varphi : \mathbb{R}^d \to \mathbb{C}$ which are infinitely often differentiable. Furthermore, by $L^p(\mathbb{R}^d, A)$ for $A \subseteq \mathbb{C}$ and $0 < p \leq \infty$ we denote the set of all Borel-measurable functions $f : \mathbb{R}^d \to A$ such that $\int_{\mathbb{R}^d} |f(x)|^p \, \lambda^d(dx) < \infty$ for $0 < p < \infty$ and $\operatorname{ess\,sup}_{x \in \mathbb{R}^d} |f(x)| < \infty$ for $p = \infty$, where λ^d is the $d-$dimensional Lebesgue measure. We denote by $||f||_{L^p} = (\int_{\mathbb{R}} |f(x)|^p \, \lambda(dx))^{1/p}$ for $0 < p < \infty$ and $\|f\|_{L^\infty} = \operatorname{ess\,sup}_{\mathbb{R}^d} |f|$ the L^p-(quasi-)norm for a measurable function f. We write $\langle x \rangle := (1 + \|x\|^2)^{1/2}$ and $\|f\|_{L^p(\mathbb{R}^d, \rho)} := \|\langle \cdot \rangle^\rho f\|_{L^p(\mathbb{R}^d)}$ for $\rho \in \mathbb{R}$. Let $(a_k)_{k \in \mathbb{N}_0} \subset \mathbb{C}$ be a sequence and we set

$$\|(a_k)_{k \in \mathbb{N}_0}\|_{l^q} := \left(\sum_{k \in \mathbb{N}_0} |a_k|^q \right)^{\frac{1}{q}}$$

for $0 < q < \infty$. For $q = \infty$ the norm is given by $\|(a_k)_{k \in \mathbb{N}_0}\| = \sup_{k \in \mathbb{N}_0} |a_k|$. By d_f we denote the distribution function of $f : \mathbb{R}^d \to \mathbb{C}$, which means that

$$d_f(\alpha) := \lambda^d(\{x \in \mathbb{R}^d : |f(x)| > \alpha\}), \, \alpha \geq 0. \tag{5.4}$$

The space $\mathcal{D}(\mathbb{R}^d)$ denotes the set of all infinitely differentiable functions $f : \mathbb{R}^d \to \mathbb{R}$ with compact support with its usual topology (e.g. [32, Section 2.1]), where we denote the support of f by supp f. The topological dual space of $\mathcal{D}(\mathbb{R}^d)$ will be denoted by $\mathcal{D}'(\mathbb{R}^d)$, where an element $u \in \mathcal{D}'(\mathbb{R}^d)$ is called a distribution. The space $\mathcal{S}(\mathbb{R}^d)$ denotes the Schwartz space equipped with its usual topology, see [21, Section 1, p. 4391] and $\mathcal{S}'(\mathbb{R}^d)$ its topological dual with its strong topology. We sometimes write $\mathcal{S}$ and $\mathcal{S}'$, if the dimension is clear. We will write $\langle u, \varphi \rangle := u(\varphi)$ for $\varphi \in \mathcal{D}(\mathbb{R}^d)$ (or $\mathcal{S}(\mathbb{R}^d)$) and $u \in \mathcal{D}'(\mathbb{R}^d)$ (or $\mathcal{S}'(\mathbb{R}^d)$). We say that a function $a : Y \to \mathbb{R}$ from some function space Y acts as a Fourier multiplier for some function space X to a function space R with well-defined Fourier transform $\mathcal{F}$ if $a : X \to R$ is defined by $a(u) := \mathcal{F}^{-1}(a\mathcal{F}u)$, where $(a\mathcal{F}(u))(t) = a(t)\mathcal{F}(u)(t)$ such that the inverse Fourier transform $\mathcal{F}^{-1}$ is well-defined. For a function $f \in L^1(\mathbb{R}^d, \mathbb{C}^d)$ we set $\mathcal{F}f(x) = \int\limits_{\mathbb{R}^d} e^{-i\langle z, x \rangle} f(z) \lambda^d(dz)$ and the L^2-Fourier transform likewise. A polynomial p is a function given by $p(z) = \sum\limits_{|\alpha| \leq m} p_\alpha z^\alpha$, $\alpha \in \mathbb{N}_0^d$, $m \in \mathbb{N}$, $z^\alpha = z_1^{\alpha_1} \ldots z_d^{\alpha_d}$ and $|\alpha| := \alpha_1 + \ldots + \alpha_d$. We set $D^\alpha = \partial_{x_1}^{\alpha_1} \ldots \partial_{x_d}^{\alpha_d}$ for $\alpha \in \mathbb{N}_0^d$. We denote by A^* the adjoint of the operator A.
We introduce weighted Besov spaces and follow [64]. Let $\varphi_0 \in \mathcal{S}(\mathbb{R}^d)$ such that $\varphi_0(x) = 1$ if $\|x\| \leq 1$ and $\varphi_0(x) = 0$ if $\|x\| \geq 3/2$, and we set

$$\varphi_k(x) = \varphi_0(2^{-k}x) - \varphi_0(2^{-k+1}x), \, x \in \mathbb{R}^d, \, k \in \mathbb{N}.$$

As $\sum_{k=0}^{\infty} \varphi_k(x) = 1$ for all $x \in \mathbb{R}^d$, it is clear that $(\varphi_k)_{k\in\mathbb{N}_0}$ is a dyadic decomposition of unity in $\mathbb{R}^d$. We set

$$\Delta_k f := \mathcal{F}^{-1}\varphi_k \mathcal{F} f$$

for every $f \in \mathcal{S}'(\mathbb{R}^d)$. Observe that this object is a well-defined function, which can be evaluated pointwise, see [63, Remark 1, p. 37]. A weighted Besov space $B^l_{r,t}(\mathbb{R}^d, \rho)$ is a subspace of $\mathcal{S}'(\mathbb{R}^d)$ which is characterized by four parameters $l, \rho \in \mathbb{R}$ and $r, t > 0$, where $f \in B^l_{r,t}(\mathbb{R}^d, \rho)$ if and only if

$$\|f\|_{B^l_{r,t}(\mathbb{R}^d,\rho)} := \|(2^{lk}\|\Delta_k f\|_{L^r(\mathbb{R}^d,\rho)})_{k\in\mathbb{N}_0}\|_{l^t} < \infty.$$

For $r = t = 2$ and $l > 0$ we identify the weighted Sobolev space $W^l_2(\mathbb{R}^d, \rho)$ with $B^l_{2,2}(\mathbb{R}^d, \rho)$, i.e. there exists a continuous and bijective mapping ξ from $B^l_{2,2}(\mathbb{R}^d, \rho)$ to $W^l_2(\mathbb{R}^d, \rho)$ such that for all $f \in B^l_{2,2}(\mathbb{R}^d, \rho)$

$$f(\varphi) = \int_{\mathbb{R}^d} \xi(f)(x)\varphi(x)\lambda^d(dx) \text{ for all } \varphi \in \mathcal{S}, \tag{5.5}$$

see [63, Theorem 2.5.6, p. 88]. Moreover, ξ is also continuous from $B^l_{r,r}(\mathbb{R}^d, \rho)$ to $L^r(\mathbb{R}^d, \rho)$ for $l > 0$, $r \geq 2$. From now on we write for $\xi(f)$ simply f.
An interesting property of the Besov spaces are their embeddings, which are described as follows:

Proposition 5.1 (see [31, Proposition 3, p. 1605])**.** *Let $p_0, p_1 \in (0, \infty]$ and $\tau_0, \tau_1, \rho_0, \rho_1 \in \mathbb{R}$ with $\tau_0 \geq \tau_1$. It holds that $B^{\tau_0}_{p_0,p_0}(\mathbb{R}^d, \rho_0)$ is continuously embedded in $B^{\tau_1}_{p_1,p_1}(\mathbb{R}^d, \rho_1)$ if $\tau_0 - \tau_1 \geq \frac{d}{p_0} - \frac{d}{p_1}$, $p_1 \geq p_0$ and $\rho_0 \geq \rho_1$. If the inequalities are strict, the embeddings are compact.*

5.3 Linear stochastic partial differential equations in the spaces of tempered distributions, tempered ultradistributions, Fourier hyperfunctions and Besov spaces with polynomial weights

At first we give a short introduction to generalized processes on more general distributional space A', which is the dual space of a suitable function space A over $\mathbb{R}$. For example, A can be the Schwartz space $\mathcal{S}(\mathbb{R}^d)$, the space of test functions $\mathcal{D}(\mathbb{R}^d)$ or even weighted Besov spaces $B^s_{p,q}(\mathbb{R}^d, \rho)$ for suitable s, p and ρ.

Definition 5.2 (see [32], Definition 2.1)**.** An *A'-valued generalized random process s* is a measurable mapping from $(\Omega, \mathcal{F})$ to a distributional space $(A', C(A'))$, where $C(A')$

denotes the σ-field generated by the cylindrical sets

$$\{u \in A' : \langle u, \varphi_j \rangle \in B \text{ for every } j = 1, \dots, n\}$$

for every $\varphi_1, \dots, \varphi_n \in A$ and $B \in \mathcal{B}(\mathbb{R})$.

Since in our cases under consideration A' will be nuclear or even a Hilbert space, it follows from [42, p.6] that $C(A')$ is equal to the $\sigma-$algebra $B^*(A')$ generated by the weak-$*$-topology in A'.
The probability law of a generalized random process s is given by

$$\mathcal{P}_s(B) := \mathcal{P}(s \in B)$$

for $B \in \mathcal{B}^*(A')$. The characteristic functional $\widehat{\mathcal{P}}_s$ is then defined by

$$\widehat{\mathcal{P}}_s(\varphi) = \int\limits_{A'} \exp(i\langle u, \varphi \rangle) d\mathcal{P}_s(u),\ \varphi \in A.$$

We will work with Lévy white noise, which is a generalized random process, where the characteristic functional satisfies a Lévy-Khintchine representation.

Definition 5.3. A *Lévy white noise* $\dot{L}$ on A' is an A'-valued generalized random process, where the characteristic functional is given by

$$\widehat{\mathcal{P}}_{\dot{L}}(\varphi) = \exp\left(\int\limits_{\mathbb{R}^d} \psi(\varphi(x)) \lambda^d(dx) \right)$$

for every $\varphi \in A$, where $\psi : \mathbb{R} \to \mathbb{C}$ is given by

$$\psi(z) = i\gamma z - \frac{1}{2} a z^2 + \int\limits_{\mathbb{R}} (e^{ixz} - 1 - ixz \mathbf{1}_{|x| \leq 1}) \nu(dx)$$

where $a \in \mathbb{R}^+$, $\gamma \in \mathbb{R}$ and ν is a *Lévy-measure*, i.e. a measure such that $\nu(\{0\}) = 0$ and

$$\int\limits_{\mathbb{R}} \min(1, x^2) \nu(dx) < \infty.$$

We say that $\dot{L}$ has the *characteristic triplet* (a, γ, ν).

In the case that A is a complex function space given by $A = A_{real} + iA_{real}$, where $A_{real} = A \cap \{\varphi \in A : \varphi \text{ is real-valued }\}$, we construct a Lévy white noise on A_{real} and set for $\varphi \in A$

$$\langle \dot{L}, \varphi \rangle = \langle \dot{L}, \operatorname{Re} \varphi \rangle + i \langle \dot{L}, \Im \varphi \rangle.$$

The Lévy white noise is stationary in the following sense:

Definition 5.4. A generalized random process s is called *stationary* if for every $t \in \mathbb{R}^d$, $s(\cdot + t)$ has the same law as s. Here, $s(\cdot + t)$ is defined by

$$\langle s(\cdot + t), \varphi\rangle := \langle s, \varphi(\cdot - t)\rangle \text{ for every } \varphi \in A.$$

It is well-known that a Lévy white noise $\dot{L}$ on the space of tempered distributions $\mathcal{S}'$ with characteristic triplet (a, γ, ν) exists if and only if there exists an $\varepsilon > 0$ such that $\int_{|r|>1} |r|^\varepsilon \nu(dr) < \infty$, see [21, Theorem 3.13, p. 4412]. As the space of tempered distributions is too small for many cases of the Lévy white noise, we will construct the Lévy white noise in another distributional space. We discuss the existence of the Lévy white noise in the space of tempered ultradistribution. The space of tempered ultradistributions is very similar to the space of tempered distributions, especially the space $\mathcal{S}'_\omega$ is nuclear, which allows us to use the Bochner-Minlos Theorem. Moreover, by similar arguments we construct Lévy white noise in the space of Fourier hyperfunctions. Furthermore, we will discuss in the spirit of Chapter 4 the solvability of the equations

$$p(D)s = q(D)\dot{L} \tag{5.6}$$

in the space of tempered distributions, tempered ultradistributions and Fourier hyperfunctions, where $p(z) = \sum_{|\alpha|\le n} p_\alpha z^\alpha$ and $q(z) = \sum_{|\alpha|\le m} q_\alpha z^\alpha$ are real multivariate polynomials. Moreover, we will study (5.6) also for Lévy white noise in Besov spaces, as these results are needed in Section 5.4 for more complex (nonlinear) stochastic partial differential equations. We start with an existence result on the space of tempered distributions of (5.6). Observe that $p(D)^* = p(-D)$ and $q(D)^* = q(-D)$.

Proposition 5.5. *Let $\dot{L}$ be a Lévy white noise on the space of tempered distributions $\mathcal{S}'$. Let p and q be two polynomials such that there exists two polynomials h and l such that $\frac{q(i\cdot)}{p(i\cdot)} = \frac{h(i\cdot)}{l(i\cdot)}$ on $\mathbb{R}^d$ and l has no zeroes on $i\mathbb{R}^d$. Then there exists a generalized process s on the space of tempered distributions solving (5.6), i.e. it holds that*

$$\langle s, p(D)^*\varphi\rangle = \langle \dot{L}, q(D)^*\varphi\rangle \tag{5.7}$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$, which is stationary. If $p(iz) \neq 0$ for all $z \in \mathbb{R}^d$, then the solution s is unique.

Proof. We observe by [40, Lemma 2] that $\varphi \mapsto \mathcal{F}^{-1}\left(\frac{q(i\cdot)}{p(i\cdot)}\mathcal{F}\varphi\right)$ defines a continuous operator from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}(\mathbb{R}^d)$ and define

$$\langle s, \varphi\rangle := \langle \dot{L}, \mathcal{F}^{-1}\left(\frac{q(i\cdot)}{p(i\cdot)}\mathcal{F}\varphi\right)\rangle. \tag{5.8}$$

We conclude that s defines a generalized process on the space of tempered distributions. That it solves (5.7) follows easily by $\mathcal{F}p(-D)\varphi = p(-i\cdot)\mathcal{F}\varphi$ for every $\varphi \in \mathcal{S}$ and the

stationarity of s follows from that of $\dot{L}$.
Now let u be another solution of the equation (5.6). One observes that

$$\langle p(D)(s-u), \varphi\rangle = 0 \tag{5.9}$$

for every $\varphi \in \mathcal{S}$. In the case that $p(iz) \neq 0$ for all $z \in \mathbb{R}^d$ it is known that only the null-solution satisfies equation (5.9), see [54, Proposition 2.4.1, p. 152], so we conclude $s = u$. $\square$

Our second distribution space is the space of tempered ultradistributions. For a detailed introduction to these spaces see [63]. We recall the definition.

Definition 5.6. Let $\omega : \mathbb{R}^d \to \mathbb{R}$ be a real-valued function such that $\omega(x) = \sigma(\|x\|)$, where $\sigma(t)$ is an increasing continuous concave function on $[0, \infty)$ with

$$\begin{aligned}
&\sigma(0) = 0,\\
&\int_0^\infty \frac{\sigma(t)}{1+t^2}\lambda^1(dt) < \infty,\\
&\sigma(t) \geq c + m\log(1+t) \text{ if } t \geq 0
\end{aligned}$$

for some $c \in \mathbb{R}$ and $m > 0$. Then the space $\mathcal{S}_\omega$ is the set of all infinitely differentiable functions $\varphi : \mathbb{R}^d \to \mathbb{C}$ such that

$$\begin{aligned}
p_{\alpha,\eta}(\varphi) &:= \sup_{x\in\mathbb{R}^d} e^{\eta\omega(x)}\|D^\alpha\varphi(x)\| < \infty,\\
\pi_{\alpha,\eta}(\varphi) &:= \sup_{x\in\mathbb{R}^d} e^{\eta\omega(x)}\|D^\alpha(\mathcal{F}\varphi)(x)\| < \infty,
\end{aligned}$$

for every multi-index α and every $\eta > 0$. The space is equipped with its seminorms given above and its topological dual $\mathcal{S}'_\omega$ is called the *space of tempered ultradistributions.*

We denote by $\omega^{\rightarrow}(\alpha) := \sup\{x \in [0, \infty) : \omega(xe_1) < \alpha\}$ for $\alpha \in (0, \infty)$, where e_1 is the unit vector $(1, 0 \ldots, 0)$.
We split a function $\varphi \in C^\infty(\mathbb{R}^d, \mathbb{C})$ in its real and imaginary part and prove the existence of a Lévy white noise on

$$\mathcal{S}_\omega^{real} = \mathcal{S}_\omega \cap \{\varphi : \mathbb{R}^d \to \mathbb{C} : \varphi(x) \in \mathbb{R} \text{ for all } x \in \mathbb{R}^d\}.$$

Observe that $\mathcal{S}_\omega^{real}$ equipped with the topology of $\mathcal{S}_\omega$ is closed and therefore nuclear. We then set $\langle \dot{L}, \varphi\rangle := \langle \dot{L}, \Re\varphi\rangle + i\langle \dot{L}, \Im\varphi\rangle$ which defines the Lévy white noise on $\mathcal{S}_\omega$.

Theorem 5.7. *Let (a, γ, ν) be a characteristic triplet and ω be a function defined as in*

Definition 5.6. If

$$\int\limits_{|r|>1} |r| \int\limits_{0}^{1/|r|} \omega^{\rightarrow}(c\log(|\alpha|^{-1}))^d \lambda^1(d\alpha)\nu(dr) < \infty$$

for some $c \in (0,\infty)$, then there exists a Lévy white noise $\dot{L} : (\Omega,\mathcal{F}) \to (\mathcal{S}'_\omega, C(\mathcal{S}'_\omega))$ with characteristic triplet (a,γ,ν).

Proof. We need to show that the function

$$\mathcal{P}(\varphi) := \exp\left(\int\limits_{\mathbb{R}^d}\left(i\gamma\varphi(u) - \frac{1}{2}a\varphi(u)^2 + \int\limits_{\mathbb{R}}(e^{ix\varphi(u)} - 1 - ix\varphi(u)\mathbf{1}_{|x|\leq 1})\nu(dx)\lambda^d(du)\right)\right)$$

defines a continuous and positive-definite mapping on $\mathcal{S}_\omega^{real}$ and $\mathcal{P}(0) = 1$. Then we conclude by the Bochner-Minlos Theorem [30, Theorem 1, p. 1186] that there exists a Lévy white noise in $(\mathcal{S}'_\omega, C(\mathcal{S}'_\omega))$ with characteristic triplet (a,γ,ν).
That $\mathcal{P}(0) = 1$ is trivial, so we start with the continuity. Therefore, let $\rho(x) := \exp(-\eta\omega(x))$ with $\eta > 0$. We see that

$$\begin{aligned} d_\rho(\alpha) :&= \lambda^d(\{x \in \mathbb{R}^d : \rho(x) > \alpha\}) \\ &= \lambda^d(\{x \in \mathbb{R}^d : \omega(x) < \log(\alpha^{-1})/\eta\}) \\ &= c_d\omega^{\rightarrow}(\log(|\alpha|^{-1})/\eta)^d \end{aligned} \tag{5.10}$$

for some constant $c_d > 0$. Now let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{S}_\omega^{real}$ such that $\varphi_n \to 0$ in $\mathcal{S}_\omega$, which implies that $\sup_{x\in\mathbb{R}^d} e^{\eta\omega(x)}|\varphi_n(x)| \to 0$ for $n \to \infty$ for all $\eta > 0$ or equivalently $|\varphi_n(x)| \leq c_n e^{-\eta\omega(x)}$ for all $x \in \mathbb{R}^d$ for a sequence $(c_n)_{n\in\mathbb{N}} \subset [0,\infty)$ converging to 0 for $n \to \infty$. We conclude by (5.10) and Lebesgue's dominated convergence theorem that

$$\int\limits_{|r|>1} |r| \int\limits_{0}^{1/|r|} d_{\varphi_n}(\alpha)\lambda^1(d\alpha)\nu(dr) \to 0,\ n \to \infty.$$

Now by similar arguments as Theorem 4.5 we see that $\mathcal{P}$ is continuous on $\mathcal{S}_\omega^{real}$. That $\mathcal{P}$ is positive definite follows by a denseness argument similar to [30, Proposition 2, p. 1187]. □

We give two examples and obtain for a special weight ω the space $\mathcal{S}(\mathbb{R}^d)$.

Example 5.8. Let $\omega(x) = m\log(1 + \|x\|)$ for $m > 0$. It is well-known that $\mathcal{S}_\omega = \mathcal{S}$, see [63, Remark 4, p. 246], and we see that

$$\omega^{\rightarrow}(\alpha) = e^{\alpha/m} - 1$$

for all $\alpha > 0$ and we obtain for $c \in (0, m)$ and $\alpha \in (0, 1)$

$$\omega^{\rightarrow}(c \log(|\alpha|^{-1})) = \omega^{\rightarrow}(\log(|\alpha^{-c}|)) \leq \alpha^{-c/m}.$$

As $c \in (0, 1)$ is arbitrary, we conclude from Theorem 5.7 that if $\int_{|r|>1} |r|^{\varepsilon} \nu(dr) < \infty$ for some $\varepsilon > 0$ there exists a Lévy white noise with characteristic triplet (a, γ, ν) on $\mathcal{S}'_{\omega} = \mathcal{S}'$, thus recovering the sufficient condition of [21, Theorem 3.13, p. 4412].

Example 5.9. Let $\omega(x) = \|x\|^{\beta}$ for $0 < \beta < 1$. It is easily seen that ω satisfies the assumptions of Definition 5.6 and furthermore,

$$\omega^{\rightarrow}(\alpha) = \alpha^{1/\beta}$$

for $\alpha \in (0, \infty)$. We conclude from Theorem 5.7 that a Lévy white noise with characteristic triplet (a, γ, ν) exists on $\mathcal{S}'_{\omega}$ if

$$\int_{|r|>1} (\log(|r|))^{d/\beta} \nu(dr).$$

In the next step we will analyze equation (5.6) in the space of tempered ultradistributions. We will obtain similar results as in Proposition 5.5.

Theorem 5.10. *Let p, q be two real polynomials and assume that the rational function $q(i\cdot)/p(i\cdot)$ has a holomorphic extension in a strip $\{z \in \mathbb{C}^d : \|\Im z\| < \varepsilon\}$ for some $\varepsilon > 0$. Furthermore, let ω be as in Definition 5.6 and $\dot{L}$ be a Lévy white noise on the space of tempered ultradistribution $\mathcal{S}'_{\omega}$ under the conditions of Theorem 5.7. Then there exists a generalized stationary process s in the space of tempered ultradistributions $\mathcal{S}'_{\omega}$ such that*

$$p(D)s = q(D)\dot{L}.$$

Moreover, if $p(i\cdot)$ has no zeroes in the strip, then the solution is unique.

Proof. Observe that for every $c > 0$ there exists an $n \in \mathbb{N}$ such that $\omega(x) \leq n + c\|x\|$ for all $x \in \mathbb{R}^d$, otherwise, the assumption of Definition 5.6 can not hold true. Now choose $\alpha \in \mathbb{N}$ such that

$$\sup_{\|\eta\| \leq \delta} \left\| \frac{q(-i \cdot + \eta)}{p(-i \cdot + \eta)\psi(\cdot + i\eta)} \right\|_{L^1(\mathbb{R}^d)} < \infty$$

for some $0 < \delta < \varepsilon$, where $\psi(z) := (1 + \sum\limits_{j=1}^{d} z_j^2)^{\alpha}$. It is not immediately clear that such an $\alpha \in \mathbb{N}$ exists, but by a similar argument as in the proof of [40, Lemma 2, p. 557] we conclude that such an α exists. We define $G := \mathcal{F}^{-1} \frac{q(-i\cdot)}{p(-i\cdot)\psi(\cdot)}$ and we observe that there exists a constant $C > 0$ such that $|G(x)| \leq C \exp\left(-\frac{\delta}{2}\|x\|\right)$ for all $x \in \mathbb{R}^d$, see [57, Theorem IX.14, p. 18]. One infers by the subadditivity of ω (see [63, Remark 2, p. 246]) for $\varphi \in \mathcal{S}_\omega$ that

$$\begin{aligned}|G*\varphi(x)| &\le \int\limits_{\mathbb{R}^d} |G(y)\varphi(x-y)|\lambda^d(dy)\\ &\le p_{0,\eta}(\varphi)\int\limits_{\mathbb{R}^d} |G(y)e^{-\eta\omega(y-x)}|\lambda^d(dy) \le p_{0,\eta}(\varphi)e^{-\eta\omega(x)}\int\limits_{\mathbb{R}^d} |G(y)e^{\eta\omega(y)}|\lambda^d(dy)\end{aligned} \tag{5.11}$$

for every $\eta > 0$. We conclude that $|G*\varphi(x)| \le Ce^{-\eta\omega(x)}$ for some constant $C > 0$ and as $D^\alpha\varphi \in \mathcal{S}_\omega$, we conclude that

$$\sup_{x\in\mathbb{R}^d} e^{\eta\omega(x)}\|D^\alpha G*\varphi(x)\| = \sup_{x\in\mathbb{R}^d} e^{\eta\omega(x)}\|G*D^\alpha\varphi(x)\| < \infty$$

for every $\eta > 0$. Moreover, for the Fourier-transform of $G*\varphi$ it is easy to see that

$$\sup_{z\in\mathbb{R}^d} e^{\eta\omega(z)}\|D^\alpha\mathcal{F}(G*\varphi)(z)\| = \sup_{z\in\mathbb{R}^d} e^{\eta\omega(z)}\|\sum_{|\beta|\le|\alpha|} p_\beta(z)D^\beta\mathcal{F}\varphi(z)\| < \infty, \tag{5.12}$$

which follows by [40, Lemma 2, p. 557] for every $\eta > 0$, where p_β are rational functions well-defined on $\{z \in \mathbb{C}^d : \|\Im z\| < \varepsilon\}$ for every $|\beta| \le |\alpha|$. So by similar estimates as in (5.11) and (5.12) one sees that

$$\varphi \mapsto \mathcal{F}^{-1}\frac{q(-i\cdot)}{p(-i\cdot)\psi(\cdot)}\mathcal{F}\psi(\cdot)\varphi \tag{5.13}$$

defines a continuous operator from $\mathcal{S}_\omega$ to $\mathcal{S}_\omega$. Hence

$$\langle s,\varphi\rangle = \langle \dot{L}, \mathcal{F}^{-1}\frac{q(-i\cdot)}{p(-i\cdot)\psi(\cdot)}\mathcal{F}\psi(\cdot)\varphi\rangle,$$

defines a generalized random process $s : (\Omega,\mathcal{F}) \to (\mathcal{S}'_\omega, C(\mathcal{S}'_\omega))$. That is solves (5.6) follows as in Theorem 4.16.
The uniqueness of the solution s follows by the proof of [44, Proposition 2.2]. □

We have shown so far that for every Lévy white noise $\dot{L}$ with characteristic triplet (a,γ,ν) living in S'_ω for every ω satisfying the assumptions of Definition 5.6 there exists a unique solution s of the equation

$$p(D)s = q(D)\dot{L},$$

if $p(i\xi)$ has no zeroes in a strip around $\mathbb{R}^d$. In Theorem 4.5 we have seen that we obtain a solution s in the space of distributions $\mathcal{D}'$ in the case that

$$\int_{|r|>1} \log(|r|)^d\nu(dr) < \infty, \tag{5.14}$$

but it seems difficult to find such ω such that $\dot{L}$ would be living in S'_ω if the Lévy white noise satisfies (5.14), as (5.14) does not imply the condition of Theorem 5.7 for any suitable weight functions ω. Therefore, we use another more suitable space for the analysis of (5.6) under the assumption of (5.14), the space of analytic functions with rapid decay and its topological dual, the Fourier hyperspace.

Definition 5.11. The space $\mathcal{P}_*$ consists of all functions $\varphi \in C^\infty(\mathbb{R}^d, \mathbb{C})$ which have an analytic continuation on a strip

$$A_\delta := \{z \in \mathbb{C}^d : \|\Im z\| < \delta\}$$

for some $\delta > 0$ and it holds that

$$\sup_{z \in A_l} |\exp((\delta - \varepsilon)\|z\|)\varphi(z)| < \infty \tag{5.15}$$

for every $0, \varepsilon, l < \delta$. The space $\mathcal{P}_*$ is nuclear with its inductive topology, i.e. a sequence $(\varphi_n)_{n\in\mathbb{N}} \subset \mathcal{P}_*$ converges to 0 if and only if there exists a $\delta > 0$ such that φ_n has an analytic continuation in A_δ for every $n \in \mathbb{N}$ and

$$\sup_{z \in A_{\delta/2}} |\exp(\delta/2\|z\|)\varphi_n(z)| \to 0 \text{ for } n \to \infty,$$

see [43, p. 408]. We denote by $\mathcal{Q}$ its topological dual and call it the *space of Fourier hyperfunctions*.

We show first that there exists a Lévy white noise on $\dot{L}$ with characteristic triplet (a, γ, ν) on $\mathcal{Q}$ if (5.14) holds. Observe that we split a function $\varphi \in C^\infty(\mathbb{R}^d, \mathbb{C})$ in its real and imaginary part and prove on each part separately the existence of the Lévy white noise. Then $\langle \dot{L}, \varphi\rangle := \langle \dot{L}, \Re\varphi\rangle + i\langle \dot{L}, \Im\varphi\rangle$.

Proposition 5.12. *Let (a, γ, ν) be a characteristic triplet such that (5.14) holds true. Then there exists a Lévy white noise on $(\mathcal{Q}, C(\mathcal{Q}))$.*

Proof. The proof is very similar to that of Theorem 5.7. At first we observe for $\rho(x) := D\exp(-\delta\|x\|)$ for some $D, \delta > 0$ that

$$\begin{aligned}
d_\rho(\alpha) &= \lambda^d(\{x \in \mathbb{R}^d : \rho(x) > \alpha\}) \\
&= \lambda^d(\{x \in \mathbb{R}^d : \|x\| < -\frac{1}{\delta}\log(\alpha/D)\}) \\
&= -C\log(\alpha/D)^d
\end{aligned}$$

for some constant $C > 0$ for all $\alpha < D$. So for every sequence $(\varphi_n)_{n\in\mathbb{N}} \subset \mathcal{P}_*$ convgerging to 0 we obtain similar to Example 4.6 that

$$\int_{|r|>1} |r| \int_0^{1/|r|} d_{\varphi_n}(\alpha)\lambda^1(d\alpha)\nu(dr) \to 0,\ n \to \infty.$$

By following the same argumentation as in the proof of Theorem 5.7 one infers that there exists a Lévy white noise $\dot{L}$ on $\mathcal{Q}$. $\square$

As a final step we prove the unique solvability of equation (5.6) in $\mathcal{Q}$.

Theorem 5.13. *Let $\dot{L}$ be a Lévy white noise on $\mathcal{Q}$. Assume that p, q be two real polynomials such that the rational function $q(i\cdot)/p(i\cdot)$ has a holomorphic extension in a strip $\{z \in \mathbb{C}^d : \|\Im z\| < \varepsilon\}$ for some $\varepsilon > 0$. Then there exists a generalized stationary process s in $\mathcal{Q}$ such that*

$$p(D)s = q(D)\dot{L}.$$

Moreover, if p has no zeroes in the strip, than the solution is unique.

Proof. The uniqueness when p has no zeroes on the strip follows in the same manner as in Proposition 5.5 by the proof of [44, Proposition 2.2].
For the existence of the stationary solution s, let G, ψ and α be as in the proof of Theorem 5.10. Since $(\mathcal{F}^{-1}\psi(\cdot)\mathcal{F}\varphi) = (1-\Delta)^{\alpha}\varphi$, it follows similarity to the proof of Theorem 4.16 that it is sufficient to show that

$$T : \mathcal{P}_* \to \mathcal{P}_*,\ \varphi \mapsto G * (1-\Delta)^{\alpha}\varphi$$

is continuous. To see this let $(\varphi_n)_{n\in\mathbb{N}} \subset \mathcal{P}_*$ be converging to 0, i.e. there exists a $\delta > 0$ such that φ_n has an analytic continuation in A_δ for every $n \in \mathbb{N}$ and $\sup_{z\in A_\delta} |\exp(\delta\|z\|)\varphi_n(z)| \to 0$ for $n \to \infty$. Then it holds by Cauchy's integral formula for derivatives that $(1-\Delta)^{\alpha}\varphi_n \in \mathcal{P}_*$ for every $n \in \mathbb{N}$ and $(1-\Delta)^{\alpha}\varphi_n \to 0$ for $n \to \infty$ in $\mathcal{P}_*$. So it is sufficient to show that $\tilde{T} : \mathcal{P}_* \to \mathcal{P}_*$ defined by $\tilde{T}(\varphi) := G * \varphi$ is continuous. This follows easily by the same method as in the proof of Theorem 5.10. Therefore we obtain a mapping $s : \Omega \to \mathcal{Q}$ defined by $s(\varphi) := \langle \dot{L}, G * (1-\Delta)^{\alpha}\varphi\rangle$ for every $\varphi \in \mathcal{P}_*$, which solves (5.6) and is stationary. $\square$

As a Lévy white noise $\dot{L}$ on $\mathcal{S}'(\mathbb{R}^d)$ with characteristic triplet (a, γ, ν) such that $\int_{|r|>1} |r|^{\varepsilon}\nu(dr) < \infty$ lives in $\mathcal{S}'(\mathbb{R}^d)$, it is only natural to ask if the Lévy white noise can be constructed on certain negative Sobolev spaces with some weights. In [31] it was shown that $P(\dot{L} \in W_2^{\tau}(\mathbb{R}^d, \rho)) = 1$ for $\tau < -d/2$ and $\rho < -d/\min\{\varepsilon, 2\}$. Indeed, there exists even a generalized process on $(W_2^{\tau}(\mathbb{R}^d, \rho), \mathcal{B}^*(W_2^{\tau}(\mathbb{R}^d, \rho)))$ which follows by [42, Theorem 1.2.4, p.6]. Moreover, $\dot{L}$ can be seen as a random variable on the space $(W_2^{\tau}(\mathbb{R}^d, \rho), \mathcal{B}(W_2^{\tau}(\mathbb{R}^d, \rho)))$, which is just the Borel $\sigma-$field generated by the strong topology on $W_2^{\tau}(\mathbb{R}^d, \rho)$, see [42, p. 6]. In this case, the solution s of (5.6) can be identified with a random variable on a weighted Sobolev space, or more generally weighted Besov spaces, too:

Lemma 5.14. *Let p, q be polynomials in d variables such that there exists $\kappa \in (0, \infty)$ such that*

$$\left| D^\gamma \frac{q(i\xi)}{p(i\xi)} \right| \leq c_\gamma \langle \xi \rangle^{-\kappa - |\gamma|} \tag{5.16}$$

for every $\gamma \in \mathbb{N}_0^d$, where $c_\gamma \geq 0$. Let $\dot{L}$ be a Lévy white noise on $\mathcal{S}'$ with characteristic triplet (a, γ, ν) such that $\int_{|r|>1} |r|^\varepsilon \nu(dr) < \infty$ for some $\varepsilon > 0$. Let $\rho < -d/\min\{2, \varepsilon\}$, $l < -\frac{d}{2}$ and choose a version of $\dot{L}$ in the Sobolev space $W_2^l(\mathbb{R}^d, \rho)$ as described above. Then there exists a solution s of (5.6) in $\mathcal{S}'$ which almost surely lies in $B_{r,r}^{\tau+\kappa}(\mathbb{R}^d, \rho)$ whenever $r \in [2, \infty]$ and $\tau \leq l + d\left(\frac{1}{r} - \frac{1}{2}\right)$, and even is a random variable in $(B_{r,r}^{\tau+\kappa}(\mathbb{R}^d, \rho), \mathcal{B}(B_{r,r}^{\tau+\kappa}(\mathbb{R}^d, \rho)))$.

Proof. By [28, Theorem 5.4.2, p. 224] we conclude that $\varphi \mapsto \mathcal{F}^{-1} \frac{q(i\cdot)}{p(i\cdot)} \mathcal{F} \varphi$ defines a continuous operator both from $\mathcal{S}'$ to $\mathcal{S}'$ and from $W_2^l(\mathbb{R}^d, \rho)$ to $W_2^{l+\kappa}(\mathbb{R}^d, \rho)$. We conclude by construction of s in (5.8) that we have a solution in $\mathcal{S}'$ which is also in $(W_2^{l+\kappa}(\mathbb{R}^d, \rho), \mathcal{B}(W_2^{l+\kappa}(\mathbb{R}^d, \rho)))$. The rest follows easily by Proposition 5.1. □

Observe that if $\kappa > d\left(1 - \frac{1}{r}\right)$, with $r \geq 2$ we can choose l and τ from above such that $\tau + \kappa > 0$. In this case, s has positive regularity and can be identified with a random field on $\mathbb{R}^d$ via the mapping of (5.5).

Example 5.15. Let $p(D) = (\lambda - \Delta)^\alpha$ for $\alpha \in \mathbb{N}$ and $\lambda > 0$ and $q(D) = 1$. Then (5.16) is satisfied for $\kappa = 2\alpha$, see [37, Example 6.2.9, p. 449].

5.4 Semilinear stochastic partial differential equations

Our goal of this section is to study the semilinear stochastic partial differential equation

$$p(D)s = g(\cdot, s) + \dot{L}, \tag{5.17}$$

where $\dot{L}$ is a Lévy white noise on $\mathcal{S}'$ with characteristic triplet (a, γ, ν) such that $\int_{|r|>1} |r|^\varepsilon \nu(dr) < \infty$ for some $\varepsilon > 0$, p is a polynomial in d variables and $g : \mathbb{R}^d \times \mathbb{C} \to \mathbb{R}$ a sufficiently nice function. We assume that the Lévy white noise is the modified version on the measurable space $(B_{2,2}^l(\mathbb{R}^d, \rho), \mathcal{B}(B_{2,2}^l(\mathbb{R}^d, \rho)))$, where $l < -d/2$ and $\rho < -\frac{d}{\min\{2,\varepsilon\}}$. We are looking for a $B_{r,r}^\beta(\mathbb{R}^d, \rho) \subset \mathcal{S}'$-valued solution s, where $r \geq 2$ and $\beta > 0$. Observe that since $r \geq 2$ and $\beta > 0$ every $f \in B_{r,r}^\beta(\mathbb{R}^d, \rho)$ can be identified with a function $\xi(f) \in L^r(\mathbb{R}^d, \rho)$ in a continuous way via (5.5). We again denote by f the function $\xi(f)$. Then $g(\cdot, s)$ means the function

$$g(\cdot, s) : \mathbb{R}^d \to \mathbb{R},\ x \mapsto g(x, s(x)),$$

which in turn can again be identified with a distribution via

$$\langle g(\cdot,s),\varphi\rangle := \int\limits_{\mathbb{R}^d} g(x,s(x))\varphi(x)\,\lambda^d(dx).$$

By a $\mathcal{B}^\beta_{r,r}(\mathbb{R}^d,\rho)\subset\mathcal{S}'$-valued solution of (5.17) we mean a measurable mapping

$$s:(\Omega,\mathcal{F})\to(B^\beta_{r,r}(\mathbb{R}^d,\rho),\mathcal{B}(B^\beta_{r,r}(\mathbb{R}^d,\rho)))$$

such that

$$\langle s,p(D)^*\varphi\rangle=\int_{\mathbb{R}^d} g(x,s(x))\varphi(x)\lambda^d(dx)+\langle\dot{L},\varphi\rangle$$

for every $\varphi\in\mathcal{S}$.

Proposition 5.16. *Let $r\in[2,\infty]$, $\rho<-\frac{d}{\min\{2,\varepsilon\}}$, $\kappa>d(1-1/r)+\beta$ for some $\beta>0$ and $p(D)$ be a partial differential operator satisfying*

$$\left|D^\gamma\frac{1}{p(i\xi)}\right|\le c_\gamma\langle\xi\rangle^{-\kappa-|\gamma|} \tag{5.18}$$

for every $\gamma\in\mathbb{N}_0^d$, where $c_\gamma\ge 0$. Furthermore, let $g:\mathbb{R}^d\times\mathbb{C}\to\mathbb{R}$ be a Lipschitz function such that

$$|g(x,y)|\le C(1+|y|)$$

for some constant $C>0$ for all $x\in\mathbb{R}^d$ and $y\in\mathbb{C}$ and assume that

$$\begin{aligned}\|g\|_{Lip}:=&\sup_{x\in\mathbb{R}^d}\sup_{z,y\in\mathbb{C}}\frac{|g(x,y)-g(x,z)|}{|y-z|}\\ <&(\|p(D)^{-1}\|_{L^r(\mathbb{R}^d,\rho)\to B^\beta_{r,r}(\mathbb{R}^d,\rho)}\|id\|_{B^\beta_{r,r}(\mathbb{R}^d,\rho)\to L^r(\mathbb{R}^d,\rho)})^{-1}<\infty.\end{aligned} \tag{5.19}$$

Let $\dot{L}$ be a Lévy white noise on $\mathcal{S}'$ with characteristic triplet (a,γ,ν) such that $\int_{|r|>1}|r|^\varepsilon\nu(dr)<\infty$. Let $l=\beta-\kappa+d\left(\frac{1}{2}-\frac{1}{r}\right)<-\frac{d}{2}$ and choose a version of $\dot{L}$ in the Sobolev space $B^l_{2,2}(\mathbb{R}^d,\rho)$ as described above.
Then there exists a unique measurable mapping $s:(\Omega,\mathcal{F})\to(B^\beta_{r,r}(\mathbb{R}^d,\rho),\mathcal{B}(B^\beta_{r,r}(\mathbb{R}^d,\rho)))$, which solves the equation (5.17). Especially, it holds that $s\in L^r(\Omega,B^\beta_{r,r}(\mathbb{R}^d,\rho))$ if $\varepsilon>r\ge 2$.

Remark 5.17. Observe that

$$\begin{aligned}&p(D)^{-1}:B^{\beta-\kappa}_{r,r}(\mathbb{R}^d,\rho)\to B^\beta_{r,r}(\mathbb{R}^d,\rho)\text{ and}\\ &p(D)^{-1}:L^r(\mathbb{R}^d,\rho)\to B^\beta_{r,r}(\mathbb{R}^d,\rho)\end{aligned}$$

are well-defined and continuous linear operators by assumption (5.18) and that $B^l_{r,r}(\mathbb{R}^d,\rho)$ is continuously embedded into $B^j_{r,r}(\mathbb{R}^d,\rho)$ for every $l > j$ and $L^r(\mathbb{R}^d,\rho)$ is continuously embedded into $B^a_{r,r}(\mathbb{R}^d,\rho)$ for every $a < 0$, see [28, Theorem 5.4.2, p. 224].

Proof. We set u to be the unique solution of

$$p(D)u = \dot{L}. \tag{5.20}$$

From Lemma 5.14 we see that u has a measurable version from $(\Omega,\mathcal{F})$ to $(B^\beta_{r,r}(\mathbb{R}^d,\rho),\mathcal{B}(B^\beta_{r,r}(\mathbb{R}^d,\rho)))$. We see that in order to solve (5.17) we need to solve the equation

$$p(D)v = g(\cdot, u+v) \tag{5.21}$$

with $v \in B^\beta_{r,r}(\mathbb{R}^d,\rho)$, where u is defined in (5.20), as $s := u+v$ solves (5.17). Since $u \in B^\beta_{r,r}(\mathbb{R}^d,\rho)$, we have $g(\cdot,u+v) \in L^r(\mathbb{R}^d,\rho)$, as

$$\begin{aligned}\int\limits_{\mathbb{R}^d} \langle x\rangle^{r\rho}|g(x,u(x)+v(x))|^r\lambda^d(dx) &\le \int\limits_{\mathbb{R}^d}\langle x\rangle^{r\rho}C(1+|u(x)|+|v(x)|)^r\lambda^d(dx)\\ &\le C'(1+\|u\|^r_{L^r(\mathbb{R}^d,\rho)}+\|v\|^r_{L^r(\mathbb{R}^d,\rho)})\end{aligned}$$

for some suitable constants C and $C' > 0$. Moreover, we see from Remark 5.17 that $p(D)^{-1}g(\cdot,u+v) \in B^\beta_{r,r}(\mathbb{R}^d,\rho)$. Therefore let $\tilde{u} \in B^\beta_{r,r}(\mathbb{R}^d,\rho)$ and we define $\Psi_{\tilde{u}} = \Psi : B^\beta_{r,r}(\mathbb{R}^d,\rho) \to B^\beta_{r,r}(\mathbb{R}^d,\rho)$ by

$$\Psi(\varphi) = p(D)^{-1}g(\cdot,\tilde{u}+\varphi)$$

for all $\varphi \in B^\beta_{r,r}(\mathbb{R}^d,\rho)$. We show that there exists a fixed point of Ψ, which is especially the solution of (5.21) for the fixed $\tilde{u} \in B^\beta_{r,r}(\mathbb{R}^d,\rho)$. We see that

$$\begin{aligned}\|\Psi(\varphi_1)-\Psi(\varphi_2)\|_{B^\beta_{r,r}(\mathbb{R}^d,\rho)} &\le \|p(D)^{-1}\|_{L^r(\mathbb{R}^d,\rho)\to B^\beta_{r,r}(\mathbb{R}^d,\rho)}\|g(\cdot,\tilde{u}+\varphi_1)-g(\cdot,\tilde{u}+\varphi_2)\|_{L^r(\mathbb{R}^d,\rho)}\\ &\le \|p(D)^{-1}\|_{L^r(\mathbb{R}^d,\rho)\to B^\beta_{r,r}(\mathbb{R}^d,\rho)}\|g\|_{Lip}\|id\|_{B^\beta_{r,r}(\mathbb{R}^d,\rho)\to L^r(\mathbb{R}^d,\rho)}\|\varphi_1-\varphi_2\|_{B^\beta_{r,r}(\mathbb{R}^d,\rho)}.\end{aligned}$$

It follows that Ψ is a strict contraction and by Banach's fixed point theorem we conclude that for every $\tilde{u} \in B^\beta_{r,r}(\mathbb{R}^d,\rho)$ there exists a unique solution $v \in B^\beta_{r,r}(\mathbb{R}^d,\rho)$ of

$$p(D)\tilde{v} = g(\cdot,\tilde{u}+\tilde{v}).$$

By a small calculation we see that $\tilde{v}$ depends continuously on $\tilde{u}$. Namely let u_1 and u_2 be in $B^\beta_{r,r}(\mathbb{R}^d,\rho)$ and let v_1 and v_2 be the corresponding fixed points. We see that

$$\begin{aligned}&\|v_1-v_2\|_{B^\beta_{r,r}(\mathbb{R}^d,\rho)}\\ \le&\|p(D)^{-1}\|_{L^r(\mathbb{R}^d,\rho)\to B^\beta_{r,r}(\mathbb{R}^d,\rho)}\|g(\cdot,u_1+v_1)-g(\cdot,u_2+v_2)\|_{L^r(\mathbb{R}^d,\rho)}\\ \le&\|p(D)^{-1}\|_{L^r(\mathbb{R}^d,\rho)\to B^\beta_{r,r}(\mathbb{R}^d,\rho)}\|g\|_{Lip}\|id\|_{B^\beta_{r,r}(\mathbb{R}^d,\rho)\to L^r(\mathbb{R}^d,\rho)}(\|u_1-u_2\|_{B^\beta_{r,r}(\mathbb{R}^d,\rho)}+\|v_1-v_2\|_{B^\beta_{r,r}(\mathbb{R}^d,\rho)}),\end{aligned}$$

which implies that

$$\|v_1 - v_2\|_{B^\beta_{r,r}(\mathbb{R}^d,\rho)} \leq \frac{\|p(D)^{-1}\|_{L^r(\mathbb{R}^d,\rho)\to B^\beta_{r,r}(\mathbb{R}^d,\rho)}\|g\|_{Lip}\|id\|_{B^\beta_{r,r}(\mathbb{R}^d,\rho)\to L^r(\mathbb{R}^d,\rho)}}{1-\|p(D)^{-1}\|_{L^r(\mathbb{R}^d,\rho)\to B^\beta_{r,r}(\mathbb{R}^d,\rho)}\|g\|_{Lip}\|id\|_{B^\beta_{r,r}(\mathbb{R}^d,\rho)\to L^r(\mathbb{R}^d,\rho)}}\|u_1 - u_2\|_{B^\beta_{r,r}(\mathbb{R}^d,\rho)}.$$

We conclude that there exists a measurable solution v of (5.21) in the space $(B^\beta_{r,r}(\mathbb{R}^d,\rho), \mathcal{B}(B^\beta_{r,r}(\mathbb{R}^d,\rho)))$. As the solution s of (5.17) is then given by $s = u + v$, we conclude that we find a unique measurable solution of (5.17).
Now let $\varepsilon > r \geq 2$. We compute as above that

$$\begin{aligned}\|s\|_{B^\beta_{r,r}(\mathbb{R}^d,\rho)} =&\|p(D)^{-1}p(D)(u+v)\|_{B^\beta_{r,r}(\mathbb{R}^d,\rho)}\\ \leq&\|p(D)^{-1}\|_{L^r(\mathbb{R}^d,\rho)\to B^\beta_{r,r}(\mathbb{R}^d,\rho)}\|g(\cdot,s)\|_{L^r(\mathbb{R}^d,\rho)} + C\|p(D)u\|_{B^{\beta-\kappa}_{r,r}(\mathbb{R}^d,\rho)}\\ \leq&\|p(D)^{-1}\|_{L^r(\mathbb{R}^d,\rho)\to B^\beta_{r,r}(\mathbb{R}^d,\rho)}\|g(\cdot,s)\|_{L^r(\mathbb{R}^d,\rho)} + C\|\dot{L}\|_{B^{\beta-\kappa}_{r,r}(\mathbb{R}^d,\rho)}\end{aligned}$$

for some constant C, from which we conclude that

$$\|s\|_{B^\beta_{r,r}(\mathbb{R}^d,\rho)} \leq C'(1 + \|\dot{L}\|_{B^{\beta-\kappa}_{r,r}(\mathbb{R}^d,\rho)})$$

for some constant C' and by [3, Proposition 5] we infer that $\mathbb{E}\|s\|^r_{B^\beta_{r,r}(\mathbb{R}^d,\rho)} < \infty$. □

Example 5.18. Let $d = 1, 2$ or 3, $\lambda > 0$ and $\dot{L}$ be a Lévy white noise in $\mathcal{S}'$ with $\int_{|x|>1} |x|^\varepsilon \nu(dx) < \infty$ for some $\varepsilon > 0$. Let $\rho < -\frac{d}{\min 2,\varepsilon}$ and choose a modification of $\dot{L}$ which is $(W^\alpha_2(\mathbb{R}^d,\rho), \mathcal{B}(W^\alpha_2(\mathbb{R}^d,\rho))$ measurable with $\alpha \in (-2, -3/2)$. Then for every $\beta \in (0, 2+\alpha]$ there exists some $\varepsilon_\beta > 0$ such that

$$(\lambda - \Delta)s + c\sin(s) = \dot{L}$$

has a unique and measurable solution in $W^\beta_2(\mathbb{R}^d,\rho)$ for all $c \in (0, \varepsilon_\beta)$.

Proof. By [37, Example 6.2.9, p. 449] we know that $(\lambda - \Delta)$ satisfies (5.18) for $\kappa = 2$ and u is even a real-valued distribution. Moreover, we see that sin is Lipschitz-continuous with Lipschitz constant equal to 1 and by Proposition 5.16 we conclude that there exists a unique pathwise solution. □

6 Central Limit Theorems for Moving Average Random Fields with Non-Random and Random Sampling On Lattices

For a Lévy basis L on $\mathbb{R}^d$ and a suitable kernel function $f : \mathbb{R}^d \to \mathbb{R}$, consider the continuous spatial moving average field $X = (X_t)_{t\in\mathbb{R}^d}$ defined by $X_t = \int_{\mathbb{R}^d} f(t-s)\, dL(s)$. Based on observations on finite subsets Γ_n of $\mathbb{Z}^d$, we obtain central limit theorems for the sample mean and the sample autocovariance function of this process. We allow sequences (Γ_n) of deterministic subsets of $\mathbb{Z}^d$ and of random subsets of $\mathbb{Z}^d$. The results generalise existing results for time indexed stochastic processes (i.e. $d = 1$) to random fields with arbitrary spatial dimension d, and additionally allow for random sampling. The results are applied to obtain a consistent and asymptotically normal estimator of $\mu > 0$ in the stochastic partial differential equation $(\mu - \Delta)X = dL$ in dimension 3, where L is Lévy noise.

6.1 Introduction

Many statistical models with more than one spatial dimension are described by a linear stochastic partial differential equation with some additive noise, which means that we have a random field X on $\mathbb{R}^d$ satisfying

$$\mathcal{L}(\mu)X = dL, \tag{6.1}$$

where $\mathcal{L}(\mu)$ is a linear partial differential operator depending on some parameter μ and dL denotes some noise, for example Gaussian or stable noise. If $\mathcal{L}(\mu)$ has an integrable fundamental solution G_μ the mild solution of (6.1) can be written as

$$X_t = \int_{\mathbb{R}^d} G_\mu(t-s)dL(s), \tag{6.2}$$

where dL denotes the additive noise, see for example [5], [55], [65] and Chapters 4 and 5. The solution (6.2) is a so called continuous moving average random field.

The additive noise dL studied in this chapter will be a Lévy white noise, where the Gaussian white noise and stable noise are included. A detailed study of Lévy white noise can be found in [32], where it is also shown that a Lévy white noise defines a Lévy basis in the sense of Rajput and Rosinski [56]. Random fields of the form

$$X_t = \int_{\mathbb{R}^d} f(t-s)\, dL(s), \tag{6.3}$$

with a suitable kernel function $f : \mathbb{R}^d \to \mathbb{R}$ and a Lévy basis L on $\mathbb{R}^d$ (as in (6.2) with $f = G_\mu$) can be seen as a continuous and spatial extension of the discrete time moving average processes $Z = (Z_t)_{t\in\mathbb{Z}}$, defined by

$$Z_t = \sum_{k\in\mathbb{Z}} a_{t-k} W_k, \tag{6.4}$$

where $(W_k)_{k\in\mathbb{Z}}$ is an independent and identically distributed sequence and a_k, $k \in \mathbb{Z}$, are real coefficients.

In many cases one is interested in estimating the parameter μ of the equation (6.1). If we know how the fundamental solution G_μ depends on the parameter μ, it is sometimes possible to give moment estimators for μ. Of particular interest are estimators of the mean $\mathbb{E}(X_t)$ and the autocovariance $\operatorname{cov}(X_t, X_{t+h})$ for $t, h \in \mathbb{R}^d$. In most applications only discrete spatial data is available, for example observations based on a finite subset Γ_n of the lattice $\mathbb{Z}^d$. A natural estimator for $\mathbb{E}X_t$ is then the sample mean $\frac{1}{|\Gamma_n|}\sum_{s\in\Gamma_n} X_s$, while a natural estimator for the autocovariance $\operatorname{cov}(X_t, X_{t+h})$ is the (adjusted) sample autocovariance

$$\gamma_n^*(h) := \frac{1}{|\Gamma_n|} \sum_{s\in\Gamma_n} X_s X_{s+h}, \quad h \in \mathbb{Z}^d \tag{6.5}$$

(assuming that the Lévy basis and hence X have mean zero and that for each $s \in \Gamma_n$, both X_s and X_{s+h} are observed). Motivated by this, in this chapter we will provide central limit theorems for the sample mean and sample autocovariance function as defined in (6.5) for continuous spatial moving average random fields as defined in (6.3) (equivalently, (6.2)), when the kernel function f decays sufficiently fast and the Lévy basis has finite variance or finite fourth moment and mean zero, respectively. The sampling sequence $(\Gamma_n)_{n\in\mathbb{N}}$ will be a nested sequence of finite subsets of $\mathbb{Z}^d$ satisfying $|\Gamma_n| \to \infty$ and some extra conditions, and it will be either a sequence of deterministic subsets (referred to as non-random sampling) or a sequence of random subsets (referred to as random-sampling), more precisely of the form $\Gamma_n = \{t \in [-n, n)^d \cap \mathbb{Z}^d | Y_t = 1\}$, where $(Y_t)_{t\in\mathbb{Z}^d}$ is a $\{0, 1\}-$valued stationary ergodic random field on $\mathbb{Z}^d$. In the case of non-random sampling, we will need slightly higher moment conditions on the Lévy basis.

Central limit theorems for the sample mean and the sample autocovariance of (6.4) are classic and can be found e.g. in Chapter 7 of the book [12] (for $d = 1$). On the other hand, central limit theorems for Lévy driven moving average processes based on discrete low-frequency observations have only recently attracted attention, and this also only in dimension $d = 1$, i.e. for continuous time series and not spatial data. In [18], the asymptotics of the sample mean and sample autocovariance are studied when f decays sufficiently fast and L has finite second or fourth moment, respectively. [59] studies the situation when f decays slowly leading to a long-memory process X, while [27] considers the heavy tailed situation when the Lévy process L is in the domain of attraction of a stable non-normal distribution, and in [9] the case of random sampling when the process X is sampled at a renewal sequence is treated. Observe that all these results are in dimension $d = 1$ only. The results of this chapter can be seen as a generalization of the results of [18], who have $d = 1$ and $\Gamma_n = \{1, 2, \dots, n\}$, to arbitrary spatial dimensions $d \in \mathbb{N}$ and more general sets Γ_n, and additionally allowing random sampling as described above.

The chapter is organized as follows. In the next section, we fix notation and recall the notion of Lévy bases. Then, in Section 6.3, we state the main results of the present chapter. These are central limit theorems for the sample mean as described above for non-random and random sampling (Theorems 6.2 and 6.7, respectively), and central limit theorems for the sample autocovariance as described above for non-random and random sampling (Theorems 6.9 and 6.10, respectively). In Section 6.4 we apply the results to a random field given as a solution as in (6.1), more specifically, we consider the stochastic partial differential equation

$$(\mu - \Delta)X = dL$$

in dimension $d = 3$, where Δ denotes the Laplace operator, and obtain a consistent and asymptotically normal estimator of $\mu > 0$ based on the sample mean. Finally, Sections 6.5 and 6.6 contain the proofs of the main theorems for the sample mean and the sample autocovariance, respectively.

6.2 Notation and Preliminaries

To fix notation, by a distribution on $\mathbb{R}$ we mean a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with $\mathcal{B}(\mathbb{R})$ being the Borel $\sigma-$algebra on $\mathbb{R}$. By a measure on $\mathbb{R}^d$, d a natural number, we always mean a positive measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, i.e. an $[0, \infty]$-valued $\sigma-$additive set function on $\mathcal{B}(\mathbb{R}^d)$, the Borel $\sigma-$algebra on $\mathbb{R}^d$, that assigns the value 0 to the empty set. The set $\mathcal{B}_b(\mathbb{R}^d)$ is the set of all bounded Borel measurable sets. The Dirac measure at a point $b \in \mathbb{R}$ will be denoted by δ_b, the Gaussian distribution with mean $a \in \mathbb{R}$ and variance $b \geq 0$ by $N(a, b)$ and the Lebesgue measure by λ^d on $\mathbb{R}^d$. If a random vector X has law $\mathcal{L}$ we write $X \sim \mathcal{L}$. Weak convergence of measures will be denoted by "$\xrightarrow{d}$". We write $\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{Z}$, $\mathbb{R}$ for the set of integers and real numbers respectively. The indicator function of a set $A \subset \mathbb{R}$ is denoted by $\mathbf{1}_A$. By $L^p(\mathbb{R}^d, A)$ for $1 \leq p < \infty$ and $A \subset \mathbb{C}$ we denote the set of all Borel-measurable functions $f : \mathbb{R}^d \to A$

such that $\int_{\mathbb{R}^d} |f(x)|^p \, \lambda^d(dx) < \infty$. If $A = \mathbb{R}$ we simply write $L^p(\mathbb{R}^d)$. For two different sets $A, B \subset \mathbb{R}^d$, we denote by $dist(A, B) := \inf\{\|x - y\| : x \in A \text{ and } y \in B\}$, where $\|\cdot\|$ is the euclidean norm. We write 'a.e.' to denote almost everywhere and 'a.s.' to denote almost surely. $|A|$ denotes the number of elements of the set A.
We are interested in integrals of the form $\int_{\mathbb{R}^d} f(u)\, dL(u)$, where dL denotes the integration over a Lévy basis. A Lévy basis can be understood in the following way:

Definition 6.1 (see [56, p. 455])**.** A *Lévy basis* is family $(L(A))_{A \in \mathcal{B}_b(\mathbb{R}^d)}$ of real valued random variables such that

i) $L(\bigcup_{n=0}^{\infty} A_n) = \sum_{n=0}^{\infty} L(A_n)$ *a.s.* for pairwise disjoint sets $(A_n)_{n\in\mathbb{N}_0} \subset \mathcal{B}_b(\mathbb{R}^d)$ with $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{B}_b(\mathbb{R}^d)$,

ii) $L(A_i)$are independent for pairwise disjoint sets $A_1, \dots, A_n \in \mathcal{B}_b(\mathbb{R}^d)$ for every $n \in \mathbb{N}$,

iii) there exist $a \in [0, \infty)$, $\gamma \in \mathbb{R}$ and a Lévy measure ν on $\mathbb{R}$ (i.e. a measure ν on $\mathbb{R}$ such that $\nu(\{0\}) = 0$ and $\int\limits_{\mathbb{R}} \min\{1, x^2\}\nu(dx) < \infty$) such that

$$\mathbb{E}e^{izL(A)} = \exp\left(\psi(z)\lambda^d(A)\right)$$

for every $A \in \mathcal{B}_b(\mathbb{R}^d)$, where

$$\psi(z) := i\gamma z - \frac{1}{2}az^2 + \int\limits_{\mathbb{R}} (e^{ixz} - 1 - ixz\mathbf{1}_{[-1,1]}(x))\nu(dx), \quad z \in \mathbb{R}.$$

The triplet (a, ν, γ) is called the *characteristic triplet* of L and ψ its *characteristic exponent.* By the Lévy-Khintchine formula, $L(A)$ is then infinitely divisible.

It can be shown that the characteristic triplet is unique; conversely, to every $a \in [0, \infty)$, $\gamma \in \mathbb{R}$ and Lévy measure ν there exists a Lévy basis with (a, ν, γ) as characteristic triplet. It follows from the general theory of infinitely divisible distributions that for a Lévy basis L with characteristic triplet (a, ν, γ) and $p \in [1, \infty)$, we have $\int\limits_{|x|>1} |x|^p \nu(dx) < \infty$ if and only if $\mathbb{E}|L(A)|^p < \infty$ for some (equivalently, all) $A \in \mathcal{B}_b(\mathbb{R}^d)$ with $\lambda^d(A) > 0$. In that case,

$$\mathbb{E}L(A) = \lambda^d(A)\mathbb{E}L([0, 1]^d).$$

Integration of deterministic functions with respect to Lévy bases is described by Rajput and Rosinski [56]; in particular for *simple functions* f of the form $f = \sum\limits_{j=1}^{n} x_j \mathbf{1}_{A_j}$ with $x_j \in \mathbb{R}$ and $A_j \in \mathcal{B}_b(\mathbb{R}^d)$, the integral $\int\limits_A f(u)dL(u)$ for $A \in \mathcal{B}(\mathbb{R}^d)$ is defined as $\sum\limits_{j=1}^{n} x_j L(A_j \cap A)$. A general Borel-measurable function $f : \mathbb{R}^d \to \mathbb{R}$ is called *integrable with respect to* L, if there exists a sequence of simple functions $(f_n)_{n\in\mathbb{N}}$ such that $f_n \to f$ λ^d–a.e.

and such that $\int_A f_n(u)dL(u)$ converges in probability as $n \to \infty$ for every $A \in \mathcal{B}(\mathbb{R}^d)$, in which case this limit is denoted by $\int_A f(u)dL(u)$, see [56, p.460]. Rajput and Rosinski also characterize integrability of functions. In particular, if $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ and $\mathbb{E}L([0,1]^d)^2 < \infty$, or if $f \in L^2(\mathbb{R}^d)$, $\mathbb{E}L([0,1]^d)^2 < \infty$ and $\mathbb{E}L([0,1]^d) = 0$, then the integral $\int f(u)dL(u)$ is well-defined and satisfies $\mathbb{E}\left(\int f(u)dL(u)\right)^2 < \infty$. This follows by standard calculations. Moreover, for two such functions f, g we have

$$\operatorname{cov}\left(\int_{\mathbb{R}^d} f(u)dL(u), \int_{\mathbb{R}^d} g(u)dL(u)\right) = \sigma^2 \int_{\mathbb{R}^d} f(u)g(u)\lambda^d(du), \tag{6.6}$$

where $\sigma^2 = \mathbb{E}L([0,1]^d)^2$. For a stationary random field $X = (X_t)_{t\in\mathbb{R}^d}$ with finite second moment we write $\gamma_X(t) := \operatorname{cov}(X_t, X_0)$.

6.3 Main results

In this section, we formulate our main results. To specify the sampling grid, throughout we fix some orthogonal $d \times d-$matrix $A \in O(d)$ and some $\Delta > 0$, and consider the set

$$\Delta A\mathbb{Z}^d = \{\Delta A v : v \in \mathbb{Z}^d\}.$$

Our sampling sets Γ_n will then be subsets of $\Delta A\mathbb{Z}^d$. The process under consideration is given by $X_t = \int_{\mathbb{R}^d} f(t-s)dL(s)$, where $f : \mathbb{R}^d \to \mathbb{R}$ is integrable with respect to the Lévy basis L. By homogeneity of the Lévy basis, it is easy to see that $(X_t)_{t\in\mathbb{R}^d}$ is a strictly stationary random field, meaning that its finite dimensional distributions are shift invariant.

6.3.1 Central limit theorems for the sample mean

In this and the next section, we give central limit theorems (CLTs) for the sample mean.

Theorem 6.2. *Let L be a Lévy basis with $\mathbb{E}(L([0,1]^d)^2 < \infty$ and $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, and let*

$$X_t := \int_{\mathbb{R}^d} f(t-u)dL(u), \quad t \in \mathbb{R}^d.$$

Let $\Delta > 0$, $A \in O(d)$, and $(\Gamma_n)_{n\in\mathbb{N}}$ be a sequence of finite subsets of $\Delta A\mathbb{Z}^d$ such that

a) $\Gamma_n \subset \Gamma_{n+1}$ *for every* $n \in \mathbb{N}$, *b)* $|\Gamma_n| \to \infty$ *as* $n \to \infty$, *and*

c) $a_l^n := \frac{|\{(t,s)\in\Gamma_n\times\Gamma_n : t-s=l\}|}{|\Gamma_n|}$ *converges as* $n \to \infty$ *to some* a_l *for each* $l \in \Delta A\mathbb{Z}^d$.

Assume that

$$\sum_{t\in\Delta A\mathbb{Z}^d} \sup_{n\in\mathbb{N}} a_t^n \int_{\mathbb{R}^d} |f(-u)f(t-u)|\lambda^d(du) < \infty. \tag{6.7}$$

Then

$$\sum_{t\in\Delta A\mathbb{Z}^d} a_t |\mathrm{cov}\,(X_t, X_0)| < \infty,$$

and

$$\frac{1}{\sqrt{|\Gamma_n|}} \sum_{t\in\Gamma_n} \left(X_t - \mathbb{E}L([0,1]^d) \int_{\mathbb{R}^d} f(u)\lambda^d(du) \right) \xrightarrow{d} N\left(0, \sum_{t\in\Delta A\mathbb{Z}^d} a_t \mathrm{cov}\,(X_t, X_0)\right)$$

as $n \to \infty$.

Remark 6.3. From the definition of a_l^n it is obvious that $0 \le a_l^n \le 1$, hence necessarily also $a_l \in [0,1]$ for each $l \in \mathbb{N}$.
A sufficient condition for (6.7) to hold is hence that

$$\sum_{t\in\Delta A\mathbb{Z}^d} \int_{\mathbb{R}^d} |f(-u)f(t-u)|\lambda^d(du) < \infty.$$

Denoting

$$F(u) := \sum_{t\in\mathbb{Z}^d} |f(u + \Delta At)|, \quad u \in \mathbb{R}^d,$$

it is easy to see that F is periodic and that

$$\begin{aligned}
\sum_{t\in\Delta A\mathbb{Z}^d} \int_{\mathbb{R}^d} |f(-u)f(t-u)|\lambda^d(du) &= \int_{\mathbb{R}^d} |f(u)|F(u)\lambda^d(du) \\
&= \int_{\Delta A([0,1]^d)} \sum_{t\in\Delta A\mathbb{Z}^d} |f(u+t)|F(u)\lambda^d(du) \\
&= \int_{\Delta A([0,1]^d)} F(u)^2\lambda^d(du),
\end{aligned}$$

so that $F \in L^2(\Delta A([0,1]^d))$ is a sufficient condition for (1.22) to hold. Observe however that there also other cases when (1.22) holds but $F \notin L^2(\Delta A([0,1]^d))$. For example, when the sets Γ_n are contained in some hyperplane of $\mathbb{R}^d$, then many of the a_l^n will be 0.

Example 6.4. Let $\Gamma_n = \Delta A(-n,n]^d \cap \Delta A\mathbb{Z}^d$. Then it is clear that a_l^n in Theorem 1.5 will converge to 1 as $n \to \infty$ for each $l \in \Delta A\mathbb{Z}^d$. Sequences that satisfy $\lim_{n\to\infty} a_l^n = 1$ for each l are called *Følner*. They play an important role in ergodic theorems in the theory of amenable groups, see [50].

Another example of sequences (Γ_n) satisfying the assumptions of Theorem 6.2 can be obtained as realisations of certain random subsets, in which also the limits a_l may be non-trivial (i.e. different from 0 or 1). This follows from the next lemma, where we use the concept of ergodicity on $\Delta A\mathbb{Z}^d$, see [61, Definition 1.1, p. 52].

Lemma 6.5. *Let $(Y_t)_{t\in\Delta A\mathbb{Z}^d}$ be a $\{0,1\}-$valued stationary ergodic random field such that $\mathbb{E}Y_0 \neq 0$ (i.e. $P(Y_0 = 0) < 1$). We define*

$$\Gamma_n := \{t \in \Delta A[-n,n)^d \cap \Delta A\mathbb{Z}^d \; : \; Y_t = 1\}.$$

Then $(\Gamma_n)_{n\in\mathbb{N}}$ satifies

$$\frac{\{(t,s) \in \Gamma_n \times \Gamma_n : t - s = l\}}{|\Gamma_n|} \to \frac{\mathbb{E}Y_l Y_0}{\mathbb{E}Y_0} \qquad a.s. \text{ for } n \to \infty.$$

Especially, $(\Gamma_n)_{n\in\mathbb{N}}$ satisfies almost surely the assumptions of Theorem 6.2.

Proof. This is an easy application of the ergodic properties of Z_t. We write

$$\begin{aligned}
&\frac{\{(t,s) \in \Gamma_n \times \Gamma_n : t - s = l\}}{|\Gamma_n|} \\
=&\frac{\sum\limits_{t\in\Delta A[-n,n)^d\cap\Delta A[-n-l,n-l)^d\cap\Delta A\mathbb{Z}^d} Y_t Y_{t+l}}{|\Delta A[-n,n)^d \cap \Delta A[-n-l,n-l)^d \cap \Delta A\mathbb{Z}^d|} \\
&\cdot \frac{|\Delta A[-n,n)^d \cap \Delta A\mathbb{Z}^d|}{\sum\limits_{t\in\Delta A[-n,n)^d\cap\Delta A\mathbb{Z}^d} Y_t} \cdot \frac{|\Delta A[-n,n)^d \cap \Delta A[-n-l,n-l)^d) \cap \Delta A\mathbb{Z}^d|}{|\Delta A[-n,n)^d \cap \Delta A\mathbb{Z}^d|}.
\end{aligned}$$

Letting n go to infinity we obtain the assertion from the ergodic theorem for random fields (e.g. Lindenstrauss [50, Theorem 1.3]). □

Example 6.6. Let $(Z_t)_{t\in\Delta A\mathbb{Z}^d}$ be a random field of independent and identically distributed random variables. A typical example of an ergodic random field is the moving average random field $M_t := \sum\limits_{l\in\Delta A\mathbb{Z}^d} a_l Z_{t-l}$, where $(a_l)_{l\in\Delta A\mathbb{Z}^d} \in \mathbb{R}^{A\Delta\mathbb{Z}^d}$ such that the sum is well-defined. Let $\varphi : \mathbb{R} \to \{0,1\}$ be a measurable function, then the random field $\varphi(M_t)$ is an ergodic and stationary random field. Assuming that $\varphi(M_t) > 0$ with probability greater than 0, $\varphi(M_t)$ satisfies the assumption of Lemma 6.5.

6.3.2 From Non-Random Sampling to Random Sampling

We obtain a CLT on sequences $(\Gamma_n)_{n\in\mathbb{N}}$ similar to the construction as in Lemma 6.5 under the assumption that $(Y_t)_{t\in\Delta A\mathbb{Z}^d}$ is α-mixing, which means that

$$\alpha_Y(k;u,v) := \sup\{\alpha(\sigma(Y_t, t\in A), \sigma(Y_t, t\in B)) \,:\, dist(A,B)\geq k, |A|\leq u, |B|\leq v\} \to 0$$

for $k\to\infty$ for every $u,v\in\mathbb{N}$, where for two σ-fields $\mathcal{F}$ and $\mathcal{G}$, $\alpha(\mathcal{F},\mathcal{G})$ is defined by

$$\sup\{|P(A)P(B)-P(A\cap B)| \,:\, A\in\mathcal{F}, B\in\mathcal{G}\}.$$

A related but much stronger condition is h-dependence. A stationary random field $Y=(Y_t)_{t\in\mathbb{Z}^d}$ or $Y=(Y_t)_{t\in\mathbb{R}^d}$ is h-dependent ($h>0$), if for every two finite subsets $A,B\subset\mathbb{Z}^d$ ($\subset\mathbb{R}^d$, resp.) the two σ-fields $\sigma(Y_s : s\in A)$ and $\sigma(Y_s : s\in B)$ are independent if $dist(A,B)>h$.

Theorem 6.7. *Let $(Y_t)_{t\in\Delta A\mathbb{Z}^d}$ be a $\{0,1\}-$valued $\alpha-$mixing random field, which is independent of the Lévy basis L and satisfies $P(Y_0=1)>0$. Moreover, assume there exists a $\delta>0$ such that Y satisfies*

i) for every $u,v\in\mathbb{N}$ it holds $\alpha_Y(k;u,v)k^d\to 0$ for $k\to\infty$,

ii) for every $u,v\in\mathbb{N}$ such that $u+v\leq 4$ it holds $\sum_{k=0}^{\infty} k^{d-1}\alpha_Y(k;u,v)<\infty$ and especially $\sum_{k=0}^{\infty} k^{d-1}\alpha_Y(k;1,1)^{\delta/(2+\delta)}<\infty$.

Let Γ_n be as in Lemma 6.5 and $X=(X_t)_{t\in\mathbb{R}^d}$ be a moving average random field with $X_t=\int_{\mathbb{R}^d} f(t-u)\,dL(u)$ with $\mathbb{E}|L([0,1]^d)|^{2+\delta}<\infty$ and $f\in L^1(\mathbb{R}^d)\cap L^{2+\delta}(\mathbb{R}^d)$. If

$$\sum_{t\in\Delta A\mathbb{Z}^d} \mathbb{E}Y_0Y_t \int_{\mathbb{R}^d} |f(-u)|\,|f(t-u)|\,\lambda^d(du)<\infty,$$

then we have that

$$\frac{1}{\sqrt{|\Gamma_n|}}\sum_{t\in\Gamma_n}\left(X_t-\mathbb{E}L([0,1]^d)\int_{\mathbb{R}^d} f(u)\lambda^d(du)\right)\xrightarrow{d} N\left(0, \sum_{t\in\Delta A\mathbb{Z}^d}\frac{1}{\mathbb{E}Y_0}\operatorname{cov}(Y_tX_t, Y_0X_0)\right)$$

as $n\to\infty$. In the special case that Y is $h-$dependent for some finite $h>0$, it is enough to assume that $\mathbb{E}|L([0,1]^d)|^2<\infty$ and $f\in L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$.

Example 6.8. Every h-dependent random field Y is α-mixing with $\alpha_Y(k;u,v)=0$ for $|k|>h$. Other examples of (non-h-dependent) random fields Y with suitable mixing rates can be constructed by [26 ,Theorem 2, p. 58].

6.3.3 Non-Random Sampling of the Autocovariance

Our object of interest is the estimator

$$\gamma_n^*(t) := \frac{1}{|\Gamma_n|} \sum_{s \in \Gamma_n} X_s X_{s+t}$$

for some $(\Gamma_n)_{n\in\mathbb{N}} \subset \Delta A\mathbb{Z}^d$. We assume that Γ_n satisfies the same conditions as in Theorem 1.5. We state a central limit theorem for the sample autocovariance which can be proven similar to Theorem 1.5. Netherless, the calculations are a little bit longer.
We assume that

$$\mathbb{E}\, L([0,1]^d)^4 < \infty,\ \mathbb{E}\, L([0,1]^d) = 0,\ \sigma^2 := \mathbb{E}\, L([0,1]^d)^2 > 0 \tag{6.8}$$

and denote

$$\eta := \sigma^{-4}\mathbb{E}\, L([0,1]^d)^4.$$

Theorem 6.9. *Let $m \in \mathbb{N}$ and $\Delta_1, \dots, \Delta_m \in \Delta A\mathbb{Z}^d$, Γ_n as in Theorem 6.2, and let $(X_t)_{t\in\mathbb{R}^d} = \left(\int_{\mathbb{R}^d} f(t-s)dL(s)\right)_{t\in\mathbb{R}^d}$ be a moving average random field such that it satisfies the assumptions (6.8), $f \in L^2(\mathbb{R}^d) \cap L^4(\mathbb{R}^d)$ and*

$$\sum_{l\in\Delta A\mathbb{Z}^d} \int_{\mathbb{R}^d} \sup_{n\in\mathbb{N}} a_l^n |f(u)f(u+l)f(u+\Delta_p)f(u+l+\Delta_d)| \lambda^d(du) < \infty$$

for every $p, d \in \{1, \dots, m\}$ and

$$\sum_{l\in\Delta A\mathbb{Z}^d} \sup_{n\in\mathbb{N}} a_l^n \gamma_X(l)^2 < \infty.$$

Then

$$\sqrt{|\Gamma_n|}(\gamma_n^*(\Delta_1) - \gamma_X(\Delta_1), \dots, \gamma_n^*(\Delta_m) - \gamma_X(\Delta_m)) \xrightarrow{d} N(0,V) \text{ as } n \to \infty, \tag{6.9}$$

where $N(0,V)$ is the multivariate normal distribution with mean 0 and covariance matrix $V = (v_{pq})_{p,q\in\{1,\dots,m\}}$ given by

$$\begin{aligned} v_{pq} = \quad & \sum_{l\in\Delta A\mathbb{Z}^d} a_l \Bigg((\eta-3)\sigma^4 \int_{\mathbb{R}^d} f(u)f(u+\Delta_p)f(u+l)f(u+l+\Delta_q)\, \lambda^d(du) \\ & + \gamma_X(l)\gamma_X(l+\Delta_q-\Delta_p) + \gamma_X(l+\Delta_q)\gamma_X(l-\Delta_p) \Bigg). \end{aligned}$$

6.3.4 Random Sampling of the Autocovariance

Now we present a theorem similiar to Theorem 6.7.

Theorem 6.10. *Let $(Y_t)_{t\in\Delta A\mathbb{Z}^d}$ be a $\{0,1\}$-valued $\alpha-$mixing random field with mixing rates as in Theorem 6.7 ($\delta > 0$), which is independent of the Lévy basis L. Let $X = (X_t)_{t\in\mathbb{R}^d}$ be a moving average random field with $X_t = \int_{\mathbb{R}^d} f(t-u)\,dL(u)$ such that (6.8) holds with $\mathbb{E}|L([0,1]^d)|^{4+\delta} < \infty$ and $f \in L^2(\mathbb{R}^d) \cap L^{4+\delta}(\mathbb{R}^d)$. Let $\Delta_1, \dots, \Delta_m \in \Delta A\mathbb{Z}^d$ and for every $p, d \in \{1, \dots, m\}$ assume that*

$$\sum_{t\in\Delta A\mathbb{Z}^d} \mathbb{E}Y_0Y_t \int_{\mathbb{R}^d} |f(u)f(u+t)f(u+\Delta_p)f(u+t+\Delta_d)|\lambda^d(du) < \infty$$

and

$$\sum_{l\in\Delta A\mathbb{Z}^d} \mathbb{E}Y_0Y_l\gamma_X(l)^2 < \infty.$$

Then for $\Gamma_n := \{t \in \Delta A[-n,n)^d \cap \Delta A\mathbb{Z}^d \,:\, Y_t = 1\}$ we have

$$\sqrt{|\Gamma_n|}(\gamma_n^*(\Delta_1) - \gamma_X(\Delta_1), \dots, \gamma_n^*(\Delta_m) - \gamma_X(\Delta_m)) \xrightarrow{d} N(0,V) \tag{6.10}$$

as $n \to \infty$, with covariance matrix $V = (v_{pq})_{p,q\in\{1,\dots,m\}}$ given by

$$\begin{aligned} v_{pq} = \quad & \sum_{l\in\Delta A\mathbb{Z}^d} \frac{\mathbb{E}Y_0Y_l}{\mathbb{E}Y_0}\Bigg((\eta-3)\sigma^4 \int_{\mathbb{R}^d} f(u)f(u+\Delta_p)f(u+l)f(u+l+\Delta_q)\,\lambda^d(du) \\ & + \gamma_X(l)\gamma_X(l+\Delta_p-\Delta_q) + \gamma_X(l+\Delta_p)\gamma_X(l+\Delta_q)\Bigg). \end{aligned} \tag{6.11}$$

6.4 Applications

In this section we present an application of our theorems before. We fix the dimension $d = 3$ and estimate the parameter $\mu > 0$ of the equation

$$(\mu - \Delta)X = dL, \tag{6.12}$$

where L is a Lévy basis with $\mathbb{E}L([0,1]^3)^2 < \infty$. The mild solution of (6.12) can be written as

$$X(x) = \int_{\mathbb{R}^d} G_\mu(x-z)dL(z), \tag{6.13}$$

where $G_\mu(x) := \frac{\exp\left(-\sqrt{\mu}\|x\|\right)}{\|x\|}$ for $x \neq 0$, see [22, Definition 3.5] for the notion of the mild solution. That G_μ is a fundamental solution of $(\mu - \Delta)X = \delta_0$ follows e.g. from [41, Section 2.1, Equation (21)]. We see that $G_\mu \in L^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$, so X exists since $\mathbb{E}L([0,1]^3)^2 < \infty$.
Calculating the mean we obtain

$$\mathbb{E}X(x) = \mathbb{E}X(0) = \frac{\mathbb{E}L([0,1]^3)}{\mu}.$$

Our moment estimator is then given by

$$\widehat{\mu}_n = \mathbb{E}L([0,1]^3)\frac{|\Gamma_n|}{\sum\limits_{k\in\Gamma_n} X(k)}. \tag{6.14}$$

Corollary 6.11. *Let $\widehat{\mu}_n$ be defined as in* (6.14), $\mathbb{E}L([0,1]^3) \neq 0$ *and* $\Gamma_n \subset \Delta A\mathbb{Z}^3$ *satisfying the assumptions of Theorem 6.2. Then $\widehat{\mu}_n$ defines a consistent and asymptotically normal estimator.*

Proof. By Theorem 6.2 we conclude that $\widehat{\mu}_n$ is asymptotically normal, as

$$\begin{aligned}
&\sum_{t\in\Delta A\mathbb{Z}^d}\int_{\mathbb{R}^d} |G_\mu(-u)G_\mu(t-u)|\lambda^d(du)\\
&= \sum_{t\in\Delta A\mathbb{Z}^d}\int_{\mathbb{R}^d} |G_\mu(-u)G_\mu(t-u)|\exp\left(\varepsilon\|u\| + \varepsilon\|t-u\|\right)\exp\left(-\varepsilon\|u\| - \varepsilon\|t-u\|\right)\lambda^d(du)\\
&\leq \sum_{t\in\Delta A\mathbb{Z}^d} \exp\left(-\varepsilon\|t\|\right)\|G_\mu \exp(\varepsilon\|\cdot\|)\|_{L^2}^2,
\end{aligned}$$

which is finite for $0 < \varepsilon < \sqrt{\mu}$. As asymptotical normality implies consistency, we are done. □

If in the situation above, additionally Γ_n is a tempered Følner sequence, which means that

$$\lim_{n\to\infty} \frac{((k+\Gamma_n)\setminus\Gamma_n)\cup(\Gamma_n\setminus(\Gamma_n+k))}{|\Gamma_n|} = 0 \text{ for all } k \in \Delta A\mathbb{Z}^3 \text{ and} \tag{6.15}$$

$$\left|\bigcup_{k<n}(-\Gamma_k + \Gamma_n)\right| \leq C|\Gamma_n| \text{ for some constant } C > 0, \tag{6.16}$$

then the estimator $\hat{\mu}_n$ is strongly consistent by [50, Theorem 1.2, p. 260]. A simple example of a tempered Følner sequence is $(-n,n]^d \cap \mathbb{Z}^d$.

6.5 Proof of Theorems 6.2 and 6.7

Since

$$X_t = \int\limits_{\mathbb{R}^d} f(t-u)dL'(u) + \mathbb{E}(L([0,1]^d)) \int\limits_{\mathbb{R}^d} f(u)\lambda^d(du),$$

where the mean zero Lévy basis L' is defined by

$$L'(A) := L(A) - \mathbb{E}L([0,1]^d)\lambda^d(A),\ A \in \mathcal{B}_b(\mathbb{R}^d),$$

and since

$$\operatorname{cov}(Y_t X_t, Y_0 X_0) = \operatorname{cov}(Y_t(X_t - \mathbb{E}X_t), Y_0(X_0 - \mathbb{E}X_0))$$

in Theorem 6.7 by independence of X and Y, we may and do assume for rest of this section that $\mathbb{E}L([0,1]^d) = 0$.

Proof of Theorem 6.2. For every $h \in \mathbb{N}$ we define a new random field $(X_t^{(h)})_{t\in\Delta A\mathbb{Z}^d}$ by

$$X_t^{(h)} := \int\limits_{\mathbb{R}^d} f(t-u)\mathbf{1}_{\Delta A[-h,h)^d}(t-u)\,dL(u).$$

It is obvious that $(X_t^{(h)})_{t\in\Delta A\mathbb{Z}^d}$ is $2\sqrt{d}\Delta h+1$-dependent.
We want to use [38, Theorem 2, p. 135]. We set $U_t^{(n,h)} := \frac{1}{\sqrt{|\Gamma_n|}} X_t^{(h)}$. We calculate that

$$\mathbb{E}\left(\sum_{t\in\Gamma_n} U_t^{(n,h)}\right)^2 = \frac{1}{|\Gamma_n|}\sum_{t,s\in\Gamma_n} \mathbb{E}X_t^{(h)}X_s^{(h)} = \frac{1}{|\Gamma_n|}\sum_{t,s\in\Gamma_n} \gamma_{X^{(h)}}(t-s) = \sum_{l\in\Delta A\mathbb{Z}^d} a_l^n \gamma_{X^{(h)}}(l). \tag{6.17}$$

Letting n go to infinity, we obtain

$$\mathbb{E}\left(\sum_{t\in\Gamma_n} U_t^{(n,h)}\right)^2 \to \sum_{t\in\Delta A\mathbb{Z}^d} a_t \gamma_{X^{(h)}}(t).$$

Furthermore, we immediately see that

$$\sum_{t\in\Gamma_n} \mathbb{E}(U_t^{(n,h)})^2 = \frac{1}{|\Gamma_n|}\sum_{t\in\Gamma_n} \mathbb{E}(X_t^{(h)})^2 = \gamma_{X^{(h)}}(0) < \infty$$

and

$$\sum_{t\in\Gamma_n}\mathbb{E}\left((U_t^{(n,h)})^2\mathbf{1}_{|U_t^{(n,h)}|\geq\varepsilon}\right)=\frac{1}{|\Gamma_n|}\sum_{t\in\Gamma_n}\mathbb{E}(X_t^{(h)})^2\mathbf{1}_{|X_t^{(h)}|\geq\varepsilon\sqrt{|\Gamma_n|}}$$
$$=\mathbb{E}(X_0^{(h)})^2\mathbf{1}_{|X_0^{(h)}|\geq\varepsilon\sqrt{|\Gamma_n|}}\to 0 \qquad \text{for } n\to\infty.$$

Hence all conditions of [38, Theorem 2, p. 135] are satisfied and we conclude that

$$\frac{1}{\sqrt{|\Gamma_n|}}\sum_{t\in\Gamma_n}X_t^{(h)}\xrightarrow{d}Y^{(h)}$$

for $n\to\infty$ with $Y^{(h)}\sim N(0,\sum_{t\in\Delta A\mathbb{Z}^d}a_t\gamma_{X^{(h)}}(t))$.
Observe that $\lim_{h\to\infty}\gamma_{X^{(h)}}(t)=\gamma_X(t)$ for all $t\in\Delta A\mathbb{Z}^d$ by (6.6) and dominated convergence and $|\gamma_{X^{(h)}}(t)|\leq\sigma^2\int_{\mathbb{R}^d}|f(-u)|\,|f(t-u)|\lambda^d(du)$, hence we conclude by dominated convergence that

$$\lim_{h\to\infty}\sum_{t\in\Delta A\mathbb{Z}^d}a_t\gamma_{X^{(h)}}(t)=\sum_{t\in\Delta A\mathbb{Z}^d}a_t\gamma_X(t)$$

and hence

$$Y^{(h)}\xrightarrow{d}Y\sim N(0,\sum_{t\in\Delta A\mathbb{Z}^d}a_t\gamma_X(t))\text{ for }h\to\infty.$$

As in (6.17), we obtain

$$\mathbb{E}\left(\frac{1}{\sqrt{|\Gamma_n}}\sum_{t\in\Gamma_n}(X_t-X_t^{(h)})\right)^2=\sum_{l\in\Delta A\mathbb{Z}^d}a_l^n\gamma_{X-X^{(h)}}(l)$$
$$=\sum_{l\in\Delta A\mathbb{Z}^d}a_l^n\int_{\mathbb{R}^d}f(l-u)\mathbf{1}_{\mathbb{R}^d\setminus\Delta A[-h,h)^d}(t-u)f(-u)\mathbf{1}_{\mathbb{R}^d\setminus\Delta A[-h,h)^d}(-u)\lambda^d(du),$$

hence

$$\lim_{h\to\infty}\lim_{n\to\infty}\mathbb{E}\left(\frac{1}{\sqrt{|\Gamma_n|}}\left(\sum_{t\in\Gamma_n}X_t-X_t^{(h)}\right)\right)^2=0$$

from Lebesgue's dominated convergence theorem for series. An application of Chebyshev's inequality gives for $\varepsilon>0$,

$$\lim_{h\to\infty}\lim_{n\to\infty}P\left(\frac{1}{\sqrt{|\Gamma_n|}}\left|\sum_{t\in\Gamma_n}X_t-X_t^{(h)}\right|>\varepsilon\right)=0.$$

The claim then follows by a variant of Slutsky's theorem, e.g. [12, Proposition 6.3.9, pp. 207-208]. □

Proof of Theorem 6.7. The proof is very similiar to the proof of Theorem 6.2. Let us start with approximating X_t by $X_t^{(h)}$ as above. Observe that

$$\frac{1}{\sqrt{|\Gamma_n|}}\sum_{t\in\Gamma_n} X_t^{(h)} = \frac{(2n)^{d/2}}{\sqrt{|\Gamma_n|}}\frac{1}{(2n)^{d/2}}\sum_{t\in\Delta A((-n,n]^d\cap\mathbb{Z}^d)} X_t^{(h)} Y_t.$$

We know that $\frac{(2n)^{d/2}}{\sqrt{|\Gamma_n|}} \to (\sqrt{\mathbb{E}Y_0})^{-1}$, which follows from the ergodic theorem. Furthermore, as $(X_t^{(h)})$ is $(2\sqrt{d}\Delta h+1)-$dependent and Y is $\alpha-$mixing, we obtain that $(X_t^{(h)}Y_t)_{t\in\Delta A\mathbb{Z}}$ is $\alpha-$mixing with the same rate as Y. From this we conclude by [26, Theorem 3, p. 48] that

$$\frac{1}{(2n)^{d/2}}\sum_{t\in\Gamma_n} X_t^{(h)} \xrightarrow{d} N\left(0, \sum_{t\in\Delta A\mathbb{Z}^d}\frac{1}{\mathbb{E}Y_0}\operatorname{cov}(X_t^{(h)}Y_t, X_0^{(h)}Y_0)\right) \quad \text{for } n\to\infty.$$

Now by the same arguments as above we conclude that this theorem holds true when Y is α-mixing. When Y is even h'-dependent for some h', then $(X_t^{(h)}Y_t)_{t\in\Delta A\mathbb{Z}^d}$ is $\max\{h', 2\sqrt{d}\Delta h+1\}$-dependent and we can use [38, Theorem 2, p. 135] instead of [26, Theorem 3, p. 48] and hence need weaker moment conditions. □

6.6 Proof of Theorems 6.9 and 6.10

Proposition 6.12. *Let $f_1,\dots,f_4 \in L^4(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$. It holds true that*

$$\begin{aligned}
\mathbb{E}\prod_{i=1}^{4}\int_{\mathbb{R}^d} f_i(t)dL(t) = \quad & (\eta-3)\sigma^4\int_{\mathbb{R}^d} f_1(u)f_2(u)f_3(u)f_4(u)\lambda^d(du)\\
&+\sigma^4\int_{\mathbb{R}^d}\prod_{i=1,2} f_i(u)\lambda^d(du)\int_{\mathbb{R}^d}\prod_{i=3,4} f_i(u)\lambda^d(du)\\
&+\sigma^4\int_{\mathbb{R}^d}\prod_{i=1,3} f_i(u)\lambda^d(du)\int_{\mathbb{R}^d}\prod_{i=2,4} f_i(u)\lambda^d(du)\\
&+\sigma^4\int_{\mathbb{R}^d}\prod_{i=1,4} f_i(u)\lambda^d(du)\int_{\mathbb{R}^d}\prod_{i=2,3} f_i(u)\lambda^d(du).
\end{aligned}$$

Proof. Follows directly from the proof of [9, Lemma 4.1]. □

Proposition 6.13. *Under the assumptions of Theorem 6.9, for $\Delta_p,\Delta_q\in\Delta A\mathbb{Z}^d$, we have*

$$|\Gamma_n|\operatorname{cov}(\gamma_n^*(\Delta_p),\gamma_n^*(\Delta_q)) \to \sum_{l\in\Delta A\mathbb{Z}^d} a_l T_l \quad \textit{for } n\to\infty,$$

where

$$T_l := (\eta - 3)\sigma^4 \int_{\mathbb{R}^d} f(u)f(u+l)f(u+\Delta_p)f(u+l+\Delta_q)\lambda^d(du)$$
$$+ \gamma_X(l)\gamma_X(l+\Delta_q-\Delta_p) + \gamma_X(l+\Delta_q)\gamma_X(l-\Delta_p).$$

Proof. A direct calculation gives us

$$\begin{aligned}
|\Gamma_n|\mathrm{cov}\left(\gamma_n^*(\Delta_p), \gamma_n^*(\Delta_q)\right) &= \frac{1}{|\Gamma_n|}\sum_{s,t\in\Gamma_n} \mathrm{cov}\left(X_t X_{t+\Delta_p}, X_s X_{s+\Delta_q}\right) \\
&= \frac{1}{|\Gamma_n|}\sum_{s,t\in\Gamma_n} \mathbb{E}(X_t X_s X_{t+\Delta_p} X_{s+\Delta_q}) - \gamma_X(\Delta_p)\gamma_X(\Delta_q) \\
&= \frac{1}{|\Gamma_n|}\sum_{s,t\in\Gamma_n} \mathbb{E}(X_0 X_{s-t} X_{\Delta_p} X_{s-t+\Delta_q}) - \gamma_X(\Delta_p)\gamma_X(\Delta_q) \\
&= \frac{1}{|\Gamma_n|}\sum_{s,t\in\Gamma_n} T_{s-t},
\end{aligned}$$

which follows from Proposition 6.12, and we get that

$$\frac{1}{|\Gamma_n|}\sum_{s,t\in\Gamma_n} T_{s-t} = \sum_{l\in\Delta A\mathbb{Z}^d} a_l^n T_l.$$

By our assumptions and Lebesgue's dominated convergence theorem for series we conclude that

$$|\Gamma_n|\mathrm{cov}\left(\gamma_n^*(\Delta_p), \gamma_n^*(\Delta_q)\right) \to \sum_{l\in\Delta A\mathbb{Z}^d} a_l T_l \quad \text{for } n\to\infty.$$

□

Proof of Theorem 6.9. Let $h\in\mathbb{N}$ and $X_t^{(h)}$ be given by

$$X_t^{(h)} := \int_{\mathbb{R}^d} f^{(h)}(t-u)\,dL(u),$$

where $f^{(h)}(u) := f(u)\mathbf{1}_{\Delta A[-h,h)^d}(u)$. We define

$$U_t^{(h)} := (X_t^{(h)}X_{t+\Delta_1}^{(h)}, \ldots, X_t^{(h)}X_{t+\Delta_m}^{(h)}).$$

Now observe that $U_t^{(h)}$ is $(2\sqrt{d}\Delta h + 2\sup_{i=1,\ldots,m}\|\Delta_i\| + 1)$-dependent. We want to show that

$$\frac{1}{\sqrt{|\Gamma_n|}}\sum_{t\in\Gamma_n}(U_t^{(h)} - (\gamma_{X^{(h)}}(\Delta_1), \ldots, \gamma_{X^{(h)}}(\Delta_m))) \xrightarrow{d} Y^{(h)} \stackrel{d}{=} N(0, V^{(h)}) \tag{6.18}$$

as $n \to \infty$, where $V^{(h)} = (v_{pq}^{(h)})_{p,q\in\{1,\dots,n\}}$ is defined by (6.11) with f replaced by $f^{(h)}$. Let $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{R}^m \setminus \{0\}$. Definde $K_t^{(h)} := \alpha(U_t^{(h)} - (\gamma_{X^{(h)}}(\Delta_1), \dots, \gamma_{X^{(h)}}(\Delta_m)))^T$, which is also $(2\sqrt{d}\Delta h + 2\sup_{i=1,\dots,m} \|\Delta_i\| + 1)$-dependent. Then we see that $\mathbb{E}K_t^{(h)} = 0$ and

$$
\begin{aligned}
\frac{1}{|\Gamma_n|}\mathbb{E}\left(\sum_{t\in\Gamma_n} K_t^{(h)}\right)^2 &= \frac{1}{|\Gamma_n|}\sum_{t,s\in\Gamma_n} \mathbb{E}\, K_t^{(h)} K_s^{(h)} \\
&= \frac{1}{|\Gamma_n|}\sum_{t,s\in\Gamma_n} \mathbb{E}(\alpha(U_t^{(h)} - (\gamma_{X^{(h)}}(\Delta_1), \dots, \gamma_{X^{(h)}}(\Delta_m)))^T \\
&\qquad\qquad \alpha((U_s^{(h)} - (\gamma_{X^{(h)}}(\Delta_1), \dots, \gamma_{X^{(h)}}(\Delta_m)))^T) \\
&= \frac{1}{|\Gamma_n|}\sum_{t,s\in\Gamma_n} \mathbb{E}\sum_{i,j=1}^m \alpha_i\alpha_j (X_t^{(h)} X_{t+\Delta_i}^{(h)} - \gamma_{X^{(h)}}(\Delta_i))(X_s^{(h)} X_{s+\Delta_j}^{(h)} - \gamma_{X^{(h)}}(\Delta_j)) \\
&= \frac{1}{|\Gamma_n|}\sum_{t,s\in\Gamma_n} \sum_{i,j=1}^m \alpha_i\alpha_j \mathrm{cov}\,(X_t^{(h)} X_{t+\Delta_i}^{(h)}, X_s^{(h)} X_{s+\Delta_j}^{(h)}).
\end{aligned}
$$

By Proposition 6.13 we conclude that

$$
\frac{1}{|\Gamma_n|}\mathbb{E}\left(\sum_{t\in\Gamma_n} K_t^{(h)}\right)^2 \to \sum_{i,j=1}^m \alpha_i\alpha_j v_{ij}^{(h)}
$$

for $n \to \infty$. Furthermore, for every $\varepsilon > 0$ we have

$$
\begin{aligned}
&\lim_{n\to\infty} \frac{1}{|\Gamma_n|}\sum_{t\in\Gamma_n} \mathbb{E}(K_t^{(h)})^2 \mathbf{1}_{|K_t^{(h)}|\geq|\Gamma_n|\varepsilon} \\
&= \lim_{n\to\infty} \mathbb{E}(K_0^{(h)})^2 \mathbf{1}_{|K_0^{(h)}|\geq|\Gamma_n|\varepsilon} = 0
\end{aligned}
$$

and

$$
\frac{1}{|\Gamma_n|}\sum_{t\in\Gamma_n} \mathbb{E}(K_t^{(h)})^2 = \mathbb{E}(K_0^{(h)})^2 < \infty.
$$

By [38, Theorem 2, p. 135] we conclude that

$$
\frac{1}{\sqrt{|\Gamma_n|}}\sum_{t\in\Gamma_n} K_t^{(h)} \xrightarrow{d} N(0, \sum_{i,j=1}^m \alpha_i\alpha_j v_{ij}^{(h)}),\ n \to \infty.
$$

By the Crámer-Wold Theorem we see that (6.18) holds true. Next we have to show that $V^{(h)} \to V$ for $h \to \infty$. But this follows from dominated convergence, since $f^{(h)} \to f$ in $L^4(\mathbb{R}^d)$ and in $L^2(\mathbb{R}^d)$ as $h \to \infty$, since $|f^{(h)}| \leq |g|$ and by (6.6). Hence we get

$$
Y^{(h)} \xrightarrow{d} Y \sim N(0, V) \text{ as } h \to \infty.
$$

The claim will now follow by [12, Proposition 6.3.9, pp. 207-208] if we can show that for any $\varepsilon > 0$,

$$\lim_{h\to\infty}\lim_{n\to\infty} P\left(\sqrt{|\Gamma_n|}\left|\gamma_n^*(\Delta_i) - \gamma_X(\Delta_i) - \frac{1}{|\Gamma_n|}\sum_{t\in\Gamma_n} X_t^{(h)}X_{t+\Delta_i}^{(h)} + \gamma_{X^{(h)}}(\Delta_i)\right| > \varepsilon\right) = 0. \tag{6.19}$$

Let us first observe that

$$\begin{aligned}
&\mathbb{E}(\sqrt{|\Gamma_n|}((\gamma_n^*(\Delta_i) - \gamma_X(\Delta_i) - \frac{1}{|\Gamma_n|}\sum_{t\in\Gamma_n} X_t^{(h)}X_{t+\Delta_i}^{(h)} + \gamma_{X^{(h)}}(\Delta_i)))^2\\
=&|\Gamma_n|\left(\operatorname{var}\left(\gamma_n^*(\Delta_i)\right) + \operatorname{var}\left(\frac{1}{|\Gamma_n|}\sum_{t\in\Gamma_n} X_t^{(h)}X_{t+\Delta_i}^{(h)}\right) - 2\operatorname{cov}\left(\gamma_n^*(\Delta_i), \frac{1}{|\Gamma_n|}\sum_{t\in\Gamma_n} X_t^{(h)}X_{t+\Delta_i}^{(h)}\right)\right).
\end{aligned}$$

From Proposition 6.12 we see that

$$\begin{aligned}
&|\Gamma_n|\operatorname{var}\left(\gamma_n^*(\Delta_i)\right)\\
=&\frac{1}{|\Gamma_n|}\mathbb{E}\sum_{t,s\in\Gamma_n} X_tX_sX_{t+\Delta_i}X_{s+\Delta_i} - \gamma_X(\Delta_i)^2\\
=&\sum_{l\in\Delta A\mathbb{Z}^d} a_l^n\Big((\eta-3)\sigma^4\int_{\mathbb{R}^d} f(u)f(u+l)f(u+\Delta_i)f(u+l+\Delta_i)\lambda^d(du)\\
&+\gamma_X(l)^2 + \gamma_X(l+\Delta_i)\gamma_X(l-\Delta_i)\Big),
\end{aligned}$$

$$\begin{aligned}
&|\Gamma_n|\operatorname{var}\left(\frac{1}{|\Gamma_n|}\sum_{t\in\Gamma_n} X_t^{(h)}X_{t+\Delta_i}^{(h)}\right)\\
=&\frac{1}{|\Gamma_n|}\mathbb{E}\sum_{t,s\in\Gamma_n} X_t^{(h)}X_s^{(h)}X_{t+\Delta_i}^{(h)}X_{s+\Delta_i}^{(h)} - \gamma_{X^{(h)}}(\Delta_i)^2\\
=&\sum_{l\in\Delta A\mathbb{Z}^d} a_l^n\Big((\eta-3)\sigma^4\int_{\mathbb{R}^d} f^{(h)}(u)f^{(h)}(u+l)f^{(h)}(u+\Delta_i)f^{(h)}(u+l+\Delta_i)\lambda^d(du)\\
&+\gamma_{X^{(h)}}(l)^2 + \gamma_{X^{(h)}}(l+\Delta_i)\gamma_{X^{(h)}}(l-\Delta_i)\Big)
\end{aligned}$$

and

$$\begin{aligned}
&|\Gamma_n|\operatorname{cov}\left(\gamma_n^*(\Delta_i), \frac{1}{|\Gamma_n|}\sum_{t\in\Gamma_n} X_t^{(h)}X_{t+\Delta_i}^{(h)}\right)\\
=&\frac{1}{|\Gamma_n|}\mathbb{E}\sum_{t,s\in\Gamma_n} X_tX_s^{(h)}X_{t+\Delta_i}X_{s+\Delta_i}^{(h)} - \gamma_{X^{(h)}}(\Delta_i)\gamma_X(\Delta_i)
\end{aligned}$$

$$
\begin{aligned}
= \sum_{l\in\Delta A\mathbb{Z}^d} a_l^n \Big(& (\eta-3)\sigma^4 \int_{\mathbb{R}^d} f(u) f^{(h)}(u+l) f(u+\Delta_i) f^{(h)}(u+l+\Delta_i)\lambda^d(du) \\
& + \sigma^4 \int_{\mathbb{R}^d} f(u) f^{(h)}(u+l)\lambda^d(du) \int_{\mathbb{R}^d} f(u+\Delta_i) f^{(h)}(u+l+\Delta_i)\lambda^d(du) \\
& + \sigma^4 \int_{\mathbb{R}^d} f(u+\Delta_i) f^{(h)}(u+l)\lambda^d(du) \int_{\mathbb{R}^d} f(u) f^{(h)}(u+l+\Delta_i)\lambda^d(du)\Big).
\end{aligned}
$$

By inserting all the terms and by Lebesgue's dominated convergence theorem we conclude that

$$
\lim_{h\to\infty}\lim_{n\to\infty} \mathbb{E}(\sqrt{|\Gamma_n|}(\gamma_n^*(\Delta_i) - \gamma_X(\Delta_i) - \frac{1}{|\Gamma_n|}\sum_{t\in\Gamma_n} X_t^{(h)} X_{t+\Delta_i}^{(h)} - \gamma_{X^{(h)}}(\Delta_i)))^2 = 0.
$$

An application of Chebyshev's inequality then gives (6.19) and hence the claim. □

Proof of Theorem 6.10. We observe that

$$
\sum_{t\in\Gamma_n} (X_t X_{t+\Delta_i} - \gamma_X(\Delta_i) = \sum_{t\in\Delta A([-n,n)^d\cap\mathbb{Z}^d)} Y_t(X_t X_{t+\Delta_i} - \gamma_X(\Delta_i))
$$

and

$$
\begin{aligned}
& \operatorname{cov}\left(Y_t(X_t^{(h)} X_{t+\Delta_i}^{(h)} - \gamma_{X^{(h)}}(\Delta_i)), Y_s(X_s^{(h)} X_{s+\Delta_j}^{(h)} - \gamma_{X^{(h)}}(\Delta_j))\right) \\
= & \mathbb{E} Y_t(X_t^{(h)} X_{t+\Delta_i}^{(h)} - \gamma_{X^{(h)}}(\Delta_i)) Y_s(X_s^{(h)} X_{s+\Delta_j}^{(h)} - \gamma_{X^{(h)}}(\Delta_j) \\
& - \mathbb{E} Y_t(X_t^{(h)} X_{t+\Delta_i}^{(h)} - \gamma_{X^{(h)}}(\Delta_i)) \mathbb{E} Y_s(X_s^{(h)} X_{s+\Delta_j}^{(h)} - \gamma_{X^{(h)}}(\Delta_j)) \\
= & \mathbb{E} Y_t Y_s \mathbb{E}(X_t^{(h)} X_{t+\Delta_i}^{(h)} - \gamma_{X^{(h)}}(\Delta_i))(X_s^{(h)} X_{s+\Delta_j}^{(h)} - \gamma_{X^{(h)}}(\Delta_j).
\end{aligned}
$$

Repeating the same steps as in the proof of Theorem 6.7 gives the claim. □

Bibliography

[1] L. Ambrosio, N. Fusco and D. Pallara, *Functions of Bounded Variation and Free Discontinuity*, Oxford University Press, (2000).

[2] M. A. Armstrong, *Basic Topology*, Springer, (1983).

[3] S. Aziznejad, J. Fageot and M. Unser, *Wavelet analysis of the Besov regularity of Lévy white noise*, (2018) arXiv:1801.09245.

[4] O.E.Barndorff-Nielsen, A.Basse-O'Connor, *Quasi Ornstein-Uhlenbeck processes*, Bernoulli (2011) 17 , no. 3, 916–941.

[5] O.E. Barndorff-Nielsen, F. Benth and A. Veraart, *Modelling electricity futures by ambit fields*, Adv. Appl. Prob., (2014) 46(3), 719-745.

[6] D. Berger, *On quasi-infinitely divisible distributions with a point mass*, Mathematische Nachrichten, (2019) 292, 1674–1684. https://doi.org/10.1002/mana.201800073

[7] P. Billinsgley, *Convergence of Probability Measures*, Second Edition, Wiley Series in Probability and Statistics, (1999).

[8] E. Bolthausen, *On the central limit theorem for stationary mixing random fields*, Ann. of. Prob., (1982) Vol. 10, No. 4, 1047-1050.

[9] D. Brandes, I.V. Curator, *On the sample autocovariance of a Lévy driven moving average process when sampled at a renewal sequence*, (2018) arXiv:1804.02254.

[10] W. O. Bray and M. A. Pinsky, *Growth properties of Fourier transforms via moduli of continuity*, J. Func. Anal. 225 (2008) 2265-2285.

[11] P.J. Brockwell,*Recent results in the theory and applications of CARMA processes*, Ann. Inst. Stat. Math. (2014) 66, 647-685.

[12] P.J. Brockwell and R.A. Davis, *Time Series: Theory and Methods*, 2nd edition, Springer (1990).

[13] P.J. Brockwell, V. Ferrazzano, C. Klüppelberg, *High-frequency sampling and kernel estimation for continuous-time moving average processes*, J. Time Ser. Anal. (2013), 34, 385-404.

[14] P.J. Brockwell and J. Hannig, *CARMA(p,q) generalized random processes*, Journal of Statistical Planning and Inference (2010), 140(12), 3613-3618.

[15] P.J. Brockwell. and A. Lindner, *Existence and uniqueness of stationary Lévy-driven CARMA processes*, Stoch. Proc. Appl. (2009) 119, 2660-2681.

[16] P.J. Brockwell and Y. Matsuda, *Continuous auto-regressive moving average random fields on* $\mathbb{R}^n$,J. R. Stat. Soc. B (2016), 79, 833-857.

[17] H. Chhaiba, N. Demni and Z. Mouayn *Analysis of generalized negative binomial distributions attached to hyperbolic Landau levels* , J. Math. Phys. (2016) 57 (7), 072103, 14pp.

[18] S. Cohen and A. Lindner, *A central limit theorem for the sample autocorrelations of a Lévy driven continuous time moving average process*, J. Stat. Plan. Inference (2013) 143, 1295-1306.

[19] R. Cuppens, *Quelques nouveaux résultats en arithmétique des lois de probabilité* (French, with English summary), Les probabilités sur les structures algébriques (Actes Colloq. Internat. CNRS, No. 186, Clermont-Ferrand, 1969), Éditions Centre Nat. Recherche Sci., Paris, 1970, pp. 97-112. Avec commentaire par K. R. Parthasarathy.

[20] R. Cuppens, *Decomposition of Multivariate Probabilities*. Academic Press, New York, (1975).

[21] R. Dalang and T. Humeau, *Lévy processes and Lévy white noise as tempered distributions*, Ann. Probab. (2017), 45, 4389-4418.

[22] R.C. Dalang and T. Humeau, *Random field solutions to linear SPDEs driven by symmetric pure jump Lévy space-time white noise*, (2018) arXiv:1809.09999.

[23] S. Datta, *Some nonasymptotic bounds for* L_1 *density estimation using kernels*, Ann. Stat., (1992), Vol. 20, No.3, 1658-1667.

[24] J. Dedeckker, *A central limit theorem for stationary random fields*, Prob. Th. Rel. Fields, (1998) 110 (3), 397-426.

[25] N. Demni and Z. Mouayn *Analysis of generalized Poisson distributions associated with higher Landau levels*, Infin. Dimens. Anal. Quant. Prob. Relat. Top., (2015) 18 (4), 1550028, 13pp.

[26] P. Doukhan, *Mixing: Properties and Examples*, Lecture Notes in Statistics, Springer (1994).

[27] M. Drapatz, *Limit theorems for the sample mean and sample autocovariances of continuous time moving averages driven by heavy-tailed Lévy noise*, ALEA, Lat. Am. J. Probab. Math. Stat., (2017) 14 , 403-426.

[28] D. Edmunds and H. Triebel, *Function spaces, entropy numbers, differential operators*, Cambridge University Press, (1996).

[29] S. Eyheramendy, F. Elorrieta and W. Palma *An autoregressive model for irregular time series of variable stars*, Proceedings of the International Astronomical Union (2016), 12(S325), 259-262.

[30] J. Fageot, A. Amini, M. Unser, *On the Continuity of the Characteristic Functionals and Sparse Stochastic Modeling*, J. Fourier. Anal. Appl., (2014), vol. 20, no. 6, pp. 1179-1211.

[31] J. Fageot, A. Fallah and M. Unser, *Multidimensional Lévy white noises in weighted Besov space*, Stoch. Proc. Appl. (2017) 127, 1599-1621.

[32] J. Fageot and T. Humeau, *Unified view on Lévy white noise: general integrability conditions and applications to linear SPDE*, (2018) arXiv:1708.02500.

[33] I. Garcia, C. Klüppelberg and G. Müller, *Estimation of stable CARMA models with an application to electricitiy spot prices*, Statistical Modelling (2011) 11, 447-470.

[34] I.M. Gelfand and N.Y. Vilenkin, *Generalized Functions, Vol. 4: Applications of Harmonic Analysis*, Academic Press, New York and London, (1964).

[35] M. Ghahramani, A. Thavaneswaran, *Financial applications of ARMA models with GARCH errors*, The Journal of Risk Finance (2006), Vol. 7 (5), pp.525-543.

[36] V. V. Gorodetskii, *On the Strong Mixing Property for Linear Sequences*, Theory Probab. Appl., 22(2), 411-413.

[37] L. Grafakos, *Classical Fourier Analysis*, Second edition, Springer, (2008).

[38] L. Heinrich, *Asymptotic behaviour of an empirical nearest-neighbour distance function for stationary Poisson cluster processes*, Math. Nachr., (1988) 136 (1), 131-148.

[39] S. Hencl, P. Koskela, *Lectures on Mappings of Finite Distortion*, Springer, (2013).

[40] L. Hörmander, *On the division of distributions by polynomials*, Ark. Mat. (1958) 3, 555-568.

[41] R.M. Höfer and J.J.L. Velázquez, *The Method of Reflections, Homogenization and Screening for Poisson and Stokes Equations in Perforated Domains*, Arch Rational Mech Anal (2018) 227, 1165-1221.

[42] K. Itō, *Foundations of Stochastic Differential Equations in Infinite-Dimensional Spaces*, CBMS–NSF Regional Conference Series in Applied Mathematics (1984), vol. 47, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA.

[43] A. Kaneko, *Introduction to hyperfunctions*, Kluwer Academic Publishers, (1988).

[44] A. Kaneko, *Liouville type theorem for solutions of linear partial differential equations with constant coefficients*, Annales Polonici Mathematici (2000) 74 , 143-159.

[45] D. Kelker *Infinite divisibility and variance mixtures of the normal distribution*, Ann. Math. Statist., (1971) Vol. 42, 2 , 802-808.

[46] J. Klepsch, C. Klüppelberg, T. Wei, *Prediction of functional ARMA processes with an application to traffic data*, Econometrics and Statistics (2017), Vol. 1, 128-149.

[47] C. Klüppelberg and V.S. Pham, *Estimation of causal CARMA random fields*, (2019) arXiv:1902.04962v1.

[48] M.G. Krein, *Integral Equations of a half-line with kernel depending upon the difference of the arguments*, Amer. Math. Soc. Transl., (1962) 22 , 163-288.

[49] E. Liflyand, S. Samko and R. Trigub, *The Wiener algebra of absolutely convergent Fourier integrals: an overview*, Anal. Math. Phys. (2012) 2, 1-68.

[50] E. Lindenstrauss, *Pointwise Theorems for Amenable Groups*, Invent. Math., (2001) 146, 259-295.

[51] A. Lindner, L. Pan, K. Sato, *On quasi-infinitely divisible distribution*, Trans. Amer. Math. Soc., (2018) 370, 8483-8520.

[52] Ju. V. Linnik and I. V. Ostrovskiĭ, *Decomposition of Random Variables and Vectors*,Translations of Mathematical Monographs, Vol. 48, American Mathematical Society, Providence, R. I., (1977).

[53] C. Muscalu and W. Schlag *Classical and Multilinear Harmonic Analysis Volume 1*, Cambridge University Press, (2013).

[54] N. Ortner and P. Wagner, *Fundamental solutions of linear partial differential operators*, Springer, (2015).

[55] V.S. Pham, *Lévy driven causcal CARMA random fields*, (2018) arXiv:1805.08807.

[56] B. S. Rajput and J. Rosinski, *Spectral Representations of Infinitely Divisible Processes*, Probab. Th. Rel. Fields, (1989) 82, 451-487.

[57] M. Reed and B. Simon, *Methods of modern mathematical physics II: Fourier anaylsis, self-adjointness*, Academic Press, New York, (1975).

[58] K. Sato, *Lévy Processes and Infinitely Divisble Distributions*, Cambridge studies in advanced mathematics, (2007).

[59] F. Spangenberg, *Limit theorems for the sample autocovariance of a continous time moving average process with long memory*, (2015) arXiv:1502.04851.

[60] F. W. Steutel, K. Van Harn *Infinite Divisibility of Probability Distributions on the Real Line*, Marcel Dekker, Inc., (2004).

[61] A. Tempelman, *Ergodic Theorems for Group Actions: Informational and Thermodynamical Aspects*, Mathematics and Its Applications, Springer-Science+Business Media (1992).

[62] M. Trabs, *On infinitely divisible distributions with polynomially decaying characteristic functions*, Stat. Prob. Letters, (2014) 94.

[63] H. Triebel, *Theory of Function Spaces*, Modern Birkhäuser Classics, (2000).

[64] H. Triebel, *Theory of Function Spaces III*, Modern Birkhäuser Classics, (2000).

[65] J.B. Walsh, *An introduction to stochastic partial differential equations*, École d'Été de Probabilités de Saint Flour (1986) XIV-1984, pages 265-439. Springer.

[66] H. Zhang, Y. Liu and B. Li *Notes on discrete compound Poisson model with applications to risk theory*, Insurance Math. Econom., (2014) 59, 335-336.

[67] V. Zvaritch and E. Glazkova, *Application of linear AR and ARMA processes for simulation of power equipment diagnostic systems information signals*, 2015 16th International Conference on Computational Problems of Electrical Engineering (CPEE) (2015), Lviv, pp. 259-261.

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Erklärung

Hiermit versichere ich, David Berger, dass ich die vorliegende Arbeit selbständig angefertigt habe und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt sowie die wörtlich oder inhaltlich übernommenen Stellen als solche kenntlich gemacht habe. Ich erkläre außerdem, dass diese Arbeit weder im In- noch im Ausland in dieser oder ähnlicher Form in einem anderen Promotionsverfahren vorgelegt wurde.

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List of preprints

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- D. Berger (2019) *Central limit theorems for moving average random fields with non-random and random sampling on lattices*, 17 pp. Submitted, https://arxiv.org/pdf/1902.01255.pdf.
- D. Berger (2019) *Lévy driven CARMA generalized processes and stochastic partial differential equations*, 17 pp. Submitted, https://arxiv.org/pdf/1904.02928.pdf.
- D. Berger (2019) *Lévy driven linear and semilinear stochastic partial differential equations*, 17 pp. Submitted, https://arxiv.org/pdf/1907.01926.pdf.

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